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Irregular C^2 Surface Construction Using Bi-polynomial Rectangular Patches

by

John A. Gregory and Jianwei Zhou

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John A. Gregory and Jianwei Zhou

Department of Mathematics and Statistics

Brunei University, Uxbridge, UBS 3PH, England

Abstract. The construction of C^2 surfaces using bi-polynomial parametric rectangular patches is studied. In particular, the analysis of the C^2 continuity conditions for the case of n patches meeting at an n-vertex is developed.

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1. Introduction

The theory of splines over a partition in JR^2 is concerned with the construction of bivariate piecewisepolynomial(scalar) surfaces of a given degree and a given order of continuity. The theory of such splines involves a careful analysis of continuityconditions around vertices of the partition. There is an analogous problem in Differential Geometry and CAGD (Computer Aided Geometric Design) concernedwith the construction of smooth complexes of parametric surface patches. Here, however, the $\mathbb{R}^2$ domains of the patches need not be considered as belonging to one common partition of $\mathbb{R}^2$. Indeed, the construction of closed surfaces cannot be achieved as a mapping from a single domain.

In this paper, we are concerned with the particular problem of the analysis of the C^2 continuity conditions which arise for the case of n rectangular patches meeting at an n-vertex. Each patch is a bi-polynomial mapping of the unit square $[0,1]^2$ into $\mathbb{R}^m$, where m = 3 is of specific interest to CAGD. For $n \neq 4$, we clearly have a situation which cannot be analysed using componentwise parametric C^2 continuity. It is necessary to view the continuity between two adjacent patches in terms of 'contact of order 2' conditions. Such conditions are called 'geometric continuous GC^2 conditions' in the CAGD literature.

The case of parametric patches meeting with GC^1, i.e. tangent plane, continuity has been considered by a number of authors, a good reference for which is the review article of Peters [7]. Also, in a recent paper [3] we studied the problem of n bicubic patches meeting at an n-vertex subject to GC^1 constraints. Here we wish to extend the method of analysis for the C^1 case to that of the construction of C^2 surfaces. We are interested in the use of bi-polynomials of minimal degree which results in a C^2 surface about the n-vertex. It will be proved that bi-polynomial patches which are quartic in position, first and second derivatives along the interior edges about the n-vertex are sufficient for the purposes of constructing C^2 surfaces.

Although the paper is primarily concerned with the proof of existence in Sections 2-6, we conclude in section 7 with some observations about the use of the theory in the practical setting of CAGD.

2. Description of the Problem

Consider an n-sided hole surrounded by a bi-polynomial C^2 rectangular patch complex as shown in figure 1. The hole is partitioned as shown in figure 2, where we wish to use n additional bi-polynomial rectangular patches to fill the hole with C^2 continuity for the resulting surface. Each additional patch thus has a common central 'n-vertex', a '4-vertex' at a corner of the hole and two 'mid-point' vertices on the boundary of the hole. We refer to an edge joining the central vertex and a boundary mid-point vertex as an 'interior edge'.

