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A two-grid, fourth order method for nonlinear fourth order boundary value problems.

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ABSTRACT

A fourth order convergent finite difference method is developed for the numerical solution of the nonlinear fourth order boundary value problem $y^{(iv)}(x) = f(x,y)$, $a < x < b$, $y(a) = A_0$, $y''(a) = B_0$, $y(b) = A_1$, $y''(b) = B_1$.

The method is based on a second order convergent method which is used on two grids, fourth order convergence being obtained by considering a linear combination of the individual results relating to the two grids.

Special formulas are developed for application to grid points adjacent to the boundaries $x = a$ and $x = b$, the principal parts of the local truncation errors of these formulas being the same as that of the second order method used at other points of each grid.

Modifications to these special formulas are noted for problems with boundary conditions of the form $y(a) = A_o$, $y'(a) = C_o$, $y(b) = A_1$, $y'(b) = c_1$.

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1. THE SECOND ORDER METHOD

Consider the general non-linear fourth order boundary value problem given by

$$y^{(iv)}(x) = f(x,y)\ ,\ \ a < x < b\ ,\ a,b,x \in \mathbb{R} \tag{1}$$

with the functional and second order derivative boundary conditions

$$y(a) = A_o\ , y''(a) = B_o\ ,\ y(b) = A_1\ , y''(b) = B_1\ . \tag{2}$$

It is assumed that $f(x,y)$ is real and continuous on $[a,b]$ with $\partial f/\partial y < 0$, and that A_0, A_1, B_0, B_1 are real finite constants. For a detailed discussion of the existence and uniqueness of the real valued function $y(x)$ which satisfies (1) and (2), the reader is referred to Agarval and Akrivis [1].

Suppose the interval $x \in [a, b]$ is discretized into N+ 1 subintervals each of width $h = (b- a) /(N+1)$, where $N \geq 3$ is a positive integer. The solution $y(x)$ will be computed at the points $x_n = a + nh$ $(n = 1,2,...,N)$ and the notation y will be used to denote the solution of an approximating difference scheme at the grid point x_n ; clearly $y_0 = A_0$ and $y_{N+1} = A_1$.

It was noted by Twizell and Tirmizi [2] that the solution $y(x)$ of (1) and (2) satisfies the recurrence relation

$$y(x-2h) - Ry(x-h) + Sy(x) - Ry(x+h) + y(x+2h) = 0\ , \tag{3}$$

where the operators R and S are given by

$$R = \exp(hD) + \exp(-hD) + \exp(ihD) + \exp(-ihD) \tag{4}$$

and

$$S = 2 + \{\exp(hD) + \exp(-hD)\}\{\exp(ihD) + \exp(-ihD)\} \tag{5}$$

with $i = +\sqrt{-1}$ and $D = d/dx$. One of the numerical methods discussed in [2] was developed by replacing the exponential terms in (4) and (5) by their (1,2) Pade approximates. This leads to the numerical method

$$y_{n-2} - 4y_{n-1} + 6y_n - 4y_{n+1} + y_{n+2}$$
$$- \frac{h^4}{81}(f_{n-2} + 14f_{n-1} + 51f_n + 14f_{n+1} + f_{n+2}) = 0\,, \qquad (6)$$

where $f_s \equiv f(x_s, y_s)$, $s = 1,2,\ldots,N$, which has local truncation error

$$t_n^{(1)} = -\frac{1}{18}h^6 y^{(vi)}(x_n) - \frac{119}{6480}h^8 y^{(viii)}(x_n) - \ldots \qquad (7)$$

at the grid point $X = x$.

Equation (6) is applicable only to the N-2 mesh points x_n $(n = 2,\ldots,N-1)$ and clearly special formulas are needed for the mesh points x_1 and x_N. In order to make possible the application of the procedure to be discussed in §2, these special formulas must be second order accurate and their local truncation errors must be of the form $t_i^{(1)} = -\frac{1}{18}h^6 y^{(vi)}(x_i) + 0(h^8)$ for $i = 1, N$.

