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Abstract simplicial complexes and group presentations

by

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Introduction

Given a graph Γ with vertex set $\underline{\underline{x}}$ say, a group presentation can be obtained from Γ as follows. Let ϕ be a function which assigns to each edge e of Γ, e having endpoints x ,y say, a non-empty set of cyclically reduced words on x, y involving both x and y. Then $G(\Gamma,\phi)$ is the group with generating set $\underline{\underline{x}}$ and defining relators $\bigcup\phi$ (e) where the union is taken over all the edges e of Γ. Thus the presentation varies according to the function ϕ , although observe that each defining relation in the presentation for $G(\Gamma,\phi)$ will always involve exactly 2 generators. Groups having presentations of this form (e.g. Artin and Higman groups) have been studied in recent work of A .K .Napthine and S.J.Pride [4.] and of S. J. Pride [5], [6,] .

In this paper we replace the graph Γ by an abstract n-dimensional simplicial complex C_n($n \geq 2$) to obtain the groups $G(c_n,\phi_n)$.Each defining relator in the presentations now obtained will involve precisely n-generators. Our aim is to generalise to these groups a Freiheitssatz for $G(\Gamma,\phi)$ due to Pride which we now describe.

For e an edge of Γ, the group G(e) given by the presentation <x,y; ϕ(e)>, where x ,y are the endpoints of e, is called an edge group of $G(\Gamma,\phi)$.

A 2-generator group with generators a, b say, has property - W_k, (with respect to a, b) if no word of the form $a^{\alpha_1}b^{\beta_1}...a^{\alpha_k}b^{\beta_k}(\alpha_i,\beta_i \in \mathbb{Z})$ is equal to 1 in the group unless the word is freely equal to 1.

THEOREM (Pride [6, Theorem 4]) The natural homomorphlsm $G(\Gamma_o,\phi)\to G(\Gamma,\phi)$ is injective for each full subgraph Γ_0 of Γ if one of the following conditions is satisfied.

(1) Each edge group of $G(\Gamma,\phi)$ has property – W_2 .
(2) Γ has no triangles and each edge group of $G(\Gamma,\phi)$ has property – W_1 .

Our main results are stated in §1 and the proofs are given in §3 and §4. In §2 we describe a modification at interior vertices of certain small cancellation diagrams and in §4 a geometric result (Lemma 4) is proven for small cancellation diagrams whose almost interior regions each have degree at least 3 and whose interior vertices each have degree at least 6. In §5 we give examples and discuss consequences of the theorems.

We will assume that the reader is familiar with the basic definitions and results of small cancellation theory [3, pp.235-252], frequent use of which is made throughout this paper. (It should be noted however that there are differences in our definitions to some of those given in [3].)

For the rest of this paper we drop the term abstract and merely write simplicial complex without any fear of confusion.

§1 Statement of results

Let X be a set and let there be a collection of subsets of X the maximum number of elements of X contained in any of these subsets being n where $n \geq 2$. Let c_n denote the full (n-dimensional) simplicial complex generated by these subsets. Thus C_n consists of the sets together with all their non-empty subsets. The ℓ- element sets ($n \geq \ell \geq 1$) are called ℓ-simplices.

Let ϕ_n be a function which assigns to each n-simplex $\underline{\underline{x}} = \{x_1,....,x_n\}$ in C_n a non-empty set of cyclically reduced words each involving $\{x_1,....,x_n\}$ and no other element of X. We define $G(C_n,\phi_n)$ to be the group with generating set those elements of X appearing in some ℓ-simplex $\underline{\underline{x}}$ of C_n and defining relators

$$\{\phi_n(\underline{\underline{x}}) : \underline{\underline{x}} \text{ an n-simplex of } C_n\}.$$

For the n-simplex $\underline{\underline{x}}$ of C_n we define $G(\underline{\underline{x}})$ to be the group with presentation $<\underline{\underline{x}};\phi_n(\underline{\underline{x}})>$ and call such a group a face group of $G(C_n,\phi_n)$.

Let $X_0 \subset \underline{X}$ and let C_{n_0} be the full sub complex generated by all the ℓ-simplicies ($n > \ell > 1$) of C_n that are contained in X_0. Then there is a natural homomorphism

$$G(C_{n_0},\phi_n) \to G(C_n,\phi_n).$$

If this homomorphism is injective for any choice of X_0 then we shall say that the Freiheitssatz (see §1 in [6]) holds for $G(C_n,\phi_n)$

Let $\underline{\underline{x}}$ be an n-simplex whose members are in X and let $\underline{\underline{x}}_1,.....\underline{\underline{x}}_k$ ($k > 1$) be (n-1)-simplices contained in $\underline{\underline{x}}$. We shall say that the group having presentation $<\underline{\underline{x}}_n : \phi_n(\underline{\underline{x}})>$ has property B_k (with respect to $\underline{\underline{x}}$) provided that there is no word of the form $w_1(\underline{\underline{x}}_1)...w_k(\underline{\underline{x}}_k)$ equal to 1 in the group unless it is freely equal to 1. Here $w_i(\underline{\underline{x}}_i)$ denotes a word involving some subset of $\underline{\underline{x}}_i$ and no other members of X.

Let $x \in X$. We define the map $d_x : c_n \to c_n$ as follows: if $\underline{\underline{y}}$ is an ℓ-simplex $(n \geq \ell \geq 1)$ of C_n then $d_x(y) = \begin{cases} \underline{\underline{y}} - \{x\} & \text{if } x \in \underline{\underline{y}} \\ \underline{\underline{y}} & \text{if } x \notin \underline{\underline{y}}. \end{cases}$

We shall say that c_n has property N(p) $(p \geq 3)$ provided that there cannot be found p n- simplices $\underline{\underline{X}}_1, \ldots, X_p$ of c_n together with a sequence of maps of the form d_x such that the image set of $\{\underline{\underline{X}}_1, \ldots, X_p\}$ is $\{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{p-1}, x_p\}\{x_p, x_1\}\}$ where $x_i \neq x_j$ for $1 \leq i \leq j \leq p$.

<u>THEOREM 1.</u> <u>The Freiheitssatz holds for $G(c_n, \phi_n)$ if one of the following conditions is satisfied.</u>

(1.1) <u>Each face group has property B_5.</u>

(1.2) <u>Each face group has property B_3 and C_n has property N(3).</u>

We remark that if n=2 then properties B_5, B_3 reduce to the properties W_2, W_1 respectively which are given in the introduction; moreover, the geometrical realisation of C_2 has no triangles if and only if C_2 has property N (3). Consequently Theorem 1 is a generalisation of Theorem 4 of [6].

The condition (1.1) corresponds to when use is made of the small cancellation hypothesis C (6); and the condition (1.2) corresponds to the C(4), T(4) situation. The national question to ask is what can be said about the C(3), T(6) case? Here the defining relators can have shorter length and it has been necessary to introduce a further restriction.

For a given work $w(\underline{\underline{x}})$ in some. ℓ-simplex $\underline{\underline{x}}$ of we c_n shall write $g(w(x))$ to denote those elements of x inavolved in $w(\underline{\underline{x}})$.

THEOREM 2. The Freihestssatz holds for $G(C_n, \phi_n)$ if the following conditions are satisfied.

