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CLASSICAL AND BAYESIAN INFERENCES IN STEP-STRESS PARTIALLY ACCELERATED LIFE TESTS FOR INVERSE WEIBULL DISTRIBUTION UNDER TYPE-I CENSORING

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This paper deals with the classical and Bayesian estimations of step-stress partially accelerated life test model under type-I censoring for the inverse Weibull lifetime distribution. In classical estimation, the maximum likelihood estimates of the distribution parameters and the acceleration factor were obtained. In addition, approximate confidence intervals of the parameters were constructed based on the asymptotic distribution of the maximum likelihood estimators. Under Bayesian inference, besides the Lindley and Tierney–Kadane approximation posterior expectation methods, which yielded point estimates of the distribution parameters and the acceleration factors under square error loss function, we also applied the Gibbs sampling method, in order to construct credible intervals of these parameters together with their point estimates. Finally, Monte Carlo simulations were conducted to compare the performances of the above estimation methods.

Keywords: step-stress partially accelerated life test, inverse Weibull distribution, type-I censoring, maximum likelihood estimation, Bayesian estimation, Gibbs sampling.

Introduction. The high-reliability devices have become an integral part of our lives with technological and industrial improvements. Therefore, the pressure on the manufacturer to produce high-quality products has increased day by day. It is crucial for manufacturers to test the lifetime of their products before launch to the market. However, testing the products under their normal-use conditions can be very costly and take a long time. For this reason, accelerated life tests (ALT) are preferred to obtain enough failure data in a short period [1]. In ALT, the products are tested under stresses, such as temperature, pressure, vibration amplitude, cycling rate, load, etc. The underlying assumption of ALT is that the mathematical model related to the lifetime of the unit and the stress are known. Nevertheless, life-stress relations are not always known, and ALT is not available [2, 3]. In this case, a partially accelerated life test (PALT) is used, in which items are firstly tested under normal conditions until the prefixed time. Then, the survived ones are subjected to accelerated test/stress conditions [4].

According to Nelson [5], stress application can be reduced to step-stress and constant-stress schemes. In step-stress PALT (SSPALT), firstly, the tested item is run under normal conditions. If it does not fail for a specified time, then it is run under accelerated condition until the test terminates. However, in constant-stress PALT (CSPALT), each unit is run at constant stress level until the test is terminated. The objective of these methods is to collect more failure data in a limited time without applying high stresses to all test units [6]. It should be noted that both SSPALT and CSPALT are used to shorten the test time. However, they would be long-term if they proceed until the failure of all units. Therefore, it is necessary to consider the impact of censoring schemes.

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Goel [7] and DeGroot and Goel [8] introduced the concept of SSPALT. Since then, SSPALT has been considered by several authors, especially under conventional type-I and type-II censoring schemes. For example, the problem of the estimation of the acceleration factor and distribution parameters were handled under type-I, or type-II censored data when the lifetime of the components are Burr type-II, Weibull, Gompertz, truncated logistic, Pareto, and Lomax by [9–14], respectively. In addition to these studies, Ismail [6, 15–17] and Zhang et al. [4] considered estimation of model parameters based on SSPALT using hybrid censoring, type-I progressive hybrid censoring, type-II progressive hybrid censoring, type-I progressive hybrid censoring with competing risk and adaptive type-I progressive hybrid censoring schemes under the assumption of Weibull lifetimes.

The Weibull distribution is one of the most popular distributions in life testing and reliability studies. Its popularity stems from the wide variety of shapes. Depending on the choice of the shape parameter, Weibull distribution is described by a decreasing or increasing hazard function (hf). However, if the data represent non-monotone hf, such as unimodal, Weibull distribution may not be suitable. In this case, the inverse Weibull (IW) distribution is a more appropriate model than the Weibull distribution with its unimodal or decreasing hf, see [18, 19] for the details. Also, the IW distribution has a larger right-tail probability than the Weibull one [20]. This property provides the flexibility of IW distribution for modeling the data set with extremes or outliers in the direction of the right tail.

The purpose of this paper is to focus on the estimation of the acceleration factor and parameters of IW distribution when the data are type-I censored under SSPALT. To this end, the maximum likelihood (ML) estimates of the model parameters are obtained by using iterative methods. Also, the approximated Fisher information matrix is derived from conducting the approximate confidence intervals (ACI) of the model parameters. Then, we obtain the Bayesian estimators of the model parameters under square error loss (SEL) function based on informative and non-informative priors by using three different methods. As expected, the Bayesian estimators cannot be obtained explicitly. Therefore, we implement Lindley's approximation, Tierney–Kadane approximation, and Gibbs sampling methods to compute the Bayesian estimates and also use the Gibbs sampling method to construct the Bayesian credible intervals (BCI).

The rest of this paper is organized as follows. The description of the model is elaborated in Section 1. In Section 2, the ML estimates of the SSPALT model parameters are obtained, and ACI for the model parameters are constructed. In Section 3, the Bayesian estimates of the model parameters are obtained using Lindley's approximation, Tierney–Kadane approximation, and Gibbs sampling methods. Furthermore, BCIs are constructed. Section 4 includes the Monte Carlo simulation study. Final comments and conclusions are also provided.

1. Description of the Model. IW distribution was proposed by Keller and Kamath [21] as a suitable model for describing the degeneration phenomena of mechanical components, especially in the dynamic components (pistons, crankshaft, etc.) of diesel engines. Erto and Rapone [22] explored that IW distribution provides a good fit for survival data such as the times to breakdown of an insulating fluid subject to the action of constant tension, see also [23]. IW distribution has many applications in different areas, for example, geology [24], wind energy [25], medical studies [26]. Also, in recent years, it has been used as a lifetime distribution for the ALT model by Hakamipour and Rezaei [27] and Ismail and Tamimi [28].

Assume that the random variable Y representing the lifetime of a product has IW distribution with shape and scale parameters α and λ, respectively. The probability density function (pdf) of Y is defined as

$$f_Y(y,\alpha,\lambda) = \alpha\lambda y^{-(\alpha+1)} e^{-\lambda y^{-\alpha}}, \qquad y > 0,\ \alpha,\lambda > 0, \tag{1}$$

cumulative distribution function (cdf), survival function and hf are

$$F(y,\alpha,\lambda) = e^{-\lambda y^{-\alpha}}, \qquad y > 0, \tag{2}$$

$$S(y,\alpha,\lambda) = 1 - e^{-\lambda y^{-\alpha}}, \qquad y > 0, \tag{3}$$

and

$$h(y,\alpha,\lambda)=\frac{\alpha\lambda y^{-(\alpha+1)}e^{-\lambda y^{-\alpha}}}{1-e^{-\lambda y^{-\alpha}}},\qquad y>0, \tag{4}$$

respectively.

In SSPALT, all of the n items are firstly tested under normal conditions. If the item does not fail for a pre-specified time τ, the test is switched to a higher level of stress and continues until the items fail. The effect of this switch is to multiply the remaining lifetime of the item by the inverse of the acceleration factor β. The total lifetime of Y under SSPALT is expressed as follows:

$$Y=\begin{cases}T, & \text{if}\quad T\le\tau,\\ \tau+\beta^{-1}(T-\tau), & \text{if}\quad T>\tau,\end{cases} \tag{5}$$

where T is the lifetime of an item at normal condition, τ is the stress change time, and β is the acceleration factor, which is the ratio of mean life under normal conditions to that under accelerated ones, usually $\beta>1$. This model is called the tampered random variable (TRV) model.

Assume that the lifetime of the test item follows the IW distribution with parameters α and λ. Then, the pdf of the total lifetime Y of an item under SSPALT is given by

$$f(y)=\begin{cases}0, & \text{if}\quad y\le 0,\\ f_1(y), & \text{if}\quad 0<y\le\tau,\\ f_2(y), & \text{if}\quad y>\tau,\end{cases} \tag{6}$$

where

$$f_1(y)=\alpha\lambda y^{-(\alpha+1)}e^{-\lambda y^{-\alpha}},$$

$$f_2(y)=\alpha\lambda\beta(\tau+\beta(y-\tau))^{-(\alpha+1)}e^{-\lambda(\tau+\beta(y-\tau))^{-\alpha}}.$$

It is obvious that $f_2(y)$ is obtained by using the variable transformation defined in (5).

2. Maximum Likelihood Estimation. Under the type I censoring scheme, the test terminates when the censoring time η is reached. The observed values of lifetime Y are given by

$$y_{(1)}\le\cdots\le y_{(n_a)}\le\tau\le y_{(n_a+1)}\le\cdots\le y_{(n_u+n_a)}\le\eta,$$

where n_a and n_u are the number of items that failed under normal and accelerated conditions, respectively. Let $n_0=n_u+n_a$. For simplification, we express $y_{(i)}$ by y_i.

