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Numerical techniques for conformal mapping onto a rectangle

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ABSTRACT

This paper is concerned with the problem of determining approximations to the function F which maps conformally a simply-connected domain Ω onto a rectangle R, so that four specified points on $\partial\Omega$ are mapped respectively onto the four vertices of R. In particular, we study the following two classes of methods for the mapping of domains of the form $\Omega := \{ z = x + iy : 0 < x < 1, T_1(x) < y < T_2(x) \}$. (i) Methods which approximate $F : \Omega \to R$ by $\overline{F} = S\,0\,\bar{f}$, where $\bar{f}$ is an approximation to the conformal map of Q onto the unit disc, and S is a simple Schwarz-Christoffel transformation. (ii) Methods based on approximating the conformal map of a certain symmetric doubly-connected domain onto a circular annulus.

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Abbreviated Title: Conformal mapping onto a rectangle.

1. Introduction.

Let Ω be a finite simply-connected domain with boundary $\partial\Omega$ in the complex z-plane, where $\partial\Omega$ is a closed piecewise analytic Jordan curve. Also, let Zj ; j=1 (1)4, be four points in counter-clockwise order on $\partial\Omega$, and let R_h denote the rectangle

$$h := \{ w = \varepsilon + in \ : \ 0 < \varepsilon < 1, \quad 0 < n < h\} \tag{1.1}$$

in the w-plane. Then, it follows from the Riemann mapping theorem that, for a certain h, there exists a unique conformal map $F : \Omega \to R_h$ which takes the four boundary points z_j ; j=l(1)4, respectively onto the four vertices $w_1 = 0$, $w_2 = 1$, $w_3 = 1 + ih$ and $w_4 = ih$ of R_h. This conformal map has many practical applications, and in these the height h of the rectangle is often of special significance In fact, h is an important domain functional known as the conformal module of the quadrilateral$\{\Omega; z_1, z_2, z_3, z_4\}$. (See e.g. [5]f [8,§16.11] and [13], and observe that some authors define h^{-1}, rather than h, as the conformal module.}

This paper is oncerned with a study of the following two well-known classes of procedures for determining approximations to the conformal map $F : \Omega \to R_h$.

(I) Procedures which approximate F by $\widetilde{F} = S \; 0 \; \widetilde{f}$ where $\widetilde{f}$ is an approximation to the conformal map of Ωonto the unit disc D := { ζ: |ζ|<1),and S : D $\to$ R_h is a simple Schwarz-Christoffel transformation.

In theory, procedures of this type depend only on the availability of a suitable approximation $\widetilde{f}$, and their use is not otherwise restricted by the geometry of & and the position of the points z_j on $\partial\Omega$. It is well-known however that in practice the application of such procedures is restricted considerably by a form of ill-conditioning which is caused by a certain crowding phenomenon. (See [4, p.179], [8, p.428] and the remarks of Trefethen in his preface of [14, p.4]. See also the paper by Zemach [15], which concerns a similar crowding phenomenon.) □

(II) Procedures in which the approximation to F is obtained by approximating the conformal map of a certain doubly-connected domain Ω_d onto a circular annulus of the form $A_q := \{ \zeta : q<|\zeta|<1 \}$.

Procedures of this type can be used only in cases where the quadrilateral {Ω ft; Z_1,z_2,z_3,z_4) has one of the two special forms illustrated in Fig, 1. We note however that the mapping of such geometries has received considerable attention recently. {See e.g. [2, 9, 12].) □

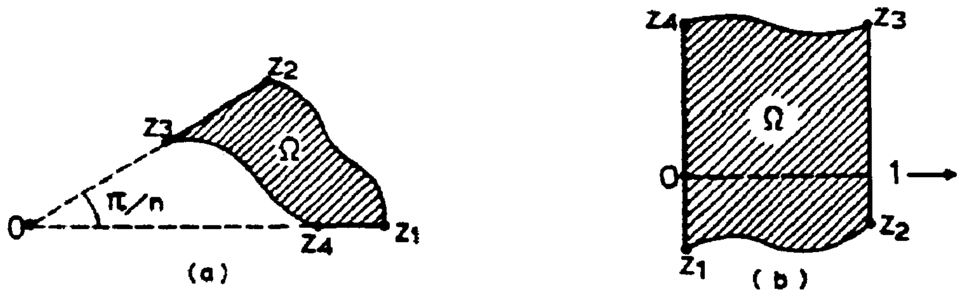

FIGURE 1

The specific objectives of the paper are as follows:

(i) To consider in detail the effects of the crowding phenomenon and of the resulting ill-conditioning, associated with the use of procedures of type (I).