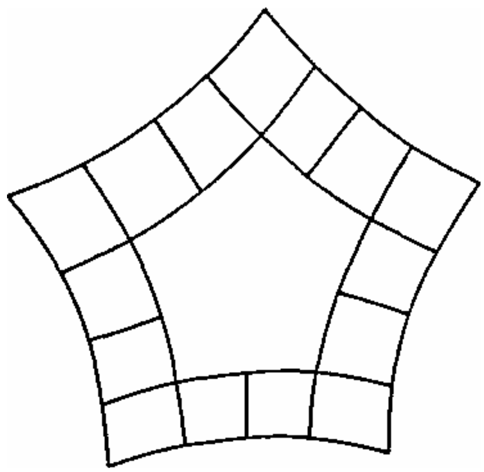

Figure 1 The polygonal hole problem

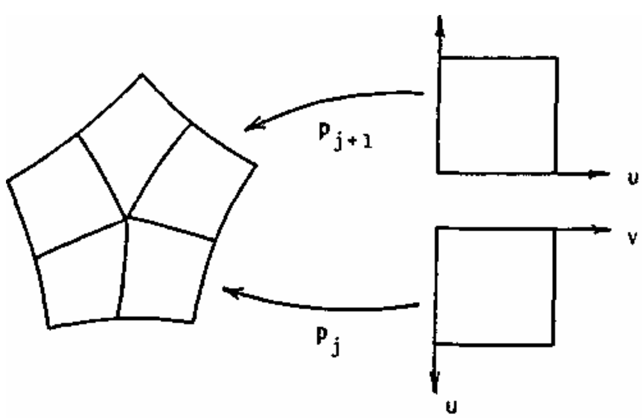

Figure 2 Filling the hole with rectangular patches

Let P_j, $j = 0,\ldots,n-1$, be the n additional bi-polynomial patches parameterized as shown in figure 2. Then $P_j(s,1)$ and $P_{j+1}(1,s)$ defines the j'th boundary edge of the hole; $\partial_{0,1} P_j(s,1)$ and $\partial_{1,0} P_{j+1}(1,s)$ defines the cross boundary first derivative; and $\partial_{0,2} P_j(s,1)$ and $\partial_{2,0} P_{j+1}(1,s)$ defines the cross boundary second derivative.

To achieve C^2 continuity inside the hole, the bi-polynomial patches must be constrained such that they join with 'geometric' GC^2 continuity across interior edges, that is, the patches join with C^2 continuity under a reparameterization. Such geometric continuity conditions between patches have been extensively studied, see for example DeRose[1], Hahn[4] and Höllig[5]. In our case, the conditions for a GC^2 join of the patches P_j across the interior edges give the constraints

$$(2.1) \qquad P_{j+1}(s,0) - P_j(0,s) = 0,$$

$$(2.2) \qquad \alpha_j(s)\partial_{10}p_j(0,s) + \beta_j(s)\partial_{01}p_j(0,s) + \gamma_j(s)\partial_{01}p_{j+1}(s,0) = 0,$$

where $\alpha_j(s) > 0$ and $\gamma_j(s) > 0$, and

$$\gamma_j^2(s)\partial_{02}p_{j+1}(s,0) = \sigma_j(s)\partial_{10}p_j(0,s) + \rho_j(s)\partial_{01}p_j(0,s) +$$

$$(2.3) \qquad \alpha_j^2(s)\partial_{20}p_j(0,s) + 2\alpha_j(s)\beta_j(s)\partial_{11}p_j(0,s) + \beta_j^2(s)\partial_{02}p_j(0,s),$$

for $j = 0,\ldots, n-1$. Here, the scalar functions $\alpha_j(s)$ etc. are polynomial unknowns. Condition (2.1) is that for continuity of position, (2.2) is that for tangent plane continuity and (2.3) is that for curvature continuity.

We wish to analyse the constraints (2.1)-(2.3) about the n-vertex, using bi-polynomial patches of minimal degree consistent with the existence of a solution. We thus consider bi-polynomial patches whose position, first and second cross boundary derivatives along the interior edges are no higher than quartic. The use of quintics gives more constraint equations than is necessary and the use of cubics proves to be too restrictive. Indeed, we hypothesise that for a C^K construction, the minimal degree of first cross boundary derivatives should be at least $k + 2$, see Zhou [8].

The interior edge conditions will be analysed using quartic Hermite representations which leads to some simplification of the theory. Thus, we denote the relevant Hermite data in these GC^2 conditions as

(2.4) $$B_j^{u^l u^m} = (-1)^{l+m} \partial_{l,m}\, p_j(0,1) = (-1)^m \partial_{m,l}\, p_{j+1}(1,0), \quad 0 \le l,m \le 2,$$

for the data at the boundary mid-point vertices and

(2.5) $$\begin{cases} Q &= p_j(0,0), \\ Q_j &= \partial_{01} p_j(0,0) = \partial_{10} p_{j+1}(0,0), \\ Q_{j-1,j} &= \partial_{11} p_j(0,0), \end{cases}$$

at the central n-vertex. Here, we note that since the boundary mid-point vertices are regular 4-vertices, $C^{2,2}$ continuity is assumed, consistent with a regular rectangular patch complex, see (2.4). These notations enable us to write the following polynomials:

(2.6) $$p_{j+1}(s,0) = p_j(0,s) = QH_o(s) + Q_j H_1(s) + B_j^{u^2}\ H_2(s) - B_j^v H_3(s) + B_j H_4(s),$$

