TR/12/87 November 1987

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by

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Abstract

This paper describes an algorithm for locating stationary points of n-forms. Use is made of the associated n-linear form, the stationary points of which are seen to coincide with those of the n-form. Conditions of convergence are established using the concept of Liapunov stability, and it is seen that the scheme can always be made to converge to the global maximum of the n-form over unit vectors.

1. Introduction

Many applications exist where the problem of finding stationary points of n-forms arises. In particular when a local analysis is made of a function of several variables, study can be made of the terms in a Taylor series expansion which is itself a sum of n-forms with n = 0,1,2,... . The nature of singularities of such functions can be classified according to properties of the appropriate n-form. One example of this in differential geometry is the problem of finding the directions of lines of curvature through an umbilic point on a manifold or surface, see for example [1,2,3]. Another example arises in the context of structural mechanics where the problem is to find the directions of post-buckling equilibrium paths for elastic buckling in coincident modes, see [1,2,3,4,5,6,7,8]. In this application the post-buckling path associated with the global maximum value of an n-form contained in the potential energy function is of particular significance since it identifies a lowest imperfect failure load [1,5,8]. Extensive study has been made in the structural mechanics context by Samuels and Bousfield using trigonometric polynomials [1,2,3,4,7,8] and some of the results used can be seen to have parallels in terms of multilinear forms [1].

The algorithm introduced in this paper is intended to find stationary points of n-forms for unit vectors. In order to do this the algorithm locates stationary points of the associated symmetric n-linear form. This is made possible by a theorem which in essence indicates that a stationary point of a symmetric n-linear form evaluated for unit vectors occurs when the vectors are parallel. (We say parallel rather than equal because the vectors need not have the same sense). Consider for example the symmetric bilinear form T given by

$$T(\underline{x},\underline{y}) = \underline{x}^t A\underline{y}.$$

where A is a symmetric m x m matrix and $\underline{x}$ and $\underline{Y}$ are m-vectors. The associated quadratic form P is then

$$P(\underline{x}) - \underline{x}^t A\underline{x}$$

The theorem states that apart from an exceptional case (discussed subsequently), if T is stationary for unit vectors $\underline{x}^0, \underline{y}^0$ then, $\underline{y}^0 = \pm\underline{x}^0$ and thus ± $\underline{x}^0$ also makes P stationary over unit vectors [9]. As is well known for quadratic forms P will be stationary if $\underline{x}$ is an eigenvector of A . The algorithm attempts to exploit the theorem by optimising T over the larger space of independent variables, unit vectors (x,y), rather than optimising P over the unit vector $\underline{x}$. This clearly has computational disadvantages but is motivated by the fact that T is linear in each vector separately, whereas P is clearly non-linear. The algorithm uses an iterative scheme which guarantees to increase the value of T at each step by replacing one of its argument vectors. This can be seen for bilinear forms using matrix notation. T can be thought of as the dot product T(x,y) = (x^tA). $\underline{y}$. The value of T can be increased by replacing the unit vector $\underline{y}$ with a unit vector in the direction of $\underline{x}^t A$ i.e. $\underline{z} = A\underline{x} /_1 A\underline{x}1$ (recall that A is symmetric so that A = A^t). That is

$$T(\underline{x}, \underline{y}) - (\underline{x}^t A).\underline{y} \leq (\underline{x}^t A).((A\underline{x}) / | A\underline{x} |) - T(\underline{x}, \underline{z}) \quad .$$

On the next iteration $\underline{x}$ is replaced in a similar way and then $\underline{z}$ and so on. The value of T is thus increased until a maximum is reached. Following the algorithm through using the matrix notation reveals that for bilinear forms the method is equivalent to the power method for finding the eigenvector of A associated with its largest eigenvalue [10]. The eigenvalue is equal to the value of T at the stationary point and is thus equal to the global maximum of T over unit vectors. It will be seen that for n-linear forms the algorithm will always converge to the global maximum of T over unit vectors given a sufficiently close starting value.

Since T is a continuously differentiable function over unit vectors, the convergence or otherwise of the scheme can easily be studied by local analysis of the stationary points. The concept of Liapunov stability of a discrete dynamical system is used to investigate the convergence and the n-linear form T will be seen to provide a suitable Liapunov function for this purpose.

In an attempt to simplify the subsequent notation, the following sets are defined

$$Q = \{ \underline{X} \varepsilon R^{m} ; |\underline{x}| = 1\}. \quad (1)$$

That is the set of unit m-vectors where $|\underline{x}|$ indicates the Euclidian norm of $\underline{x}$.

$$S_n = \{x \in R^{nm} ; x = (\underline{x}^1,......,\underline{x}^n) \text{ where } \underline{x}^i \in Qi = 1,......., n\} \quad (2)$$

$$S_1 = \{X \in S_n ; X = (\underline{x},.......\underline{x})\} \quad (3)$$

For each $\underline{x} \in S$ a set $S\underline{x}$ is defined by

$$S_x = \{x-(\underline{x}_1,.......\underline{x}_n) \in S_n ; \underline{x}_i = \pm \underline{x}\, i = 1,......, n\}.$$

S_x is a subset of $\overset{s}{n}$ containing 2^n elements.