(ii) To show that the ill-conditioning mentioned above can be avoided by using procedures of type (II), and to illustrate that such procedures are well-suited for the mapping of domains of the form illustrated in Fig. 1.

2. Procedures based on approximating $f : \Omega \to D := \{ \zeta : |\zeta| <1 \}$.

With the notation of Section 1, let f be the function which maps confermally Ω onto the unit disc D so that $f(z_0)=0$ and $f'(z_0)>0$, where z_0 is some fixed point in Ω Then, $F : \Omega \to R_h$ can be expressed as

$$F = S \circ f, \tag{2.1}$$

where $S : D \to R_h$ is a simple Schwarz-Christoffel transformation. In particular S can be formulated as

$$S = S_3 \circ S_2 \circ S_1, \tag{2.2}$$

where Sj ; j=1,2,3 are the three elementary conformal maps illustrated in Fig. 2. The details of these maps are as follows:

(i) Let $\varsigma_j = f(z_j)$; j=1(1)4, and let

$$S_1(\varsigma) := e^{i\alpha}\{ (\varsigma - \varsigma_o) / (1 - \bar{\varsigma}_o \varsigma) \}, \tag{2.3}$$

where the point $\varsigma_O \in$ and the rotation $e^{i\alpha}$ are such that

$$S_1 (\varsigma_1) + S_1 (\varsigma_3) = 0, \qquad S_1 (\varsigma_2) + S_1 (\varsigma_4) = 0,$$

and

$$\arg\{ (S_1(\varsigma_1) + S_1 (\varsigma_4))/2 \} = 0.$$

Then, the bilinear transformation $\hat{\varsigma} = S_1(\varsigma)$ maps the unit disc D onto itself and arranges the points $\hat{\varsigma} = S_1(\varsigma_j)$; j=1(1)4, on $|\hat{\varsigma}| = 1$ so that $\hat{\varsigma}_1$ and $\hat{\varsigma}_2$ are diametrically opposite to $\hat{\varsigma}_3$ and $\hat{\varsigma}_4$ respectively, and the mid-point of the arc $\hat{\varsigma}_1 \hat{\varsigma}_4$ coincides with the point $\hat{\varsigma} = 1$; see Fig. 2(b).

(ii) Let

$$e := \arg \hat{\varsigma}_1, \tag{2.4}$$

and let

$$S_2(\hat{\varsigma}) := -i\{ (\hat{\varsigma} - i) / (\sqrt{k} - \hat{\varsigma} + i) \}, \tag{2.5}$$

where

$$k = (1 - \text{sine}) / (1 + \text{sine}). \tag{2.5a}$$

Then, the bilinear transformation $t = S_2(\hat{\varsigma})$ maps the unit disc $\hat{} = \hat{\varsigma}$: $|\hat{\varsigma}| < 1$ } onto the half-plane H = Imt > 0 } so that

$$\left.\begin{aligned} \varsigma_1 &\to t_1 = -1, & \varsigma_2 &\to t_2 = -1, \\ \varsigma_3 &\to t_3 = -1/k, & \varsigma_4 &\to t_4 = -1/k; \end{aligned}\right\} \tag{2.6}$$

and

see Fig. 2(c)

(iii) Let

$$S_3(t) := \{ 1 + \mathrm{sn}^{-1}(K(k)t,\ k) \} \ / \ 2, \tag{2.7}$$

where sn(., k) and K(k) denote respectively the Jacobian elliptic sine and the complete elliptic integral of the first kind, each with modulus k. Then, the transformation $w = S_3(t)$ maps H onto the rectangle R_h, where

$$h = K\{ (1-k^2)^{1/2}\} / 2K(k), \tag{2.8}$$

so that

$$t_1 \to w_1 = 0,\quad t_2 \to w_2 = 1,\quad t_3 \to w_3 = 1+ih,\quad t_4 \to w_4 = ih. \tag{2.9}$$