(2.7) $$\partial_{1,0} p_j(0,s) = Q_{j-1} H_0(s) + Q_{j-1}, j H_1(s) - B_j^{uv^2} H_2(s) + B_j^{uv} H_3(s) - B_j^u H_4(s),$$

(2.8) $$\partial_{0,1} p_{j+1}(s,0) = Q_{j+1} H_0(s) + Q_{j,j+1} H_1(s) + B_j^{uv^2} H_2(s) - B_j^{uv} H_3(s) + B_j^u H_4(s),$$

(2.9) $$\partial_{2,p} p_j(0,s) = Q'_{j+1} H_0(s) + Q''_{j+1} H_1(s) + B_j^{u^2v^2} H_2(5) - B_j^{u^2v} H_3(s) - B_j^{u^2} H_4(s),$$

And

(2.10) $$\partial_{0,2} p_{j+1}(s,0) = Q'_{j+1} H_0(s) + Q'''_{j+1} H_1(s) + B_j^{u^2v^2} H_2(s) - B_j^{u_2 v} H_3(s) - B_j^{u^2} H_4(s),$$

for j = 0, ...,n- 1, where

$$(2.11)\qquad \begin{cases} Q_j^{'} = -12Q + 12B_j - 6Q_j 6B_j^{v} + B_j^{v^2}, \\ Q_j^{''} = -12Q_{j+1} + 12B_j^{u} - 6Q_{j,j+1} + 6B_j^{uv} + B_j^{uv^2}, \\ Q_j^{'''} = -12Q_{j-1} - 12B_j^{u} - 6Q_{j-1,j} - 6B_j^{uv} - B_j^{uv^2}. \end{cases}$$

Here, the quartic Hermite basis functions are given by

$$(2.12)\qquad \begin{cases} H_0(s) = 1 - 6s^2 + 8s^3 - 3s^4, \\ H_1(s) = s - 3s^2 + 3s^3 - s^4, \\ H_2(s) = s^2/2 - s^3 + s^4/2, \\ H_3(s) = -3s^2 + 5s^2 - 2s^4, \\ H_4(s) = 6s^2 - 8s^3 + 3s^4, \end{cases}$$

The constraints (2.1) are automatically satisfied by the identification of Hermite data along interior edges. In the remaining sections, we shall analyse and solve the tangent plane constraints (2.2) and the curvature constraints (2.3).

3. Simplified Continuity Constraints

From (2.2) and (2.3), together with (2.4), we derive

$$(3.1)\qquad \alpha_j(1) - \gamma_j(1) = \alpha_j^{'}(1) - \gamma_j^{'}(1) = \alpha_j^{''}(1) - \gamma_j^{''}(1) = 0, \quad \beta_j(1) = \beta_j^{'}(1) = \beta_j^{''}(1) = 0$$

and

$$(3.2)\qquad \sigma_j(1) = \sigma_j^{'}(1) = \sigma_j^{''}(1) = 0, \quad \rho_j(1) = \rho_j^{'}(1) = \rho_j^{''}(1) = 0$$

The minimum degree scalar polynomials consistent with (3.1) and (3.2) are

$$(3.3)\qquad \alpha_j(s) = 1 = \gamma_j(s), \quad \beta_j(s) = \beta_{0,j}(1-s)^3$$

and

$$(3.4)\qquad \sigma_j(s) = \sigma_{0,j}(1-s)^3, \quad \rho_j(s) = \rho_{0,j}(1-s)^3.$$

Here we assume

(3.5) $$\alpha_j(0) = 1, \quad j = 0,\dots,n-1,$$

without loss of generality. Further simplification is made by imposing symmetric coefficients

(3.6) $$\beta_{0,j} = \beta_{0,} \ \sigma_{0,j} = \sigma_{0,} \ \rho_{0,\varphi} = \rho_{0,} \ j = 0,\dots,n-1.$$

Using these minimum degree scalar polynomials for the continuity constraints (2.2) and (2.3) and removing the common factor $(1 - s)^3$, we obtain the following simplified continuity constraints