An n-linear form T can be thought of as a map $T : R^{nm} \to R$ defined by

$$T(\underline{x}^1,.......\underline{x}^n) = T_{i_1,........i_n}\ x^1_{i_1}, X^2_{i_2} ......x^n_{i_n} \quad (4)$$

where $\underline{x}^i = \{x^i_1......x^i_m\}^t$ and the summation convention is employed over all repeated indices from 1 to m. It is clear that T is linear hi each vector separately.

An n-linear form is symmetric if and only if it is invariant under permutations of its argument vectors. This is equivalent to the coefficients Ti,j…..n being identical for all permutations of their indices (as is the case where coefficients are obtained by evaluating partial derivatives). Symmetry will be assumed throughout the paper.

A useful notation which will be employed is defined as follows. A vector $\underline{t} = \{t_{1......,} t_m\}^t$ can be defined by

$$\underline{t} = T(\underline{x}^1,.. ..,\underline{x}^{j-1},., \underline{x}^{j+1},...,\underline{x}^n) \quad (5)$$

where t_k, the kth component of $\underline{t}$ is

$$t_k = T_{i_1,\ldots\ldots,i_{j-1},k,i_{j+1},\ldots\ldots i_n} X^1_{i1} \ldots\ldots X^{j-1}_{i_{j-1}} X^{j-1}_{i_{j+1}} \ldots\ldots x^n_{i_n} \qquad (6)$$

That is t_k is the multiplier of the k^{th} component of $\underline{x}^j$ in the n-linear form $T(x^1,\ldots,x^n)$. Thus the n-linear form can be thought of as a dot product of any of its argument vectors and the appropriate $\underline{t}$, i.e.

$$T(\underline{x}^1,\ldots\ldots\underline{x}^n) = T(\underline{x}^1,\ldots,\underline{x}^{j-1},.,\underline{x}^{j+1},\ldots,\underline{x}^n).\underline{x}^j \qquad (7)$$

In a similar way a square m x m matrix M can be defined by extracting two argument vectors from T so that

$$T(\underline{x}^1,\ldots\ldots x^n) = \underline{x}^{kt} M \underline{x}^j \qquad (8)$$

where

$$M = T(\underline{x}^1,\ldots\ldots,\underline{x}^{j-1},.,\underline{x}^{j+1},\ldots\ldots\underline{x}^{j+1},\ldots\ldots\ldots\underline{x}^{k-1},.,\underline{x}^{k+1},\ldots\ldots\underline{x}^n) \qquad (9)$$

One further notation which will be used is for a normalised vector of the form of $\underline{t}$ This is

$$\overline{T}(\underline{x}^1,\ldots\ldots\underline{x}^{j-1},.,\underline{x}^{j+1},\ldots\ldots\underline{x}^n) = \frac{t(\underline{x}^1\ldots\ldots\underline{x}^{j-1},.,\underline{x}^{j+1},.,\ \underline{x}^n)}{|T(\underline{x}^1,\ldots\ldots,\underline{x}^{j-1},.,\underline{x}^{j+1},\ldots\ldots x^n)|} \qquad (10)$$

So it is always the case that $\tilde{T} \in Q$ (Note that $\tilde{T}$ is not linear). Since T is symmetric, the dots in the above notation can be located at the position of any argument vector and the order of the vectors themselves rearranged.

2. The Stationary Values of an n-Linear Form

The motivation for the algorithm described below is the following theorem, a proof of which can be found in [9].

Theorem 2.1

Let T be any symmetric n-linear form with n $\geq$ 2.

Let T be stationary over S_n at A = $(\underline{a}^1,\ldots\ldots\underline{a}^n) \in S_n$) with $T^* = T(\underline{a}^1,\ldots\ldots\underline{a}^n)$ and

assume $T^* \neq 0$.

(i) there exists $X \in S_1$ such that T(X) is stationary

and T(X) = T*. (11)

and

(ii) if for any neighbourhood of A there exists $X \in S_n$

but $X \in S_n$ for any $\underline{x}$ such that T(X) = T* then there

exists a pair of orthogonal unit vectors $\underline{V}$ and $\underline{V}^1$ such that

$$T(\underline{V},.......,\underline{V},\underline{V}^1) = \pm\cos\,(i\pi/2)\,|\,T^*\,| \quad i = 0,......., n \qquad (12)$$

In fact v and y' can be found to achieve either choice of sign.