These requirements are met by the pair of

$$5y_1 - 4y_2 + y_3 - \frac{1}{360}h^4(205f_1 + 76f_2 + f_3)$$
$$= 2A_0 - h^2 B_0 + \frac{2}{15}h^4 f_0 \qquad (8)$$

formulas

and

$$y_{N-2} - 4y_{N-1} + 5y_N - \frac{1}{360}h^4(f_{N-2} + 76f_{N-1} + 205f_N)$$
$$= 2A_1 - h^2 B_2 + \frac{2}{15}h^4 f_{N+1}\,, \qquad (9)$$

for which the local truncation errors are, respectively,

$$t_1^{(1)} = -\frac{1}{18}h^6 y^{(vi)}(x_1) - \frac{521}{60480}h^8 y^{(viii)}(x_1) - \ldots \qquad (10)$$

and

$$t_N^{(1)} = -\frac{1}{18}h^6 y^{(vi)}(x_N) - \frac{521}{60480}h^8 y^{(viii)}(x_N) - \ldots \qquad (11)$$

The second order algorithm on which the procedure of §2 is based, is therefore described by {(8),(6),(9)} and the solution vector

$\underline{Y}^{(1)} = [y_1^{(1)}, y_2^{(1)}, ..., y_N^{(1)}]^T \equiv [y_1, y_2, ..., y_N]^T$, T denoting transpose, is obtained by solving a nonlinear algebraic system of order N which has the form

$$A_h \underline{Y}^{(1)} - \frac{1}{81} h^4 M_h \underline{f}(1) (\underline{Y}^{(1)}) = \underline{r}^{(1)} \tag{12}$$

In (12) the matrices A_h, and M_h, are both quindiagonal and of order N and are given by

$$A_h = \begin{bmatrix} 5 & -4 & 1 & & & & & & & 0 \\ -4 & 6 & -4 & 1 & & & & & & \\ 1 & -4 & 6 & -4 & 1 & & & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & & 1 & -4 & 6 & -4 & 1 & \\ & 0 & & & & 1 & -4 & 6 & -4 & \\ & & & & & & 1 & -4 & 5 & \end{bmatrix} \tag{13}$$

and

$$M_h = \begin{bmatrix} 46.125 & 17.1 & 0.225 & & & & 0 \\ 14 & 51 & 14 & 1 & & & \\ 1 & 14 & 51 & 14 & 1 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & 1 & 14 & 51 & 14 & 1 \\ 0 & & & 1 & 14 & 51 & 14 \\ & & & & 0.225 & 17.1 & 46.125 \end{bmatrix} \tag{14}$$

The vectors $\underline{f}^{(1)}$ and $\underline{r}^{(1)}$ each have N components and are given by

$$\underline{f}^{(1)} = [f_1^{(1)}, f_2^{(1)}, \ldots, f_N^{(1)}]^T \tag{15}$$

where $f_n^{(1)}(\underline{Y}^{(1)}) = f(x_n, y_n^{(1)})$, $n = 1,2,\ldots,N$, and

$$\underline{r}^{(1)} = [2A_0 - h^2 B_0 + \frac{2}{15} h^4 f_0, -A_0 + \frac{h^4}{81} f_0, 0, \ldots, 0, -A_1 + \frac{h^2}{81} f_{N+1}, 2A_1 - h^2 B_1 + \frac{2}{15} h^4 f_{N+1}]^T \tag{16}$$