Let $\underline{y}=(y_1,y_2,\dots,y_n)$ be the total lifetimes of n items, then the likelihood function of them is given below:

$$L(\alpha,\lambda,\beta|\underline{y})=\prod_{i=1}^{n_a}f_1(y_i)\prod_{i=n_a+1}^{n_0}f_2(y_i)(S_2(\eta))^{n-n_0}$$

$$=\alpha^{n_0}\lambda^{n_0}\beta^{n_u}\prod_{i=1}^{n_a}y_i^{-(\alpha+1)}e^{-\lambda\sum_{i=1}^{n_a}y_i^{-\alpha}}\prod_{i=n_a+1}^{n_0}\psi_i^{-(\alpha+1)}e^{-\lambda\sum_{i=n_a+1}^{n_0}\psi_i^{-\alpha}}(1-e^{-\lambda\psi_\eta^{-\alpha}})^{n-n_0}, \tag{7}$$

where $\psi_i=\tau+\beta(y_i-\tau)$ and $\psi_\eta=\tau+\beta(\eta-\tau)$.

The log-likelihood function $\ell(\alpha,\lambda,\beta|\underline{y})=\log L(\alpha,\lambda,\beta|\underline{y})$ is then given by

$$\ell(\alpha,\lambda,\beta|\underline{y})=n_0\ln\alpha+n_0\ln\lambda+n_u\ln\beta$$

$$-(\alpha+1)\left[\sum_{i=1}^{n_a}\log y_i+\sum_{i=n_a+1}^{n_0}\log\psi_i\right]-\lambda\left[\sum_{i=1}^{n_a}y_i^{-\alpha}+\sum_{i=n_a+1}^{n_0}\psi_i^{-\alpha}\right]+(n-n_0)\log(1-e^{-\lambda\psi_\eta^{-\alpha}}). \tag{8}$$

By taking the derivatives of (8) with respect to unknown parameters α, λ, and β, respectively, the likelihood equations are obtained as follows:

$$\frac{\partial\ell}{\partial\alpha}=\frac{n_0}{\alpha}-\left[\sum_{i=1}^{n_a}\log y_i+\sum_{i=n_a+1}^{n_0}\log\psi_i\right]+\lambda\left[\sum_{i=1}^{n_a}y_i^{-\alpha}\log y_i+\sum_{i=n_a+1}^{n_0}\psi_i^{-\alpha}\log\psi_i\right]$$

$$-(n-n_0)\frac{\lambda\psi_\eta^{-\alpha}\log(\psi_\eta)e^{-\lambda\psi_\eta^{-\alpha}}}{1-e^{-\lambda\psi_\eta^{-\alpha}}}=0, \tag{9}$$

$$\frac{\partial\ell}{\partial\lambda}=\frac{n_0}{\lambda}-\left[\sum_{i=1}^{n_a}y_i^{-\alpha}+\sum_{i=n_a+1}^{n_0}\psi_i^{-\alpha}\right]+(n-n_0)\frac{\psi_\eta^{-\alpha}e^{-\lambda\psi_\eta^{-\alpha}}}{1-e^{-\lambda\psi_\eta^{-\alpha}}}=0, \tag{10}$$

$$\frac{\partial\ell}{\partial\beta}=\frac{n_u}{\beta}-(\alpha+1)\sum_{i=n_a+1}^{n_0}\frac{(y_i-\tau)}{\psi_i}+\alpha\lambda\sum_{i=n_a+1}^{n_0}\frac{(y_i-\tau)}{\psi_i^{\alpha+1}}-(n-n_0)\frac{(\eta-\tau)\alpha\lambda\psi_\eta^{-(\alpha+1)}e^{-\lambda\psi_\eta^{-\alpha}}}{1-e^{-\lambda\psi_\eta^{-\alpha}}}=0. \tag{11}$$

The solutions of the likelihood equations (9)–(11) are the ML estimators of the parameters, but there are no explicit forms of these equations. Therefore, we resort to iterative methods such as Newton–Raphson to solve them and obtain the ML estimates of α, λ, and β.

In the context of interval estimation, the ACI of the parameters is constructed based on the asymptotic distribution of the ML estimators of unknown parameters. The variance-covariance matrix for the ML estimates of the parameters $\theta=(\alpha,\lambda,\beta)$ is derived from the inverse of the following Fisher information matrix

$$I_{ij}(\theta)=-E\left[\frac{\partial^2\ell}{\partial\theta_i\partial_j}\right],\qquad i,j=1,2,3. \tag{12}$$

The second derivatives of the log-likelihood function are given in Appendix 1.

Insofar as the derivation of exact mathematical expressions for the expectations (12) is quite problematic and cumbersome, the approximated (observed) Fisher information matrix can be used. Therefore, by writing the approximated variance-covariance matrix as

$$I^{-1}=\begin{bmatrix}-\frac{\partial^2\ell}{\partial\alpha^2} & -\frac{\partial^2\ell}{\partial\alpha\partial\lambda} & -\frac{\partial^2\ell}{\partial\alpha\partial\beta}\\ -\frac{\partial^2\ell}{\partial\lambda\partial\alpha} & -\frac{\partial^2\ell}{\partial\lambda^2} & -\frac{\partial^2\ell}{\partial\lambda\partial\beta}\\ -\frac{\partial^2\ell}{\partial\beta\partial\alpha} & -\frac{\partial^2\ell}{\partial\beta\partial\lambda} & -\frac{\partial^2\ell}{\partial\beta^2}\end{bmatrix}^{-1}_{(\hat\alpha,\hat\lambda,\hat\beta)}=\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix}_{\hat\alpha,\hat\lambda,\hat\beta}, \tag{13}$$

we have the approximate $100(1-\gamma)\%$ two-sided confidence intervals for α, λ, and β as follows:

$$\hat\alpha\pm z_{\gamma/2}\sqrt{\hat\sigma_{11}},\qquad \hat\lambda\pm z_{\gamma/2}\sqrt{\hat\sigma_{22}},\qquad \hat\beta\pm z_{\gamma/2}\sqrt{\hat\sigma_{33}}. \tag{14}$$

Here $z_{\gamma/2}$ is the upper ($\gamma/2$)th percentile of a standard normal distribution and $\hat{\sigma}_{ii}$ $(i = 1, 2, 3)$ is the ML estimates of σ_{ii}.

3. The Bayesian Estimation. We compute Bayesian estimates and credible intervals of the model parameters α, λ, and β. To this end, independent priors of the parameters α, λ, and β are considered as $Gamma(a_1, b_1)$, $Gamma(a_2, b_2)$, and $Gamma(a_3, b_3)$, respectively. Then, the joint prior distribution function is written as follows:

$$\pi(\alpha,\lambda,\beta) \propto \alpha^{a_1-1} e^{-b_1\alpha} \lambda^{a_2-1} e^{-b_2\lambda} \beta^{a_3-1} e^{-b_3\beta}, \tag{15}$$

where (a_1, b_1), (a_2, b_2), and (a_3, b_3) are assumed to be known and positive. Thus, using the likelihood function (7) and the joint prior distribution function (15), the joint posterior distribution function is given by

$$\pi^*(\alpha,\lambda,\beta|\underline{y}) \propto \alpha^{n_0+a_1-1} \lambda^{n_0+a_2-1} \beta^{n_u+a_3-1} e^{-b_1\alpha - [b_2 + \sum_{i=1}^{n_a} y_i^{-\alpha} + \sum_{i=n_a+1}^{n_0} \psi_i^{-\alpha}]\lambda - b_3\beta}$$

$$\times \prod_{i=1}^{n_a} y_i^{-(\alpha+1)} \prod_{i=n_a+1}^{n_0} \psi_i^{-(\alpha+1)} (1 - e^{-\lambda\psi_\eta^{-\alpha}})^{n-n_0}. \tag{16}$$

Therefore, Bayesian estimators of any function of the parameters, say $u(\alpha,\lambda,\beta)$, under squared error loss (SEL) function are obtained from the following expression:

$$E(u(\alpha,\lambda,\beta)|\underline{y}) = \int\int\int u(\alpha,\lambda,\beta)\,\pi^*(\alpha,\lambda,\beta|\underline{y})d\alpha d\lambda d\beta = \frac{\int\int\int u(\alpha,\lambda,\beta)L(\alpha,\lambda,\beta|\underline{y})\,\pi(\alpha,\lambda,\beta)d\alpha d\lambda d\beta}{\int\int\int L(\alpha,\lambda,\beta|\underline{y})\,\pi(\alpha,\lambda,\beta)d\alpha d\lambda d\beta}. \tag{17}$$

However, it is clear from (17) that there is no explicit form of the Bayesian estimators of the model parameters. So, we consider three different approximation methods, namely Lindley's approximation, Tierney–Kadane approximation, and Gibbs sampling for computing the corresponding Bayesian estimators. The details of these methods are explained in the following subsections.