(1.3) (i) Each face group $G(\underline{\underline{x}})$ has property B_2 and C_n has properties $N(3)$, $N(4)$ and $N(5)$;

(1.3) (ii) For any pair of distinct n-simplices $\underline{\underline{x}}$, $\underline{\underline{y}}$ of C_n whenever there are words of the form $w_1(\underline{\underline{x}}_1)w_2(\underline{\underline{x}}_2)w_3(\underline{\underline{x}}_3), w_1(\underline{\underline{y}}_1)u_2(\underline{\underline{y}}_2)u_3(\underline{\underline{y}}_3)$ equal (but not freely equal) to 1 in the face groups $G(\underline{\underline{x}})$, $G(\underline{\underline{y}})$ respectively, where $\underline{\underline{x}}_j, \underline{\underline{y}}_j$, are (n-1)-simplices of $\underline{\underline{x}}, \underline{\underline{y}}$ respectively and $\underline{\underline{x}}_1, \underline{\underline{y}}_1$, then the following holds: $g(w_3(\underline{\underline{x}}_3)) \cup g(u_3(\underline{\underline{v}}_3)) \supseteq \underline{\underline{x}}$ or $\underline{\underline{y}}$.

We remark that conditions (1.1), (1.2) and (1.3)(i) reduce the question of whether or not the Freiheitssatz holds for the group $G(C_n, \phi_n)$ to an analysis of the face groups. The condition (1.3)(ii) is concerned with how pairs of face groups combine; we shall see how this condition arises from the geometry in §4.

§2 Preliminaries

Let F be a free group with free basis $\underset{\sim}{u}$ and let $\underline{\underline{s}}$ be a symmetrised set of non-empty words on $\underset{\sim}{u}$. Let M be a connected, simply connected $\underline{\underline{s}}$-diagram. If Δ is a region of $\underline{M}$ then $g(\Delta)$ denotes the set of elements of $\underset{\sim}{u}$ which occur in a label of Δ. If $\underline{L}$ is a subdiagram of $\underline{M}$ then $g(\underline{L})$ - $\bigcup g(\Delta)$ where Δ ranges over the regions of $\underline{L}$. Also, we denote by $g(\partial\underline{L})$ the set of elements of $\underset{\sim}{u}$ which occur as labels of boundary edges of $\underline{L}$.

In general $\underline{\underline{s}}$-diagrams may have vertices of degree 1. It can and will be assumed here (apart from exceptions where indicated) that our $\underline{\underline{s}}$-diagrams have no such vertices.

Let $\tilde{x}$ denote the underlying set of the simplicial complex C_n. Let $\underline{\underline{r}}$ denote the smallest symmetrised subset in the free group with basis $\tilde{X}$ containing

$$U\ \phi_n(\underline{\underline{x}})$$

where the union is taken over all the n-simplices $\underline{\underline{x}}$ of C_n.

Let $\hat{\underline{\underline{r}}}$ denote the set

$$U\ \widehat{\phi_n(\underline{\underline{x}})}$$

where $\widehat{\phi_n(\underline{\underline{x}})}$ consists of all words not freely equal to 1 which are in the normal closure of $\phi_n(\underline{\underline{x}})$ in the free group whose basis is the underlying set of $\underline{\underline{x}}$ and where again the union is taken over all the n-simplices of C_n.

In this paper we shall not, unlike the usual definition, demand that each member of a symmetrised set be freely reduced. Thus observe that we have that $\hat{\underline{\underline{r}}}$ is a symmetrised set.

Let $\hat{M}$ be a connected, simply connected $\hat{\underline{\underline{r}}}$-diagram. We are interested in modifying such diagrams. We begin by making the following definition.

Let v be an interior vertex of $\hat{M}$ of degree $m \geq 3$ (as shown in Fig.2.1) and suppose that the $\hat{\Delta}_j$ $(1 \leq j \leq m)$ are distinct simply connected regions of $\hat{M}$; and that each vertex on the line segments $\overline{vw_j}$ have degree 2 apart from v and w_j $(1 \leq j \leq m)$ Then v is a c_m-vertex if for at least one $j \in \{1,\ldots,m\}$ there exists some $i \in \{1,\ldots,j-2,j+1,\ldots,m\}$ such that the label on each edge which occurs on the line segment $\overline{vw_j}$ belongs to the set $g(\hat{\Delta}_i)$.

Figure 2.1

We shall be interested in removing c_m vertices. Our modification can be described as follows. Cut along the line segment $\overline{vw_j}$ .to obtain a new diagram $\hat{M}$ * (see Fig. 2. 2).

Figure 2.2.

Observe that the labels on each region remain unchanged apart from the boundary label of $\hat{\Delta}_i$ which is now a conjugate by a word in $g(\hat{\Delta}_i)$ of some cyclic permutation of the original label. Moreover $\hat{M}$ * will also be a connected, simply connected diagram. Thus $\hat{M}$ * is an $\hat{\underline{\underline{r}}}$-diagram with the same number of regions as $\hat{M}$ * but with fewer c_m -vertices (provided that w is not then a c_m-vertex).

Lemma 1. Let $\hat{M}$ be a connected, simply connected $\hat{\underline{\underline{r}}}$ -diagram such that for each region $\hat{\Delta}$ of $\hat{M}$, $\hat{\Delta}$ is simply connected and $g(\hat{\Delta})$ is an n-simplex of c_n. Then the following hold: (i) if has property N(3) then every interior vertex $\hat{M}$ of of degree 3 is a C_3-vertex; (ii) if c_n has properties N(3) and N(4) then every interior vertex of $\hat{M}$ of degree 4 is a C_4-vertex; (iii) if C_n has properties N(3), N(4) and N(5) then every interior vertex of $\hat{M}$ of degree 5 is a C_5 -vertex.

Proof We give the proof for (iii) only; parts (i) and (ii) can be proved similarly.

Let v be an interior vertex of $\hat{M}$ of degree 5. Then v can be illustrated as in Fig. 2.3 where each vertex on the line segment $\overline{vw_j}$. has degree 2 apart from v and w. ($1 \leq i \leq 5$).

Figure 2.3

Observe that it follows from the assumption no vertices of degree 1 and the statement of the lemma that the $\hat{\Delta}_i$ are distinct, simply connected regions. (Note however that the vertices w_i. may not be distinct - this makes no difference to our arguments.)

Suppose, by way of contradiction, that the vertex v in Fig. 2.3 is not a C_5--vertex. Then there must be $a_i, b_i, c_i \in g(\overline{vwi})$ such that

$a_i \notin g(\Delta_{i+1})$, $b_i \notin g(\hat{\Delta}_{i+2}), c_i \notin g(\hat{\Delta}_{i+3})$ for $1 \le i \le 5$. (Throughout the proof subscripts shall be reduced mod 5 and we take 5 as 0.) Thus we have

$$g(\hat{\Delta}_i) = \{a_i, b_i, c_i, a_{i+1}, b_{i+1}, c_{i+1}, \ldots\} \mid \ni c_{i+2}, b_{i+3}, a_{i+4} \; (1 \le i \le 5)$$

If $a_{i+1} \in g(\hat{\Delta}_{i+3})$ then there is a sequence of d_X maps with $a(\hat{\Delta}_{i+3}), a(\hat{\Delta}_{i+1}), a(\hat{\Delta}_{i+2})$ having images $\{c_{i+3}, a_{i+1}\}, \{a_{i+1}, a_{i+2}\}$ $\{a_{i+2}, c_{i+3}\}$ respectively, contradicting the fact that c_n has property N(3). This forces

$$g(\hat{\Delta}_i) = \{a_i, b_i, c_i, c_{i+1}, b_{i+1} c_{i+1}, \ldots\} \not\ni c_{i+2}, b_{i+3}, a_{i+4}, a_{i+3} \; (1 \le i \le 5).$$