The vector $\underline{y}^{(1)} = [y(x_1), y(x_2), \ldots, y(x_N)]^T$ clearly satisfies the equation

$$A_h \underline{y}^{(1)} - \frac{1}{81} h^4 M \underline{f}^{(1)}(\underline{y}^{(1)}) = \underline{r}^{(1)} + \underline{t}^{(1)}, \tag{17}$$

where $\underline{t}^{(1)} = [t_1^{(1)}, t_2^{(1)}, \ldots, t_N^{(1)}]^T$ is the vector of order N of local truncation errors. Defining $\underline{E}^{(1)} = \underline{y}^{(1)} - \underline{Y}^{(1)} = [e_1^{(1)}, e_2^{(1)}, \ldots, e_N^{(1)}]^T$, it is seen that $\underline{E}^{(1)}$ satisfies

$$\underline{f}^{(1)}(\underline{y}^{(1)}) - \underline{f}^{(1)}(\underline{Y}^{(1)}) = F_h(\underline{y}^{(1)}) \underline{E}^{(1)} \tag{18}$$

where $F_h = F_h(\underline{y}^{(1)}) = \text{diag}\{\partial f_i^{(1)} / \partial y_i^{(1)}\}$ is a diagonal matrix of order N. It follows that $\underline{E}^{(1)}$ satisfies

$$\underline{E}^{(1)} = P_h \underline{t}^{(1)} \tag{19}$$

where P_h. is the matrix of order N given by

$$P_h = (I_h - \frac{1}{81} h^4 A_h^{-1} M_h F_h)^{-1} A_h^{-1} \tag{20}$$

and I_h, is the identity matrix of order N.

It is shown in Usmani and Marsden [3] that

$$\| A_h^{-1} \|_\infty \le \frac{5}{384 h^4} (b-a)^2 Z_h, \tag{21}$$

where $Z_h, = (b-a)^2 + 0.8h^2$ (the norm referred to is the L_∞ norm and from this point onwards the subscript will be omitted), and it is easy to see from (14) that $\| M_h \| \le 81$. Using these norms, it was verified in [2] that {(8),(6),(9)} is a second order convergent method provided

$$U < 76.8 / \{(b-a)^2 Z_h\} \tag{22}$$

where $U = \max|\partial f / \partial y|$.

2. THE FOURTH ORDER METHOD

Suppos e now that a second, finer, grid of step size $\frac{1}{2}h$ is used. The interval $a < x < b$ is thus divided into 2N+2 subintervals each of width ½h and the points x_n ($n = 1,2,\ldots,N$) of the coarse grid of §t are named x_m ($m= 2,4,..,2N$) with respect to the fine grid for which the finite difference formulas (8), (6) and (9) are modifed to

$$5y_1 - 4y_2 + y_3 - \frac{h^4}{5760}(205f_1 + 76f_2 + f_3) = 2A_0 - \frac{1}{4}h^2 B_0 + \frac{h^4}{120} f_0 , \qquad (23)$$

$$y_{m-2} - 4y_{m-1} + 6y_m - 4y_{m+1} + y_{m+2}$$

$$- \frac{h^4}{1296}(f_{m-2} - 14f_{m-1} + 51f_m - 14f_{m+1} + f_{m+2}) = 0 , \ m-1, \ldots, 2N \qquad (24)$$

and

$$y_{2N-1} - 4y_{2N} + 5y_{2N+1} - \frac{h^4}{5760}(f_{2N-1} + 76f_{2N} + 205_{2N+1})$$

$$= 2A_1 - \frac{1}{4}h^2 B_1 + \frac{h^4}{120} f_{2N+2} , \qquad (25)$$

The solution vector of these 2N+2 nonlinear algebraic equations will be denoted by $\underline{Y}^{(2)}$ so that $\underline{Y}^{(2)}$ satisfies the equation

$$A_{\frac{1}{2}h}\underline{y}^{(2)} - \frac{1}{1296}h^4 M_{\frac{1}{2}h}\underline{f}^{(2)}(\underline{Y}^{(2)}) = \underline{r}^{(2)} . \qquad (26)$$