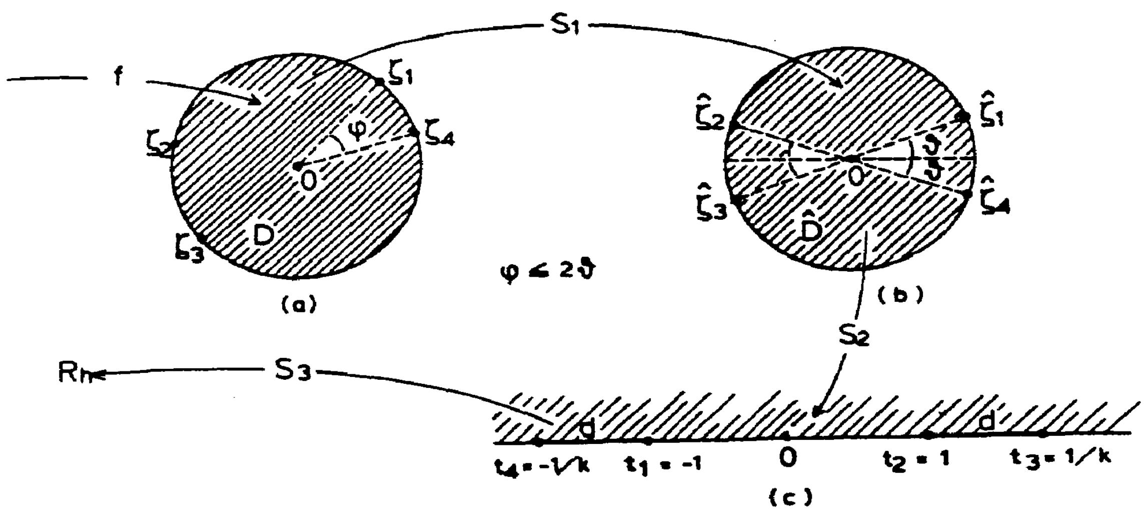

FIGURE 2

It follows easily from the above that the main computational requirement for the construction of S by means of (2.2) is the calculation of two incomplete elliptic integrals of the first kind for each transformed point. Thus, in theory the problem of approximating $F: \Omega \to R_h$, may be regarded as solved once a suitable approximation to $f: \Omega \to D$ is found. In practice however, the use of (2.1) is restricted by a well-known numerical difficulty which can be explained as follows:

With reference to Figs 2(a) and 2(b), let Φ be the length of the smaller of the two aros $\varsigma_1 \varsigma_4$ and $\varsigma_2 \varsigma_3$, and observe that both aros

$\hat{\varsigma}_1 \hat{\varsigma}_4$ and $\hat{\varsigma}_2 \hat{\varsigma}_3$ are of length $2\theta \geq \Phi$. Also, with reference to Fig. 2(c), let d be the distance between the points t_1, t_4 (and t_1, t_3), i.e.

$$d = 1/k - 1 = 2\sin\theta \;/\; (1 - \sin\theta) \;. \tag{2.10}$$

Then, the numerical difficulty mentioned above is due to the fact that the three lengths Φ, θ and d become very small even when the height h of the rectangle R_h is only moderately small, for example when h = 0.1. In fact, it can be shown easily by using (2.5a), (2.8) and (2.10) that

$$d \approx 2\theta \approx 8\exp\{-\pi / (2h)\}; \tag{2.11}$$

see also [1, p.21]. Therefore, if h is small then some of the images of the points z_j ; j=1(1)4, on the two circles $|\varsigma|=1$ and $|\hat{\varsigma}|=1$ and on the real axis of the t-plane will be very close to each other. This crowding of points may be regarded as a form of ill-conditioning, in the sense that a numerical procedure based on (2.1) - (2.2) may fail to provide a meaningful approximation to $F: \Omega \to R_h$, even if an "accurate" approximation to $f: \Omega \to D$ is used. For example, if h = 1/12 then $\Phi <$ $5.3\text{x}10^{-8}$ and the procedure will fail on a computer with precision 10^{-7}, even if the conformal map f is performed "exactly".

