College of Engineering, Design and Physical Sciences

Reversibility, Equivariance and Bifurcations of Mixed Functional Differential Equations

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by

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Abstract

We study the equivariance and reversibility of Neutral Mixed Functional Differential Equations (NMFDEs). Those equations are ill-posed but can behave properly on a reduced phase space which we define. We construct solutions of MFDEs with asymmetrical constant deviating arguments and extend this to MFDEs with distributed arguments on such a phase space and study the infinitesimal generator of the semi-group associated with the solution operator.

We develop a theory for reversible-equivariant NMFDEs, laying emphasis on $\mathbb{D}_n$-reversible-equivariant systems. We apply the results to a system of ring networks of cyclically arranged identical cells with forward and backward coupling.

Equivariant Lyapounov-Schmidt reduction is used to analyse Hopf bifurcation in equivariant NMFDEs. Equivariant centre manifold reduction theory is developed and we carry out an unfolding of an NMFDE having a Bogdanov-Takens bifurcation.

We determine the necessary and sufficient conditions for optimality in variational problems with generalised delayed arguments. We obtain the critical points of symmetric functionals with distributed delays from which the resulting Euler-Lagrange equations yield MFDEs. The Euler-Lagrange equations ensuing from the optimisation of the logistic equation yields a difference equation.

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I dedicate this work to my mother, Theresia. To God be the glory.

Contents

1 Introduction **6**

1.1 Review of Functional Differential Equations . . . 8

1.2 Mixed Functional Differential Equations (MFDEs) . . . 10

1.2.1 Mathematical Techniques for MFDEs . . . 11

1.2.2 Neutral MFDEs . . . 17

1.3 Thesis Overview . . . 19

2 Solution of Mixed Functional Differential Equations **22**

2.1 Introduction to the Method of Steps . . . 22

2.2 Construction of Solutions of MFDEs . . . 24

2.2.1 Construction of Solution to MFDEs with Distributed Delays . . . 29

2.3 Existence of Solutions to MFDEs . . . 30

2.3.1 Possible Initial Functions $\phi(\theta)$ that satisfy the MFDE . . . 32

2.4 Semigroup Associated to an MFDE . . . 33

2.4.1 Infinitesimal Generator . . . 36

2.5 Spectral Analysis of the Infinitesimal Generator . . . 38

3 Symmetries and Reversibility of Neutral MFDEs **41**

3.1 Symmetry and Equivariance of Vector Fields . . . 42

3.2 Reversibility . . . 43

3.3 Reversible Equivariant NMFDEs (REN MFDEs) . . . 44

3.3.1 Linear Reversible Equivariant Neutral (REN) MFDEs . . . 45

3.4 Group Actions in One Dimension . . . 46

3.5 Group Actions in Two Dimensions . . . 47

3.6 Group Actions in n Dimensions . . . 48

3.7 Reversible Equivariance . . . 50

3.7.1 $\mathbb{Z}_{\frac{n}{2}}$ Action . . . 50

3.7.2 $\mathbb{Z}_n$ Equivariance . . . 52

3.7.3 $\mathbb{Z}_n$ Reversibility . . . 53

3.7.4 $\mathbb{Z}_2$ Equivariance . . . 54

3.7.5 $\mathbb{Z}_2$ Reversibility 55
3.8 $\mathbb{D}_n-$Reversible Equivariance 56
3.8.1 The case when $\chi(\rho) = 1$ and $\chi(\kappa) = 1$ 56
3.8.2 The case when $\chi(\rho) = 1$ and $\chi(\kappa) = -1$ 57
3.8.3 The case when $\chi(\rho) = -1$ and $\chi(\kappa) = 1$ 58
3.8.4 The case when $\chi(\rho) = -1$ and $\chi(\kappa) = -1$ 60
3.9 The Spectral Analysis of the matrix, $d\eta$ 61
3.10 Ring Networks 67
3.10.1 Neutral MFDE in a Ring Network 68
3.10.2 The $\mathbb{D}_n$ Actions on the Operators T_i 70
3.10.3 Another look at $\mathbb{D}_n$ actions on the ring network neutral MFDE 72
3.10.4 Reversibility 73
3.11 Group Actions in a Three-cell Network 76

4 Centre Manifold Theory of Nonlinear MFDEs **77**
4.1 Centre Manifold Theory 78
4.1.1 Some Centre Manifold Theorems 79
4.1.2 Local Centre Manifold 82
4.2 Center Manifolds for MFDEs 83
4.2.1 The Effects of Symmetry on the Center Manifold 84
4.3 Versal Unfoldings 85
4.3.1 Versal Unfolding of a DDE 86
4.3.2 Bogdanov-Takens Bifurcation 87
4.3.3 Unfolding a Neutral Mixed Functional Differential Equation 90

5 Hopf Bifurcation via Lyapounov-Schmidt Reduction **94**
5.1 Setting up the Scene 96
5.2 Hopf Bifurcation on Loop Spaces 105
5.3 Lyapounov-Schmidt Reduction 110
5.4 Bifurcation Direction 113
5.5 One Dimensional Lyapounov-Schmidt Reduction for Neutral MFDEs 115
5.5.1 Lyapounov-Schmidt Reduction of $1-$D NMFDEs 116
5.5.2 Some Degenerate Cases of the 1-D NMFDE 118

6 Bifurcation of Neutral MFDE in a Ring Network **122**
6.0.1 Examples of Networks with Nearest-neighbour Coupling 124
6.1 A Ring Network Example 125
6.2 Equivariant Hopf Bifurcation 127
6.2.1 Hopf Bifurcation 129

7 REN Optimisation with Delays **133**
7.0.1 Examples . . . 134
7.0.2 Critical Points . . . 135
7.0.3 Example . . . 138
7.0.4 Quadratic Objective Functions . . . 139
7.0.5 Optimising One Dimensional DDEs with Harvesting . . . 141
7.0.6 The Logistic Equation . . . 142
7.0.7 Periodic Solutions . . . 143

8 Conclusions and Future Work **145**
8.1 Results . . . 145
8.2 Future Work . . . 146

A Functions of Bounded Variation **147**
A.1 The Riesz Representation Theorem . . . 149
A.2 Examples . . . 149
A.2.1 Remarks on the Example . . . 150

B Circulant Diagonalisation **152**
B.1 Shift and Fourier Matrices . . . 152
B.2 Circulant Matrices . . . 153
B.2.1 Commutativity of M and $\mathbb{D}_n$ . . . 155

C The Implicit Function Theorem and the Lyapounov-Schmidt Reduction **156**
C.1 Implicit Function Theorem (IFT) . . . 156
C.2 Bifurcation and Lyapunov-Schmidt (L-S) Reduction . . . 157
C.2.1 Equivariant L-S Reduction . . . 159
C.2.2 Bifurcation Equivalence and L-S Reduction . . . 160

D Linear Group Actions **162**
D.1 Representations . . . 162
D.2 Group Actions . . . 163
D.3 Integration on Compact Groups . . . 163
D.4 Irreducibility . . . 164
D.4.1 Isotypic Decomposition . . . 165
D.5 Invariant and Equivariant Maps . . . 166
D.5.1 Equivariant Linear Mappings . . . 167
D.6 Orbits and Isotropy Subgroups . . . 167
D.6.1 Isotropy Subgroup . . . 168
D.7 Fixed-Point Subspaces . . . 168

Chapter 1

Introduction

In this work we study and extend some aspects of reversibility, equivariance, equivariant Hopf bifurcation and Bogdanov-Takens bifurcation of Mixed Functional Differential Equations (MFDEs). We also develop a step derivative method to solve MFDEs with asymmetrical and distributed arguments, extending the work of Iakovleva et al. in [35]. Furthermore, we extend the work of Hughes [33], providing the necessary and sufficient conditions for optimality in variational problems with delayed arguments from which the resulting Euler-Lagrange equations yield MFDEs.

Our main interest in mixed functional differential equations arises because they generalise the delay differential equations which in turn generalise differential equations in the sense that the rate of change of a system is allowed to depend on future states as well as past states. MFDEs have attracted considerable attention over the past few decades since time delays are intrinsic in many real systems and therefore must be properly accounted for when developing mathematical models. Delay is a common feature of many real processes and with a growing demand for more precise predictions, control and performance, there is a greater need for models to behave as close to real systems as possible. MFDEs are important in the study of travelling wave solutions to differential equations posed on lattices, see Mallet-Paret [49], Ma et al. [47], and as well as in economic theory, biological sciences and engineering amongst others. MFDEs have a richer mathematical framework than ordinary differential equations and display better consistency with the nature of some engineering and biological processes.