Now suppose that, for some i, $a_{i+1} \in g(\hat{\Delta}_{i+4})$. If we also have $a_{i+4} \in g(\hat{\Delta}_{i+2})$ then there is a sequence of d_x maps with $g(\hat{\Delta}_{i+4})$, $g(\hat{\Delta}_{i+1})$, $g(\hat{\Delta}_{i+2})$ having images $\{a_{i+4}, a_{i+1}\}, \{a_{i+1}, a_{i+2}\}, \{a_{i+2}, a_{i+4}\}$ respectively; this contradiction means that $a_{i+4} \notin g(\hat{\Delta}_{i+2})$. Suppose now that we also have $a_{i+3} \in g(\hat{\Delta}_{i+1})$ Then there is a sequence of d_x maps with $g(\hat{\Delta}_{i+4})$, $g(\hat{\Delta}_{i+1})$, $g(\hat{\Delta}_{i+3})$ having images $\{a_{i+4}, a_{i+1}\}, \{a_{i+1}, a_{i+3}\}, \{a_{i+3}, a_{i+4}\}$ respectively; this contradiction means that $a_{i+3} \notin g(\hat{\Delta}_{i+1})$ We conclude from all this that there is a sequence of d_x maps with $g(\hat{\Delta}_{i+4})$, $g(\hat{\Delta}_{i+1})$, $g(\Delta_{i+2}), g(\Delta_{i+3})$ having images $\{a_{i+4}, a_{i+1}\}, \{a_{i+1}, a_{i+2}\}, \{a_{i+2}, a_{i+3}\}$, $\{a_{i+3}, a_{i+4}\}$ respectively, contradicting the fact that C_n has property N(4).

Thus $a_{i+1} \notin g(\hat{\Delta} 1+4)$ consequently $g(\hat{\Delta}_i) \neq \{a_i, a_{i+1}, \ldots\} \not\ni$ $a_{i+2}, a_{i+3}, a_{i+4}$ for $1 \leq i \leq 5$ But this contradicts the assumption that $\mathcal{C}_n$ has property N(5) .

□

§3 Proof of Theorem 1

We require some further comments on $\underline{\underline{s}}$-diagrams. Recall that $\underline{\underline{s}}$ is a symmetrised set of non-empty words on the free basis $\underset{\sim}{u}$ of the group F.

A region Δ of an $\underline{\underline{s}}$-diagram $\underline{M}$ will be called a boundary region if $\partial\Delta \cap \partial\underline{M}$ contains at least one edge; Δ will be called almost interior otherwise. A boundary region Δ for which $\partial\Delta \cap \partial\underline{M}$ is a consecutive part [3,p.248] of $\underline{M}$ will be called a simple boundary region.

If the word W lies in the normal closure of $\underline{\underline{s}}$ in F then W is equal in F to a product $\prod_{i=1}^{r} T_i^{-1} s_i T_i$ where $r \leq 0, s_i \in \underline{\underline{s}}$., and T_i is a word on $\underset{\sim}{u}$ $(1 \leq i \leq r)$. The least value of r over all such expressions equal in F to W is denoted by deg (W).

A connected, simply connected $\underline{\underline{s}}$-diagram with boundary label W is said to be minimal if it has deg (W) regions.

Proof of Theorem 1. Let $x_0 \subseteq x$ and let $\mathcal{C}_{n_0}$ be the full subcomplex generated by all the ℓ-simplices $(n \geq \ell \geq 1)$ of $\mathcal{C}_n$ which are contained in X_0 , Let $\tilde{x}_0$ denote the underlying set of $\mathcal{C}_{n_0}$ and let W be a word on $\tilde{x}_0$ equal to 1 in $G(\mathcal{C}_n, \phi_n)$. We must show that W equals 1 in $G(\mathcal{C}_n, \phi_n)$.

The proof is by induction on deg(W) . If deg(W) = O then W is freely equal to 1 and the result follows. So assume deg(W) > 0. Let $\overline{W}$ be a cyclically reduceed word freely conjugate to W. Then there is a connected, simply connected minimal $\underline{\underline{r}}$-diagram *M* with boundary label $\overline{W}$ [3, pp. 237-238] . Let us assume that *M* has a boundary region Δ with $g(\Delta) \subseteq g(\partial M)$. Let *M*' be obtained from *M* by removing the interior of *M* and one edge of $\partial\Delta \cap \partial M$ (note that this may create vertices of degree 1), and let W' be a boundary label of *M*'(reading $\partial M'$ in the same orientation as ∂M). Then W' is a word on $\widetilde{X}_0$ conjugate to W in $G(C_{n_0}, \phi_n)$. Moreover, W' equals 1 in and $G(C_n, \phi_n)$ deg(W') = deg(W)-l. But a connected, simply connected subdiagram of a minimal $\underline{\underline{s}}$-diagram is minimal [2, Lemma 2.4]. We can therefore apply our inductive hypothesis to obtain the results.

□

In the remainder of this section we justify the assumption made about *M* in the above proof.

Define as equivalence relation ~ on the regions of *M* by $\Delta = \Delta'$ if and only if there are regions $\Delta = \Delta_0, \Delta_1, \ldots, \Delta_n = \Delta'$ such that $g(\Delta_0) = g(\Delta_1) = \ldots g(\Delta_n)$ and with Δ_i, Δ_{i+1} having an edge in common for i = O,...,n-1. The regions in an ~-equivalence class give rise to a connected subdiagram of *M* called a federation.

Lemma 3. (i) Let *M* be a connected, simply connected minimal $\underline{\underline{r}}$-diagram and assume that each federation in *M* is simply connected. If (1.1) or (1.2) holds the *M* has a boundary region Δ with $g(\Delta) \subseteq g(\partial M)$.
(ii) If (1.1) or (1.2) holds then any federation contained in a connected, simply connected minimal $\underline{\underline{r}}$-diagram is simply connected.

This lemma completes the proof of the theorem. Let us assume (i) is true and we shall prove (ii).

Assume (ii) is false and let $\underline{K}$. be a counterexample with as few regions as possible. Let $\underline{F}$ be a federation in $\underline{K}$. which is not simply connected, and let $\underline{N}$ be a bounded component of $\underline{K}$ - $\underline{F}$. Then by (i) $\underline{N}$ has a boundary region Δ with $g(\Delta) \subseteq g(\partial \underline{N}) = g(\underline{F})$. Hence $g(\Delta) = g(\underline{F})$ contradicting the fact $\underline{F}$ is a federation.

Before proving (i).we need some further discussion.

Since we are now assuming that each federation is simply connected the boundary labels of federations are elements of $\underset{=}{\tilde{r}}$. We can therefore obtain from M an r-diagram $\hat{M}$ whose regions are the federations with all their interior edges and vertices removed.

A connected, simply connected r-diagram $\hat{M}^*$ is now obtained from $\hat{M}$ as follows: firstly, by repeated use of the modification described in §2, remove all the C_3 - vertices of $\hat{M}$; then remove each C_4. – vertex taking care to remove, as one proceeds, any C_3 - vertex each modification may produce; next remove all C_5 - vertices again removing, as one proceeds, any C_4 - vertex and, in turn, C_3 - vertex which may be produced; finally remove, in the usual way, all interior vertices of degree 2.