(3.7) $$\Phi_j(s) := (Q_{j-1} + Q_{j+1})(1+3s) + (Q_{j-1,j} + Q_{j,j+1})s + \beta_0 \delta_{10} P_{j+1}(s,0) = 0$$

and

$$\psi_j(s) := (Q'_{j+1} - Q_{j-1})(1+3s) + (Q'''_{j+1} + Q''_{j-1})s + \beta_0^2 (1-s)^3 \partial_{20} P_{j+1}(s,0)$$

(3.8) $$+ 2\beta_0 \partial_{10} P_{j+1}(s,0) + \sigma_0 \partial_{10} P_{j+1}(s,0) + \rho_0 \partial_{01} P_{j+1}(s,0) = 0$$

for tangent plane and curvature continuities across the interior edges $B_j Q$

$j = 0,\dots n-1$

In the following sections, we shall analyse these simplified continuity con straints.

4. Analysing the Tangent Plane Constraints

Equating coefficients in (3.7) gives

(4.1)

$$\begin{cases} Q_{j-1} + \beta_0 Q_j + Q_{j+1} = 0, \\ 3(Q_{j-1} + Q_{j+1}) + (Q_{j-1,j} + Q_{j,j+1})\beta_0(-12Q + 12B_j - 6Q_j + 6B_j^{\nu} + B^{\nu^2}) = 0 \\ \beta_0(24Q - 24B_j + 9Q_j - 15B_j^{\nu} - 3B^{\nu^2}) = 0, \\ \beta_0(-12Q + 12B_j - 4Q_j + 8B_j^{\nu} + 2B^{\nu^2}) = 0, \end{cases}$$

The first equation of (4.1) gives a cyclic system of difference equations for the tangent vectors at the central vertex. An analysis of this system, see Gregory and Zhou [3], gives

$$\beta_0 = -2\cos(2\pi / n) \tag{4.2}$$

and

$$Q_j = X\cos(2j\pi / n) + Y\sin(2j\pi / n)\, j = 0,...,n-1, \tag{4.3}$$

where X and Y are two linearly independent vectors.

Obviously, $\beta \neq 0$ is concluded from (4.2) since the case n = 4 is excluded from our polygonal hole problems. Therefore, we can write

$$\begin{cases} Q_{j-1} + \beta_0 Q_j + Q_{j+1} = 0, \\ 2Q - 2B + Q_j - B_j^{\nu} = 0, \\ -6Q + 6B_j - 2Q_j + 4B_j^{\nu} + B^{\nu^2}) = 0, \\ Q_{j-1,j} + Q_{j,j+1} + \beta_0 (B_j^{\nu^2} - 3Q_j) = 0, \end{cases} \tag{4.4}$$

as an equivalent form of (4.1).

The following proposition is a simple consequence of (4.4).

Proposition 1. Given $Q, \{Q_{j,j+1}\}_{j=0}^{n-1}$ *and* $\{Q_j\}_{j=0}^{n-1}$ *satisfying* (4.3) so that

$$Q_{j-1} + \beta_0 Q_j + Q_{j+1} = 0,\ j = 0,....,n-1, \tag{4.5}$$

then a solution to the tangent plane constraints (3.7) is

$$\begin{cases} B_j = Q + \dfrac{5}{2}Q_j - \dfrac{1}{2\beta_0}(Q_{j-1,j} + Q_{j,j+1}), \\ B_j^{\nu} = -4Q + \dfrac{1}{\beta_0}\left(Q_{j-1,j} + Q_{j,j+1}\right), \qquad j = 0,...,n-1. \\ B_j^{\nu^2} = 3Q_j - \dfrac{1}{\beta_0}\left(Q_{j-1,j} + Q_{j,j+1}\right) \end{cases} \tag{4.6}$$

5. Analysing Curvature Conditions at Central Vertex

In this and the following section, we assume that the tangent plane constraints (3.7) for j=0 ..., n-1 have been satisfied. Therefore (4.5) and (4.6) hold for j= 0,…,n- 1.