The forms of the matrices $A_{\frac{1}{2}H}$, $M_{\frac{1}{2}h}$. and of the vector $\underline{f}^{(2)}$ are obvious from (13), (14) and (15) while the vector $\underline{r}^{(2)}$ is given by

$$\underline{r}^{(2)} = [2A_0 - \frac{1}{4}h^2 B_0 + \frac{h^4}{120}f_0, -A_0 + \frac{h^4}{1296}f_0, 0, \ldots, 0, -A_1 + \frac{h^2}{1296}f_{2N+2}, 2A_1 - \frac{1}{4}h^2 B_1 + \frac{h^4}{120}f_{2N+2}]^T \qquad (27)$$

The matrix analogous to P_h, is $P_{\frac{1}{2}h}$, and this is defined by

$$P_{\frac{1}{2}h} = (I_{\frac{1}{2}h} - \frac{1}{1296}h^4 A^{-1}_{\frac{1}{2}h} M_{\frac{1}{2}h} F_{\frac{1}{2}h})^{-1} A^{-1}_{\frac{1}{2}h} \qquad (28)$$

in which $I_{\frac{1}{2}h}$, is the identity matrix of order 2N+1and $F_{\frac{1}{2}h} = \mathrm{diag}\{\partial f_i^{(2)} / \partial y_i^{(2)}\}$ is of order 2N+1.

The local truncation errors of (23), (24) and (25) are, respectively,

$$t_1^{(2)} = -\frac{1}{1152}h^6 y^{(vi)}(x_1) - \frac{521}{15482880}h^8 y^{(viii)}(x_1) - \dots , \tag{29}$$

$$t_m^{(2)} = -\frac{1}{1152}h^6 y^{(vi)}(x_m) - \frac{521}{1658880}h^8 y^{(viii)}(x_m) - \dots ,$$

$$(m = 1, \dots, 2N) \quad , \tag{30}$$

$$t_{2N+1}^{(2)} = -\frac{1}{1152}h^6 y^{(vi)}(x_{2N+1}) - \frac{521}{15482880}h^8 y^{(viii)}(x_{2N+1}) - \dots , \tag{31}$$

and the theoretical solution vector $\underline{y}^{(2)}$, relating to the fine gird, satisfies the equation

$$A_{\frac{1}{2}h}\underline{y}^{(1)} - \frac{16}{81}h^4 M_{\frac{1}{2}h}\underline{f}^{(2)}(\underline{y}^{(2)}) = \underline{r}^{(2)} + \underline{t}^{(2)} , \tag{32}$$

where $\underline{t}^{(2)} = [t_1^{(2)}, t_2^{(2)}, \dots, t_{2N+1}^{(2)}]^T$.

Suppose now that $I_{\frac{1}{2}h}^{h}$ is a fine-to-coarse grid restriction operator defined by

$$I_{\frac{1}{2}h}^{h}\underline{Y}^{(2)} = [y_2^{(2)}, y_4^{(2)}, \dots, y_{2N}^{(2)}] . \tag{33}$$

Then the components of $I_{\frac{1}{2}h}^{h}\underline{Y}^{(2)}$ give second approximations to y(x) at the N points x_n (n = 1,2,...,N) of the original (coarse) grid used in § 1. To develop a numerical method which is fourth order convergent, a parameter α must be determined such that the vector $\underline{E}^*$ of order N defined by

$$\underline{E}^* = \alpha I_{\frac{1}{2}h}^{h}\underline{E}^{(2)} + (1+\alpha)\underline{E}^{(1)} = \underline{y} - [\alpha I_{\frac{1}{2}h}^{h}\underline{Y}^{(2)} + (1+\alpha)\underline{Y}^{(1)}] , \tag{34}$$

where $\underline{E}^{(2)} = \underline{y}^{(2)} - \underline{Y}^{(2)} = [e_1^{(2)}, e_2^{(2)}, \dots, e_{2N+1}^{(2)}]^T$, has norm $\| E^* \| = 0(h^4)$.
In (34) it is clear that

$$\underline{E}^{(2)} = P_{\frac{1}{2}h}\underline{t}^{(2)} \tag{35}$$

where $\underline{t}^{(2)} = [t_1^{(2)}, t_2^{(2)}, \dots, t_{2N+1}^{(2)}]^T$. is the vector of local truncation errors given by (29),(30) and (31), an that $I_{\frac{1}{2}h}^{h}\underline{E}^{(2)} = [e_4^{(2)}, e_2^{(2)}, \dots, e_{2N}^{(2)}]^T$.