The version of this theorem to be found in [9] guarantees the existence of a stationary point in S_x for some $\underline{x}\,\varepsilon\,Q$. However, it can easily be shown that if $X\ \varepsilon\ S_x$ for some $\underline{x}\,\varepsilon\,Q$ is a stationary point of T then all the other elements of S_x also make T stationary. The above statement of the theorem is thus valid since for any $\underline{x}\,\varepsilon\,Q$ the elements $(\underline{x},.......\underline{x})$ and $(-\underline{x}......-\underline{x})$ are contained in both S_x and S_1.

Part (ii) of the theorem highlights a rather exceptional case where a stationary value for T can be found for $X \in S_X$ for any $\underline{x}$ part (i) indicates however that in every case a value of $X \in S_1$ can be found for which T is stationary over S_n. Note that if the exceptional case holds then putting i=0 and 2 in equation (12) gives

$$T(\underline{v},........\underline{v}) = \pm\,|\,T^*| \qquad (13)$$

and

$$T(\underline{v},...,\underline{v},\underline{v}',\underline{v}') = \mp\,|\,T^*\,| \qquad \mathbf{(14)}$$

So then the symmetric m x m matrix T(v,...,v, . , .) has a pair of equal and opposite eigenvalues ± |T*|. In fact it can be shown using the details of the proof of theorem 2.1 in [9] that in the exceptional case where (12) holds

$$T(\underline{\alpha},\underline{\beta}.........\underline{\beta}) = |\,T^*\,| \qquad (15)$$

for all φ where $\underline{\alpha} = \cos\varphi\underline{v} + \mathrm{Sin}\,\varphi\underline{v1}$ and $\underline{\beta} = \mathrm{Cos}((n-1)\varphi)\underline{v1} - \sin((n-1)\varphi)\underline{v1}$

Since $\underline{\alpha} \to \underline{v}$ and $\underline{\beta} \to \underline{v}$ as $\varphi \to 0$ it is clear that the stationary point at X = $(\underline{v},...,\underline{v})$ is not isolated. Indeed φ parametrizes a whole line of points in S_n at which T is stationary. It may thus be concluded that for any isolated stationary point at $x\ \varepsilon\ S_n$ it must be the case that there exists an isolated stationary point for some $X \in S_1$.

In order to classify the types of stationary point of T the following local analysis is required. Stationary points of T(X) subject to the constraint that $x \in s_n$ are stationary points of

$$F(x)\text{-}T(x)+\lambda_i \underline{x}_i \cdot \underline{x}_i \qquad (16)$$

where λ_i are Lagrange multipliers, $X = (\underline{x}_1 ......\underline{x}_n)$ and a summation is implied over i=l,...,n. Since T is a continuously differentiable function its stationary points will always satisfy

$$\frac{\partial F}{\partial x_i} = \underline{0} \qquad i = 1,......., n. \qquad (17)$$

That is

$$T(\underline{X}_1.......\underline{x}_{i-1},.,\underline{x}_{i+1},...,\underline{x}_n) + 2\lambda_i \underline{x}_i = 0 \qquad i = 1,........n. (18)$$

Taking the dot product of these with $\underline{x}_i$ gives

$$\lambda_1 = ... = \lambda_n = -\frac{1}{2} T(\underline{X}_1.......\underline{X}_n) \qquad . \qquad (19)$$

Denoting $T(x_1,...,\underline{x}_n) = T^*$ as before, equations (18) become

$$T(\underline{x}_1,......\underline{x}_{i-1},.,\underline{x}_i.......\underline{x}_n) - T^* \underline{x}_i \qquad i = 1,....., n \qquad (20)$$

It is clear then that X = ($\underline{x}$,...,$\underline{x}$) 6 S, makes T stationary if and only if

$$T(\underline{x},......\underline{x}) - T^* \underline{x} \qquad (21)$$

It will be assumed that the stationary point of T is isolated and also $X \in S_1$.

Consider now the m x m symmetric matrix

$$\mathcal{T} = T(\underline{x},........\underline{x},.,.)/T^* \qquad (22)$$

It can be seen from (21) that one eigenvector of $\mathcal{T}$ is $\underline{x}$ and is associated with the eigenvalue $t_0 = 1$. Since $\mathcal{T}$ is symmetric its other eigenvalues, $t_1,...,t_{m-1}$ will also be real and then- associated eigenvectors $\underline{x}'_i$ i=l.....m-1 will be orthogonal to $\underline{x}$. A condition for T to have a positive maximum over S_n at X* is given by the following theorem.

<u>Theorem 2.2</u>

If $-1 < t_i < l/(n-l)$ for all i=l,...,m-l then there exists a neighbourhood, N, of X* in S_n such that in any $X \in N$

$$T(X)/T(X^*) < 1 \quad (23)$$

If in addition $T(X^*) > 0$ then

$$T(X) < T(X^*) \quad (24)$$

and T has a positive local maximum at X*.