It should be observed that the crowding of points described above is caused by the conformal map $f: \Omega \to D$ For this reason, any procedure based on the use of (2.1) is subject to serious numerical difficulties in cases where h is small. It should also be observed that the situation cannot be improved by altering the formulation (2.2) of S. In fact the use of other standard formulations for S may lead to much more severe crowding. For example, let S be expressed in the well-known form

$$S = \hat{S}_2 \text{ o } \hat{S}_1 \,, \tag{2.12}$$

where $\hat{S}_1$ is a bilinear transformation mapping D onto $\hat{H} := \{ \hat{t} : \text{Im}\hat{t} > 0 \}$ so that

$$\varsigma_1 \to \hat{t}_1 = 0, \quad \varsigma_2 \to \hat{t}_2 = 1, \quad \varsigma_3 \to \hat{t}_3 = 1, \quad \varsigma_4 \to \hat{t}_4 = \infty, \tag{2.13}$$

and is $\hat{S}_2 \; : \; \hat{H} \to R_h$ given by

$$\hat{S}_2(t) := \{ \operatorname{sn}^{-1}(\hat{t}^{1/2}, \hat{k}) \} / K(\hat{k}) ; \quad \hat{k} = \hat{t}^{-1/2} ; \tag{2.14}$$

see e.g. {1, p.57}. Also, let $\hat{d}$ be the distance between the points $\hat{t}_2$ and $\hat{t}_3$, i.e. $\hat{d} = 1/\hat{k}^2 - 1$. Then, it can be shown that

$$\hat{d} \approx 16\exp\{-\pi/h\}, \tag{2,15}$$

i.e. $\hat{d} \approx d^2/4$, were d is the measure of crowding assocated with the use of (2.2). This means that if the formulation (2.12) is used then the crowding on $\mathrm{Im}\hat{t} = 0$ can be much more severe than on $|\varsigma| = 1$. For example, if $h = 1/12$ then $d < 6.8\mathrm{x}10^{-16}$ and a procedure based on (2.1) and {2.12} will fail on a computer with precision 10^{-5}, even if the conformal map f is performed "exactly".

We end this section by considering the use of (2.1) for the mapping of domains having one of the two special forms illustrated in Fig. 1. Clearly, in such applications, severe crowding will occur when the domain under consideration is "thin", i.e. when the two arcs z_1z_2 and z_3z_4 are close to each other. Equivalently, for domains of the form of Fig. l(b) severe crowding will also occur when the two arcs are "far" from each other. For example, with reference to Fig. l(b), let the arcs z_1z_2 and z_3z_4 have cartesian equations $y = \tau_j(x) ; j = 1,2,$ respectively, where $\tau_1(x) < \tau_2(x),\ x \in \{0,1\}$. Also, let

$$\alpha = \min | \tau_1(S) - \tau_2(t) |, \ \beta = \max | \tau_1(S) - \tau_2(t) | ; \quad s,t \in [0,1]. \tag{2.16}$$

Then, since

$$0 \leq \alpha \leq h \leq \beta \tag{2.17}$$

a procedure based on (2.1) will be subject to serious numerical difficulties when β is "small" or α is "large".

3. Procedures based on approximating $g : \Omega_d \rightarrow A_q := \{ \varsigma : q<|\varsigma| < 1 \}$.

Let Ω be of the form illustrated in Fig. l(a), where the straight line z_2z_3 is inclined at an angle π / n to the real axis, with $n \geq l$ an integer, and where $\Gamma_1 := \mathrm{arc}(z_1z_2)$ and $\Gamma_2 := \mathrm{aro}(z_3z_4)$ are given in polar co-ordinates by

$$\Gamma_j = \{ z : z = \rho_j(\theta) e^{i\theta}, \ O \le \theta \le \pi/n \} ; \quad j = 1,2, \tag{3.1}$$

with $0 < \rho_2(\theta) < \rho_1(\theta)$, $\theta \in [0, \pi/n]$. Also, let Ω_d be the 2n-fold symmetric doubly-connected domain obtained by first reflecting Ω about the straight line $z = e^{i\pi/n}$. That is,

$$\Omega_d := \mathrm{Int}\,(\partial\Omega_1) \ \cap \ \mathrm{Ext}\,(\partial\Omega_1)\,, \tag{3.2}$$

where

$$\partial\Omega_j := \{ z : z = \rho_j(\theta) e^{i\theta}, \ O \le \theta \le 2\pi \ \} ; \quad j = 1,2, \tag{3.2a}$$

with

$$\left.\begin{aligned} &\hat{\rho}_j(\theta) = \rho_j(\theta), \quad \theta \in [0, \pi/n], \\ &\text{and} \\ &\hat{\rho}_j(k, \pi/n + \theta) = \hat{\rho}_j(k, \pi/n - \theta) \quad \theta \in [0, \pi/n] ;\ k = 1(1)2n-1 \end{aligned}\right\} \tag{3.2b}$$

Then, for a certain value q , 0<q<1 , Ω_d is conformally equivalent to the annulus

$$A_q := \{ \varsigma : q < |\varsigma| < 1 \}, \tag{3.3}$$

and the reciprocal of the inner radius

$$M := 1/q, \tag{3.4}$$

is called the conformal module of Ω_d.