3.1. Lindley's Approximation. Lindley's approximation is proposed to calculate the approximate ratio of two integrals, such as (17) by Lindley [29]. Now, let $\theta = (\theta_1, \theta_2, \theta_3)$ be a set of parameters, then the posterior expectation of an arbitrary function $u(\theta)$ can be calculated from the following expression:

$$E(u(\theta)) = \frac{\int u(\theta) e^{\log \pi^*(\theta)} d\theta}{\int e^{\log \pi^*(\theta)} d\theta} \approx \left[u + \frac{1}{2}\sum_i \sum_j (u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_l L_{ijk}\sigma_{ij}\sigma_{kl}u_l \right]_{\hat{\theta}}, \tag{18}$$

where $u = u(\theta_i)$, $i, j, k, l = 1, 2, 3$, $u_i = \partial u/\partial\theta_i$, $u_{ij} = \partial^2 u/\partial\theta_i\partial\theta_j$, $L_{ijk} = \partial^3 \ell/\partial\theta_i\partial\theta_j\partial\theta_k$, $\rho = \log\pi(\theta)$, $\rho_j = \partial\rho/\partial\theta_j$, σ_{ij} are the elements of the inverse of the Fisher information matrix in (13), and $\hat{\theta}$ is the ML estimator of θ.

Based on Lindley's approximation, the approximate Bayesian estimates of α, λ, and β under SEL function are obtained as

$$\begin{aligned} \tilde{\alpha} &= \hat{\alpha} + \rho_1\sigma_{11} + \rho_2\sigma_{12} + \rho_3\sigma_{13} + \frac{1}{2}(\sigma_{11}A + \sigma_{12}B + \sigma_{13}C), \\ \tilde{\lambda} &= \hat{\lambda} + \rho_1\sigma_{21} + \rho_2\sigma_{22} + \rho_3\sigma_{23} + \frac{1}{2}(\sigma_{21}A + \sigma_{22}B + \sigma_{23}C), \\ \tilde{\beta} &= \hat{\beta} + \rho_1\sigma_{31} + \rho_2\sigma_{32} + \rho_3\sigma_{33} + \frac{1}{2}(\sigma_{31}A + \sigma_{32}B + \sigma_{33}C), \end{aligned} \tag{19}$$

respectively. Here

$$A=\Sigma_i\Sigma_j\sigma_{ij}L_{ij1}=\sigma_{11}L_{111}+2\sigma_{12}L_{121}+2\sigma_{13}L_{131}+\sigma_{22}L_{221}+2\sigma_{23}L_{231}+\sigma_{33}L_{331},$$
$$B=\Sigma_i\Sigma_j\sigma_{ij}L_{ij2}=\sigma_{11}L_{112}+2\sigma_{12}L_{122}+2\sigma_{13}L_{132}+\sigma_{22}L_{222}+2\sigma_{23}L_{232}+\sigma_{33}L_{332},$$
$$C=\Sigma_i\Sigma_j\sigma_{ij}L_{ij3}=\sigma_{11}L_{113}+2\sigma_{12}L_{123}+2\sigma_{13}L_{133}+\sigma_{22}L_{223}+2\sigma_{23}L_{233}+\sigma_{33}L_{333}.$$

The explicit expressions of L_{ijk} and ρ_j $(i, j, k = 1, 2, 3)$ are given in Appendix 2.

3.2. The Tierney–Kadane Approximation. Now, we derive the approximate Bayes estimates of α, λ, and β under SEL using the Tierney–Kadane approximation [30]. According to this method, the posterior expectation of an arbitrary function $u(\alpha,\lambda,\beta)$ can be written as follows:

$$E(u(\alpha,\lambda,\beta|\underline{y})=\iiint u(\alpha,\lambda,\beta)\,\pi^*(\alpha,\lambda,\beta|\underline{y})d\alpha d\lambda d\beta=\frac{\iiint e^{n\delta^*(\alpha,\lambda,\beta)}d\alpha d\lambda d\beta}{\iiint e^{n\delta(\alpha,\lambda,\beta)}d\alpha d\lambda d\beta}, \tag{20}$$

where

$$\delta(\alpha,\lambda,\beta)=(\ell(\alpha,\lambda,\beta|\underline{y})+\rho(\alpha,\lambda,\beta))/n$$

and

$$\delta^*(\alpha,\lambda,\beta)=\delta(\alpha,\lambda,\beta)+(1/n)\log u(\alpha,\lambda,\beta).$$

Here $\ell(\alpha,\lambda,\beta|\underline{y})$ is the log-likelihood function given in (8) and $\rho(\alpha,\lambda,\beta)=\log\pi(\alpha,\lambda,\beta)$. Then, the posterior expectation of $u(\alpha,\lambda,\beta)$ can be estimated as

$$\hat{u}(\alpha,\lambda,\beta)=\left(\frac{|\Sigma^*|}{|\Sigma|}\right)^{1/2}e^{n[\delta^*(\hat{\alpha}^*,\hat{\lambda}^*,\hat{\beta}^*)-\delta(\hat{\alpha},\hat{\lambda},\hat{\beta})]}, \tag{21}$$

where $(\hat{\alpha}^*, \hat{\lambda}^*, \hat{\beta}^*)$ and $(\hat{\alpha}, \hat{\lambda}, \hat{\beta})$ maximize $\delta^*(\hat{\alpha}^*, \hat{\lambda}^*, \hat{\beta}^*)$ and $\delta(\hat{\alpha}, \hat{\lambda}, \hat{\beta})$, respectively. $|\Sigma^*|$ and $|\Sigma|$ are the determinants of the inverse Hessian of $\delta^*(\hat{\alpha}^*, \hat{\lambda}^*, \hat{\beta}^*)$ and $\delta(\hat{\alpha}, \hat{\lambda}, \hat{\beta})$ at $(\hat{\alpha}^*, \hat{\lambda}^*, \hat{\beta}^*)$ and $(\hat{\alpha}, \hat{\lambda}, \hat{\beta})$, respectively.

The estimation procedure of $(\hat{\alpha}, \hat{\lambda}, \hat{\beta})$ and the calculation of the corresponding inverse Hessian matrix can be summarized as follows:

Step 1. Define $\delta(\alpha, \lambda, \beta)$ as

$$\delta(\alpha,\lambda,\beta)=\frac{1}{n}[\ell(\alpha,\lambda,\beta|\underline{y})+(a_1-1)\log\alpha+(a_2-1)\log\lambda+(a_3-1)\log\beta-b_1\alpha-b_2\lambda-b_3\beta]. \tag{22}$$

Step 2. Derive the likelihood equations $\dfrac{\partial\delta(\alpha,\lambda,\beta)}{\partial\alpha}=0$, $\dfrac{\partial\delta(\alpha,\lambda,\beta)}{\partial\lambda}=0$, and $\dfrac{\partial\delta(\alpha,\lambda,\beta)}{\partial\beta}=0$.

Step 3. Obtain $(\hat{\alpha}, \hat{\lambda}, \hat{\beta})$ by solving these likelihood equations.

Step 4. Calculate the inverse Hessian matrix Σ as given by

$$\Sigma=\begin{bmatrix}\dfrac{\partial^2\delta(\alpha,\lambda,\beta)}{\partial\alpha^2} & \dfrac{\partial^2\delta(\alpha,\lambda,\beta)}{\partial\alpha\partial\lambda} & \dfrac{\partial^2\delta(\alpha,\lambda,\beta)}{\partial\alpha\partial\beta}\\ \dfrac{\partial^2\delta(\alpha,\lambda,\beta)}{\partial\lambda\partial\alpha} & \dfrac{\partial^2\delta(\alpha,\lambda,\beta)}{\partial\lambda^2} & \dfrac{\partial^2\delta(\alpha,\lambda,\beta)}{\partial\lambda\partial\beta}\\ \dfrac{\partial^2\delta(\alpha,\lambda,\beta)}{\partial\beta\partial\alpha} & \dfrac{\partial^2\delta(\alpha,\lambda,\beta)}{\partial\beta\partial\lambda} & \dfrac{\partial^2\delta(\alpha,\lambda,\beta)}{\partial^2\beta}\end{bmatrix}^{-1}. \tag{23}$$

In terms of estimation of $\hat{\alpha}^*$, $\hat{\lambda}^*$, and $\hat{\beta}^*$, the SEL function $u(\alpha,\lambda,\beta)$ is taken as a function of α, λ, and β, respectively. Then, for each value of $u(\alpha,\lambda,\beta)$, a similar procedure is applied by incorporating $\delta(\alpha,\lambda,\beta)$ into $\delta^*(\alpha,\lambda,\beta)$, separately.

It should be noted that either Lindley's approximation or Tierner–Kadane approximation provides to obtain the Bayesian estimate of the parameters. However, it is not possible to construct credible intervals using these methods. Therefore, in the following subsection, the Bayesian estimators of model parameters together with their BCIs are obtained using posterior distributions.

3.3. Gibbs Sampling. In this subsection, we use the Gibbs sampling procedure, which is a particular case of Markov Chain Monte Carlo method to generate samples from the conditional posterior distributions of the model parameters.