The procedure described above is illustrated in Fig. 3,1 where a possible sequence of modifications is given for an interior vertex of degree 5 (We are of course assuming that the vertices involved in Fig. 3.1 are C_m - vertices, $m \in \{3,4,5\}$; and we have not removed the interior vertices of degree 2.)

Fig. 3.1

The next observations follow immediately from Lemma 1.

(3.1) If C_n has property N(3) then each interior vertex of $\hat{M}^*$ has degree at least 4;

(3.2) if C_n has properties N(3), N(4) and N(5) then each interior vertex of $\hat{M}^*$ has degree at least 6.

If $\hat{\Delta}_1, \hat{\Delta}_2$ are distinct regions of $\hat{M}$ with an edge in common then $g(\hat{\Delta}_1) \cap g(\hat{\Delta}_2)$ is contained in some (n-l)-simplex of C_n (otherwise they would not be from distinct federations of M). Recalling that if $\hat{\Delta}^*$ of $\hat{M}^*$ has been obtained from $\hat{\Delta}$ of $\hat{M}$ then $g(\hat{\Delta}^*) = g(\hat{\Delta})$, it is clear that the same property holds for $g(\hat{\Delta}_1^*) \cap g(\hat{\Delta}_2^*)$. We therefore have:

(3.3) if $\hat{e}*$ is an interior edge of $\hat{M}*$ then the labelling set of $\hat{e}*$ is contained in some (n-1)-simplex of C_n;

(3.4) if each face group $G(\underline{\underline{x}})$ of $G(c_n, \phi_n)$ has property B_k then each almost interior region of $\hat{M}*$ has degree at least k+1.

<u>Proof of Lemma 3(i).</u> Suppose (1.1) holds. If M has only one region the result is immediate so we can assume otherwise. Then $\hat{M}*$, our modified $\hat{\underline{\underline{r}}}$-diagram, has the property that each almost interior region will have degree at least 6 (by (3.4)), and that each interior vertex will have degree at least 3. Therefore $\hat{M}*$ has a simple boundary region $\hat{\Delta}*$ with at most three interior edges [3, Theorem V.4.3]. Now $\hat{\Delta}*$ arises from some federation $\underline{F}$ in M where some region Δ of $\underline{F}$ is a boundary region of M. Condition B_5 together with (3.3) now implies that

$$g(\Delta) = g(\underline{F}) = g(\hat{\Delta}*) \subseteq g(\partial\hat{\Delta}* \cap \partial\hat{M}*) \subseteq g(\partial\hat{M}*) = g(\partial M).$$

If (1.2) holds then each almost interior region of $\hat{M}*$ has degree at least 4 (by (3.4)) and each interior vertex of $\hat{M}*$ has degree at least 4 (by (3.1)) Consequently $\hat{M}*$ has a simple boundary region with at most 2 interior edges [3, Theorem V.4.3]. Now argue as above.

□

§4 Proof of Theorem 2

The proof of Theorem 2 is similar to that of Theorem 1. We do however require a further technical result in order to obtain the analogue of Lemma 3(i).

Convention: when drawing diagrams in this section double lines shall indicate that the line segment may have vertices of degree 2.

Lemma 4. Let M be a connected, simply connected s-diagram (having no vertices of degree 1) where each almost interior region has degree at least 3, each interior vertex has degree at least 6, and whose boundary is a simple closed curve. Suppose further that M has at least one interior vertex and that M does not have a boundary region having interior degree equal to 1. Then M has two simple boundary regions Δ_1, Δ_2 (see Fig.4.1) each having interior degree 2 and having a single edge in common.

Figure 4.1

Proof. We proceed by induction on the number of interior vertices of M. If M has a single interior vertex then the diagram is a "wheel" having at least six spokes and the result holds. So assume that M has more than one interior vertex.

We know that $\underline{M}$ has a simple boundary region Δ with at most 2 interior edges [3, Theorem V.4.3]. Therefore, by our hypothesis, Δ has precisely 2 interior edges e_1, e_2 say. Remove from $\underline{M}$ the interior of Δ together with $\partial\Delta \cap \partial\underline{M}$ apart from the vertices $v_1 = e_1 \cap \partial\underline{M}$, $v_2 = e_2 \cap \partial\underline{M}$ to obtain the diagram $\underline{M}'$ (see fig. 4.2). Then $\underline{M}'$ satisfies the conditions of the lemma but has one fewer interior vertex than

Figure 4.2

$\underline{M}$. Observe however that $\underline{M}'$ may have a boundary region Δ' of interior degree 1. Let v_3 denote the vertex of Δ not on ∂M. If $\partial\Delta'$ does not involve v_3 then M must have a boundary region of interior degree 1, a contradiction. If $\partial\Delta'$ involves v_3 then the assumption about the degree of almost interior regions or interior vertices of $\underline{M}$ is contradicted. We conclude that no such Δ' exists and that, by our inductive hypothesis, $\underline{M}'$ has two simple boundary regions Δ_1', Δ_2', each having interior degree 2 and having a single edge in common.

If $\partial\Delta_1', \partial\Delta_2'$ do not involve vertex v_3 then it is obvious that the conclusion of the lemma holds for $\underline{M}$; so assume otherwise. If v_1, v_2, v_3 all are involved in $\partial\Delta_1', \cup \partial\Delta_2'$ then it is easy to check that the conditions about the degree of almost interior regions or interior vertices of $\underline{M}$ are contradicted. The remaining cases are illustrated in Fig. 4.3.

Figure 4.3

The argument in each of the four cases illustrated in Fig. 4.3 is similar. We shall assume, with out any loss, that $\underline{M}'$ is as in (i) of Fig. 4.3.

Remove from $\underline{M}'$ the interior of $\Delta_1' \cup \Delta_2'$ together with $(\partial\Delta_1' \cap \partial\underline{M}') \cup (\partial\Delta_2' \cap \partial\underline{M}')$ apart from the vertices v_3, u_4 (see Fig. 4.3) to obtain the diagram $\underline{M}''$. As before we can conclude that $\underline{M}''$ has two simple boundary regions Δ_1'', Δ_2'' satisfying the conclusion of the lemma.

If $\partial\Delta_1'', \partial\Delta_2''$ do not involve the vertices u_1, v_3(see Fig. 4.3(i)) then once more we can check that the conclusion holds for $\underline{M}$. If both the vertices u_1 and v_3 are involved in $\partial\Delta_1'' \cup \partial\Delta_2''$ then a contradiction can be obtained as in the previous case. The remaining cases for $\underline{M}''$ together with the corresponding diagrams for $\underline{M}$ are illustrated in Fig. 4.4.

Figure 4.4

Once again, without loss, we can assume that $\underline{M}''$ is as in (i) of Fig. 4.4, As before we remove from $\underline{M}''$ the interior of $\Delta_1'' \cup \Delta_2''$ together with $((\partial\Delta_1'' \cap \partial\underline{M}'') \cup (\partial\Delta_2'' \cap \partial\underline{M}'')$ apart from the vertices u_1 and w_2(see Fig. 4.4(i)) and apply similar arguments to the new diagram obtained.

Proceeding in this way either the conclusion of the lemma is obtained or $\underline{M}$ is as illustrated in Fig. 4.5. (Note that the diagram for $\underline{M}$ given there corresponds to remaining with subcase (i) throughout. For the other cases the final part of our proof is the same.)