Let g be the function which maps Ω_d conformally onto A_q and observe the following:

(i) The requirement that $\partial\Omega_1$ is mapped onto $|\varsigma| = 1$ defines g uniquely, apart from a rotation in the z-plane

(ii) The transformation

$$w = n \log \varsigma / i\pi \tag{3.5}$$

maps the sector

$$S_q := \{ \varsigma : \varsigma = re^{i\varphi}, q < r < 1, 0 < \Phi < \pi / n \} \tag{3.6}$$

conformally onto the rectangle

$$h := \{ w = \xi + in : 0 < \xi < 1, \ 0 < n < h \ \}, \tag{3.7}$$

where

$$h = -(\ n \log q\ ) / \pi, \tag{3.7a}$$

so that the four corners of S_q are mapped onto those of $_h$.

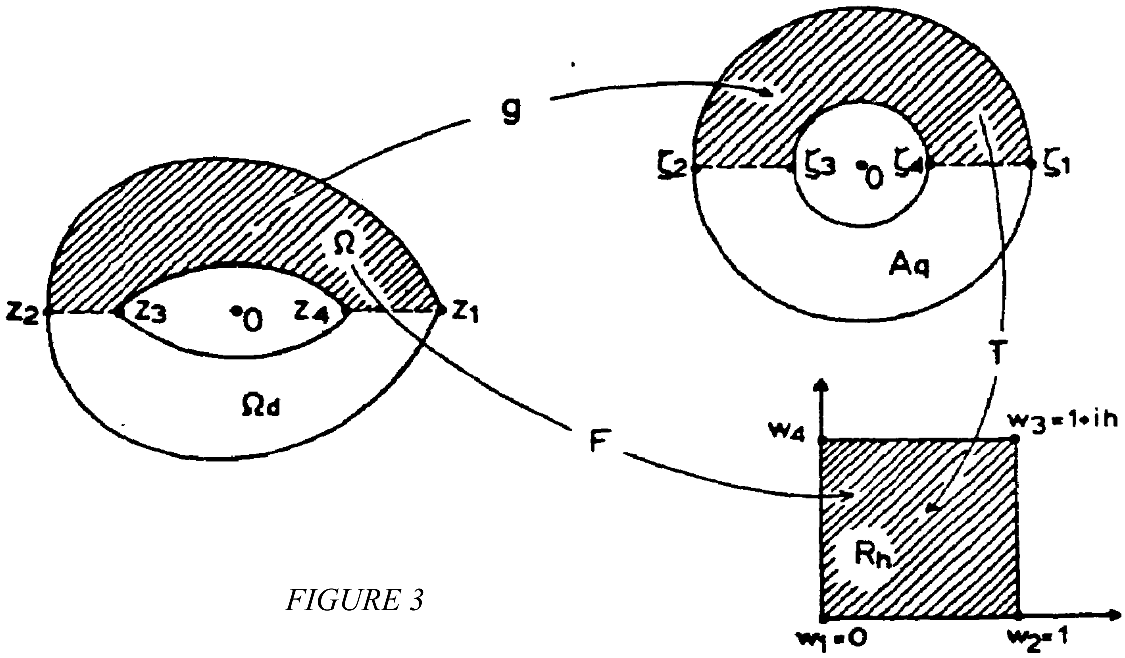

FIGURE 3

It follows from the above that

$$F(z) = \{ n\log(g(z)) \}/i\pi; \tag{3.8}$$

see the illustration of Fig. 3, which corresponds to the case n=1. Therefore, the problem of determining $F : \Omega \rightarrow R_h$ is equivalent to that of determining $g : \Omega_d \rightarrow A_q$. Furthermore, since Aq has unit outer radius and inner radius

$$q = \exp\{-\pi h/n) , \tag{3.9}$$

a procedure based on the use of (3.8) will not be affected by crowding of the form described in Section 2. More precisely, if h is small then the distance between the inner and outer circles of Aq is

$$D = 1-q \approx -\pi h/n. \tag{3.10}$$

This should be compared with the measure of crowding (2.11) assocated with the mapping via the disc described in Section 2.