The posterior distribution function of α, λ, and β in (16) can be written as

$$\pi^*(\alpha,\lambda,\beta|\underline{y}) \propto \pi_1^*(\alpha|\lambda,\beta,\underline{y})\pi_2^*(\lambda|\alpha,\beta,\underline{y})\pi_3^*(\beta|\alpha,\lambda,\underline{y}), \tag{24}$$

where conditional posterior densities of α, λ, and β are obtained as follows:

$$\pi_1^*(\alpha|\lambda,\beta,\underline{y}) \propto \alpha^{n_0+a_1-1} e^{-b_1\alpha-(\Sigma_{i=1}^{n_a} y_i^{-\alpha}+\Sigma_{i=n_a+1}^{n_0}\psi_i^{-\alpha})\lambda} \prod_{i=1}^{n_a} y_i^{-(\alpha+1)} \prod_{i=n_a+1}^{n_0} \psi_i^{-(\alpha+1)} (1-e^{-\lambda\psi_\eta^{-\alpha}})^{n-n_0}, \tag{25}$$

$$\pi_2^*(\lambda|\alpha,\beta,\underline{y}) \propto \lambda^{n_0+a_2-1} e^{-(b_2+\Sigma_{i=1}^{n_a} y_i^{-\alpha}+\Sigma_{i=n_a+1}^{n_0}\psi_i^{-\alpha})\lambda} (1-e^{-\lambda\psi_\eta^{-\alpha}})^{n-n_0}, \tag{26}$$

$$\pi_3^*(\beta|\alpha,\lambda,\underline{y}) \propto \beta^{n_0+a_2-1} e^{-b_3\beta-(\Sigma_{i=n_a+1}^{n_0}\psi_i^{-\alpha})\lambda} \prod_{i=n_a+1}^{n_0} \psi_i^{-(\alpha+1)} (1-e^{-\lambda\psi_\eta^{-\alpha}})^{n-n_0}. \tag{27}$$

It is obvious that the posteriors of α, λ, and β in (25)–(27) are unknown. Therefore, to generate samples from these distributions, we use the Metropolis-Hastings algorithm, which is proposed by Metropolis et al. [31]. Now, we propose to use the following Gibbs sampling procedure to compute the Bayesian estimates of α, λ, and β. The steps of Gibbs sampling are given as follows:

Step 1. Start with $\alpha^{(0)}=\hat{\alpha}$, $\lambda^{(0)}=\hat{\lambda}$, and $\beta^{(0)}=\hat{\beta}$ and set $j=1$.

Step 2. Use the following Metropolis-Hasting algorithm, generate $\alpha^{(j)}$, $\lambda^{(j)}$, and $\beta^{(j)}$ from $g_1(\alpha^{(j-1)}|\lambda^{(j-1)},\beta^{(j-1)},\underline{y})$, $g_2(\lambda^{(j-1)}|\alpha^{(j)},\beta^{(j-1)},\underline{y})$, and $g_3(\beta^{(j-1)}|\alpha^{(j)},\lambda^{(j)},\underline{y})$ with normal proposal distributions $Normal(\alpha^{(j-1)},\sigma_{11})$, $Normal(\lambda^{(j-1)},\sigma_{22})$, and $Normal(\beta^{(j-1)},\sigma_{33})$, where σ_{ii} $(i=1,2,3)$ is from the variance-covariance matrix in (13).

(i). Generate proposal α^* from $Normal(\alpha^{(j-1)},\sigma_{11})$, λ^* from $Normal(\lambda^{(j-1)},\sigma_{22})$, and β^* from $Normal(\beta^{(j-1)},\sigma_{33})$.

(ii). Evaluate the acceptance probabilities

$$\rho_\alpha = \min\left[1, \frac{g_1(\alpha^*|\lambda^{(j-1)},\beta^{(j-1)},\underline{y})}{g_1(\alpha^{(j-1)}|\lambda^{(j-1)},\beta^{(j-1)},\underline{y})}\right], \qquad \rho_\lambda = \min\left[1, \frac{g_2(\lambda^*|\alpha^{(j)},\beta^{(j-1)},\underline{y})}{g_2(\lambda^{(j-1)}|\alpha^{(j)},\beta^{(j-1)},\underline{y})}\right],$$

and

$$\rho_\beta = \min\left[1, \frac{g_3(\beta^*|\alpha^{(j)},\lambda^{(j)},\underline{y})}{g_3(\beta^{(j-1)}|\alpha^{(j)},\lambda^{(j)},\underline{y})}\right].$$

(iii). Generate U_1, U_2, and U_3 from *Uniform*(0, 1).

(iv). If $U_1 \le \rho_\alpha$, accept the proposal and set $\alpha^{(j)} = \alpha^*$, else set $\alpha^{(j)} = \alpha^{(j-1)}$.

(v). If $U_2 \le \rho_\lambda$, accept the proposal and set $\lambda^{(j)} = \lambda^*$, else set $\lambda^{(j)} = \lambda^{(j-1)}$.

(vi). If $U_3 \le \rho_\beta$, accept the proposal and set $\beta^{(j)} = \beta^*$, else set $\beta^{(j)} = \beta^{(j-1)}$.

Step 3. Set $j = j+1$.

Step 4. Repeat steps 1–4 M times and obtain $(\alpha_1, \lambda_1, \beta_1), \ldots, (\alpha_M, \lambda_M, \beta_M)$.

Step 5. Obtain the Bayesian estimate of α, λ, and β under SEL function as follows:

$$\tilde{\alpha} = \frac{1}{M}\sum_{j=1}^{M}\alpha_j, \qquad \tilde{\lambda} = \frac{1}{M}\sum_{j=1}^{M}\lambda_j, \qquad \tilde{\beta} = \frac{1}{M}\sum_{j=1}^{M}\beta_j. \tag{28}$$

Step 6. Order $\alpha_1, \ldots, \alpha_M$ as $\alpha_{(1)} \le \ldots \le \alpha_{(M)}$, $\lambda_1, \ldots, \lambda_M$ as $\lambda_{(1)} \le \ldots \le \lambda_{(M)}$, and $\beta_1, \ldots, \beta_M$ as $\beta_{(1)} \le \ldots \le \beta_{(M)}$.

Then, the $100(1-\gamma)\%$ BCIs of model parameters are constructed as follows:

$$(\alpha_{[[M(\gamma/2)]]}, \alpha_{[[M(1-\gamma/2)]]}), \quad (\lambda_{[[M(\gamma/2)]]}, \lambda_{[[M(1-\gamma/2)]]}), \quad (\beta_{[[M(\gamma/2)]]}, \beta_{[[M(1-\gamma/2)]]}), \tag{29}$$

where $[[\cdot]]$ represents the integer value function.

4. Simulation Study. In this section, we compare the performances of ML and Bayesian estimates of the model parameters α, λ, and β via an extensive Monte Carlo simulation study. The comparisons are made for both point and interval estimations. In the context of point estimation, the means and mean square error (MSE) of the estimator of each parameter is calculated. In terms of interval estimation, 95% ACIs of the model parameters are constructed based on the asymptotic distribution of ML estimates. Furthermore, we compute 95% BCIs of the parameters of interest. Average confidence/credible lengths (ACL) and corresponding coverage probabilities (CP) are calculated.

The simulation study is carried out according to the following settings:

(1). The sample size is taken as $n = 50$, 100, 150, and 200.

(2). The parameter settings are considered as $(\alpha, \lambda, \beta) = (3, 1, 2)$ and (1.2, 1.7, 1.5). For the first setting, τ and η are specified as (1, 1.3) and (1, 4), respectively. For the second setting, τ and η are specified as (2, 7) and (2, 15), respectively.

(3). The random sample of size n from Y is generated by using (5). To do this, let U be a random variable from *Uniform*(0, 1). If $y \le \tau$, then $Y = (-(1/\lambda)\log U)^{-1/\alpha}$. If $y > \tau$, then $Y = ((-(1/\lambda)\log U)^{-1/\alpha} - \tau)/\beta + \tau$.

(4). Bayesian estimates of the model parameters are obtained by using informative and non-informative priors. In case of informative priors, we choose $a_1 = 2$, $b_1 = a_2 = b_2 = a_3 = b_3 = 1$. We call them Prior-I. For non-informative priors, we take $a_i = b_i = 0$, $i = 1, 2, 3$, and they are referred to as Prior-II.

(5). All computations are realized in MATLAB R2013 based on 1000 Monte Carlo runs. Also, Bayesian estimates based on the Gibbs sampling and corresponding credible intervals are obtained using 1000 sampling, namely $M = 1000$.

The estimated mean and MSEs of the ML estimates and Bayesian estimates based on Lindley's approximation, Tierney–Kadane approximation, and Gibbs sampling methods for α, λ, and β are represented in Tables 1 and 2. The ACL and CP of 95% confidence/credible intervals are presented in Table 3.