It may be shown from (21) that

$$\| A^{-1}_{\frac{1}{2}h} \| \leq \frac{5}{24h^4}(b-a)^2 Z_{\frac{1}{2}h} \quad , \tag{36}$$

where $Z_{\frac{1}{2}h} = (b\text{-}a)^2 + 0.2h^2$, and it is easy to see that $\| M_{\frac{1}{2}h} \| = 81$.

It then follows from (20) and (28) that

$$\| P_{\frac{1}{2}h} \| < \| P_h \| \tag{37}$$

Defining $\tau_h = \|\underline{t}^{(1)}\| / h^4$ and $\tau_{\frac{1}{2}h} = \| \underline{t}^{(2)} \| / h^4$, it is clear from (34) that

$$\| \underline{E}^* \| < \| P_h IK(\alpha \tau_{\frac{1}{2}h} (1 - \alpha) T_h) \tag{38}$$

provided (22) is satisfied. Noting from (7), (10), (11) that $\| \underline{t}^{(1)} \| = \frac{1}{18} h^6 V_6$, and from (29), (30), (31) that $\| \underline{t}^{(1)} \| = \frac{1}{1152} h^6 V_6$, where $V_6 = \max_{a \leq x \leq b} |y^{(vi)}(x)|$, it follows that

$$\| \underline{E}^* \| = O(h^4) \tag{39}$$

when $a = \frac{4}{3}$. The numerical formulation

$$\underline{Y}^{(E)} = \frac{4}{3} I^h_{\frac{1}{2}h} \underline{Y}(2) - \frac{1}{3} \underline{Y}^{(1)} \tag{40}$$

where $\underline{Y}^{(E)}$. is avector of order N, is therefore a fourth order convergent method,

Undoubtedly the main advantage of the fourth order method just dveloped is its ease of implementation, especially in comparison with other fourth order methods (in particular, multiderivative methods [2]). The novel method requires only two applications of a second order method, the solution being obtained by taking a linear combination of the results relating to the two applications.

3. FIRST ORDER DERIVATIVE BOUNDARY CONDITIONS

In this section, the boundary value problem consisting of the differential equation (1) with the boundary conditions

$$y(a) = A_0, \quad y'(a) = C_0, \; y(b) = A_1, \; y^1(b) = C_1 \tag{41}$$

will be considered.

The special formulas (8) and (9) must be replaced by

$$9y_1 - \frac{9}{2}y_2 + y_3 - \frac{1}{2520}h^4\,(1079f_1 + 608f_2 - 9f_3) = \frac{11}{2}A_0 + 3hC_0 + \frac{53}{630}\,h^4 f_0 \tag{42}$$

and

$$y_{N-2} - \frac{9}{2}y_{N-1} + 9y_N - \frac{1}{2520}h^4\,(-9f_{N-2} + 608f_{N-1} + 1079f_N) = \frac{11}{2}A_1 + 3hC_1 + \frac{53}{630}\,h^4 f_{N+1} \tag{43}$$

for which the local truncation errors are, respectively,

$$t_1^{(2)} = -\frac{1}{18}h^6 y^{(vi)}(x_1) - \frac{271}{60480}h^8 y^{(viii)}(x_1) - \ldots, \tag{44}$$

and

$$t_N^{(2)} = -\frac{1}{18}h^6 y^{(vi)}(x_N) - \frac{271}{60480}h^8 y^{(viii)}(x_N) - \ldots, \tag{45}$$

The second order algorithm is then given by {(42),(6),(43)}.