<u>Proof</u>

Consider an isolated stationary point of T at $X^* = (\underline{x},...,\underline{x}) \;\varepsilon\; S_1$ Then by equation (21)

$$T(x,\ldots,\ \underline{x},.) \;=\; T(X^*)\,\underline{x}\ . \quad (25)$$

Consider a perturbation from X* in S_n denoted by $X = (\underline{x}_1,...,\underline{x}_n)$ where

$$\underline{x}_i - (\underline{x} + \underline{\varepsilon}_i)/|\underline{x} + \underline{\varepsilon}_i| \qquad i = 1,...,n \quad (26)$$

with $\underline{\varepsilon}i\ \varepsilon\ Q$ and $\underline{\varepsilon}_{1.}\ \varepsilon_1 + ...\underline{\varepsilon}_n.\underline{\varepsilon}_n$ small . The resulting perturbation in. T is

$$T(x) = \frac{T(x + \underline{\varepsilon}_1,...,\underline{X} + \underline{\varepsilon}_n)}{|\underline{x} + \underline{\varepsilon}_1|,...,|\underline{x} + \underline{\varepsilon}_n|} \quad (27)$$

Using the linearity of T and the smallness condition on $|\underline{\in}i|$, this can be expanded as

$$T(x) - (T(\underline{x},...,\underline{x}) + T(\underline{x},...,\underline{x},\underline{\varepsilon}_i) + \frac{1}{2}T(\underline{x},...,\underline{x}\underline{\varepsilon}_i,\underline{\varepsilon}_j) + ...) \times$$
$$(i \neq j)$$

$$(1 - \underline{x}\cdot \underline{\in}_i - \tfrac{1}{2})\underline{\in}_i \cdot \underline{\in}_i + \tfrac{3}{2}\,(\underline{x}.\in_i)(\underline{x}.\underline{\in}_i) + \tfrac{1}{2}(\underline{x}.\underline{\in}_i)(\underline{x}.\underline{\in}_j) + ...) \quad (28)$$
$$(i \neq j)$$

where a summation is implied over i and j for l,...,n. This can be simplified by defining the m x m symmetric matrix.

$$X = \underline{x}\cdot\underline{x}^t \quad (29)$$

and using (25) and (22). Then

$$T(X) = T(X^*)(1 + \frac{1}{2}\underline{\varepsilon}_i^t(x - t)\underline{\varepsilon}_i + \frac{1}{2}\underline{\varepsilon}_i^t(t - x)\underline{\varepsilon}_j$$
$$(i \neq j)$$
$$+\ 0(\underline{\varepsilon}_1.\underline{\varepsilon}_i)^{3/2} \quad (30)$$

That is

$$T(x) = T(x^*)(1 + \frac{1}{2}\underline{E}\ A\underline{E}^t + 0(E\cdot E)^{3/2}\ ) \quad (31)$$

where $\underline{E}$ is the inn-vector $\underline{E}\ \{\underline{\varepsilon}_1^t,......,\underline{\varepsilon}_n^t\}^t$ and A is the mn x mn symmetric matrix

$$A = \begin{bmatrix} X-\mathbf{I} & \tau - X & \tau - X & \dots \\ \tau - X & X-\mathbf{I} & \tau - X & \dots \\ \tau - X & \tau - X & . & \\ . & . & & . \\ . & . & & & . \\ . & . & & & \\ & & & & \end{bmatrix} \qquad (32)$$

Let λ be any eigenvalue of A and E the associated eigenvector so that

$$(A - \lambda I)\underline{E} = \underline{0}) \qquad (33)$$

Now this can be written

$$(\tau - x)(\underline{\in_1} + \dots\dots + \underline{\varepsilon_n}) + (2x - r - (\lambda+1)\underline{\in_j} = 0, \quad j = 1,\dots\dots,n) \qquad (34)$$

Adding these gives

$$((n-1)\tau - (n-r)x - (\lambda+1)I)(\underline{\in_1} + \dots\dots + \underline{\in_n}) = 0 \qquad (35)$$

Assuming X is a non-null perturbation in S_n there exists a value of j such that $\underline{\varepsilon_j} \neq$

$k\underline{x}$ for all k $k \in R$ (including k=0). From this condition it is easy to show that a non-null perturbation is only possible in two cases. Firstly if $\underline{\in_1} + \dots + \underline{\in_n} = 0$ then from (34)

$$(2x - \tau - (\lambda+1)I)\underline{\in_j} = \underline{o} \qquad j = 1,\dots\dots\dots n. \qquad (36)$$

Taking the dot product of this with one of the eigenvectors of $\tau, \underline{x_1}'$ which is orthogonal to $\underline{x}$ gives

$$(t_i + 1 + 1)(X^1_j \underline{\in_j}) \qquad (37)$$

But for a non-null perturbation there must exist values for i and j such that $\underline{x}_1{}'.\underline{\in}_j$ $\neq 0$ and so