The study of functional equations deals with seeking functions that satisfy equations such as $f(x+y) = f(x)+f(y)$. Functional equations arise in all areas of mathematics, science, engineering, and social sciences. A functional differential equation (also called a differential equation with deviating argument,) can be considered as a combination of differential and functional equations. The values of the argument in a functional differential equation can be discrete, continuous or mixed. Correspondingly, one may introduce the notions of differential difference equations, and integro-differential equations, etc. Delays are inherent in control, transport and biological systems (e.g. gestation) and as such ordinary and partial differential equations cannot capture the rich variety of dynamics observed in such complex systems, providing a strong motivation for the study of functional differential equations. However, we note that solving functional differential equations can be much more challenging than solving ordinary differential equations. For example, if we seek exponential solutions, the resulting characteristic equation gives a transcendental equation in contrast to a polynomial in the case of ODEs. In general, such characteristic equations have infinitely many solutions corresponding to an infinite family of independent solutions.

The notion of symmetry (equivariance) is a fundamental topic in many areas of mathematics: see Golubitsky et al. [28] and Field [22]. Many systems in engineering and in nature possess some symmetry, which somehow influences their functionality. Taking symmetry into account may significantly simplify the study of such systems. The symmetries of a physical system may be preserved in the mathematical tools used to model them. Mathematically, the conventional notion of symmetries (equivariances) and reversing symmetries in a system of differential equations consists of phase space transformations, including time transformations for reversing symmetries, that leave the equations of motion invariant. A vector field $\dot{x} = f(x)$ is reversible if the dynamics on the phase space is given by the time reversed vector field.

Symmetries (equivariance) and reversing symmetries affect dynamical systems in different ways. Symmetries map trajectories to other trajectories preserving their direction whilst reversing symmetries map trajectories to trajectories, reversing the time-direction of the trajectories. One difference resulting from this is the role of fixed point subspaces. The fixed point subspace of a map $F : V \to V$, where V is a vector space, is defined as $\mathrm{Fix}(F) := \{x \in V : F(x) = x\}$. Fixed point subspaces of symmetries are setwise invariant under the dynamics but the fixed point subspaces of reversing symmetries may not be invariant under the dynamics, but give rise to symmetric periodic orbits. Symmetries and reversibility in dynamical systems have been studied in relation to ODEs by numerous authors such as Lamb et al. [43, 44] , Baptistelli et al. [6] and Teixeira et al. [60].

Bifurcation or branching occurs in a nonlinear system when the state of the system depends on some parameter which when varied causes the state to branch to another state at some critical value of the parameter, usually with a change of stability. The goal of bifurcation theory is to determine the existence and stability of various branches of solutions like fixed points and periodic orbits. The various equilibria emerge from one another in a continuous manner as the bifurcation parameter varies across the bifurcation point and the local dynamics is contained in a suitably defined center manifold at the bifurcation point.

Hopf bifurcation concerns the birth of a periodic solution from an equilibrium solution through a local oscillatory instability. Hopf bifurcation theorems prove the existence of periodic solutions of a nonlinear equation, in the vicinity of a stationary solution, when a conjugate pair of distinct eigenvalues of the linearized equation crosses the imaginary axis. We study reversible equivariant Hopf bifurcation from symmetric equilibrium points in MFDEs. We make use of the approach introduced by Rustichini in [56] by adopting a purely functional analytic argument and involving a Lyapunov-Schmidt reduction (LSR). We set the problem in the space of periodic functions of fixed period. The linearization of the stationary solution of the MFDE then defines a linear operator acting on this space. It is noted that a linear operator of mixed type, when its action is restricted to the periodic functions, can be identified with an operator of the delay type. Once this is done, the task is reduced to the study of the zeros of the bifurcation functions.

We also study the versal unfolding of a Neutral Mixed Functional Differential Equation (NMFDE) under a Bogdanov-Takens (B-T) bifurcation. Recall that a B-T bifurcation is a bifurcation of an equilibrium point in a two-parameter system at which the critical equilibrium has a zero eigenvalue of algebraic multiplicity two. For nearby parameter values, the system has two equilibria (a saddle and a non-saddle) which collide and disappear via a saddle-node bifurcation. The non-saddle equilibrium then undergoes a Hopf bifurcation generating a limit cycle which degenerates into an orbit that is homoclinic to the saddle and disappears via a saddle-homoclinic bifurcation. Elements of bifurcation theory can be found in the books by Kielhofer [38] and Kuznetsov [41].

1.1 Review of Functional Differential Equations

A dynamical or evolutionary system may be represented by a function $t \mapsto x(t)$ taking values in some state space X. The independent variable t does not necessarily represent time but may also represent some spatial or spatio-temporal continuum. When the variation of x depends instantaneously on the current state of the system, the usual evolution is described using the differential equation

$$\dot{x}(t) = f(t, x(t)),$$

an ordinary differential equation (ODE) where $f : \Omega \subset \mathbb{R}^{1+n} \to \mathbb{R}^n$ is a non linear function, or a partial differential equation (PDE) when X is an appropriately chosen infinite dimensional function space.

In a functional differential equation (FDE), the evolution of a system links states at different values of t, including derivatives. In this thesis we consider mixed functional differential equations (MFDEs), which are sometimes called forward-backward FDEs. An MFDE links the values of the function to be determined with the values of its derivatives over an interval of the independent variable, with the initial interval containing 0, thereby using delayed and advanced arguments. For instance, given some function $f : \Omega \subset \mathbb{R}^{1+3n} \to \mathbb{R}$, the nonlinear differential equation

$$\dot{x}(t) = f(t, x(t), x(t + \theta_{\min}), x(t + \theta_{\max})), \quad \theta_{\min} \leq \theta_{\max},$$

is retarded when $\theta_{\min} \leq \theta_{\max} < 0$, is advanced when $0 < \theta_{\min} \leq \theta_{\max}$ and mixed when $\theta_{\min} < 0 < \theta_{\max}$.

In this work we shall see many different types of differential equations, some with rather long names, and we therefore define their acronyms. As we have seen, when a model does not incorporate a dependence on its (past or future) history, we get ODEs or PDEs. Models incorporating past and current history include delay differential equations (DDEs) or, more generally, functional differential equations (FDEs). When the highest derivative of a FDE is also evaluated with delay, we add the adjective neutral, giving rise to neutral functional differential equations (NFDEs).

The dynamical system approach to the study of delay differential equations can be seen by considering the following nonlinear delay differential equation (DDE)

$$\dot{x} = F(x(t), x(t-\tau)), \tag{1.1}$$

in which $F : \mathbb{R}^{2n} \to \mathbb{R}^n$, with a single delay τ where $\tau > 0$ is a constant. The dynamical system approach to DDEs is to associate with it a semi-flow on the space of continuous functions $C([-\tau, 0], \mathbb{R}^n)$ defined by the time evolution of segments of solutions of (1.1) of length τ. Such segments are defined introducing the notation

$$x_t(\theta) = x(t+\theta), \quad \theta \in [-\tau, 0],$$

so that $x(t) = x_t(0)$ and we may write $F(x(t), x(t-\tau)) = F(x_t(0), x_t(-\tau))$. Furthermore, given an initial function $\phi \in C([-\tau, 0], \mathbb{R}^n)$, we define $f(\phi) = F(\phi(0), \phi(-\tau))$ so that we may therefore write (1.1) as

$$\dot{x} = f(x_t) \tag{1.2}$$

with $x_t \in C([-\tau, 0], \mathbb{R}^n)$ being that segment of the function $x(\delta)$ defined by letting the real value δ range in the interval $t - \tau \leq \delta \leq t$, and where f is a continuous function mapping $C([-\tau, 0], \mathbb{R}^n)$ into $\mathbb{R}^n$.

If ϕ is any given function in $C([-\tau, 0], \mathbb{R}^n)$ and $x(\phi)$ is the solution of (1.2) with initial function ϕ at zero, we define the operator $T(t)$ mapping $C([-\tau, 0], \mathbb{R}^n)$ into $C([-\tau, 0], \mathbb{R}^n)$ by

$$T(t)\phi = x_t(\phi),$$

where for each fixed $t \geq 0$, $x_t(\phi)$ is the function in $C([-\tau, 0], \mathbb{R}^n)$ determined by

$$x_t(\phi)(\theta) = x(\phi)(t+\theta), \; -\tau \leq \theta \leq 0.$$

The operator $T(t)$, $t \geq 0$ defined on $C([-\tau, 0], \mathbb{R}^n)$ satisfies the following properties:

1. $T(t)$ is a bounded linear operator for each $t \geq 0$;

2. $T(t)$ is strongly continuous on $(0, \infty)$; i.e. $T(0) = I$ and $\lim_{s \to t} ||T(s)\phi - T(t)\phi|| = 0$ for all $t, s \geq 0$, $\phi \in C([-\tau, 0], \mathbb{R}^n)$.