Figure 4.5

Let us assume that $\underline{M}$ is indeed as in Figure 4.5. Then remove the boundary layer [3 ,p.260] of $\underline{M}$ to get the diagram $\hat{\underline{M}}$ all of whose regions have degree at least 3 and whose interior vertices each have degree at least 6. But each boundary vertex of $\hat{\underline{M}}$ has degree at least 4; this contradicts Corollary V.33 of [3]. This final contradiction completes the proof. □

Lemma 5. Let M be a connected, simply connected, minimal $\underline{\underline{r}}$-diagram. Assume that each federation in M is simply connected. If (1.3) holds then M has a boundary region Δ with $g(\Delta) \subseteq (\partial M)$

Proof. Let $\hat{M}^*$ be a modified r-diagram obtained from M as described in §3. It follows from (1.3) (i), (3.2) and (3.4) that each almost interior region of $\hat{M}^*$ has degree at least 3 and that each interior vertex has degree at least 6. Let $\hat{D}^*$ be an external disc [3, p.247] of $\hat{M}^*$ obtained from the external disc D of M.

If $\hat{D}^*$ has no interior vertices then the same will hold for D and the result follows. If $\hat{D}^*$ has a boundary region of interior degree 1 then we can use property B_2 and (3.3) and argue as in Lemma 3(i) to obtain the result. It can be assumed therefore $\hat{D}^*$ that has at least one interior vertex and has no boundary regions of interior degree 1, whence, by Lemma 4, $\hat{D}^*$ has two simple boundary regions $\hat{\Delta}_1^*, \hat{\Delta}_2^*$ say, each having interior degree 2 and having a single edge in common. Consequently there are distinct n- siniplices $\underline{\underline{x}}, \underline{\underline{y}}$ of c_n such that

$$g(\hat{\Delta}_1^*) \subseteq \underline{\underline{x}} \quad \text{and} \quad g(\hat{\Delta}_2^*) \subseteq \underline{\underline{y}}$$

We want to show that either $g(\hat{\Delta}_1^*) \subseteq g(\partial \hat{M}^*)$ or $g(\hat{\Delta}_2^*) \subseteq g(\partial \hat{N}^*)$ and argue as in lemma 3(i) to obtain the result. Let us assume neither holds. Then using (3.3) together with the fact that the face groups $G(\underline{\underline{x}})$, $G(\underline{\underline{y}})$ have property B_2, it can be deduced that the boundary labels of $\hat{\Delta}_1^*, \hat{\Delta}_2^*$, (reading from the vertex $\hat{e}^* \cap \partial\hat{M}^*$ inboth cases) are of the form $w_1(\underline{\underline{x}}_1)w_2(\underline{\underline{x}}_2)w_3(\underline{\underline{x}}_3)w_1(\underline{\underline{y}}_1)u_2(\underline{\underline{y}}_2)u_3(\underline{\underline{y}}_3)$ respectively, where the $\underline{\underline{x}}_j, \underline{\underline{y}}_j (1 \leq j \leq 3)$ are (n-1)- simplices of $\underline{\underline{x}}, \underline{\underline{y}}$ respectively and the labelling set of $\hat{e}^*$ is a subset of $\underline{\underline{x}}_1 = \underline{\underline{y}}_1$. But (1.3) (ii) now implies that

$$g(\partial \hat{M}^*) \supseteq g(w_3(\underline{\underline{x}}_3)) \cup g(u_3(\underline{\underline{y}}_3)) \supseteq \underline{\underline{x}} \text{ or } \underline{\underline{y}}$$

This contradiction provides us with the result.

$\square$

In order to complete the proof of Theorem 2 it suffices to argue as in the proof of Theorem 1. We omit the details.

§5

In order to obtain presentations $G(c_n, \phi_n)$ for which the Freiheitssatz will hold we require some examples of face groups having some B_k property.

Example 5.1 Let $G(\underline{\underline{x}}) = \langle \underline{\underline{x}}; \phi_n(\underline{\underline{x}}) \rangle$ and suppose that $\phi_n(\underline{\underline{x}})$ consists of a single element $w(\underline{\underline{x}})^m$ where m is a positive integer and $w(\underline{\underline{x}})$ is a cyclically reduced word. If $m > 1, 2, 4$ (respectively) then $G(\underline{\underline{x}})$ has property B_2, B_3, B_5 (respectively).

For suppose that Z is a non-empty cyclically reduced word which is equal to 1 in $G(\underline{\underline{x}})$. By a theorem of Gurevich [1] (see, for example, Theorem A of [7]), Z contains a subword of the form $T^{m-1}T_1$ where T is a cyclic permutation of $w(\underline{\underline{x}})^{\pm 1}$, $T \equiv T_1T_2$, and T_1 involves every member of $\underline{\underline{x}}$ Let $\underline{\underline{x}} = \{x_1 \ldots, x_n\}$ where x_i is the ith element of $\underline{\underline{x}}$ to appear in T for $1 \leq i \leq n$. If $m>1$ then Z contains a subword of the form

$$T\,T_1 \equiv x_1u_1x_2u_2 \ldots x_nu_nx_1v_1x_2v_2 \ldots x_nv_n$$

where u_i, v_i are words in some subset of $\underline{\underline{x}}$ for $1 \leq i \leq n$. It is clear that in this case we can conclude that $G(\underline{\underline{x}})$ has property B_2. The other two cases $m > 2$ and $m > 4$ are similar.

Let $w(\underline{\underline{x}})$ be a reduced word in the n-simplex $\underline{\underline{x}}$ of c_n. Then $w(\underline{\underline{x}})$ is freely equal to expressions of the form $w_1(\underline{\underline{x}}_1)\ldots w_r(\underline{\underline{x}}_r)$ where $w_i(\underline{\underline{x}}_i)$ is a word in the (n-1)-simple $\underline{\underline{x}}_i \subseteq \underline{\underline{x}}$ for $1 \le i \le r$. We shall say that $w(\underline{\underline{x}})$ has simplex length q, and write s. l.$(w(\underline{\underline{x}})) = q$, if q is the minimum value r takes over all such expressions.

Thus s. l.$(w(\underline{\underline{x}}))>1$ if and only if $w(\underline{\underline{x}})$ involves every member of the n-simplex $\underline{\underline{x}}$. Observe also that the face group $G(\underline{\underline{x}})$ has property B_k, if and only if s.l.$(Z^*) \ge k+1$ for all conjugates Z^* of each non-empty cyclically reduced word Z equal to 1 in $G(\underline{\underline{x}})$.

Let F be a free group and let R be a word in F. Let symm R denote the smallest symmetrised subset of F containing R. If s is a word in F we write $s > c$ symmR, c a rational number, to mean that there is a $u \in$ symmR with $u \equiv st$ in reduced form and $|s| > c|u|$ (here $|.|$ denotes the length of the word).

Lemma 6. Let $\underline{\underline{x}}$ be an n-simplex of c_n and suppose that symm $(\phi_n(\underline{\underline{x}}))$ satisfies the condition $C'(1/6)$. Then the following hold:

(i) If s. l. $(w) > 2$ for each word $w \in$ symm$(\phi_n(\underline{\underline{x}}))$; s.l. $(s) > 1$ for each word $s > 1/2$ symm $(\phi_n(\underline{\underline{x}}))$; and s.l.$(s)>2$ for each word $s > 5/6$ symm $(\phi_n(\underline{\underline{x}}))$; then $G(\underline{\underline{x}})$ has property B_2.