It is clear from Tables 1 and 2 that in terms of the means of the estimates, the Bayesian estimates based on Lindley's approximation have the largest bias under both informative and non-informative priors. Furthermore, Bayesian estimates based on Tierney–Kadane approximation and Gibbs sampling outperform other estimates in all cases. The ML estimates do not perform well as much as Bayesian estimates based on Tierney–Kadane approximation and Gibbs sampling. It should also be stated that the biases of all estimates decrease as the sample size n or the censoring value η increases as expected.

TABLE 1. The Mean and MSE Values of α, λ, and β ($\alpha = 3,\ \lambda = 1,\ \beta = 2$) for Given ($\tau = 1,\ \eta = 1.3$) and ($\tau = 1,\ \eta = 4$)

n	Parameter	ML	Prior-I			Prior-II		
			Lindley	TK	GS	Lindley	TK	GS
1	2	3	4	5	6	7	8	9
$\alpha = 3,\ \lambda = 1,\ \beta = 2,\ \tau = 1,\ \eta = 1.3$								
50	α	3.1475 (0.3574)	3.2663 (0.2857)	3.1129 (0.2342)	3.1089 (0.2402)	3.3663 (0.5737)	3.0989 (0.3418)	3.0862 (0.3478)
	λ	1.0106 (0.0328)	0.9524 (0.0208)	1.0138 (0.0277)	1.0141 (0.0285)	0.9097 (0.0344)	1.0232 (0.0338)	1.0241 (0.0350)
	β	2.0508 (0.5324)	1.7013 (0.2928)	2.0036 (0.2873)	2.0059 (0.2990)	1.8359 (0.5782)	2.2685 (0.8050)	2.2824 (0.8430)
100	α	3.0802 (0.1568)	3.2112 (0.1905)	3.0699 (0.1251)	3.0713 (0.1279)	3.0957 (0.1555)	3.0260 (0.1414)	3.0264 (0.1446)
	λ	1.0022 (0.0169)	0.9410 (0.0164)	0.9998 (0.0147)	0.9988 (0.0149)	0.9847 (0.0139)	1.0125 (0.0156)	1.0119 (0.0158)
	β	2.0394 (0.2477)	1.7918 (0.2049)	2.0042 (0.1757)	2.0008 (0.1848)	1.9972 (0.2476)	2.1395 (0.3059)	2.1332 (0.3133)
150	α	3.0508 (0.0995)	3.1245 (0.1081)	3.0234 (0.0789)	3.0225 (0.0798)	3.1460 (0.1237)	3.0406 (0.0869)	3.0376 (0.0897)
	λ	1.0041 (0.0109)	0.9607 (0.0110)	1.0024 (0.0101)	1.0023 (0.0101)	0.9656 (0.0116)	1.0083 (0.0110)	1.0087 (0.0113)
	β	2.0204 (0.1545)	1.8764 (0.1555)	2.0389 (0.1401)	2.0379 (0.1424)	1.9002 (0.1848)	2.0697 (0.1756)	2.0724 (0.1836)
200	α	3.0434 (0.0709)	3.1251 (0.0864)	3.0389 (0.0631)	3.0376 (0.0646)	3.1032 (0.0858)	3.0172 (0.0658)	3.0152 (0.0659)
	λ	1.0016 (0.0079)	0.9667 (0.0081)	1.0007 (0.0073)	1.0008 (0.0075)	0.9703 (0.0090)	1.0044 (0.0085)	1.0047 (0.0086)
	β	2.0050 (0.1123)	1.8666 (0.1178)	2.0066 (0.0981)	2.0053 (0.0985)	1.9146 (0.1270)	2.0588 (0.1204)	2.0583 (0.1215)
$\alpha = 3,\ \lambda = 1,\ \beta = 2,\ \tau = 1,\ \eta = 4$								
50	α	3.1487 (0.3543)	3.1683 (0.8526)	3.1374 (0.2321)	3.1324 (0.2412)	3.1290 (1.1270)	3.1035 (0.3048)	3.1004 (0.3240)
	λ	1.0114 (0.0374)	0.9610 (0.0487)	1.0022 (0.0282)	1.0030 (0.0287)	0.9893 (0.0720)	1.0242 (0.0350)	1.0231 (0.0364)
	β	2.1113 (0.6097)	1.8780 (0.5202)	2.0533 (0.2926)	2.0599 (0.3086)	2.1454 (1.8198)	2.2903 (0.8833)	2.2910 (0.9361)
100	α	3.0729 (0.1513)	3.0786 (0.2664)	3.0710 (0.1204)	3.0703 (0.1247)	3.0925 (0.2724)	3.0650 (0.1469)	3.0644 (0.1495)
	λ	1.0076 (0.0156)	0.9858 (0.0179)	1.0040 (0.0134)	1.0039 (0.0138)	0.9891 (0.0203)	1.0117 (0.0170)	1.0108 (0.0171)
	β	2.0379 (0.2456)	1.9065 (0.2965)	2.0335 (0.1779)	2.0323 (0.1813)	1.9887 (0.3952)	2.1297 (0.2861)	2.1256 (0.2873)

Table 1 continued

1	2	3	4	5	6	7	8	9
150	α	3.0621 (0.0946)	3.0705 (0.1179)	3.0438 (0.0829)	3.0428 (0.0844)	3.0822 (0.1344)	3.0465 (0.0922)	3.0471 (0.0946)
	λ	1.0066 (0.0113)	0.9861 (0.0108)	1.0041 (0.0101)	1.0045 (0.0103)	0.9851 (0.0113)	1.0048 (0.0109)	1.0042 (0.0111)
	β	2.0448 (0.1700)	1.9175 (0.1898)	2.0459 (0.1384)	2.0441 (0.1420)	1.9257 (0.2062)	2.0674 (0.1597)	2.0639 (0.1627)
200	α	3.0358 (0.0693)	3.0708 (0.0784)	3.0354 (0.0613)	3.0346 (0.0632)	3.0708 (0.0876)	3.0366 (0.0676)	3.0371 (0.0698)
	λ	0.9984 (0.0079)	0.9842 (0.0073)	1.0018 (0.0072)	1.0015 (0.0074)	0.9843 (0.0079)	1.0014 (0.0079)	1.0013 (0.0081)
	β	2.0185 (0.1193)	1.8986 (0.1281)	2.0231 (0.1039)	2.0225 (0.1074)	1.9096 (0.1428)	2.0395 (0.1210)	2.0375 (0.1249)

Notes. Here and in Table 2: 1. MSEs are reported within brackets. 2. Tierney–Kadane approximation and Gibbs sampling are represented as TK and GS, respectively, for brevity.

TABLE 2. The Mean and MSE Values of α, λ, and β ($\alpha = 1.2$, $\lambda = 1.7$, $\beta = 1.5$) for Given ($\tau = 2$, $\eta = 7$) and ($\tau = 2$, $\eta = 15$)

n	Parameter	ML	Prior-I			Prior-II		
			Lindley	TK	GS	Lindley	TK	GS
1	2	3	4	5	6	7	8	9
$\alpha = 1.2$, $\lambda = 1.7$, $\beta = 1.5$, $\tau = 2$, $\eta = 7$								
50	α	1.2530 (0.0398)	1.3037 (0.0518)	1.2870 (0.0398)	1.2822 (0.0386)	1.2345 (0.0534)	1.2241 (0.0400)	1.2301 (0.0360)
	λ	1.7545 (0.0753)	1.6692 (0.0581)	1.7182 (0.0577)	1.7186 (0.0577)	1.6951 (0.0679)	1.7463 (0.1171)	1.7404 (0.0687)
	β	1.5868 (0.5549)	1.2802 (0.2984)	1.4914 (0.2367)	1.5050 (0.2252)	1.6804 (1.0028)	1.8596 (0.8973)	1.8092 (0.7292)
100	α	1.2241 (0.0166)	1.2538 (0.0200)	1.2389 (0.0159)	1.2381 (0.0162)	1.2295 (0.0216)	1.2161 (0.0171)	1.2150 (0.0172)
	λ	1.7240 (0.0339)	1.6730 (0.0316)	1.7048 (0.0318)	1.7044 (0.0320)	1.6921 (0.0299)	1.7232 (0.0312)	1.7239 (0.0316)
	β	1.5238 (0.2037)	1.3564 (0.2002)	1.5005 0.1410)	1.5042 (0.1470)	1.5028 (0.3154)	1.6503 (0.2706)	1.6536 (0.2786)
150	α	1.2119 (0.0118)	1.2406 (0.0143)	1.2256 (0.0117)	1.2243 (0.0116)	1.2215 (0.0134)	1.2074 (0.0113)	1.2067 (0.0114)
	λ	1.7210 (0.0214)	1.6824 (0.0212)	1.7061 (0.0214)	1.7065 (0.0216)	1.6877 (0.0205)	1.7110 (0.0210)	1.7119 (0.0212)
	β	1.5427 (0.1525)	1.3986 (0.1540)	1.5285 (0.1201)	1.5310 (0.1250)	1.4873 (0.1915)	1.6195 (0.1751)	1.6079 (0.1830)