3. The family of transformations $\{T(t), \, t \geq 0\}$ is a semigroup i.e. $T(s+t) = T(t)T(s)$ for all $t, s \geq 0$.

4. $T(t)$ is completely continuous (compact) for $t \geq \tau$; i.e. $T(t)$, $t \geq \tau$ maps closed bounded sets into compact sets.

For any semigroup of transformations $T(t)$ of a Banach space into itself, the infinitesimal generator A of $T(t)$ is defined by the relation $A\phi = \lim_{t \to 0^+} \frac{1}{t}[T(t)\phi - \phi]$ for every value of ϕ for which this limit exists. The infinitesimal generator of $\{T(t)\}$ is given by

$$A\phi(\theta) = \dot{\phi}(\theta), \qquad -r \leq \theta \leq 0,$$

and the domain of definition of A is given by

$$\operatorname{dom}(A) = \{ \ \phi \in C([-\tau, 0], \mathbb{R}^n) \ : \ A\phi \in C([-\tau, 0], \mathbb{R}^n), \quad (D_0\phi)(0) = Df(\phi(0), \phi(-\tau)) \ \},$$

where Df is the linearisation of f at zero.

The DDE can be viewed as a transport equation $\dot{u} = Au$, $u(0) = \phi$ with nonlocal boundary conditions in $\operatorname{dom}(A)$. The solutions of the DDE are in one-to-one correspondence with the solutions of an abstract nonlinear ODE in $C([-\tau, 0], \mathbb{R}^n)$ where the correspondence is given by $u(t)(\theta) = x(t + \theta)$. This observation originated with Krasovskii [40] and has been crucial in the development of the qualitative theory of DDEs.

1.2 Mixed Functional Differential Equations (MFDEs)

Mixed functional differential equations (MFDEs), allow us to describe the dynamics of a variable whose 'time' derivative depends on its past and future (anticipatory) values of the state variable. The idea of an interaction from the future might raise doubts about the usefulness of MFDEs in modelling applications but not when the independent variable is spatial. Historically, the primary motivation for the study of MFDEs comes from the study of travelling waves for lattice differential equations (LDEs), which are systems of ODEs or PDEs indexed by points on an (infinite) spatial lattice. Including the structure of the underlying space into models, as a first approximation, PDEs are concerned with continuous media and LDEs with discrete media. A more detailed exposition can be found in Chow et al. [14].

To find travelling waves solutions of FDEs on lattices or study the dynamics of nerve conduction in humans or in crystals necessitate the use of lattice functional differential equations that reflect their spatial discreteness. For instance, consider the well-known reaction-diffusion PDE

$$u_t(t, x) = \alpha \Delta u(t, x) - f(u(t, x)), \qquad x \in \Omega \subset \mathbb{R}^2,$$

where the subscript denotes partial differentiation, α is a positive constant and Δ is the Laplacian in $2-$D. A travelling wave solution takes the form $u(t, x) = \varphi(\sigma \cdot x - ct)$ for some $\varphi : \mathbb{R} \to \mathbb{R}$ where

$\sigma \in \mathbb{R}^n$ is a unit vector, $|\sigma| = 1$, representing the direction of motion of the wave and $c \in \mathbb{R}$ is the wave speed. The spatially discrete version is

$$\dot{u}_\eta = \alpha(\Delta_2 u)_\eta - f(u_\eta), \qquad \eta \in \Omega \subset \mathbb{Z}^2,$$

where we may denote η by $\eta = (i, j)$ and also where Δ_2 is the standard 5 points discretisation of the Laplacian, that is,

$$(\Delta_2 u)_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}. \tag{1.3}$$

In higher dimensions, the travelling wave solution is given by $u(t, x) = \varphi(\sigma \cdot \eta - ct)$. Substituting the following ansatz, (in which $\sigma = (\sigma_1, \sigma_2)$ is a unit vector representing the direction of motion of the wave),

$$u_{i,j}(t) = \varphi(i\sigma_1 + j\sigma_2 - ct), \qquad c \in \mathbb{R}, \tag{1.4}$$

into (1.3) yields the MFDE

$$-c\dot{\varphi}(\xi) = \alpha(\varphi(\xi + \sigma_1) + \varphi(\xi - \sigma_1) + \varphi(\xi + \sigma_2) + \varphi(\xi - \sigma_2) - 4\varphi(\xi)) - f(\varphi(\xi)),$$

where $\xi = i\sigma_1 + j\sigma_2 - ct$ is called the moving coordinate. For example, let

$$u_{i,j}(t) = \varphi(i\sigma_1 + j\sigma_2 - ct),$$

then

$$\begin{aligned} u_{i+1,j} &= \varphi((i+1)\sigma_1 + j\sigma_2 - ct) \\ &= \varphi(i\sigma_1 + j\sigma_2 - ct + \sigma_1) \\ &= \varphi(\xi + \sigma_1). \end{aligned}$$

If the wave speed c in the ansatz (1.4) is equal to 0, then we have a difference equation on the lattice and only discrete values φ_i for $i \in \mathbb{Z}$ are relevant, and the solutions are constant in time.

1.2.1 Mathematical Techniques for MFDEs

In Chapter 2 we shall investigate in more detail the solutions of the linear equation

$$\dot{x}(t) = ax(t) + bx(t + \theta_{\min}) + cx(t + \theta_{\max}), \quad \theta_{\min} < 0 < \theta_{\max} \tag{1.5}$$

and address some of the challenging issues encountered when solving MFDEs.

An important challenge that has to be overcome in an infinite dimensional setting, which is the case with functional differential equations, is the fact that Banach spaces, in general, do not possess some of the desirable properties that are taken for granted in finite dimensional spaces. For example, in order to study functional differential equations we need to provide information (an initial function or initial history) on the entire interval i.e. $\phi : [\theta_{\min}, \theta_{\max}] \to \mathbb{R}^n$. Each such initial function determines a unique solution to the functional differential equation. If we require the initial functions to be continuous, then the space of solutions will be infinite dimensional. If we seek exponential solutions to the functional differential equations, and compute a characteristic equation, we obtain a transcendental equation, in fact, infinitely many linearly independent solutions $e^{\alpha t}$, where, $\alpha \in \mathbb{C}$ is an eigenvalue. The transcendental equation can have multiple zeros on the imaginary axis, giving rise to complicated critical cases, in contrast to the polynomial equations encountered when dealing with ODEs.

Another problem is that a Banach space need not have a natural inner product associated with its norm. For instance, the supremum norm cannot be given by an inner product. However, Hale et al. in [32] defines a bilinear form that acts like an inner product and it is this form that we use in this work.

MFDEs in general are not well posed if we demand that their solutions be continuous. To see this, consider the following example

$$\dot{x}(t) = x(t-1) + x(t+1) = x_t(-1) + x_t(1), \quad x_0 = \phi. \tag{1.6}$$

If we set $\phi(\theta) = 1$ for $\theta \in [-1, 1]$, then the solution to equation (1.6) is found to be discontinuous and simply oscillates between the constant values ± 1.

A logical first step in the development of the mathematical theory of MFDEs would of course be to identify the parts of the powerful finite dimensional toolbox that can be utilised in Banach space settings. In the twentieth century the foundations for linear semigroup theory were already being laid in an effort to generalize the matrix exponentials that now appear when studying ODEs. The theory for DDEs is now well established. Linear DDEs define in general a semi-group on the (infinite dimensional) space of initial data, and the whole of modern nonlinear evolution theory, including center manifold and bifurcation theory, applies.

Though the use of semigroups has been quite successful, there is still a wide class of systems to which the machinery cannot be so readily applied. As an important example, we mention our situation, MFDEs, in which the linear operator describing the infinitesimal change of a system has unbounded spectrum both to the left and right of the imaginary axis. One cannot define a strongly continuous semigroup that behaves as the exponential of such an operator. This difficulty may be circumvented by splitting the state space of the system into two separate parts, that both do allow the construction of a semigroup. One of these will however only be defined in backward time. Such a splitting is referred to as an exponential dichotomy. The main piece of work on this subject in finite dimensions is Coppel [19] and results on exponential splittings in infinite dimensional systems were obtained by various authors (see reference in Sandstede and

Scheel [57]). As in the finite dimensional situation, invariant manifolds play a fundamental role in the study of nonlinear systems. A very important structure in this respect is the so-called center manifold, which according to Vanderbauwhede and Iooss [64] forms one of the cornerstones of the theory of infinite dimensional dynamical systems.