Secondly if $\underline{\in_1} + \dots + \underline{\in_n} \neq k\underline{x}$ for any $k \in R$ then the dot product of equation (35) with $\underline{x_1'}$ gives

$$((n-1)t_i - \lambda - 1)(\underline{x^1_i}.(\underline{\in_1} + \dots\dots + \underline{\in_n})) \quad -0 \qquad (39)$$

But by the assumption for this case there must exist a value of i such that $\underline{x_1}(\underline{\in_1} + \dots\dots + \underline{\in_n}) \neq 0$ and so

$$\lambda = (n-1)t_i - 1 \text{ for some value of i=1....,m-1} \qquad (40)$$

Now equations (38) and (40) give the only possible values for λ for a non-null perturbation and so it is clear that if $-1 < ti < 1/(n-1)$ for all i=1,...,m-1 then $\lambda < 0$. Since λ is any eigenvalue of A it can be seen from (31) that there exists a neighbourhood, N, of X^* in S_n such that for any $X \in N$

$$T(X)/T(X^*) < 1. \tag{41}$$

If $T(X^*) > 0$ then this gives $T(X)<T(X^*)$ and so T has a positive local maximum at X^*. (Similarly if $T(X^*)<0$ then T has a negative local minimum at X^*). □

It is interesting to note that equations (38) and (40) do not allow the possibility of all the eigenvalues of A being positive. This leads to the conclusion that T can never have negative maxima or positive minima on S_n.

3. The Algorithm

The algorithm investigated here attempts to find stationary values of T by generating a sequence $(\underline{x}_i)$ of $\underline{x}_i \in Q$ using the recursive formula

$$\underline{x}_{i+1} = \widetilde{T}(\underline{x}_i, \underline{x}_{i+1}, \ldots, \underline{x}_{i-(n-2)}). \tag{42}$$

Each new term in the sequence depends on the previous n-1 terms. It should be noted that $\underline{x}_{i+1}$ becomes undefined if

$$T(\underline{x}_i, \ldots, \underline{x}_{i-(n-2)}) = \underline{0} \tag{43}$$

It will be shown however that this will never be the case (for exact arithmetic at least) provided that the (n-1) starting values of the sequence, $\underline{x}_0$, $\underline{x}_1 \ldots, \underline{x}_{2-n}$ do not satisfy (41).

The conditions under which the algorithm converges are now investigated. For the purpose of this analysis it is found easiest to study n steps of the sequence at a time. So a sequence (X_i) of $x_1 \in S_n$ is defined where $X_i = \underline{x}_i, \ldots, \underline{x}_{i-(n-1)}$. Each term of this new sequence is thus generated by a map $\varphi : Sn \rightarrow S_n$ such that $X_{i+1} = \phi(x_i)$ for all i=0,l,... . A map $\psi : Sn \rightarrow S_n$ can then be defined by $\psi \equiv \phi^n$ which moves n terms along the original sequence on each application. So then

$$X_{i+n} = \psi = (X_i) \quad \text{for all } i = 0,1,\ldots\ldots \tag{44}$$

Now the generation of the sequence (X_i) can be thought of as a discrete dynamical system with ϕ or Ψ as the iterating map. In this way convergence of the sequence to a fixed point of the map can be studied by way of the Liapunov stability of the dynamical system. It will be seen that the n-linear form T provides a Liapunov function which leads to an indication of the stability or otherwise of the fixed points. The link between T and the map Ψ is provided by the following theorem.

Theorem 3.1

X is a stationary point of T with $T(X) > 0$ if and only if X is a fixed point of Ψ and $T(X) > 0$.

(It will be seen later that the algorithm is such as to ensure that $T(x_i) > 0$ for $i > 0$ and so the condition of positive T will not be a restriction. In fact it can easily be shown that If $T(X) < 0$ then the theorem can be satisfied with X replaced by Ψ (X).)

Proof

First the sufficiency condition is proved. Assume that $X = x_n \ldots x_1 \in S_n$ is a stationary point of T with $T(X) > 0$. Then X must satisfy equation (20) and so

$$T(\underline{x}_n,\ldots,\underline{x}_{i+1},\underline{x}_{i+1},\ldots\underline{x}_1)T^*\underline{x}_i \qquad i = 1,\ldots,n \qquad (45)$$

where $T^* = T(\underline{x}_n,\ldots,\underline{x}_{i+1},\ldots\underline{x}_1) > 0$. Now since $|\underline{x}_i| = 1$ this equation can be written

$$\tilde{T}(\underline{x}_n,\ldots,\underline{x}_{i+1},\underline{x}_{i+1},\ldots\underline{x}_1)\underline{x}_i \qquad i = 1,\ldots\ldots,n \qquad (46)$$