Let now ft be a domain of the form illustraded in Fig. 1(b) and, as in Section 2, let the arcs z_1z_2 and $z_3\,z_4$ have cartesian equations $y = T_j(x) : j = 1,2$, i.e. let

$$\Omega = \{ z = x+iy : 0 < x < 1, \; T_1(x) < y < T_2(x) \}. \tag{3.11}$$

Then, the transformation

$$\hat{z} = \exp(i\pi z) \tag{3.12}$$

maps Ω conformally onto the upper half of a symmetric doubly-connected domain Ω_d which has the form (3.2) with n = 1 and

$$\theta_j(\theta) = \exp\{-\pi T_j(\theta/\pi)\}; \qquad j = 1,2. \tag{3.13}$$

Therefore,

$$F(z) = \log\{ f(\exp(i\pi z)) \} / i\pi , \tag{3.14}$$

and the equivalence of the conformal maps $F : \Omega \to R_h$ and $g : \Omega_d \to A_q$ persists.

4. Numerical examples

In this section we illustrate the theory of the previous sections by considering the computation of the conformal modules of two domains of the type illustrated in Fig. 1. In both examples we compute approximations to the modules by using each of the following three methods:

Method 1: Based on approximating $f : \Omega \to D$. The approximation to f is determined by using the Bergman kernel method (BKM), i.e. an orthonormalization method based on the properties of the Bergman kernel function of Ω. {Full details of the BKM procedure used can be found in [10,11] and the reference cited there.)

Method 2: Based on approximating $g : \Omega_d \to A_q$.The approximation to g is determined by an orthonormalization method (ONM), which may be regarded as the generalization of the BKM to the mapping of doubly-connected domains. (The details of the ONM can also be found in [10,11] and the references cited there.)

Method 3. This method determines the inverse conformal map $F^{[-1]} : {}_h \to \Omega$, by approximating $g^{[-1]} : A_q \to \Omega_d$. (That is the method is of the type described in Sect. 3.) The approximation to $g^{[-1]}$ is obtained by using the well-known method of Garrick; [7], [4, p.p. 194-207], [8, p.p. 4 76-4 78]. (For the actual Garrick algorithm used see [6, Alg. 4.1], and observe that for the two domains considered here this algorithm is equivalent to that of Challis and Burley [2]; see [6, §5].)

In presenting the numerical results we use the notations $E_1, \tilde{\Phi}, \tilde{\theta}$ and C_T in connection with Meth. 1, and E_2 in connection with Meth. 2. The meaning of these notations is as follows:

E_1, E_2: These denote respectively the estimates of the maximum error in modulus in the BKM approximation to $f : \Omega \to D$ and the ONM approximation to $g : \Omega_d \to A_q$.

$\tilde{\phi}, \tilde{\theta}, C_T$: With reference to Fig. 2, $\tilde{\Phi}$, and $\tilde{\theta}$, denote respectively the computed values of ϕ and θ, whilst C_T denotes the theoretical measure of crowding, i.e.

$$C_T = 8\exp\{-\pi/(2h)\}; \qquad (4.1)$$

see Eq. (2.11).

The computations were performed on a CRAY I computer using programs in single precision Fortran, Single length working on the CRAY I is between 14 and 15 significant figures.

Example 1. Ω is the trapezium with corners at the points $z_1 = 1,$, $z_2 = 1+i$, $z_3 = (1+i)(1-i)$ and $z_4 = 1-\alpha$, where $0 < \alpha < 1$; see Fig. 4. (That is Ω is of the form illustrated in Fig. l(a).)

The numerical results obtained in the three cases where a = 0.2, 0.15 and 0.1 are listed in Tables 1 and 2. Table 1 contains the values E_1, E_2, $\tilde{\Phi}$, $\tilde{\theta}$ and C_T concerning the accuracy of the BKM and ONM approximations in Meths 1 and 2, and the crowding in Meth. 1. Table 2 contains the approximations to the conformal module h, obtained by each of the three Methods 1, 2 and 3, The exact values of h, which are also listed in this table, were computed by using the formulae of Bowman [1, p.104].

In the case $\alpha = 0.1$, the measure of crowding C_T is smaller than the BKM error E_t and, not surprisingly, the BKM does not give the approximate images of the points z; in the correct order. For this reason, Meth. 1 fails completely. In the other two cases Meth. 1 does not "fail", but the resulting approximations to h are much less accurate than those computed by Meths 2 and 3. This is due to the damaging effect of crowding and also to the fact that, in general, numerical methods for the conformal map $g : \Omega_d \to A_q$ (or $g^{[-1]} : A_q \to \Omega_d$) approximate q more accurately than g (or $g^{[-1]}$). □

Example 2. Ω is the domain bounded by the straight lines $y = 0$, $x = 1$ and $x = 0$, and a circular arc of unit radius and centre at the point $0.5 + i\alpha$, $\alpha > 1$; see Fig. 5. (That is, Ω is of the form illustrated in Fig. 1(b).)