It is well known that delay differential equations (DDEs) tend to smooth out irregularities in the initial values but that no such smoothing occurs for neutral equations where the solutions retain the degree of regularity of the initial values. Equations of advanced type tend to destroy the regularity of an initial function and the solution will only extend to $+\infty$ when the initial values comprise a function of class C^∞. See Bellman and Cooke [7] for a discussion on such properties.

In this work, we use real and complex valued functions. For many definitions and results, this distinction does not make any difference, so we denote by $\mathbb{K}$ either of $\mathbb{R}$ or $\mathbb{C}$. We represent norms in finite or infinite dimension by $\| \ \|$, with some index depending on the context. For instance, we denote the sup-norm in $\mathbb{K}^n$ by $\| \ \|_\infty$. If $f : [a, b] \to \mathbb{K}^n$ is a continuous function, we denote by $\|f\|_\infty$ the sup-norm for continuous $\mathbb{K}$-valued functions but $\|f(t)\|_\infty$ will represent the norm of $f(t) \in \mathbb{K}^n$. The 'phase space' of our MFDEs will be functions defined on closed intervals $[\theta_{\min}, \theta_{\max}]$ where, in general, $\theta_{\min} \leq 0 \leq \theta_{\max}$. In principle, we could allow any (or both) of the bounds to be infinite. Given $\mathcal{I} = [a, b]$ and a function $x : [a + \theta_{\min}, b + \theta_{\max}] \to \mathbb{K}^n$, for any $t \in [a, b]$, we denote by x_t the function defined by

$$x_t(\theta) = x(t + \theta), \quad \theta \in [\theta_{\min}, \theta_{\max}]. \tag{1.7}$$

The function x_t represents the segment of $x(t)$ on $t + [\theta_{\min}, \theta_{\max}]$. It is clear that x is continuous on $[a + \theta_{\min}, b + \theta_{\max}]$ if and only if x_t is continuous for each $t \in [a, b]$.

To avoid confusion with the classical delay differential equation, DDE-notation, we denote by $D_x f$ the (partial) derivative of a function f as a function (of one) of its variables, x. The notation $D_x^o f$ indicate that we consider $D_x f$ at the origin. The time right-derivative of a function x is often denoted by $\dot{x}$ and the first derivative of a function f by $\dot{f}$. Given A and B subsets of normed spaces, we denote by $C^n(A, B)$ the set of n-times continuously differentiable functions $f : A \to B$. We often write C for C^0.

To introduce simplified notations for function spaces, we denote by $X = C([\theta_{\min}, \theta_{\max}], \mathbb{K}^n)$ the space of $\mathbb{K}^n$-valued functions continuous on $[\theta_{\min}, \theta_{\max}]$. When $\theta_{\min} \leq 0 \leq \theta_{\max}$, we need to distinguish between the retarded and advanced parts: $X^- = C([\theta_{\min}, 0], \mathbb{K}^n)$ and $X^+ = C([0, \theta_{\max}], \mathbb{K}^n)$. Equipped with their respective supremum norms, they are Banach spaces. For the special situation of reversible FDEs $\theta_{\min} = -\theta_{\max}$ and so we use the notation $\mathcal{I}_r = [-r, r]$ with $0 < r \leq \infty$.

For variational problems we need to refer to PWC[a,b], the space of piecewise continuous functions on a compact interval $[a, b]$ (where the limits on each interval are finite), and PWS[a,b], the space of piecewise smooth functions on $[a, b]$ that are continuous on $[a, b]$ and have a derivative in PWC$[a, b]$. A function x that maps a closed interval $[a, b]$ into $\mathbb{R}^n$ is said to be piecewise smooth if the following hold: x is continuous in $[a, b]$, there exist points t_i's, the corners of x, so that

$a = t_0 < t_1 < \cdots < t_N = b$, $\dot{x}(t)$ exists at all $t \in [a,b]/\{t_i\}_{i=0}^N$, $\dot{x}$ is continuous at each open subinterval (t_i, t_{i+1}), and $\dot{x}$ has one-sided limits at all $t \in [a,b]$. Notice that x is a piecewise smooth function if and only if there exists a function $v \in \mathrm{PWC}[a,b]$ so that

$$x(t) = x(a) + \int_a^t v(s)\, ds. \tag{1.8}$$

Also, notice that absolutely continuous functions, or arcs, are functions for which (1.8) holds with $v \in L^1[a,b]$.

For Banach spaces X and Y, we denote by $L(X,Y)$ the Banach space of bounded linear mappings from X to Y with the operator topology. We use $\lambda \in \Lambda \subset \mathbb{R}^k$ for the bifurcation parameter(s). Let $L(\lambda) \in L(C, \mathbb{R}^n)$, $\lambda \in \Lambda$; then, the Riesz Representation Theorem implies that there is an $n \times n$-matrix function η on $[\theta_{\min}, \theta_{\max}]$ of bounded variation such that

$$L(\lambda)\phi = \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda,\theta)\, \phi(\theta),$$

where the integration variable is θ. For such an η, we always regard it as extended to $\mathbb{R}$ so that

$$\begin{aligned} \eta(\lambda,\theta) &= \eta(\lambda,\theta_{\min}), \quad \theta \le \theta_{\min}, \\ \eta(\lambda,\theta) &= \eta(\lambda,\theta_{\max}), \quad \theta \ge \theta_{\max}. \end{aligned}$$

We denote by $\Delta_L(\alpha) = 0$ the characteristic equation of a linear operator L with α denoting the critical values (eigenvalues when Δ is the characteristic polynomial).

We examine a notion which is important in the forward and backward continuation (i.e. that the solution exists to the left and the right of the initial $t-$value), of solutions of FDEs. We consider the functional differential equation

$$\dot{x}(t) = f(t, x_t). \tag{1.9}$$

A function x is a solution of (1.9) on an interval $[\sigma - r,\, \sigma + c]$ if there are $\sigma \in \mathbb{R}$ and $c > 0$ such that $x \in C([\sigma - r,\, \sigma + c], \mathbb{R}^n)$ and $x(t)$ satisfies (1.9) for $t \in [\sigma,\, \sigma + c]$.

Let $\theta \in \mathbb{R}$ and define the matrix

$$A(\lambda, \theta, L) = \eta(\lambda, \theta^+) - \eta(\lambda, \theta^-).$$

We say L is atomic at θ_0 at λ_0 if $A(\lambda_0, \theta_0, L)$ is non singular. If A is non singular on a set $K \subset \Lambda$, we say that L is atomic at θ on K.

For non linear mappings, we proceed in the following way: let $\Omega \subset \mathbb{R} \times C$ be open. A function $h : \Omega \to \mathbb{R}^n$ is said to be atomic at θ on Ω if h is continuous together with its first and second Fréchet derivatives with respect to $\phi \in C$, and $D_\phi h$ is atomic at $\theta \in \Omega$.

If $H(t,\phi)$ is linear in ϕ and continuous in $\mathbb{R}\times C$,

$$H(t,\phi)=\int_{\theta_{\min}}^{\theta_{\max}} d\eta(t,\theta)\,\phi(\theta),$$

then $A(t,\phi,\theta)=A(t,\theta)$ is independent of ϕ and

$$A(t,\theta)=\eta(t,\theta^{+})-\eta(t,\theta^{-}).$$

Thus, H is atomic at θ on $\mathbb{R}\times C$ if $\det A(t,\theta)\neq 0$ for all $t\in\mathbb{R}$.

Examples. 1. If $\theta\in[\theta_{\min},\theta_{\max}]$,

$$H(t,\phi)=\phi(0)+B(t)\phi(\theta),$$

then $A(t,\theta)=B(t)$ and H is atomic at θ on $\mathbb{R}\times C$ if $\det B(t)\neq 0$ for all $t\in\mathbb{R}$; also, $A(t,0)=I$ and H is atomic at zero for all $t\in\mathbb{R}$.

2. Consider the MFDE system

$$\dot{x}(t)=A(t)x(t+a)+B(t)x(t-b) \tag{1.10}$$

where A and B are matrices. We may re-write (1.10) as

$$A(t)x(t+a)=\dot{x}(t)+B(t)x(t-b), \tag{1.11}$$

or as $x(t+a)=A^{-1}(t)[\dot{x}(t)+B(t)x(t-b)]$ if A is non-singular (or atomic) at all values t.