Now let $Y = \Psi(X)$ where $Y = y_n,\ldots..y_1$ then by the definition of Ψ

$$\underline{y}_1 = \tilde{T}(\underline{y}_{i-1}, \underline{y}_1, X_n,\ldots,\underline{X}_{i+1},\ldots,.) \qquad i = 1,\ldots,n \qquad (47)$$

In order to prove that X=Y the proposition P_j will be proved by complete induction for j=1,...,n where

$$p : \underline{y}_j = \underline{x}_j \qquad (48)$$

Put i=l in equation (47) to give

$$\underline{y} = \tilde{T}(\underline{x}_n,\ldots,\underline{x}_2,.)$$

then by equation (46) with i=l

$$\underline{y}_1 = \underline{x}_1$$

which proves P_1 true. Now assume P_k true for all k=1,,2,...,j=1 for some j=1,...,n. That is

$$\left.\begin{array}{l} \underline{y}_1 - \underline{x}_1 \\ \underline{y}_2 - \underline{x}_2 \\ \\ \underline{y}_{j-1} - \underline{x}_{j-1} \end{array}\right\} \qquad (49)$$

Now put i=j in equation (47) and use (49) to substitute for $\underline{v}_i$ to get

$$\begin{aligned} \underline{v}_i &= \widetilde{T}(\underline{v}_{i-1}, \ldots, \underline{v}_1, \underline{x}_n, \ldots, \underline{x}_{j+1}, \cdot) \\ &= \widetilde{T}(\underline{x}_{j-1}, \ldots, \underline{x}_1, \underline{x}_n, \ldots, \underline{x}_{j+1}, \cdot) \end{aligned} \qquad (50)$$

Using the symmetry of T and equation (46) with i=j, it can be seen that

$$\underline{v}_j = \underline{x}_j. \qquad (51)$$

So if P_k is true for k=l,...,j-l then P_j is also true. Thus by complete induction P_j: is true for all j=l,...,n .

Now the necessary condition is proved. Consider a term hi the sequence (X_i). $X_i = (\underline{x}i, \underline{x}_{i-1}......, \underline{x}i(n-1))$. Assume that x_i is a fixex point of Ψ. Then $\Psi(x_i) = X_i$ so $X_{i+n} = X_i$, or

$$\underline{x}_{i+j+1} = \underline{x}_{i+j-(n-1)} \text{ for all } j = 0, ...n-1 \qquad (52)$$

By the recursion formula (42)

$$\underline{x}_{i+j+1} - \overline{T}(\underline{x}_{i+j}, \underline{x}_{i+j-1}, \ldots, \underline{x}_{i+j-(n-2)}, \cdot) \qquad j = 1, ...n \quad (53)$$

Now using (52) to replace the first j argument vectors hi the right hand side this becomes

$$X_{i+j+1} - \widetilde{T}(x_{i+j-1}, \underline{x}_{i+j-(n+1)}, \ldots, \underline{x}_{i-(n-1)}, \underline{x}_i, \underline{x}_{i-1}, ...\underline{x}_{i+j-(n-2)}, \cdot)$$

$$j \text{ terms} \qquad j = 1, \ldots, n \qquad (54)$$

Using symmetry the argument vectors can be reordered and replacing the left hand side using (52) gives

$$\widetilde{T}(x_i, x_{i-1}, ...x_{i+j-(n-2)}, \cdot, \underline{x}_{i-(n-1)}) - \underline{x}_{i+j-(n-1)}$$

$$j = 1.......,n \qquad (55)$$

Comparison with equation (20) shows that $X_i = (\underline{x}_i, \ldots, \underline{x}_{i-(n-1)}$ makes T stationary provided $T(X_i) > 0$. Thus theorem 3.1 is proved. □

Theorem 3.1 indicates that if the algorithm converges then it will have found a stationary point of T . Some results are now stated and proved which are required for the construction of the Liapunov function.

Define a sequence $(T_i), T_i \in R$, by $T_i = T(X_i)$ for $X_i \in S_n$ in the sequence (X_i) defined above. So

$$T_i = T(\underline{x}_i, \underline{x}_{i-1}, \ldots, \underline{x}_{i-(n-2)}) \qquad i = 0,1,\ldots \qquad (56)$$

Consider

$$T_{i+1} = T(\underline{x}_{i+1}, \underline{x}_i, \ldots, \underline{x}_{i-(n-2)})$$
$$= T(\underline{x}_i, \ldots, \underline{x}_{i-(n-2)}, \cdot) \cdot \underline{x}_{i+1} \qquad i = 0,1,\ldots \quad . \qquad (57)$$

Using the recursion formula (42) to replace $\underline{x}_{i+1}$, gives

$$T_{i+1} = \left| T(\underline{x}_i, \ldots, \underline{x}_{i-(n-2)}, \cdot) \right| \qquad i = 0,1,\ldots \qquad (58)$$

Returning to equation (56)

$$T_i = T(\underline{x}_i, \ldots, \underline{x}_{i-(n-2)}, \cdot) \quad \underline{x}_{i-(n-1)} \qquad i = 0,1,\ldots \qquad (59)$$