The curvature conditions, at the central vertex,

$$(5.1)\qquad -B_{j-1}^{v^2}+\beta_0^2 B_j^{v^2}+B_{j+1}^{v^2}+2\beta_0 Q_{j,j+1}+\sigma_0 Q_j+\rho_0 Q_{j+1}=0,\ j=0,\ldots n-1,.$$

are derived by requiring Ψ, (0)= 0 for j=0,n-1 Using

$$(5.2)\qquad B_j^{v^2}=2Q_j-\frac{1}{\beta_0}\left(Q_{j-1,j}+Q_{j,j+1}\right),Q_{j-1}+Q_{j+1}=-\beta_0 Q_j$$

see (4.6) and (4.5), we rewrite (5.1) as

$$Q_{j-2,j-1}+\left(1-\beta_0^2\right)Q_{j,j+1}-\left(1-\beta_0^2\right)Q_{j,j+1}-Q_{j,j+1}=$$

$$(5.3)\qquad \beta_0\left(6+\rho_0\right)Q_{j-1}-\beta_0\left(3\beta_0^2-3\beta_0-\rho_0\beta_0+\sigma_0\right)Q_j,\quad j=0,\ldots.,n-1.$$

In the case $n=3$, we have $\beta_0=1$ and

$$Q_{j-2,j-1}=Q_{j+1,j+2},\ j=0,1,2.$$

Thus, the left hand side of (5.3) is identically zero for $n=3$. The linear independence of Q_{j-1} and Q_j then requires

$$(5.4)\qquad \sigma_0=\rho_0=-6$$

for n = 3.

In the case $n\geq 5$, we shall prove that only n-3 equations of the n equations in (5.3) are linearly independent and (5.3) can be satisfied by solving the 'twists' $\left\{Q_{j,j+1}\right\}_{j=0}^{n-1}$ from an underdetermined system. As we have assumed symmetric coefficients, the following lemma about cyclic matrices is very useful for our analysis.

Lemma 2. Denote

$$(5.5)\qquad A=\left(a_{jk}\right)_{n\times n}=\begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \\ b_{n-1} & b_0 & \cdots & b_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & \cdots & b_0 \end{bmatrix}$$

and

$$(5.6)\qquad \xi_k=e^{i\frac{2k\pi}{n}}, k=0,...n-1.$$

Then, the eigenvalues of the cyclic matrix A are

$$(5.7)\qquad \lambda_k=\sum_{i=0}^{n-1} b_i\xi_k^i,\quad k=0,....,n-1,$$

and for $0\le k\le n-1$,

$$(5.8)\qquad x_k=\left(1,\xi_k^{-1},...,\xi_k^{1-n}\right)$$

is a left eigenvector of A corresponding to λ_k.

Applying Lemma 2 to the coefficient matrix of the system given by (5.3), we immediately derive

Corollary 3. The eigenvalues of the coefficient matrix of (5.3) are

$$(5.9)\qquad \lambda_k=-(\xi_k-1)\left(\xi_k-e^{i\frac{4\pi}{n}}\right)\left(\xi_k-e^{-i\frac{4\pi}{n}}\right), k=0,...,n-1.$$

Therefore, a λ_k is zero if and only if k is 0, 2 or n-2.

Proof. For the coefficient matrix of the system given by (5.3), we have

$$b_0=1, b_1=1-\beta_0^2, b_2=-(1-\beta_0^2), b_3=-1$$

and

$$b_4 = \cdots = b_{n-1} = 0.$$

Thus by the lemma, we get

$$\lambda_k = 1 + \left(1 - \beta_0^2\right)\xi_k - \left(1 - \beta_0^2\right)\xi_k^3$$

$$= -\left(\xi_k - 1\right)\left(\xi_k - e^{i\frac{4\pi}{n}}\right)\left(\xi_k - e^{-i\frac{4\pi}{n}}\right)$$

for k=0,...,n-1.

This corollary shows that the rank deficiency of the cyclic coefficient matrix of (5.3) is always 3. Hence it is always singular.