Recall that $X=C([\theta_{\min},\theta_{\max}],\mathbb{C}^n)$ will be the phase space of MFDEs and we assume that $0\in[\theta_{\min},\theta_{\max}]$. Let $F:\mathcal{U}\subset(\mathbb{R}\times X\to\mathbb{C}^n)$ be a smooth enough function. When $\theta_{\min}<0<\theta_{\max}$, it defines an MFDE

$$\dot{x}(t)=f(t,x_t), \tag{1.12}$$

where $x_t\in X$. A solution of (1.12) on an interval $0\in[t_1,t_2]\subset\mathbb{R}$ is a function $x:[t_1+\theta_{\min},t_2+\theta_{\max}]\to\mathbb{C}^n$ which is absolutely continuous on $[t_1,t_2]$, satisfies (1.12) for almost every $t\in[t_1,t_2]$ and satisfies one of the following possible (initial) boundary values:

1. (initial value problem) $x_0=\phi$, $\phi\in X$;

2. (boundary value problem) $x(t)=\phi(t-t_1)$, $t\in[t_1+\theta_{\min},t_1]$ and $x(t)=\psi(t-t_2)$, $t\in[t_2,t_2+\theta_{\max}]$, with $\phi\in X^{-}$ and $\psi\in X^{+}$;

3. (special boundary value problem) $x(t)=\phi(t)$, $t\in[t_1+\theta_{\min},t_1]$ and $x(t_2)=b$, $b\in\mathbb{C}$, with $\phi\in X^{-}$;

4. (Predetermined variable, backward-looking) $x_0 = \phi$, $\phi \in X^-$;

5. (Non-predetermined variable, forward-looking) $x_0 = \phi \in C([\theta_{\min}, 0), \mathbb{C}^n)$ and $x_0(0^-)$ exists.

When (1.12) is linear, see [66], it takes the general form

$$\dot{x}(t) = \int_{\theta_{\min}}^{\theta_{\max}} d\eta(t, \theta) x_t(\theta) + h(t), \qquad t \in I \subset \mathbb{R}, \tag{1.13}$$

where $x(t) \in \mathbb{C}^n$ and $d\eta$ is an $n \times n$ matrix of finite measures on $[\theta_{\min}, \theta_{\max}]$ (following Hale et al. [32]). When the entries of $d\eta$ are linear combinations of delta functions, we obtain the MFDE

$$\dot{x}(t) = \sum_{i=1}^{N} A_i(t) x(t + \theta_i) + h(t), \tag{1.14}$$

as a special case of (1.13) with $\theta_{\min} = \theta_1 < \ldots < \theta_N = \theta_{\max}$, a finite set of discrete shifts, and each A_i an $n \times n$-matrix with complex entries. Results about the numerical analysis of linear MFDEs can be found in [24, 25, 45, 46].

In general, the initial value problem for delay differential equations (DDEs) have solutions for any initial data because we can use the methods of steps and integrate the equation on each new interval of length τ. This means that the DDE is a dynamical system in forward time over the phase space X^-. We can associate with it a semigroup $T(t)$, defined by the time evolution of segments of solutions, acting on the Banach space of initial data X^-. This is not true anymore of MFDEs since the initial value problem is ill posed.

Recall that an operator T is a Fredholm operator if

1. its kernel, $\mathcal{K}(T)$ is finite dimensional,

2. the range, $\mathcal{R}(T)$ is closed and has finite codimension.

The Fredholm index is defined as the integer

$$\operatorname{ind}(T) = \dim \mathcal{K}(T) - \operatorname{codim} \mathcal{R}(T).$$

In [48], Mallet-Paret studied the MFDE

$$\dot{x}(t) = \sum_{i=1}^{N} A_i(t) x(t + \theta_i) + h(t), \tag{1.15}$$

where x is a mapping from $\mathbb{R}$ into $\mathbb{C}^n$ for some integer $n \geq 1$ and each A_i an $n \times n$ matrix with complex entries. The shifts $\theta_i \in [\theta_{\min}, \theta_{\max}]$ may be positive or negative, requiring that $\theta_{\min} \leq 0 \leq \theta_{\max}$. The state space is given by X. The equation (1.15) can be rewritten as

$$\dot{x}(t) = L(t) x_t + h(t),$$

where $L(t)$ denotes the linear functional

$$L(t)\varphi = \sum_{i=1}^{N} A_i(t)\varphi(\theta_i), \quad \varphi \in X.$$

When h is absent, the homogeneous version of (1.15) is

$$\dot{x}(t) = L(t)x_t. \tag{1.16}$$

The homogeneous equation

$$\dot{x}(t) = L_0 x_t \tag{1.17}$$

linearised at 0, has characteristic equation (which is obtained by substituting the ansatz $x(t) = e^{\alpha t}v$ into the equation

$$\Delta_{L_0}(\alpha) = \alpha I - \sum_{i=1}^{N} A_i(0)\, e^{\alpha\theta_i}$$

and the constant coefficient system (or L_0) is hyperbolic if

$$\det \Delta_{L_0}(i\eta) \neq 0, \quad \eta \in \mathbb{R}.$$

Associated to (1.17) is the closed operator A defined on a dense domain $\mathcal{D} \subset X$ given by

$$A\varphi = \dot{\varphi}, \quad \varphi \in \mathcal{D} = \{\ \varphi \in C^1([\theta_{\min}, \theta_{\max}], \mathbb{C}^n)\ :\ \dot{\varphi}(0) = L_0\varphi\ \}.$$

The spectrum $\sigma(A)$ of A is known to consist only of point spectrum and coincides with the solutions α of the characteristic equation $\det \Delta_{L_0}(\alpha) = 0$. For each $\alpha \in \sigma(A)$, the generalised eigenspace $E_\alpha \subset X$ of A corresponding to α is finite dimensional, and consists precisely of functions of the form

$$\varphi(\theta) = e^{\alpha\theta}\, p(\theta), \quad \theta \in [\theta_{\min}, \theta_{\max}],$$

where p is any polynomial with the property that $x(t) = e^{\alpha t}p(t)$ satisfies equation (1.17). The solutions x are called eigensolutions.

1.2.2 Neutral MFDEs

We are now ready to define a large class of NMFDEs. Suppose $\Omega \subset \mathbb{R} \times C$ is open, $f, h : \Omega \to \mathbb{R}^n$ are given continuous functions with h atomic at zero. The equation

$$\frac{d}{dt}h(t, x_t) = f(t, x_t) \tag{1.18}$$

is a NMFDE. The function h is called the difference operator for the NMFDE. For a given NMFDE, a function x is said to be a solution of the NMFDE if $h(t, x_t)$ is continuously differentiable and x is continuous, satisfying the NMFDE on an interval $[a - r, b)$.

It is known that DDEs tend to smooth out irregularities in the initial values but no such smoothing occurs for neutral equations where the solutions retain the degree of regularity of the initial values. Equations of advanced type tend to destroy the regularity of an initial function and the solution will only extend to $+\infty$ when the initial values comprise a function of class C^∞ (see Bellman and Cooke [7] for a discussion). Lamb and Van Vleck [42] extend Mallet-Paret's [48] Fredholm theory for MFDES to NMFDEs. They consider saddle-node bifurcation of a solution and we aim to explore the irregularities mentioned for MFDEs and also to investigate other types of bifurcations.

Consider the NMFDE

$$\frac{d}{dt}h(\lambda, x_t) = f(\lambda, x_t), \tag{1.19}$$

where the delays are bounded in $[-r, r]$, $\lambda \in \mathbb{R}^m$ are bifurcation parameters and $f, h : X \times \mathbb{R}^m \to \mathbb{R}^n$. We seek conditions on h, f such that (1.19) is (equivariant)-reversible. Let $H(\lambda), L(\lambda) : X \to \mathbb{R}^n$ be the two linearized operators of h, f around a steady state of (1.19). Furthermore, we assume that $H(\lambda)$ is atomic at zero. By the Riesz representation theorem, there exists $n \times n$ matrix-valued functions $\mu, \eta : [-r, r] \to \mathbb{R}^{n^2}$ whose components each have bounded variation and such that for $z \in X$,

$$H(\lambda)z = z(0) - \int_{-r}^{r} d\mu(\lambda, \theta)\, z(\theta), \quad L(\lambda)z = \int_{-r}^{r} d\eta(\lambda, \theta)\, z(\theta),$$

where $\mathrm{Var}_{[s,0]}\mu(\lambda, \theta) \to 0$ (see [32] for more details). For each fixed λ, the linear system

$$\frac{d}{dt}H(\lambda)\, x_t = L(\lambda)\, x_t,$$

generates a strongly continuous semigroup of linear operators with infinitesimal generator A_λ. The spectrum of A_λ, denoted by $\sigma(A_\lambda)$, is the point spectrum. Moreover, $\alpha \in \sigma(A_\lambda)$ if and only if α satisfies that $\det \Delta_{A_\lambda}(\alpha) = 0$, where the characteristic matrix Δ_{A_λ} is given by

$$\Delta_{A_\lambda}(\alpha) = \alpha H(\lambda)(e^{\alpha(.)}I) - L(\lambda)(e^{\alpha(.)}I).$$

It is well-known that $z \in X$ is an eigenvector of A_λ associated with the eigenvalue α if and only if $z(\theta) = e^{\alpha\theta}b$ for $\theta \in [-r, r]$ and some vector $b \in \mathbb{C}^n$ such that $\Delta_{A_\lambda}(\alpha)b = 0$. We assume that A_0 has a pair of purely imaginary eigenvalues $\pm i\beta_0$ and that the symmetry group Γ acting on the system may cause purely imaginary eigenvalues to be multiple.