Table 2 continued

1	2	3	4	5	6	7	8	9
200	α	1.2102 (0.0083)	1.2322 (0.0094)	1.2178 (0.0080)	1.2188 (0.0083)	1.2245 (0.0098)	1.2097 (0.0085)	1.2091 (0.0087)
	λ	1.7108 (0.0151)	1.6822 (0.0146)	1.7013 (0.0146)	1.7009 (0.0147)	1.6977 (0.0157)	1.7167 (0.0163)	1.7114 (0.0161)
	β	1.5240 (0.0976)	1.4005 (0.1061)	1.5200 (0.0891)	1.5205 (0.0884)	1.4417 (0.1194)	1.5655 (0.1116)	1.5669 (0.1141)
$\alpha = 1.2,\ \lambda = 1.7,\ \beta = 1.5,\ \ \tau = 2,\ \eta = 15$								
50	α	1.2472 (0.0401)	1.2940 (0.0504)	1.2761 (0.0355)	1.2704 (0.0344)	1.2572 (0.0394)	1.2606 (0.0577)	1.2452 (0.0381)
	λ	1.7627 (0.0805)	1.6504 (0.0631)	1.7082 (0.0631)	1.7080 (0.0641)	1.7404 (0.0787)	1.6823 (0.0771)	1.7343 (0.0767)
	β	1.5973 (0.4602)	1.2570 (0.3473)	1.4972 (0.2411)	1.5155 (0.2226)	1.5539 (0.4673)	1.5403 (0.8522)	1.7684 (0.6960)
100	α	1.2286 (0.0179)	1.2540 (0.0219)	1.2393 (0.0169)	1.2384 (0.0171)	1.2241 (0.0167)	1.2307 (0.0213)	1.2173 (0.0163)
	λ	1.7200 (0.0330)	1.6653 (0.0312)	1.7013 (0.0317)	1.7017 (0.0323)	1.7309 (0.0339)	1.6923 (0.0310)	1.7277 (0.0333)
	β	1.5483 (0.2282)	1.3565 (0.2264)	1.5188 (0.1555)	1.5218 (0.1580)	1.5415 (0.2100)	1.4792 (0.3157)	1.6438 (0.2629)
150	α	1.2171 (0.0109)	1.2429 (0.0132)	1.2272 (0.0105)	1.2264 (0.0107)	1.2156 (0.0109)	1.2263 (0.0131)	1.2110 (0.0107)
	λ	1.7160 (0.0215)	1.6768 (0.0204)	1.7033 (0.0206)	1.7036 (0.0209)	1.7133 (0.0187)	1.6851 (0.0179)	1.7113 (0.0185)
	β	1.5209 (0.1309)	1.3622 (0.1467)	1.5092 (0.1040)	1.5114 (0.1073)	1.5239 (0.1381)	1.4404 (0.1866)	1.5903 (0.1597)
200	α	1.2109 (0.0080)	1.2335 (0.0094)	1.2186 (0.0078)	1.2188 (0.0080)	1.2155 (0.0083)	1.2267 (0.0096)	1.2119 (0.0082)
	λ	1.7147 (0.0152)	1.6814 (0.0137)	1.7026 (0.0138)	1.7021 (0.0140)	1.7110 (0.0155)	1.6886 (0.0150)	1.7095 (0.0154)
	β	1.5168 (0.0952)	1.3772 (0.1045)	1.5102 (0.0813)	1.5098 (0.0841)	1.5091 (0.0961)	1.4244 (0.1182)	1.5580 (0.1066)

In view of the MSE values of the estimates, all Bayesian estimates under Prior-I demonstrate better performance with smaller MSEs than the ML estimates. However, when $n = 50$ and under non-informative priors, the ML estimates perform better than all Bayesian estimates. As the sample size n increases, the MSE values of the ML estimates and the Bayesian estimates under Prior-II are close to each other. It is clear from Table 1 that increasing of η does not affect the MSE values of estimates prominently. Nevertheless, according to Table 2 that if η increases significantly, the MSEs of the estimates decrease apparently for the same sample size n.

In the context of the comparisons of confidence/credible intervals, it can easily be seen from Table 3 that the ACLs based on BCIs of the model parameters under informative priors are significantly shorter than the ACLs based on ACIs. Furthermore, the ACLs based on BCIs under non-informative priors provide shorter intervals according to

TABLE 3. Comparisons of ACL and CP of 95% CIs for All Parameter Settings

n	Parameter	ML		Prior-I		Prior-II	
				GS		GS	
		ACL	CP	ACL	CP	ACL	CP
1	2	3	4	5	6	7	8
$\alpha = 3,\ \lambda = 1,\ \beta = 2,\ \tau = 1,\ \eta = 1.3$							
50	α	2.2463	0.953	1.8908	0.947	2.0722	0.934
	λ	0.7330	0.954	0.6549	0.953	0.7023	0.940
	β	3.0799	0.920	2.3619	0.946	3.2859	0.935
100	α	1.5302	0.948	1.3756	0.950	1.4229	0.934
	λ	0.5095	0.945	0.4735	0.932	0.4897	0.946
	β	2.0406	0.946	1.7586	0.945	2.0355	0.939
150	α	1.2322	0.955	1.1236	0.949	1.1717	0.945
	λ	0.4153	0.950	0.3910	0.943	0.4011	0.932
	β	1.6187	0.948	1.4855	0.944	1.5994	0.933
200	α	1.0605	0.960	0.9829	0.950	1.0010	0.946
	λ	0.3582	0.950	0.3389	0.944	0.3447	0.942
	β	1.3789	0.954	1.2711	0.951	1.3585	0.945
$\alpha = 3,\ \lambda = 1,\ \beta = 2,\ \tau = 1,\ \eta = 4$							
50	α	2.2008	0.944	1.8440	0.952	1.9999	0.934
	λ	0.7299	0.944	0.6507	0.942	0.6990	0.937
	β	3.1634	0.921	2.3705	0.950	3.1778	0.933
100	α	1.4965	0.955	1.3365	0.947	1.3984	0.941
	λ	0.5091	0.958	0.4718	0.947	0.4881	0.930
	β	2.0147	0.937	1.7424	0.940	1.9702	0.943
150	α	1.2073	0.950	1.0870	0.946	1.1409	0.935
	λ	0.4138	0.954	0.3875	0.946	0.3980	0.943
	β	1.6190	0.945	1.4452	0.946	1.5453	0.935
200	α	1.0335	0.955	0.9519	0.947	0.9810	0.945
	λ	0.3555	0.955	0.3380	0.956	0.3427	0.940
	β	1.3694	0.946	1.2562	0.942	1.3173	0.936
$\alpha = 1.2,\ \lambda = 1.7,\ \beta = 1.5,\ \tau = 2,\ \eta = 7$							
50	α	0.7502	0.944	0.6867	0.939	0.7116	0.944
	λ	1.0090	0.946	0.9506	0.959	0.9790	0.946
	β	3.0032	0.924	2.1265	0.952	3.2026	0.948
100	α	0.5135	0.966	0.4807	0.958	0.4936	0.936
	λ	0.6915	0.950	0.6700	0.948	0.6815	0.955
	β	1.8647	0.939	1.6035	0.952	1.9592	0.950

Table 3 continued

1	2	3	4	5	6	7	8
150	α	0.4154	0.950	0.3935	0.941	0.4004	0.937
	λ	0.5612	0.953	0.5474	0.944	0.5495	0.949
	β	1.5198	0.940	1.3624	0.953	1.5338	0.939
200	α	0.3570	0.952	0.3427	0.942	0.3478	0.941
	λ	0.4821	0.947	0.4732	0.946	0.4782	0.945
	β	1.2828	0.950	1.1863	0.945	1.2797	0.948
$\alpha = 1.2,\ \lambda = 1.7,\ \beta = 1.5,\ \tau = 2,\ \eta = 15$							
50	α	0.7340	0.950	0.6653	0.940	0.6876	0.944
	λ	1.0159	0.948	0.9491	0.941	0.9762	0.941
	β	2.9185	0.913	2.1435	0.957	3.0548	0.951
100	α	0.5044	0.955	0.4742	0.936	0.4851	0.943
	λ	0.6893	0.951	0.6689	0.945	0.6854	0.953
	β	1.9082	0.933	1.6270	0.953	1.9336	0.945
150	α	0.4065	0.958	0.3881	0.943	0.3927	0.943
	λ	0.5594	0.952	0.5487	0.955	0.5520	0.956
	β	1.4861	0.942	1.3490	0.948	1.4998	0.945
200	α	0.3511	0.945	0.3355	0.939	0.3409	0.945
	λ	0.4832	0.956	0.4724	0.955	0.4774	0.941
	β	1.2739	0.950	1.1788	0.948	1.2628	0.946

ACIs, except for the acceleration factor β at the sample size $n = 50$. It should be noted that for same n, the ACLs of ACIs and BCIs decrease as the censoring value η increases.