We now consider the solvability of (5.3) in the case $n \geq 5$ By Lemma 2., we only need to show

(5.10) $$\sum_{j=0}^{n-1} \xi_k^{-j} \left[\beta_0 \left(6 + \rho_0\right) Q_{j-1} - \beta_0 \left(3\beta_0^2 - 3\beta_0 + \rho_0 \beta_0 + \sigma_0\right) Q_j\right] = 0$$

for $k = 0, 2$ and n-2 to prove that the system given by (5.3) is consistent. Since

$$Q_j = \mathrm{X}\cos\left(2j\pi / n\right) + \mathrm{Y}\sin\left(2j\pi / n\right)$$

and

$$\cos\frac{2j\pi}{n} = \frac{1}{2}(\xi_1^j + \xi_1^{-j}), \sin\frac{2j\pi}{n} = \frac{1}{2i}(\xi_1^j + \xi_1^{-j}),$$

our task is thus reduced to verifying that

(5.11) $$\sum_{j=0}^{n-1} \xi_k^{-j} \xi_1^{m \pm j} = 0$$

for k=0, 2 and n-2 where m is some integer.

The left hand side of (5.11) is

$$\sum_{j=0}^{n-1} \xi_k^{-j} \xi_1^{m \pm j} = \xi_1^m \sum_{j=0}^{n-1} (\xi_k^{-1} \xi_1^{\pm 1})^j = \xi_1^m \sum_{j=0}^{n-1} \left[e^{-i\frac{2(k \mp 1)\pi}{n}} \right]^j.$$

As $e^{-i\frac{2(k\mp1)\pi}{n}} \neq 1$ for k=0, 2 and n-2 in the case $n \geq 5$, we have

$$\sum_{j=0}^{n-1}\left[e^{-i\frac{2(k\mp1)\pi}{n}}\right]^j = 0$$

and hence (5.10) is true. This demonstrates that (5.3) defines a consistent system in the case $n \geq 5$.

The following proposition summarizes our conclusion.

Proposition 4. The curvature conditions (5.3) at the central vertex can be satisfied by choosing $\sigma_0 = \rho_0 = -6$ in the case n=3 or by solving the twists $\{Q_{j,j+1}\}_{j=0}^{n-1}$ from the under determined, but consistent, system defined by (5.3) in the case $n \geq 5$.

6. Analysing Curvature Constraints along Interior Edges

In this section we further analyse the solvability of $\Psi_j(s) = 0$. For simplicity, we write

(6.1) $$\partial_{01}P_{j+1}(s,0) = Q_{j+1} + Q_{j,j+1}s + L_j s^2 + M_j s^3 + N_j s^4$$

where

(6.2) $$\begin{cases} \mathrm{L}_j = -6\mathrm{Q}_{j+1} + 6\mathrm{B}_j^u - 3\mathrm{Q}_{j,j+1} + 3\mathrm{B}_j^{uv} + (1/2)\mathrm{B}_j^{uv^2} \\ \mathrm{M}_j = 8\mathrm{Q}_{j+1} - 8\mathrm{B}_j^u + 3\mathrm{Q}_{j,j+1} - 5\mathrm{B}_j^{uv} + \mathrm{B}_j^{uv^2} \\ \mathrm{N}_j = -3\mathrm{Q}_{j+1} + 3\mathrm{B}_j^u - \mathrm{Q}_{j,j+1} + 2\mathrm{B}_j^{uv} + (1/2)\mathrm{B}_j^{uv^2} \end{cases}$$

Then, substituting (6.1) into the simplified constraints (3.8) and equating coef-ficients, we obtain

(6.3) $$\begin{cases} \rho_0\mathrm{N}_j = 0, \\ -\beta_0^2\mathrm{B}_j^{v^2} + 8\beta_0\mathrm{N}_j + \rho_0\mathrm{M}_j = 0, \\ 3\beta_0^2\mathrm{B}_j^{v^2} + 6\beta_0\mathrm{M}_j + \rho_0\mathrm{L}_j = 0, \\ 3\left(\mathrm{B}_{j+1}^{v^2} - \mathrm{B}_{j-1}^{v^2}\right) + \left(\mathrm{Q}_{j+1}^m - Q_{j-1}^n\right) - 3\beta_0^2\mathrm{B}_j^{v^2} + 4\beta_0\mathrm{L}_j + \sigma_0\mathrm{B}_j^{v^2} + \rho_0\mathrm{Q}_{j,j+1} = 0, \\ \left(\mathrm{B}_{j+1}^{v^2} - \mathrm{B}_{j-1}^{v^2}\right) + \beta_0^2\mathrm{B}_j^{v^2} + 2\beta_0\mathrm{Q}_{j,j+1} + \sigma_0\mathrm{Q}_j + \rho_0\mathrm{Q}_{j,j+1} = 0, \end{cases}$$

for j= 0, ...,n-1.