1.3 Thesis Overview

In this work we contribute to the theory of MFDES, focusing on the solution of MFDEs, symmetries, reversibility, Hopf and Bogdanov-Takens bifurcation and optimality in variational problems. We

- Construct solutions to MFDEs with asymmetrical deviating arguments and also those with distributed arguments and provide the conditions necessary for the existence and uniqueness of solutions.
- Provide a definition of reversibilty that is readily applicable to Neutral MFDEs and provide conditions under which the action of a compact Lie on the nonlinear functions in an MFDE to render it reversible-equivariant and to completely classify the actions of the dihedral group of symmetries.
- Unfold a Neutral MFDE under the Bogdanov-Takens Bifurcation.
- Develop an equivariant Hopf bifurcation theory for NMFDEs.
- Develop an equivariant Lyapunov-Schmidt reduction to consider periodic solutions of symmetric MFDEs when the standard Hopf theorem cannot be directly applied resulting from a high multiplicity of the purely imaginary eigenvalues.
- Study the Hopf bifurcation of a cell network with all-to-all coupling.
- Obtain the necessary conditions on a function which minimises a functional with distributed delayed arguments and to apply the results to a one-dimensional DDE with a harvesting term.

Chapter 2 explores the question of existence and uniqueness of solutions to MFDEs. Our first contribution is to extend the works of Iakovleva and Iakovlev in [35] and [34] by constructing solutions to a MFDEs with asymmetrical deviating arguments i.e. where $\theta_{\min} \neq -\theta_{\max}$ (the minimum and maximum delay values). We then further generalise to the rather challenging case of distributed deviating arguments, introducing a recurrence relation and providing the necessary conditions for the existence and uniqueness of solutions. We also study the conditions under which a semigroup theory can be applied to MFDEs, providing a function space whose elements satisfy the semigroup requirements.

In Chapter 3, we develop a reversible-equivariant theory for NMFDEs, laying emphasis on $\mathbb{D}_n-$reversible-equivariant systems. We obtain the matricial structures that are necessary for an NMFDE system to be $\mathbb{D}_n$, $\mathbb{Z}_n$ and $\mathbb{Z}_2$ reversible-equivariant. We apply the results to a system of ring networks of cyclically arranged identical cells with forward and backward coupling. Our effort follows the works of Golubitsky et al. [27, 28], Lamb et al. [44, 43], Antonelli et al. [4], Teixeira

[60], Roberts et al [52], Buono et al. [10] that is mainly focused on bifurcation of equilibria in reversible-equivariant vector fields. We also explore the occurrence of Hopf bifurcation resulting from the actions of $\mathbb{D}_n$, $\mathbb{Z}_n$ and $\mathbb{Z}_2$.

Chapter 4 is mainly concerned with the center manifold theory for MFDEs. The decomposition of the state space of MFDEs (Mallet-Paret's [50, 48]) into stable and unstable subspaces and their associated semigroups is useful to apply the general theory of Vauderbauwhede and Iooss [64] It is well known that there are infinitely many characteristic values and some eigenfunctions may have arbitrarily large exponential growth or decay rates. Since MFDEs are not generally well-posed and do not generate a typical dynamical system, Mallet-Paret [50] decomposed their solutions into 'forwards' and 'backwards' solutions, thereby obtaining semigroups (evolutionary processes) in terms of retarded and advanced equations whilst making use of a variation of constants formula. An autonomous MFDE is given by the equation

$$\dot{x}(t) = \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\theta)x(t+\theta)$$

where $x(t)$ is an M-vector and $d\eta(\theta)$ is an $M \times M$ matrix of finite Lebesgue-Stieltjes measures on $[-\theta_{\min}, \theta_{\max}]$. The decomposition produces the following retarded and advanced characteristic functions:

$$\Delta_{-}(\alpha) = \alpha I - \int_{\theta_{\min}}^{0} e^{\alpha\theta} d\eta_{-}(\theta), \qquad \Delta_{+}(\alpha) = \alpha I - \int_{0}^{\theta_{\max}} e^{\alpha\theta} d\eta_{+}(\theta).$$

Cao et al. [13] investigate the Bogdanov-Takens bifurcation exhibited by a neutral DDE whilst Buono et al. in [11] study the versal unfolding of a family of delay equations. We extend these works by studying the Bogdanov-Takens (double zero) bifurcation analysis of an NMFDE and its versal unfolding.

Chapter 5 is mainly concerned with the development and application of the equivariant Lyapunov-Schmidt reduction following in the steps of Rustichini [56, 55]. Since the presence of symmetries in a dynamical system may lead to the multiplicity of the purely imaginary eigenvalues, we explore the symmetries and reversing symmetries of the MFDE and develop the equivariant Hopf bifurcation theory and equivariant Lyapunov-Schmidt reduction to explore the existence of periodic solutions. We carry also out the Hopf bifurcation of NMFDEs using the Lyapunov-Schmidt reduction process.

In Chapter 6, we study the Hopf bifurcation of an NMFDE in a ring network. Studies in bifurcation in ring networks have been mainly focused on systems with nearest-neighbour coupling. We extend this by considering Neutral MFDE systems with all-to-all coupling.

Chapter 7 concerns the optimisation of functionals with deviating arguments. Hughes determinined the necessary and sufficient conditions for optimality in variational problems with a single delayed argument in [33]. We extend the result of Hughes and obtain the critical points of symmetric functionals with distributed delays from which the resulting Euler-Lagrange equations yield

MFDEs. We apply the results to the logistic equation and the Euler-Lagrange equations ensuing from the optimisation yields a difference equation.

Chapter 8 presents a conclusion, highlighting our contributions and suggestions for future work.

We provide an appendix of some well known results to clarify and act as a reference to some relevant notions addressed in this work.

Chapter 2

Solution of Mixed Functional Differential Equations

2.1 Introduction to the Method of Steps

In this chapter we develop a step derivative method (by defining recurrence relations) to solve linear mixed functional differential equations (MFDEs), generalising some explicit results of the literature and illustrating some of the challenging issues and properties about MFDEs. We extend the construction of solutions to MFDEs with asymmetrical deviating arguments and further generalise to the case of distributed deviating arguments using a method of steps (step derivative) by Iakovleva et al. [35], who give necessary and sufficient conditions for the existence and uniqueness of the solution.

To set set the scene, we briefly discuss the method of steps (step integration) that is well established in the literature and is used to solve delay differential equations (DDEs). The theory of delay differential equations can be found in Hale et al. [32] in which the existence and uniqueness of solutions of DDEs is discussed. We show the method of steps using the simple class of DDEs of the form

$$\dot{x}(t) = f(x(t-\tau)) \tag{2.1}$$

where the discrete delay τ is a positive constant and f is a functional operator from $C([-\tau, 0], \mathbb{R}^n)$ to $\mathbb{R}^n$ with initial function $x(t) = \phi$ for $\theta \in [-\tau, 0]$. In the method of steps, we think of the solution of the DDE as a mapping of functions from the interval $[t-\tau, t]$ into functions on the interval $[t, t+\tau]$. The solutions may then be seen as a sequence of functions $f_0(t), f_1(t), f_2(t), \cdots$, defined over a set of contiguous intervals of length τ. We start with the initial function ϕ and use the differential equation to obtain the interval $[0, \tau]$ and then repeat the process to generate solutions on succeeding intervals.

Suppose that $x(t) = f_{i-1}(t)$ over some interval $[t_i - \tau, t_i]$ where $i = 1, 2, 3, \cdots$. Then, over the interval $[t_i, t_i + \tau]$, we have, by separation of variables, and using dummy variables r and s,

$$\int_{f_{i-1}(t_i)}^{x(t)} dr = \int_{t_i}^{t} f_{i-1}(s-1)ds \tag{2.2}$$

with the solution $x(t)$ is given by

$$\begin{aligned} x(t) &= f_i(t) \\ &= f_{i-1}(t_i) - \int_{t_i}^{t} f_{i-1}(s-1)ds \end{aligned} \tag{2.3}$$

with $f_0(t) = \phi$. We write the DDE (2.1) as an ODE on the interval $[0, \tau]$ as

$$\begin{aligned} \dot{x}(t) &= f(x(t)) \\ &= \phi(t-\tau) \end{aligned} \tag{2.4}$$

for all $t \in [-\tau, 0]$. The integral form of its solution can be written as

$$\begin{aligned} x(t) &= x(0) + \int_0^t f(x(s))ds \\ &= x(0) + \int_0^t \phi(s-\tau)ds. \end{aligned} \tag{2.5}$$

This process can be continued on succeeding intervals and we may call it the 'method of step integration' . In the case of mixed functional differential equations (MFDEs), the first step is to re-arrange the equation, forcing the solution $x(t)$ to depend on derivative terms. The method described below, in which the solution depends on successive differentiation may be termed the method of step derivatives.