In view of the CPs, the CPs of the ACIs are close the nominal value 0.95 in most of the cases apart from the CPs of the ACI of β for $n = 50$. As the sample size n increases, it also approaches its nominal value. On the other hand, the CPs of BCIs of the model parameters under Prior-I are more or less the same with the expected value. It can also be seen that the CPs of BCIs under informative priors work quite well than their non-informative alternatives.

Conclusions. In this study, we consider the classical and Bayesian estimation of SSPALT model under type-I censoring when the lifetime distribution is IW. The ML estimates of the model parameters are obtained numerically using iterative methods. The ACIs of the parameters of interest are constructed based on the asymptotic distribution of the ML estimates. Furthermore, Bayesian estimates of the parameters are obtained under SEL function based on informative and non-informative priors. In the context of informative priors, Gamma priors are used. Since the Bayesian estimates cannot be obtained in explicit form, Lindley's approximation and Tierney–Kadane approximation and Gibbs sampling methods are used to compute these estimates, respectively. Also, BCIs are constructed based on Gibbs sampling. We compare the performances of these methods via a Monte Carlo simulation study. It is observed that the Bayesian estimates under informative priors outperform other ML estimates and Bayesian estimates under non-informative priors.

Appendix 1

The second derivatives of $\log L = \ell$ with respect to α, λ, and β are as follows:

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{n_0}{\alpha^2} - \lambda\left[\sum_{i=1}^{n_a} y_i^{-\alpha} (\log y_i)^2 + \sum_{i=n_a+1}^{n_0} \psi_i^{-\alpha} (\log \psi_i)^2\right] + (n - n_0)\lambda(\log \psi_\eta)^2 \nu_1 \nu_2,$$

$$\frac{\partial^2 \ell}{\partial\alpha\partial\lambda} = \frac{\partial^2 \ell}{\partial\lambda\partial\alpha} = \left[\sum_{i=1}^{n_a} y_i^{-\alpha} \log y_i + \sum_{i=n_a+1}^{n_0} \psi_i^{-\alpha} \log\psi_i\right] - (n-n_0)\log\psi_\eta \nu_1\nu_2,$$

$$\frac{\partial^2 \ell}{\partial\alpha\partial\beta} = \frac{\partial^2 \ell}{\partial\beta\partial\alpha} = -\sum_{i=n_a+1}^{n_0} \frac{(y_i-\tau)}{\psi_i} + \lambda\left[\sum_{i=n_a+1}^{n_0} \frac{(y_i-\tau)}{\psi_i^{\alpha+1}} - \alpha \sum_{i=n_a+1}^{n_0} \frac{(y_i-\tau)}{\psi_i^{\alpha+1}} \log\psi_i\right]$$

$$-(n-n_0)(\eta-\tau)\lambda\psi_\eta^{-1}\nu_1(1-\alpha\log\psi_\eta\nu_2),$$

$$\frac{\partial^2 \ell}{\partial\lambda^2} = -\frac{n_0}{\lambda^2} - (n-n_0)\psi_\eta^{-\alpha}\nu_1\nu_3,$$

$$\frac{\partial^2 \ell}{\partial\lambda\partial\beta} = \frac{\partial^2 \ell}{\partial\beta\partial\lambda} = \alpha \sum_{i=n_a+1}^{n_0} \frac{(y_i-\tau)}{\psi_i^{\alpha+1}} - (n-n_0)(\eta-\tau)\alpha\psi_\eta^{-1}\nu_1\nu_2,$$

$$\frac{\partial^2 \ell}{\partial\beta^2} = -\frac{n_u}{\beta^2} + (\alpha+1)\sum_{i=n_a+1}^{n_0} \frac{(y_i-\tau)^2}{\psi_i^2} - \alpha(\alpha+1)\lambda \sum_{i=n_a+1}^{n_0} \frac{(y_i-\tau)^2}{\psi_i^{\alpha+2}} + (n-n_0)(\eta-\tau)^2\alpha\lambda\psi_\eta^{-2}(1+\alpha\nu_2),$$

where

$$\nu_1 = \frac{\psi_\eta^{-\alpha} e^{-\lambda\psi_\eta^{-\alpha}}}{1-e^{-\lambda\psi_\eta^{-\alpha}}}, \qquad \nu_2 = 1-\lambda\psi_\eta^{-\alpha} - \frac{\lambda\psi_\eta^{-\alpha} e^{-\lambda\psi_\eta^{-\alpha}}}{1-e^{-\lambda\psi_\eta^{-\alpha}}}, \qquad \nu_3 = 1+\frac{e^{-\lambda\psi_\eta^{-\alpha}}}{1-e^{-\lambda\psi_\eta^{-\alpha}}}.$$

Appendix 2

The third derivatives of $\log L = \ell$ with respect to α, λ, and β are as follows:

$$L_{111} = \frac{\partial^3 \ell}{\partial\alpha^3} = \frac{2n_0}{\alpha^3} + \lambda\left[\sum_{i=1}^{n_a} y_i^{-\alpha}(\log y_i)^3 + \sum_{i=n_a+1}^{n_0} \psi_i^{-\alpha}(\log\psi_i)^3\right] - (n-n_0)\lambda(\log\psi_\eta)^2\nu_1[\log\psi_\eta\nu_2^2 - \nu_4],$$

$$L_{112} = \frac{\partial^3 \ell}{\partial\alpha^2\partial\lambda} = -\left[\sum_{i=1}^{n_a} y_i^{-\alpha}(\log y_i)^2 + \sum_{i=n_a+1}^{n_0} \psi_i^{-\alpha}(\log\psi_i)^2\right] + (n-n_0)\lambda(\log\psi_\eta)^2\nu_1\left[\frac{1}{\lambda}\nu_2^2 - (\psi_\eta^{-\alpha} + \nu_1\nu_2)\right],$$

$$L_{112} = L_{121} = L_{211},$$

$$L_{113} = \frac{\partial^3 \ell}{\partial\alpha^2\partial\beta} = \lambda\left[\alpha \sum_{i=n_a+1}^{n_0} \frac{(y_i-\tau)}{\psi_i^{\alpha+1}}(\log\psi_i)^2 - 2\sum_{i=n_a+1}^{n_0} \frac{(y_i-\tau)}{\psi_i^{\alpha+1}}\log\psi_i\right]$$

$$+(n-n_0)(\eta-\tau)\lambda\psi_\eta^{-1}\log\psi_\eta\nu_1[(2-\alpha\log\psi_\eta\nu_2)\nu_2 + \alpha\lambda\log\psi_\eta(\psi_\eta^{-\alpha} + \nu_1\nu_2)],$$

$$L_{113} = L_{131} = L_{331},$$

$$L_{222} = \frac{\partial^3 \ell}{\partial\lambda^3} = \frac{2n_0}{\lambda^3} + (n-n_0)\psi_\eta^{-2\alpha}\nu_1\nu_3[2\nu_3 - 1],$$

$$L_{221} = \frac{\partial^3 \ell}{\partial \lambda^2 \partial \alpha} = (n - n_0)\psi_\eta^{-\alpha} \log \psi_\eta \nu_1 \nu_3 [(\nu_2 + 1) - \lambda \nu_1 (\nu_3 - 1)],$$

$$L_{221} = L_{212} = L_{122},$$

$$L_{223} = \frac{\partial^3 \ell}{\partial \lambda^2 \partial \beta} = (n - n_0)(\eta - \tau)\alpha \psi_\eta^{-(\alpha+1)} \nu_1 \nu_3 [(\nu_2 + 1) - \lambda (\nu_3 - 1)],$$

$$L_{223} = L_{232} = L_{322},$$

$$L_{333} = \frac{\partial^3 \ell}{\partial \beta^3} = \frac{2n_u}{\beta^3} - 2(\alpha + 1) \sum_{i=n_a+1}^{n_0} \frac{(y_i - \tau)^3}{\psi_i^3} + \alpha (\alpha + 1)(\alpha + 2)\lambda \sum_{i=n_a+1}^{n_0} \frac{(y_i - \tau)^3}{\psi_i^{\alpha+3}}$$

$$- (n - n_0)(\eta - \tau)^3 \alpha \lambda \psi_\eta^{-3} \nu_1 [(2 + \alpha \nu_2)(1 + \alpha \nu_2) + \alpha^2 \psi_\eta^{-\alpha} (\lambda + \nu_1 (1 + \nu_2))],$$

$$L_{331} = \frac{\partial^3 \ell}{\partial \beta^2 \partial \alpha} = \sum_{i=n_a+1}^{n_0} \frac{(y_i - \tau)^2}{\psi_i^2} - \lambda \left[(2\alpha + 1) \sum_{i=n_a+1}^{n_0} \frac{(y_i - \tau)^2}{\psi_i^{\alpha+2}} - \alpha (\alpha + 1)(\alpha + 2) \sum_{i=n_a+1}^{n_0} \frac{(y_i - \tau)^3}{\psi_i^{\alpha+3}} \log \psi_i \right]$$