The last equation of (6.3) is just $\Psi_j(0) = 0$ which has been discussed in section 5. Hence only the first four equations of (6.3) are to be solved. We solve these equations for the cases $n\geq 5$ and n=3 separately.

Consider the case $n\geq 5$. Then we choose

$$\sigma_0 = \rho_0 = 0 \tag{6.4}$$

for simplicity. The first equation of (6.3) is automatically satisfied. The second and third become

$$\begin{cases} -\beta_0^2 \mathrm{B}_j^{v^2} + 8\beta_0 \mathrm{N}_j = 0, \\ 3\beta_0^2 \mathrm{B}_j^{v^2} + 6\beta_0 \mathrm{M}_j = 0, \end{cases}$$

which gives

$$\begin{cases} \mathrm{B}_j^{u} = (1/2)\mathrm{B}_j^{uv^2} + (3/8)\beta_0 \mathrm{B}_j^{v^2} + \mathrm{Q}_{j+1} + \mathrm{Q}_{j,j+1}, \\ \mathrm{B}_j^{uv} = -\mathrm{B}_j^{uv^2} - (1/2)\beta_0 \mathrm{B}_j^{v^2} - \mathrm{Q}_{j,j+1}, \end{cases} \qquad j = 0,....,n-1. \tag{6.5}$$

Using (6.5) and (5.2), the fourth equation of (6.3), for j=0...,n- 1, can be rewritten as

(6.6)

$$\mathrm{B}_{j-1}^{uv^2} - 2\beta_0 \mathrm{B}_j^{uv^2} + \mathrm{B}_{j+1}^{uv^2} = 9(1+\beta_0/2)\beta_0 \mathrm{Q}_j + 6(3+2\beta_0)\mathrm{Q}_{j+1}$$
$$+ (3/\beta_0 + 3/2)(\mathrm{Q}_{j-1,j} + \mathrm{Q}_{j,j+1}) - (3/\beta_0 + 9/2)(\mathrm{Q}_{j,j+1} + \mathrm{Q}_{j+1,j+2}), j = 0,...,n-1.$$

We treat (6.6) as a linear system in $\mathrm{B}_j^{uv^2}$, $j = 0,...,n-1$.

Proposition 5. The system defined by (6.6) is solvable in the case $n\geq 5$. It has a non-singular coefficient matrix for $n\geq 5$, $n\neq 6$. For n=6, the coefficient matrix is singular with rank deficiency 1 but the system is consistent.

Proof. By Lemma 2, the eigenvalues of the coefficient matrix are

$$\lambda_k = 1 - 2\beta_0 \xi_k + \xi_k^2 = 2[2\cos(2\pi/n) + \cos(2k\pi/n)]e^{i\frac{2k\pi}{n}} \tag{6.7}$$

for k=0,..., n-1. Thus a λ_k is zero if and only if

(6.8) $$2\cos(2\pi/n)+\cos(2k\pi/n)=0.$$

For $n > 6$, we have

$$\frac{2\pi}{n}<\frac{2\pi}{6}=\frac{2\pi}{3}$$

and hence

$$2\cos(2\pi/n)+\cos(2k\pi/n)>2\cos(\pi/3)-1=0.$$

Therefore, (6.8) cannot be true for $n > 6$. Further verification for n=5 and n=6 shows that in the case $n\geq5$, (6.8) holds if and only if $k = 3$ and $n = 6$. Thus the coefficient matrix is non-singular for $n\neq6$ and singular with rank deficiency 1 for $n = 6$. Consider the case $n = 6$. Corresponding to $\lambda_3 = 0$,

$$X_3=(1,-1,1,-1,1,-1)$$

is a left eigenvector of the coefficient matrix. Since

$$\sum_{j=0}^{5}(-1)^j Q_{m+j}=0 \text{ and } \sum_{j=0}^{5}(-1)^j(\mathrm{Q}_{m+j-1,m+j}+\mathrm{Q}_{m+j,m+j+1})=0$$

where m is some integer, we conclude that the system defined by (6.6) is consistent for $n = 6$.