(ii) If s,l. $(w) > 3$ f or each word $w \in$ symm$(\phi_n(\underline{\underline{x}}))$; and s.l.$(s)> 2$ for each word $s > 1/2$ symm$(\phi_n(\underline{\underline{x}}))$, then $G(\underline{\underline{x}})$ has property B_3 .

(iii) If s.l $(w) > 5$ for each word $w \in (\phi_n(\underline{\underline{x}}))$ symm; s.l. $(s) > 2$ for each word $s > 1/2$ symm $(\phi_n(\underline{\underline{x}}))$; and s.l. $(s) > 3$ for each word $s > 5/6$ symm $(\phi_n(\underline{\underline{x}}))$, then $G(\underline{\underline{x}})$ has property B_5.

<u>Proof.</u> Let Z be a non-empty cyclically reduced word equal to 1 in $G(\underline{\underline{x}})$. Using Theorem V.4.5 [3] we deduce that either Z belongs to symm $(\phi_n(\underline{\underline{x}}))$ or for some cyclically reduced conjugate Z* of Z either Z* contains two disjoint subwords each > 5/6 symm $(\phi_n(\underline{\underline{x}}))$ or Z* contains three disjoint subwords each > 1/2 symm $(\phi_n(\underline{\underline{x}}))$.

We shall prove (ii), the proofs of (i) and (iii) being similar. By a remark made earlier, in order to show that $G(\underline{\underline{x}})$ has property B_3 it is enough to show that any cyclically reduced conjugate of Z has simplex length at least 4.

If $Z \in$ symm $(\phi_n(\underline{\underline{x}}))$ then, according to the assumption made in the statement of (ii), each cyclically reduced conjugate of Z has simplex length at least 4 and the result holds. Otherwise Z is conjugate to Z* which has the form

$$\text{either } u_1w_1w_2w_3u_2w_4w_5w_6u_3$$
$$\text{or } u_1w_1w_2w_3u_2w_4w_5w_6u_3w_7w_8w_9u_4$$

where w_i. is a non-empty reduced word in $\underline{\underline{x}}$ satisfying s.l. $(w_i) = 1$ $(1 \le i \le 9)$, and s.l. $(w_jw_{j+1}) > 1$ for j = 1,2,4,5,7,8, and u_i is a word in $\underline{\underline{x}}$ $(1 \le i \le 4)$. It may happen that any of s.l. $(w_3u_2w_4)$, s.l. $(w_6u_3u_1w_1)$, s.l. $(w_6u_3w_7)$, s.l. $(w_9u_4u_1w_1)$ is equal to 1. It is clear however that this does not prevent the simplex length of any cyclically reduced conjugate of Z*, and therefore of Z, being at least 4.

□

<u>Example 5.2</u> $G(\underline{\underline{x}}) = \langle x_1, x_2, x_3; x_1^2x_2x_3x_1^{-1}x_2^2x_3^{-1}x_1x_2^{-1}x_3 \rangle$ It is straightforward to check that the smallest symmetrised subset of the free group on x_1, x_2, x_3 containing the relator $x_1^2x_2x_3x_1^{-1}x_2^2x_3^{-1}x_1x_2^{-1}x_3 = R$, say, has no pieces of length 2 and so satisfies C'(1/6). If s > 1/2 symm R it is an easy exercise to verify that s.l. (s) > 2. Also, each member of symm R has simplex length greater than 3. Thus, by Lemma 5 (ii), $G(\underline{\underline{x}})$ has property B_3. Observe that s.l. (R) = 5 whence $G(\underline{\underline{x}})$ does not have property B_5.

We turn our attention to cyclic presentations- Suppose the group H has a presentation on m generators $x_1 \dots x_m$, and m defining relators obtained from the single word $R(x_1, \dots, x_2\}$, $m \geq n \geq 2$, by permuting the subscripts modulo m via the powers of the permutation (12...in). If we have that $R(x_1, \dots x_n)$ is a cyclically reduced word involving each x_i $(1 \leq i \leq n)$ then $H = H(\mathcal{C}_n, \phi_n)$ where here $\mathcal{C}_n$ is the simplicial complex generated by the n-simplices $\underline{\underline{x}}_j = \{x_j, \dots, x_{j+n-1}\}$, $\leq j \leq m$, the subscripts being reduced modulo m, and where $\phi_n(\underline{\underline{x}}_j) = R(x_j, \dots, x_{j+n-1})$ for $1 \leq j \leq m$.

<u>Lemma 7. The Freiheitssatz holds for $H(C_n, \phi_n)$, the presentation described for the group H in the previous paragraph, if one of the following holds:</u>

<u>(i) $K = \langle x_1, \dots, x_n; R(x_1, \dots, x_n) \rangle$ has property B_5; or</u>

<u>(ii) K has property B and $m \geq 1 + 3(n-1)$; or</u>

<u>(iii) K has property B_2, $m \geq 1 + 5(n-1)$, and $H(\mathcal{C}_n, \phi_n)$, satisfies condition (1.3) (ii) of Theorem 2</u>.

<u>Proof.</u> If $m \geq 1 + p(n-1)$ then $\mathcal{C}_n$ has property N(p) . Now use Theorems 1 and 2. □

<u>Example 5.3</u>

$$H\ (\mathcal{C}_n, \phi_n), = \langle x_1, \dots\dots x_7; x_i^2 x_{i+1} x_{i+2} x_i^{-1} x_{i+1}^2 x_{i+2}^{-1} x_i x_{i+1}^{-1} x_{i+2} \quad (1 \leq i \leq 7, \text{ subscripts mod } 7) \rangle$$

Here m = 7, n = 3 and, by Example 5.2 and Lemma 7 (ii), the Freiheitssatz holds for $H(\mathcal{C}_n, \phi_n)$,. Observe that the smallest symmetrised subset of the free group on $x_1, \dots x_7$ containing the set of defining relators of $H(\mathcal{C}_n, \phi_n)$, given here does not satisfy C' (1/6) - for example $x_2 x_3$ is a piece.

<u>Example 5 . 4</u>

$H(C_n,\phi_n), = \langle x_1,\ldots\ldots,x_{11};(x_i x_{i+1} x_{i+2}^{-1})^2 (1 \leq i \leq 11,$ subscripts mod $11)\rangle$.

Here $m = 11$ and $n = 3$. The group $K_1 = \langle x_1, x_2 x_3; (x_1 x_2 x_3^{-1})^2$ has property B_2 (but not B_3) by Example 5.1. Therefore if we can show that condition (1.3) (ii) of Theorem 2 is satisfied then the Freiheitssatz will hold by Lemma 7 (iii).

Let $K_2 = \langle x_2, x_3, x_4; (x_2 x_3 x_4^{-1})^2\rangle$. It suffices to look at K_1 and K_2 only. If Z_1 (respectively Z_2) is a non-empty cyclically reduced word equal to 1 in K_1 (resp K_2) then, by the theorem of Gurevich mentioned in Example 5.1, some cyclically reduced conjugate of $Z_1^{\pm 1}$ (resp. $Z_2^{\pm 1}$) contains a subword of the form $x_1 x_2 x_3^{-1} x_1 x_2 x_3^{-1}$ (resp. $x_2 x_3 x_4^{-1} x_2 x_3 x_4^{-1}$). It now follows that there cannot exist two words of simplex length 3 which satisfy the conditions in the statement of (1.3)(ii) - thus this condition is trivially satisfied by $H(C_n,\phi_n)$.