Iakovleva et al. [35] employ a method of steps to obtain an iterative formula for the solution and also obtain the necessary and sufficient conditions for the existence of and uniqueness of the solution to the equation

$$\dot{x}(t) = x(t-1) + x(t+1), \tag{2.6}$$

with the initial function defined on the interval $[-1, 1]$ by

$$x(t) = \phi(t) = \begin{cases} \phi_1(t), & t \in [-1, 0] \\ \phi_2(t), & t \in [0, 1] \end{cases}$$

where $\phi \in C([-1, 1], \mathbb{R}^n)$, the space of continuous functions on the interval. The first step is to rewrite equation (2.6) in the form

$$x(t) = \dot{x}(t-1) - x(t-2) \tag{2.7}$$

and then obtain a solution over succeeding intervals of unit length by means of increasing order derivatives of the function ϕ. We develop and demonstrate this method in the ensuing sections of this chapter. The existence and uniqueness of the solution to the equation summarised in the following result:

Theorem 2.1 (Iakovleva et al, [35])**.** *The solution x of (2.6) with $\phi \in C^{\infty}([-1,1],\mathbb{C})$ exists and is infinitely differentiable if and only if*

$$\phi^{(n+1)}(0) = \phi^{(n)}(-1) + \phi^{(n)}(1), \quad n = 0, 1, 2, \cdots$$

Let $\phi \in C^{\infty}([-1,1],\mathbb{C})$. If a solution x of (2.6) exists and is differentiable, then the solution is unique.

Proof. See [35, Theorem 3.1, p. 4] □

Iakovlev et al. in [34] extend the method and results to the equation

$$\dot{x}(t) = Ax(t+a) + Bx(t-a) + Cx(t) + f(t),$$

where A, B and C are matrices and $t \in \mathbb{R}$.

2.2 Construction of Solutions of MFDEs

Before we generalise the results in and [35] and [34], we present the following transformation of a more general MFDE to the two-term form given later in equation (2.12).

Lemma 2.2. *Let A_1, B_-, B_+ be $n \times n$-matrices such that A_1 commutes with B_- and B_+, $r_1, r_2 \geq 0$ and $x : I \to \mathbb{C}^n$ where $I \subset \mathbb{R}$ is an interval containing the origin. The MFDE*

$$\dot{x}(t) = A_1 x(t) + B_- x(t-r_1) + B_+ x(t+r_2), \tag{2.8}$$

is equivalent to the MFDE

$$\dot{y}(t) = Ay(t-r_1) + By(t+r_2), \tag{2.9}$$

where $A = B_- e^{-A_1 r_1}$ and $B = B_+ e^{A_1 r_2}$, with $y(t) = e^{-A_1 t} x(t)$.

Proof. Let $y(t) = e^{-A_1 t}x(t)$. Differentiating y with respect to t :

$$\begin{aligned}
\dot{y}(t) &= -A_1 y(t) + e^{-A_1 t}\dot{x}(t) \\
&= -A_1 y(t) + e^{-A_1 t}\left(A_1 x(t) + B_- x(t-r_1) + B_+ x(t+r_2)\right) \\
&= -A_1 y(t) + A_1 y(t) + e^{-A_1 t}B_- e^{A_1 t}e^{-A_1 r_1}y(t-r_1) + e^{-A_1 t}B_+ e^{A_1 t}e^{A_1 r_2}y(t+r_2) \\
&= Ay(t-r_1) + By(t+r_2),
\end{aligned}$$

when A_1, hence $e^{A_1 t}$, commutes with B_- and B_+. □

In general we get the following result.

Corollary 2.3. *Let A_1, B_-, B_+ be $n \times n$-matrices, $r_1, r_2 \geq 0$ and $x : I \to \mathbb{C}^n$ where $I \subset \mathbb{R}$ is an interval containing the origin. The MFDE*

$$\dot{x}(t) = A_1 x(t) + B_- x(t-r_1) + B_+ x(t+r_2), \tag{2.10}$$

is equivalent to the MFDE

$$\dot{y}(t) = A(t)y(t-r_1) + B(t)y(t+r_2), \tag{2.11}$$

where

$$\begin{aligned}
A(t) &= e^{-A_1 t}B_- e^{A_1 t}e^{-A_1 r_1}, \\
B(t) &= e^{-A_1 t}B_+ e^{A_1 t}e^{A_1 r_2},
\end{aligned}$$

with $y(t) = e^{-A_1 t}x(t)$.

Note that when matrices commute, their exponentials commute. Therefore, if A and B_- commute, then B_- will commute with the exponentials $e^{A_1 t}$ and $e^{-A_1 t}$ leading to cancellations in (2.12) and Lemma 2.2 follows.

We generalise the work of Iakovleva et al. [35] and Iakovlev et al. [34] and obtain the solutions of

$$\dot{x}(t) = Ax(t+\theta_{\max}) + Bx(t+\theta_{\min}), \tag{2.12}$$

where A, B are $n \times n$-complex invertible matrices and the delays $\theta_{\min} \leq 0 \leq \theta_{\max}$, subject to the initial condition $x_0 = \phi$ where $\phi : [\theta_{\min}, \theta_{\max}] \to \mathbb{C}^n$. We also study the generalised and more challenging case of mixed functional differential equations with distributed delays.

We require that the operator A should be atomic at $\theta_{\max}$. The main tool used here is the method of steps which starts with the initial function, mapping it successively from the initial interval into subsequent intervals of equal length. When the delay and advanced arguments are asymmetrical, care must be taken to subdivide the intervals appropriately in order to apply the terms of the MFDE to the prior interval. We label the intervals $I_0, I_1 \cdots I_n$. We find that the order of the derivatives increases inside an interval and in subsequent intervals. We write (2.12) as a forward equation

$$Ax(t+\theta_{\max}) = \dot{x}(t) - Bx(t+\theta_{\min}), \tag{2.13}$$

with an initial condition $x_0 = \phi$. We write (2.13) equivalently as

$$x(t) = A^{-1}\dot{x}(t-\theta_{\max}) - A^{-1}Bx(t-\theta_{\max}+\theta_{\min}). \tag{2.14}$$

We use (2.14) to generate a solution on $[\theta_{\min}, \infty)$ using the method of steps described below.

Let $I_n = [n\theta_{\max} - (n-1)\theta_{\min}, (n+1)\theta_{\max} - n\theta_{\min}]$, $n \in \mathbb{Z}$. The union of those intervals is the whole of the real line and they intersect at their end points. To simplify notation we write here x_n for $x_{n(\theta_{\max}-\theta_{\min})}$, the restriction of x on I_n, that is, $x_n : [\theta_{\min}, \theta_{\max}] \to \mathbb{R}$ is defined by

$$x_n(\theta) = x(n(\theta_{\max} - \theta_{\min}) + \theta), \quad \theta \in [\theta_{\min}, \theta_{\max}], \tag{2.15}$$

in which $n(\theta_{\max} - \theta_{\min})$ can be visualised as a point in time at the boundary of the interval I_n, recalling the notation $x_t(\theta) = x(t+\theta)$. To obtain a solution of the equation (2.13) we employ the recurrence relation defined in (2.15) and determine the solution $Ax_n(\theta)$ in the interval I_n as follows:

$$\begin{aligned} Ax(n(\theta_{\max}-\theta_{\min})+\theta) &= \dot{x}(-n\theta_{\min} + (n-1)\theta_{\max} + \theta) - Bx((n-1)(\theta_{\max}-\theta_{\min})+\theta) \\ &= \dot{x}(n(\theta_{\max}-\theta_{\min}) + \theta - \theta_{\max}) - B(n(\theta_{\max}-\theta_{\min}) + \theta_{\min} - \theta_{\max} + \theta) \\ &= \dot{x}_{n-1}(\theta - \theta_{\min}) - Bx_{n-1}(\theta). \end{aligned} \tag{2.16}$$