The matrix A_h, becomes

$$A_h = \begin{bmatrix} 9 & -\frac{9}{2} & 1 & & & 0 \\ -4 & 6 & -4 & 1 & & \\ 1 & -4 & 6 & -4 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & 1 & -4 & 6 & -4 & 1 \\ 0 & & 1 & -4 & 6 & -4 \\ & & & 1 & -\frac{9}{2} & 9 \end{bmatrix} \tag{46}$$

for which it is known [3] that

$$\| A_h^{-1} \| \leq \frac{1}{384h^4}(b-a)^4 Z_h \ , \tag{47}$$

where, now, $Z_h = 1+8h^3(b-a)^{-3}$, and the matrix M_h. becomes

$$M_h = \begin{bmatrix} \frac{9711}{280} & \frac{684}{35} & \frac{81}{280} & & & 0 \\ 14 & 51 & 14 & 1 & & \\ 1 & 14 & 51 & 14 & 1 & \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & 1 & 14 & 51 & 14 & 1 \\ 0 & & 1 & 14 & 51 & 14 \\ & & & \frac{81}{280} & \frac{684}{35} & \frac{9711}{280} \end{bmatrix} \tag{48}$$

for which $\| M_h \| = 81$ as before. The vector $\underline{r}^{(1)}$ also requires modification; it becomes

$$\underline{r}^{(1)} = [\frac{11}{2}A_0 + 3hC_0 + \frac{53}{630}h^4 f_0, -A_0 + \frac{h^4}{81}f_0, 0, \ldots, 0, -A_1 + \frac{h^4}{81}f_{N+1}, \frac{11}{2}A_1 - 3hC_1 + \frac{53}{630}h^4 f_{N+1}]^T \tag{49}$$

The restriction on U given in (22) is no longer applicable; it must be replaced by

$$U < 384/\{(b-a)^4 Z_h\} \ , \tag{50}$$

where $Z_h = 1 + 8h^3(b-a)^{-3}$.

Considering, next, the application of the second order algorithm to the fine grid, it is evident that equations (23) and (25) need modifications. They become

$$9y_1 - \frac{9}{2}y_2 + y_3 - \frac{h^4}{40320}(1079f_1 + 608f_2 - 9f_3)$$

$$= \frac{11}{2}A_0 + 3hC_0 + \frac{53}{10080}h^4 f_0 \tag{51}$$

and

$$y_{N-2} - \frac{9}{2}y_{N-1} + 9y_N - \frac{h^4}{40320}(-9f_{N-2} + 608f_{N-1} + 1079f_N)$$
$$= \frac{11}{2}A_1 + 3hC_1 + \frac{53}{10080}h^4 f_{N+1} \quad , \tag{52}$$

respectively, for which the local truncation errors are

$$t_1^{(2)} = -\frac{1}{1152}h^6 y^{(vi)}(x_1) - \frac{271}{15482880}h^8 y^{(viii)}(x_1) - \ldots , \tag{53}$$

and

$$t_N^{(2)} = -\frac{1}{1152}h^6 y^{(vi)}(x_N) - \frac{271}{15482880}h^8 y^{(viii)}(x_N) - \ldots , \tag{54}$$

With respect to the fine grid, the second order method is defined by {(51),(24),(52)} and the elements of the vector $\underline{t}^{(2)}$ are given by (53), (30) and (54).

The forms of the revised matrices $A_{½h}$ and $M_{½h}$ are obvious from (46) and (48), and the vector $\underline{r}^{(2)}$ is now seen to take the form

$$\underline{r}^{(2)} = [\frac{11}{2}A_0 + \frac{3}{2}hC_0 + \frac{53}{10080}h^4 f_0, -A_0 + \frac{h^4}{1296}f_0, 0, \ldots, 0, -A_1\frac{h^4}{1296}f_{2N+2},$$
$$\frac{11}{2}A_1 + \frac{3}{2}hC_1 + \frac{53}{10080}h^4 f_{2N+2}]^T . \tag{55}$$

Clearly, $\| A_{\frac{1}{2}h}^{-1} \| \le \frac{5}{24h^4}(b-a)^2 Z_{\frac{1}{2}h}$, where $Z_{\frac{1}{2}h} = 1+h^3(b-a)^{-3}$, and $\| M_{\frac{1}{2}h} \| \le 81$, and convergence of the fourth order method given by (40) is established for the boundary value problem {(1),(41)} as in §2.