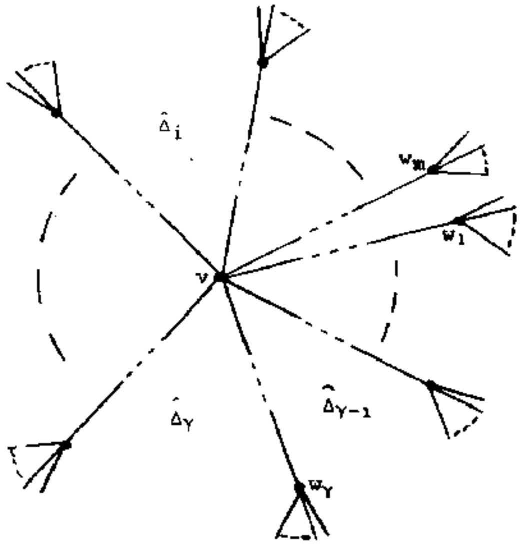


Figure 2.1

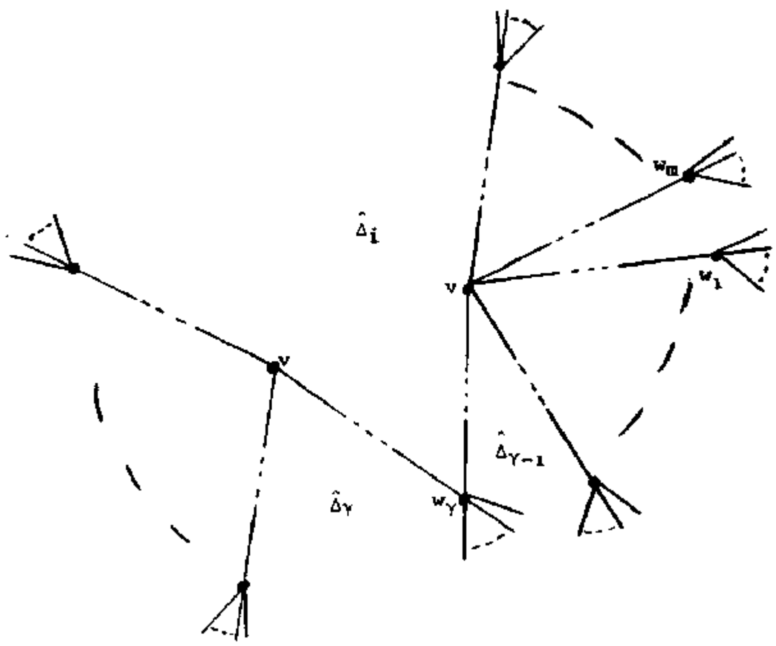


Figure 2.2

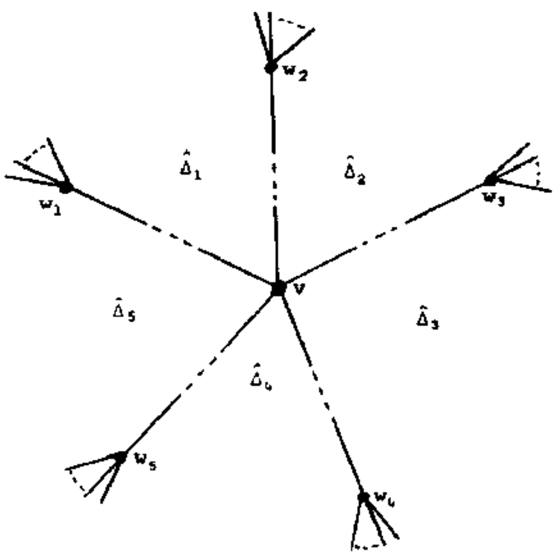
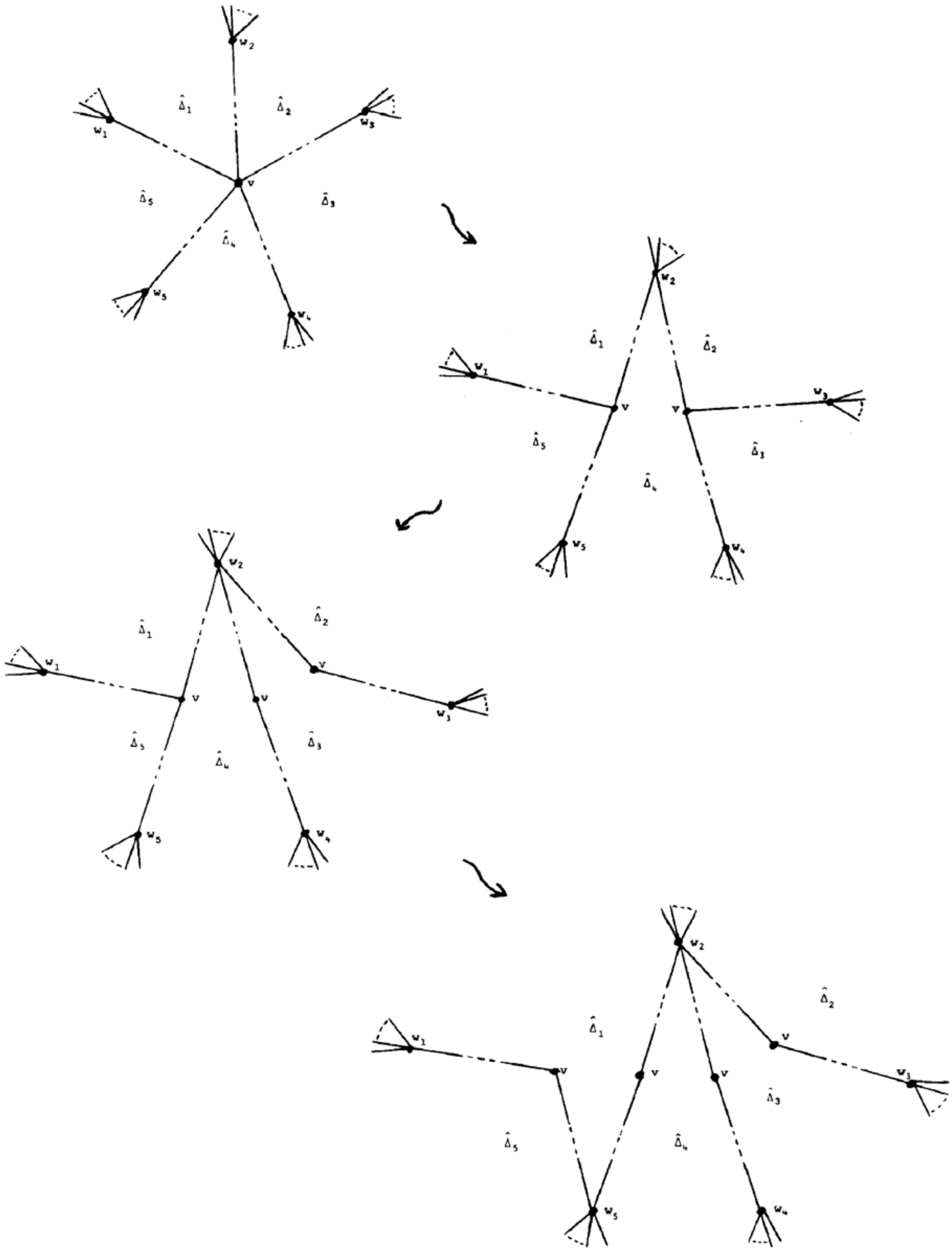