The numerical results corresponding to the four cases $\alpha = 1.2, 1.1$, 9 and 10 are listed in Tables 2(a) and 2(b). As might be expected, in the case $\alpha = 10$ the crowding on $|\xi| = 1$ is severe, and Meth. 1 fails completely. In the case $\alpha = 1.1$, although C_T is smaller than E_1, the BKM gives the approximate images of the points z_j; in the correct order. Because of this, Meth. 1 does not "fail" but, not surprisingly, the resulting approximation to h is very inaccurate.

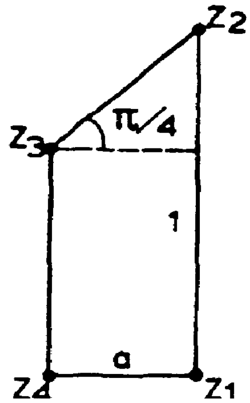

FIGURE 4

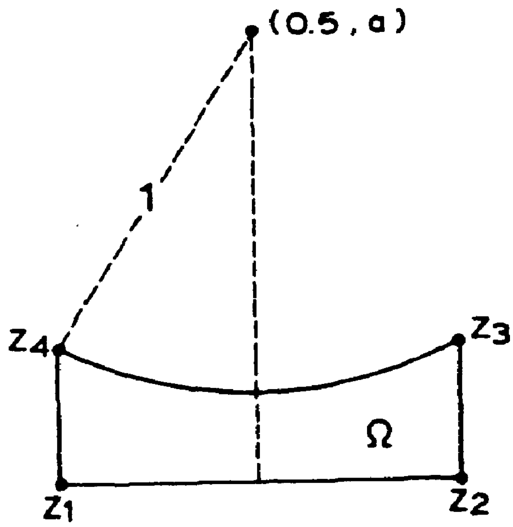

FIGURE 5

TABLE 1(a)

	Meth. 1					Meth. 2
α	E_1	$\tilde{\phi}$	$2\tilde{e}$	C_T		E_2
0.20	1.4×10^{-6}	3.1×10^{-3}	9.6×10^{-3}	9.6×10^{-3}		6.3×10^{-7}
0.15	1.3×10^{-5}	2.2×10^{-4}	6.9×10^{-4}	7.0×10^{-4}		1.6×10^{-6}
0.10	1.1×10^{-4}	*	*	3.7×10^{-6}		1.3×10^{-5}

*methods "fails".

TABLE 1(1b)

Approximation to h.

	$\alpha = 0.20$	$\alpha = 0.15$	$\alpha = 0.10$
Meth. 1	0.233 676 2. .	0.167 9.	*
Meth. 2	0.233 679 562	0.168 179 411	0.107 766 003
Meth. 3	0.233 679 564	0.168 179 411	0.107 766 002
Exact h	0.233 679 562	0.168 179 411	0.107 766 002

*methods "fails".

TABLE 2 (a)

	Meth. 1				Meth. 2
α	E_1	$\tilde{\phi} = 2\tilde{\theta}$	C_T †		E_2
1.2	1.2×10^{-5}	9.6×10^{-3}	9.6×10^{-3}		1.2×10^{-6}
1.1	4.4×10^{-4}	2.8×10^{-4}	5.2×10^{-5}		1.1×10^{-6}
9.0	3.8×10^{-6}	2.4×10^{-5}	2.6×10^{-6}		1.2×10^{-6}
10.0	1.0×10^{-5}	*	5.5×10^{-6}		1.4×10^{-6}

† Computed from (4,1), by using the Meth.3 approximation to h.

* Method "fails".

TABLE 2(b)

Approximations to h.

	α = 1. 2	α= 1.1	α = 9.0	α = 10. 0
Meth. 1	0.233 52 . . .	0.153	8.081	*
Meth. 2	0.233 498 59	0.131 422 60	8.034 180 07	9.034 180 07
Meth. 3	0.233 498 62	0.131 422 63	8.034 180 10	9.034 180 10

*Method "fails”.

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