$$+ (n - n_0)(\eta - \tau)^2 \lambda \psi_\eta^{-2} \nu_1 [(1 - \alpha \log \psi_\eta \nu_2)(1 + \alpha \nu_2) + \alpha (\nu_2 + \alpha \lambda \psi_\eta^{-\alpha} \log \psi_\eta (1 + (\nu_3 - 1)\nu_2))],$$

$$L_{331} = L_{313} = L_{133},$$

$$L_{332} = \frac{\partial^3 \ell}{\partial \beta^2 \partial \lambda} = -\alpha (\alpha + 1) \sum_{i=n_a+1}^{n_0} \frac{(y_i - \tau)^2}{\psi_i^{\alpha+2}} + (n - n_0)(\eta - \tau)^2 \alpha \psi_\eta^{-2} \nu_1 [\nu_2 (1 + \alpha \nu_2) - \alpha (\alpha \psi_\eta^{-\alpha} + \nu_1 \nu_2)],$$

$$L_{332} = L_{323} = L_{233},$$

$$L_{123} = \frac{\partial^3 \ell}{\partial \alpha \partial \beta \partial \lambda} = \sum_{i=n_a+1}^{n_0} \frac{(y_i - \tau)}{\psi_i^{\alpha+1}} - \alpha \sum_{i=n_a+1}^{n_0} \frac{(y_i - \tau)}{\psi_i^{\alpha+1}} \log \psi_i - (n - n_0)(\eta - \tau)\psi_\eta^{-1} \nu_1 [(1 - \alpha \log \psi_\eta \nu_2)\nu_2 + \alpha \nu_4],$$

$$L_{123} = L_{231} = L_{312} = L_{321},$$

where

$$\nu_4 = \lambda \psi_\eta^{-\alpha} \log \psi_\eta + \frac{\lambda \psi_\eta^{-\alpha} \log \psi_\eta e^{-\lambda \psi_\eta^{-\alpha}}}{1 - e^{-\lambda \psi_\eta^{-\alpha}}} \left(1 - \lambda \psi_\eta^{-\alpha} - \frac{\lambda \psi_\eta^{-\alpha} e^{-\lambda \psi_\eta^{-\alpha}}}{1 - e^{-\lambda \psi_\eta^{-\alpha}}} \right).$$

Also, ν_1, ν_2, and ν_3 are defined similarly as in Appendix 1.

Using the prior joint distribution (15), we obtain

$$\rho = (a_1 - 1)\log \alpha - b_1 \alpha + (a_2 - 1)\log \lambda - b_2 \lambda + (a_3 - 1)\log \beta - b_3 \beta,$$

which yields

$$\rho_1 = \frac{a_1 - 1}{\alpha} - b_1, \qquad \rho_2 = \frac{a_2 - 1}{\lambda} - b_2, \qquad \rho_3 = \frac{a_3 - 1}{\beta} - b_3.$$

REFERENCES

1. B. X. Wang, K. Yu, and Z. Sheng, "New inference for constant-stress accelerated life tests with Weibull distribution and progressively type-II censoring," *IEEE T. Reliab.*, **63**, No. 3, 807–815 (2014).
2. D. S. Bai and S. W. Chung, "Optimal design of partially accelerated life tests for the exponential distribution under type-I censoring," *IEEE T. Reliab.*, **41**, No. 3, 400–406 (1992).
3. D. Zheng and X. Fang, "Exact confidence limits for the acceleration factor under constant-stress partially accelerated life tests with type-I censoring," *IEEE T. Reliab.*, **67**, No. 1, 92–104 (2018).
4. C. Zhang, Y. Shi, and M. Wu, "Statistical inference for competing risks model in step-stress partially accelerated life tests with progressively Type-I hybrid censored Weibull life data," *J. Comput. Appl. Math.*, **297**, 65–74 (2016).
5. W. B. Nelson, *Accelerated Testing: Statistical Models, Test Plans and Data Analysis*, Wiley, New York (2004).
6. Ali A. Ismail, "Likelihood inference for a step-stress partially accelerated life test model with Type-I progressively hybrid censored data from Weibull distribution," *J. Stat. Comput. Sim.*, **84**, No. 11, 2486–2494 (2014).
7. P. K. Goel, *Some Estimation Problems in the Study of Tramped Random Variables*, PhD Thesis, Department of Statistics, Cranegie-Mellon University, Pittsburgh, PA (1971).
8. M. H. DeGroot and P. K. Goel, "Bayesian estimation and optimal design in partially accelerated life testing," *Nav. Res. Logist. Q.*, **16**, No. 2, 223–235 (1979).
9. A. M. Abd-Elfattah, A. M. Hassan, and S. G. Nassr, "Estimation in step-stress partially accelerated life tests for the Burr type XII distribution using type-I censoring," *Stat. Methodol.*, **5**, No. 6, 502–514 (2008).
10. Ali A. Ismail and H. M. Aly, "Optimal planning of failure-step stress partially accelerated life tests under type-II censoring," *J. Stat. Comput. Sim.*, **80**, No. 12, 141–148 (2010).
11. Ali A. Ismail, "Bayes estimation of Gompertz distribution parameters and acceleration factor under partially accelerated life tests with type-I censoring," *J. Stat. Comput. Sim.*, **80**, No. 11, 1253–1264 (2010).
12. P. W. Srivastava and N. Mittal, "Optimal step-stress partially accelerated life tests for truncated logistic distribution with censoring," *Appl. Math. Model.*, **34**, No. 10, 3166–3178 (2010).
13. Ali A. Ismail, "Likelihood and Bayesian estimations for step-stress life test model under Type-I censoring," *Hacet. J. Math. Stat.*, **44**, No. 5, 1235–1245 (2015).
14. Ali A. Ismail, "Optimum failure-censored step-stress life test plans for the Lomax distribution," *Strength Mater.*, **48**, No. 3, 437–443 (2016).
15. Ali A. Ismail, "Estimating the parameters of Weibull distribution and the acceleration factor from hybrid partially accelerated life test," *Appl. Math. Model.*, **36**, No. 7, 2920–2925 (2012).
16. Ali A. Ismail, "Inference for a step-stress partially accelerated life test model with and adaptive Type-II progressively hybrid censored data from Weibull distribution," *J. Comput. Appl. Math.*, **260**, 533–542 (2014).
17. Ali A. Ismail, "Statistical inference for a step-stress partially-accelerated life test model with an adaptive Type-I progressively hybrid censored data from Weibull distribution," *Stat. Pap.*, **57**, 271–301 (2016).
18. D. Kundu and H. Howlader, "Bayesian inference and prediction of the inverse Weibull distribution for Type-II censored data," *Comput. Stat. Data An.*, **54**, No. 6, 1547–1558 (2010).
19. R. M. Musleh and A. Helu, "Estimation of the inverse Weibull distribution based on progressively censored data: Comparative study," *Reliab. Eng. Syst. Safe.*, **131**, 216–227 (2014).
20. H. Rinne, *The Weibull Distribution: a Handbook*, CRC Press, Boca Raton (2009).
21. A. Z. Keller and A. R. R. Kamath, "Alternative reliability models for mechanical systems," in: Proc. of the 3rd Int. Conf. on Reliability and Maintainability, Toulouse (1982), pp. 411–415.
22. P. Erto and M. Rapone, "Non-informative and practical Bayesian confidence bounds for reliable life in the Weibull model," *Reliab. Eng.*, **7**, No. 3, 181–191 (1984).
23. W. Nelson, *Applied Lifetime Data Analysis*, Wiley, New York (1982).

24. S. Pasari and O. Dikshit, “Impact of three-parameter Weibull models in probabilistic assessment of earthquake hazards,” *Pure Appl. Geophys.*, **171**, 1251–1281 (2014).
25. F. G. Akgül, B. Şenoğlu, and T. Arslan, “An alternative distribution to Weibull for modeling the wind speed data: Inverse Weibull distribution,” *Energ. Convers. Manage.*, **114**, 234–240 (2016).
26. S. Bennette, “Log-logistic regression models for survival data,” *Appl. Stat.-J. Roy. St. C*, **32**, No. 2, 165–171 (1983).
27. N. Hakamipour and S. Rezaei, “Optimizing the simple step-stress accelerated life test with type I censored Frechet data,” *Revstat-Stat. J.*, **15**, No. 1, 1–23 (2017).
28. Ali A. Ismail and A. Al Tamimi, “Optimum constant-stress partially accelerated life test plans using type-I censored data from the inverse Weibull distribution,” *Strength Mater.*, **49**, No. 6, 847–855 (2017).
29. D. V. Lindley, “Approximate Bayesian methods,” *Trabajos de Estadistica Y de Investigacion Operativa*, **31**, 223–245 (1980), https://doi.org/10.1007/BF02888353.
30. L. Tierney and J. B. Kadane, “Accurate approximations for posterior moments and marginal densities,” *J. Am. Stat. Assoc.*, **81**, 82–86 (1986).
31. N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, et al., “Equations of state calculations by fast computing machines,” *J. Chem. Phys.*, **21**, 1087–1091 (1953).