We now study the case $n = 3$ when $\beta_0 = 1$ and (5.4) must hold. The first three equations of (6.3) become

(6.9) $$\begin{cases}-6\mathrm{N}_j=0,\\ -\mathrm{B}_j^{v^2}+8\mathrm{N}_j-6\mathrm{M}_j=0,\\ 3\mathrm{B}_j^{v^2}+6\mathrm{M}_j-6\mathrm{L}_j=0,\end{cases}$$

which gives

(6.10) $$\begin{cases}\mathrm{B}_j^{u}=(1/6)\mathrm{B}_j^{v^2}+\mathrm{Q}_{j+1}+\mathrm{Q}_{j,j+1},\\ \mathrm{B}_j^{uv}=-(1/6)\mathrm{B}_j^{v^2}-\mathrm{Q}_{j,j+1},\\ \mathrm{B}_j^{uv^2}=-(1/3)\mathrm{B}_j^{v^2}.\end{cases}$$

Using (6.10) and (5.2), we can rewrite the fourth equation of (6.3) in this case as

$$17Q_{j-1,j} - 10Q_{j,j+1} - 7Q_{j+1,j+2} = 18\,Q_j - 45\,Q_{j+1}, \quad j = 0,1,2. \tag{6.11}$$

Proposition 6. The linear system defined by (6.11) is consistent.

Proof. The eigenvalues of the cyclic coefficient matrix of the system are

$$\lambda_k = 17 - 10\xi_k - 7\xi_k^2 = (1-\xi_k)(17+7\xi_k) \tag{6.12}$$

for $k = 0,1,2$. The only zero eigenvalue is λ_0 with

$$X_0 = (1,1,1)$$

as a corresponding left eigenvector. The system is thus consistent because

$$\sum_{j=0}^{2}\left(18Q_j - 45Q_{j+1}\right) = 0.$$

7. An Application

The theory presented here provides the initial analysis needed for the construction of C^2 B-spline surfaces of irregular topology. Thus, for example, the C^1 bi-polynomial closed surface construction of Goodman[2] and Höllig and Mögerle[6] could be generalized to the C^2 case. For a topology with well separated n-vertices, the dimensionality study of Goodman, Höllig and Mögerle can be applied, using bicubic and biquartic patches. Here however, we present a simpler but important application involving the filling of a polygonal hole within a $C^{2,2}$ biquintic rectangular patch complex. The case of a $C^{2,2}$ bicubic spline complex is included in such a description as a special case. As a consequence of the analysis, we have the following basic scheme for a C^2 fill of the hole with n biquintic Hermite patches.

Given Q, X, Y and necessary boundary data then the additional Hermite data can be derived as follows:

(i) calculate $\{\mathrm{Q}_j\}_{j=0}^{n-1}$ from

$$\mathrm{Q}_j = \mathrm{X}\cos(2j\pi/n) + \mathrm{Y}\sin(2j\pi/n), \quad j = 0,\ldots,n\text{-}1,$$

(ii) calculate $\{\mathrm{Q}_{j,j+1}\}_{i=0}^{n-1}$ by solving (6.11) for $n = 3$ and (5.3) for $n\geq 5$,

(iii) calculate $\{\mathrm{B}_j, \mathrm{B}_j^v, \mathrm{B}_j^{v^2}\}_{i=0}^{n-1}$ using (4.6),

(iv) calculate $\{\mathrm{B}_j^u, \mathrm{B}_j^{uv}, \mathrm{B}_j^{uv^2}\}_{i=0}^{n-1}$ using (6.10) for $n = 3$ and by solving (6.6) and then using (6.5) for $n\geq 5$.

(v) calculate $\partial_{2,m}P_j(0,0)$ and $\partial_{m,2}P_j(0,0), 0 \leq m \leq 2, (0,0)$ from (2.9) and (2.10).

Here, it should be noted that some of the boundary mid-point data have been perturbed and hence the surrounding biquintic patches must be modified appropriately. Thus, in practice, we would suggest that some variational smoothing criterion on the surface be applied, which would constrain more of the patch complex data. We have, however, exhibited the existence of a solution for irregular C^2 surface construction using biquintic patches.

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