So if $T_{i+1} = 0$ then $T_i = 0$, otherwise

$$T_i = T_{i+1} \ \tilde{T}(\underline{x}_i, \ldots, \underline{x}_{i-(n-2)}, \cdot) \cdot \underline{x}_{i-(n-1)} \qquad (60)$$
$$= T_{i+1} \ (\underline{x}_{i+1} \underline{x}_{i-(n-1)})$$
$$\leq T_{i+1} \qquad i = 0, 1, \ldots \qquad (61)$$

So provided the starting values are chosen so that T_1 # 0 it must be the case that

$$0 < T_i < T_{i+1} \qquad \text{for all } i = 1,2,\ldots \quad . \qquad (62)$$

This then gives

$$0 < T_i < T_{i+n} \qquad \text{for all } i = 1,2,\ldots \qquad (63)$$

It will now be shown that

$$T_{i+n} = T_i \leftrightarrow X_i \text{ is a fixed point of } \Psi \quad . \qquad (64)$$

Assume $T_{i+n} = T_i$ then by result (62) it must be the case that

$$T_{i+j+1}, = T_{i+j} \quad j=0,\ldots,n-1. \qquad (65)$$

Using this and equation (60) with i replaced by i+j leads to the conclusion that

$$\underline{x}_{i+j-(n-1)} = \tilde{T}(\underline{x}_{i+j}, \ldots, \underline{x}_{i+j-(n-2)}, \cdot) \qquad j = 0,\ldots,n-1 \ .(66)$$

But from definition (42) with i replaced by i+j

$$\underline{x}_{i+j-(n-1)} = \underline{x}_{i+j+1} \qquad j = 0,\ldots,n-1 \qquad (67)$$

or

$$X_i = X_{i+n} = \Psi(X_i) \qquad (68)$$

so X_i is a fixed point of Ψ.

Now assume that X_i is a fixed point of Ψ so $X_{i+n} = X_i$ then $T(X_{i+n}) = T(X_i)$ and

$T_i = T_{i+n}$. Thus the result is proved. □

4. Conditions for Convergence

First, some definitions and Liapunov's theorems are quoted which are to be found in [11.].

Consider a region D of S_n containing the point X*. A function V: $S_n \rightarrow R$ is said to be positive definite at X* in D if and only if

$$V(X) > 0 \;\; \forall X \,\varepsilon\, D \text{ and } V(X) = 0 \leftrightarrow X = X^* . \qquad (69)$$

Consider an iterative map $F : S_n \rightarrow S_n$ which generates sequences of the form (X_i) from $X_{i+1} = F(X_i)$.

Liapunov's theorem I

Suppose that there exists a function V(X) such that both V and $-\Delta V_i$ are positive definite at X* in D. Then X* is an asymptotically stable fixed point of F in the sense of Liapunov stability. (Here Δ is the forward difference operator acting on $V_k = V(X_k)$).

Liapunov's theorem II

Suppose that there exists a function V(X) such that $-\Delta V_k$ is positive definite at X* in D but that in every neighbourhood of X* there is at least one point at which V is negative. Then X* is an unstable fixed point of F .

Using the fact that T is bounded and results (63) and (64) the Monotonic Sequence Property gives that the sequence (T_{in}) must have a limit point which will be a fixed point of Ψ . Moreover it must be that T>0 at such a point. Then from the forom 3 there must exist a stationary point of T. Liapunov's theorems can now be used to investigate the stability of such a fixed point.

Consider an isolated fixed point of Ψ at X*. From result (63) it must be that T(X*) > 0, (provided that degenerate starting values are not chosen). Since X* is isolated there must exist a region D of S_n containing X* and no other fixed points of Ψ . The required Liapunov function is

$$V(X) = T(X^*) - T(X) \qquad (70)$$

Rather than consider the sequence (X_i) for the discrete dynamical system it is better to consider the sub-sequence (X^k) where $X^k = X_{kn}$. i.e. every n^{th} term of (X_i). The reason for this choice is that an appeal can then be made to result (64). It should be noted that it is not the case that $T_i = T_{i+1} \Rightarrow X_i$ is a fixed point of Φ (or Ψ). Indeed the system could converge to iterative cycles of period n or less. It is easy to show that the attracting cycles of Φ consist of elements of S_x for which the argument vectors have a common cyclic permutation and for which T>0. For this reason the period of any such cycle must be a factor of n . Now by definition $X^{k+1} = \Psi(X^k)$. The sequence (V_k) defined by $V_k = V(X^k)$ has the following property.

$$\begin{aligned} -\Delta V_k &= V_k - V_{k+1} \\ &= T(X^{k+1}) - T(X^k) \\ &= T_{nk+n} - T_{nk} \end{aligned} \tag{71}$$