4, NUMERICAL EXPERIMENTS

The fourth order method discussed in §2 was tested on the following problem from the literature.

Problem 1

$$y^{(vi)}(x) = 6\exp[-4y(x)] - 12(1+x)^{-4}, \quad 0 < x < 1$$

with boundary conditions

$$y(0) = 0 \ , \ y(1) = ln2 \ , y''(0) = 1 \ , y''(1) = -0.25$$

for which the theoretical solution is

$$y(x) = ln(1+x)$$

The interval [0,1] was divided into N+1 equal parts each of width $h = 2^{-m}$ with m= 3,4,5 so that $N = 2^m - 1$ =7,15,31, respectively.

The value of $\| \underline{y}-\underline{Y}^{(E)} \|$, where $\underline{Y}^{(E)}$ is defined in equation (40), was calculated for each value of N. The numerical results are tabulated in Table 1 which also includes results obtained using the fourth order methods of Agarwall and Akrivis [1] as well as the second order method {(8) , (6),(9)}, with the coarse grid, on which the novel method is based.

Observing the contents of Table 1, it is evident that the fourth order method of §2 gives better numerical results than either of the other fourth order methods tested. Bearing in mind its ease of implementation, the method is clearly an economic alternative to the other fourth order methods.

The adaptation in §3 of the novel fourth order method to problems with functional and first order derivative boundary conditions was also tested. The problem on which this adaptation was tested is given by

Problem 2

$$y^{(iv)}(x) = 6\exp[-4y(x)] - 12(1+x)^{-4} \ , \ 0 < x < 1 \ ,$$

with boundary conditions

$$y(0) = 0 \ , y(1) = ln2 \ , \ \ y'(0) = 1 \ , y'(1) = 0.5 \ ,$$

for which the theoretical solution is $y = ln(1+x)$ as in Problem 1. The three discretizations used for Problem 1 were also used for Problem 2 and the value of $\| \underline{y} - \underline{Y}^{(E)} \|$ was calculated in each case.

The numerical results are given in Table 2 from which it may be observed that the method of §3 retains the accuracy attained for the problem with functional and second order derivative boundary conditions.

Only three applications of the Newton-Raphson method for nonlinear algebraic systems were need to give convergence to three significant figures for both problems,

5. SUMMARY

A fourth order convergent finite difference method has been developed and analyzed for the numerical solution of the general fourth order nonlinear boundary value problem $y^{(iv)}(x) = f(x,y)$ with boundary conditions given in the form of two functional values together with (i) two second order derivatives, or (ii) two first order derivatives.

The method was based on a second order convergent method which was used on two grids, fourth order convergence being obtained by considering a linear combination of the results determined for each grid individually. Special formulas were developed for application to points adjacent to the boundary, the principal parts of the local truncation errors of these formulas being the same as that of the second order method which was used at other points of the discretization.

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Table 1: Error norms for Problem 1.

		second order	Agarwal and Akrivis		fourth order
m	N	method	Method A	Method B	method
3	7	0.19E-3	0.14E-4	0.14E-4	0.37E-5
4	15	0.46E-4	0.83E-6	0.83E-6	0.29E-6
5	31	0.11E-4	0.54E-7	0.54E-7	0.19E-7

Table 2: Error norms for Problem 2.

		econd order	fourth order
m	N	method	method
3	7	0.15E-3	0.22E-4
4	15	0.23E-4	0.42E-5
5	31	0.27E-5	0.67E-6