Figure 2.3



w_2
$\hat{\Delta}_1$
$\hat{\Delta}_2$
w_1
w_3
v
$\hat{\Delta}_5$
$\hat{\Delta}_3$
$\hat{\Delta}_4$
w_5
w_4

Figure 3.1

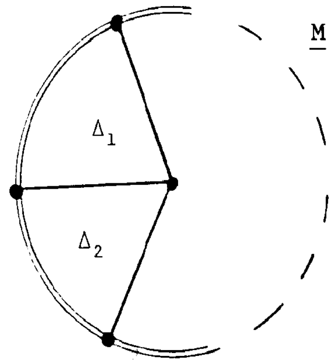


Figure 4.1

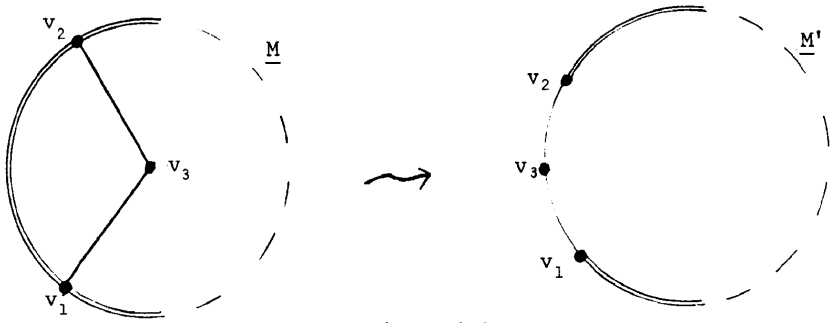


Figure 4.2

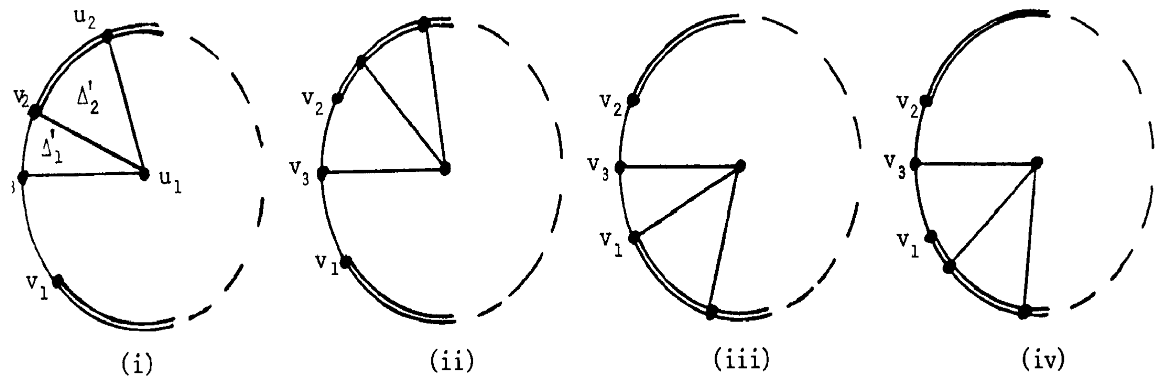
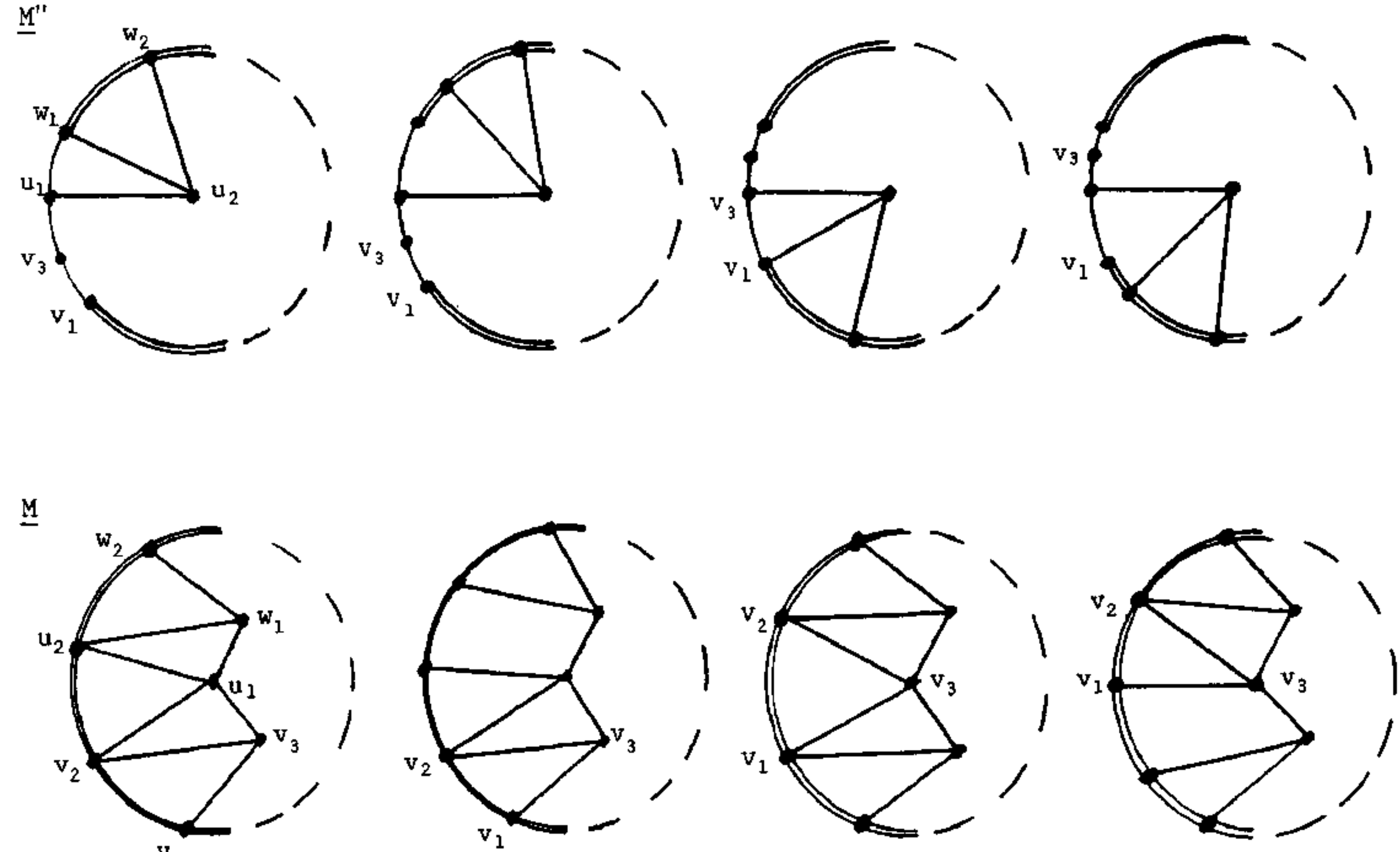


Figure 4.3



(i) (ii) (iii) (iv)

Figure 4.4

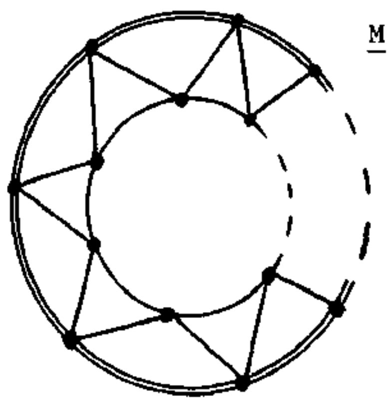


Figure 4.5

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