Then by results (63) and (64) it must be that $-\Delta V_k$ is positive definite at X* in D. So Liapunov's theorem I gives that the fixed point of Ψ at X* will be asymptotically stable if V is positive definite at X* in D . But by theorem 2 this is guaranteed if -1 < t_i < l/(n-l) for i=l,...,m-1. Also, if for some i , t_i < -1 or l/(n-l) < t_i then Liapunov's theorem II gives that the fixed point at X* will be unstable. This is summarised in the following two results:

$$\begin{matrix} -1 < t_i < 1/(n-1) \\ i = 1, \ldots, m-1 \end{matrix} \quad \Rightarrow \quad \begin{matrix} X^* \text{ is an asymptotically} \\ \text{stable fixed point of } \Psi \end{matrix} \tag{72}$$

$$\begin{matrix} \text{There exists } i = 1, \ldots, m-1 \text{ such} \\ \text{that } t_i < -1 \text{ or } t_i > 1/(n-1) \end{matrix} \quad \Rightarrow \quad \begin{matrix} X^* \text{ is an unstable fixed} \\ \text{point of } \Psi \end{matrix} \tag{73}$$

(Recall that t_i are m-1 of the m eigenvalues of $\mathcal{T}$ evaluated at X* (see (22))).

It is now shown how the above convergence condition is related to the index of the stationary point when considered over S_1. That is the index of the stationary point of the associated n-form $P(\underline{X}) = T(\underline{x},\ldots,\underline{x})$. If the perturbation of T in the proof of theorem 2 is restricted to an S_1, perturbation then equation (30) becomes

$$T(x) = T(X^*)\,(1 + (n/2)\underline{\in}^t\,((n-1)\tau - (n-2)X - I)\underline{\in} + \ldots)\ . \tag{74}$$

Now the eigenvalues σ_i say, of $(n-1)\tau-(n-2)X-I$ are given by

$$\left.\begin{aligned} \sigma_0 &= (n-1)t_0 - (n-2) - 1 = 0 \\ \sigma_0 &= (n-1)t_i - 1 \qquad i=1,\ldots,m-1 \end{aligned}\right\} \qquad (75)$$

As before, one of the eigenvalues, σ_0, is associated with $\underline{\in} = k\underline{x}$ which gives rise to a null perturbation on S_1 and so the eigenvalues of interest are $\sigma_i \;\; i=1,\ldots,m-1$. The condition for T , and hence P to have a local maximum on S_1, at $\underline{x}$ is thus t < l/(n-l) for all i=l,...,m-l. As one would expect this encompases the condition of theorem 2 for T to have a local maximum on ($-1 < t_i < 1/(n-1)$ for all i=1,…,m-1) since $S_1 \subset S_n$. So a maxhnum on S_n must be a maximum on S_1.

It can be concluded from result (72) and the above that the mapping Ψ will always converge to a stationary point of P(x) which is a local maximum. However, not all of the maxima of P($\underline{x}$) are also maxima of T(X) over S_n and those which are not will be unstable fixed points of Ψ . It is also important to note that the global maximum of P($\underline{x}$) over S_1, will always be an attracting stable fixed point of * provided that the point is an isolated local maximum. This will always be true unless the non-generic case of part (ii) of theorem 2.1 holds. For bilinear forms (n=2) where $T(\underline{x}_1,\underline{x}_2) = \underline{x}_2^t A \underline{x}_2$, this exceptional case highlights the well known fact that the power method breaks down when the matrix A has a pair of equal and opposite eigenvalues of maximum magnitude.

A point of interest to note here is that the set t_i i=l,...,m-l is identical to the parameters α_i i=l,...,m-l defined in [1] and [4]. The parameters α_i are used in the structural mechanics context to classify types of buckling behaviour.

5. Conclusions

It has been shown that the algorithm described always has an attracting fixed point at the stationary point associated with the global maximum of the n-form when evaluated for unit vectors. So a starting value can always be found which lies within the basin of attraction for this point such that the algorithm will converge to this solution. It is noted that for n>3 there may be other attracting fixed points which is the reason for the above restriction on the starting value. Attracting fixed points correspond to local maxima of the associated symmetric n-linear form and hence local maxima of the n-form. However, not all maxima of the n-form will correspond to maxima of the n-linear form and thus correspond to attractors. This highlights a possible disadvantage of optimising over the larger space, S_n, rather than S_1 which is that maxima in S_1 can be saddles in S_n.

It is also worth noting that the algorithm is not suited to finding all of the stationary points of an n-form which is sometimes desirable. This has been achieved for the case n=m=3 by Bousfield in [1] using a semi-analytical method based on resultants. This method is however, computationally very clumsy and not easily extended to higher dimensions. It is hoped that the algorithm can be developed for this purpose using the results presented here as a starting point. It may be possible to do this in the same way that the inverse iteration method is developed from the power method for the bilinear case, n=2, see for example [10].

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