Note that $\theta - \theta_{\min}$ is not necessarily in I_{n-1}, so (2.16) is not necessarily the recurrence relation we seek. Because $\theta_{\min} \leq \theta \leq \theta_{\max}$ and $\theta - \theta_{\min} \in [0, \theta_{\max} - \theta_{\min}]$ for $[\theta_{\min}, \theta_{\max}]$, we need to split $[\theta_{\min}, \theta_{\max}]$ into subintervals to write explicitly the relations linking x_n to x_{n-1}. We require that $\theta_{\min} \leq \theta - \theta_{\min} \leq \theta_{\max}$, that is, $2\theta_{\min} \leq \theta \leq \theta_{\max} + \theta_{\min}$. Therefore, the critical point occurs at $\theta = \theta_{\max} + \theta_{\min}$, beyond which $\theta - \theta_{\min} > \theta_{\max}$ and lies outside I_n and we therefore use $\dot{x_n}$ and not $\dot{x}_{n-1}$ We split the interval $[\theta_{\min}, \theta_{\max}]$ into the sub-intervals $[\theta_{\min}, \theta_{\max} + \theta_{\min}]$ (where $\theta - \theta_{\min} \in [\theta_{\min}, \theta_{\max}]$) and $[\theta_{\max} + \theta_{\min}, \theta_{\max}]$. The question is if 0 is in the first or the second interval. When $\theta > \theta_{\max}$, we reapply (2.14) to previously calculated values, increasing the order of the derivatives. We consider the two cases when $\theta_{\max} \geq |\theta_{\min}|$ or when $|\theta_{\min}| > \theta_{\max}$.

We note that when $\theta_{\max} \geq |\theta_{\min}|$ and for $\theta \in [\theta_{\min}, \theta_{\max} + \theta_{\min}]$,

$$Ax_n(\theta) = \dot{x}_n(\theta - \theta_{\min}) - Bx_{n-1}(\theta). \tag{2.17}$$

Since $\theta - \theta_{\min}$ in the derivative term is not in the interval $[\theta_{\min}, \theta_{\max}]$, we differentiate the equation to obtain

$$A\dot{x}(t) = \ddot{x}(t - \theta_{\max}) - B\dot{x}(t - \theta_{\max} + \theta_{\min}) \tag{2.18}$$

$$\begin{aligned} A\dot{x}((n-1)(\theta_{\max} - \theta_{\min}) + \theta - \theta_{\min}) &= \ddot{x}((n-2)\theta_{\max} - n\theta_{\min} + \theta)) \\ &= -B\dot{x}((n-2)\theta_{\max} - (n-1)\theta_{\min} + \theta) \end{aligned} \tag{2.19}$$

$$A\dot{x}_n(\theta - \theta_{\min}) = \ddot{x}_{n-1}(\theta - (\theta_{\min} + \theta_{\max})) - B\dot{x}_{n-1}(\theta - \theta_{\max}) \tag{2.20}$$

$$A\dot{x}_n(\theta) = \ddot{x}_{n-1}(\theta - \theta_{\min}) - B\dot{x}_{n-1}(\theta) \tag{2.21}$$

For $\theta \in [\theta_{\min}, \theta_{\max} + \theta_{\min}]$, giving

$$\begin{aligned} Ax_1(\theta) &= \dot{x}_0(\theta - \theta_{\min}) - Bx_0(\theta) \\ &= \dot{\phi}(\theta - \theta_{\min}) - B\phi(\theta) \end{aligned}$$

For $\theta \in [\theta_{\max} + \theta_{\min}, \theta_{\max}]$,

$$x_n(\theta) = A^{-2}\ddot{x}_{n-1}(\theta - (\theta_{\max} + \theta_{\min})) - A^{-2}B\dot{x}_{n-1}(\theta - \theta_{\max}) - A^{-1}Bx_{n-1}(\theta) \tag{2.22}$$

keeping with the delay notation for forward stepping. This gives, for $\theta \in [\theta_{\max} + \theta_{\min}, \theta_{\max}]$ since $\theta - \theta_{\min} > \theta_{\max}$,

$$Ax(t) = \dot{x}(t - \theta_{\max}) - Bx(t - \theta_{\max} + \theta_{\min})$$

and differentiating, we obtain,

$$\dot{x}(\theta - \theta_{\min}) = A^{-1}\ddot{x}(\theta - (\theta_{\max} + \theta_{\min})) - A^{-1}Bx(\theta - \theta_{\max}) \tag{2.23}$$

In the case when $|\theta_{\min}| > \theta_{\max}$, we note that $\theta - \theta_{\min} \in [0, \theta_{\max} - \theta_{\min}]$ for $[\theta_{\min}, \theta_{\max}]$, so we need to split $[\theta_{\min}, \theta_{\max}]$ into sub-intervals to write explicitly the relations linking x_n to x_{n-1}.

Denote by

$$\lfloor |\theta_{\min}|/\theta_{\max} \rfloor = \max\{\ k : k\theta_{\max} + \theta_{\min} > 0\},$$

$l = 1 + \lfloor |\theta_{\min}|/\theta_{\max} \rfloor$, and $\theta_k = k\theta_{\max}$, $0 \leq k \leq l$. So, let $J_k = [\theta_k + \theta_{\min}, \theta_{k+1} - \theta_{\min}]$, $0 \leq k \leq l-1$, and $J_l = [a_l + \theta_{\min}, \theta_{\max}]$. So $J_0 = [\theta_0 + \theta_{\min}, \theta_1 + \theta_{\min}] = [\theta_{\min}, \theta_{\max} + \theta_{\min}]$ and $J_1 = [\theta_{\max} + \theta_{\min}, 2\theta_{\max} + \theta_{\min}]$.

Note that the J_k are intervals of width $\theta_{\max}$ and J_l is the interval left over from removing multiples of a from $[\theta_{\min}, \theta_{\max}]$. In the interval I_n, when $\theta < \theta_{\max}$, we may apply (2.14) and then for subsequent subintervals of width $\theta_{\max}$, we reapply (2.14) to previously calculated values, increasing the order of the derivatives each time. For $\theta \in J_0$,

$$x_n(\theta) = A^{-1}\dot{x}_{n-1}(\theta - \theta_{\min}) - A^{-1}Bx_{n-1}(\theta).$$

For $\theta \in J_1$,

$$x_n(\theta) = A^{-2}\ddot{x}_{n-1}(\theta - (\theta_{\max} + \theta_{\min})) - A^{-2}B\dot{x}_{n-1}(\theta - \theta_{\max}) - A^{-1}Bx_{n-1}(\theta).$$

Lemma 2.4. *The solution to the MFDE (2.13) is given by the following recurrence relation*

$$x_n(\theta) = A^{-(k+1)}x_{n-1}^{(k+1)}(\theta - \theta_{\min} - \theta_k) - \sum_{j=0}^{k} A^{-j}Bx_{n-1}^{(j)}(\theta - \theta_j), \quad \theta \in J_k,\ 0 \leq k \leq l. \tag{2.24}$$

Proof. We prove Lemma 2.4 by induction on k.

Assume that (2.24) is true for $\theta \in J_k$. For $\theta \in J_{k+1}$ the argument $\theta - \theta_{\min} > \theta_{\max}$, so we use $\dot{x}(t) = A^{-2}\ddot{x}(t - \theta_{\max}) - A^{-2}B\dot{x}(t - \theta_{\max} + \theta_{\min})$, which has the effect of left-translating the argument by $\theta_{\max}$. Recall that $x_n(\theta) = x(n(\theta_{\max} - \theta_{\min}) + \theta)$. Taking $x(t) = x_n(\theta)$ with $t = n(\theta_{\max} - \theta_{\min}) + \theta$ we obtain

$$\begin{aligned}
\dot{x}_{n-1}(\theta - \theta_{\min}) &= A^{-2}\ddot{x}((n-1)(\theta_{\max} - \theta_{\min}) + \theta - \theta_{\min} - \theta_{\max}) \\
&- A^{-2}B\dot{x}((n-1)(\theta_{\max} - \theta_{\min}) + \theta - \theta_{\max}) \\
&= A^{-2}\ddot{x}_{n-1}(\theta - (\theta_{\max} + \theta_{\min})) - A^{-2}B\dot{x}_{n-1}(\theta - \theta_{\max}).
\end{aligned}$$

Hence for $\theta \in J_{k+1}$,

$$x_{n-1}^{(k+1)}(\theta - \theta_{\min} - \theta_k) = A^{-(k+2)}x_{n-1}^{(k+2)}(\theta - \theta_{\min} - \theta_{k+1}) - A^{-(k+2)}Bx_{n-1}^{(k+1)}(\theta - \theta_{k+1}). \tag{2.25}$$

Substituting into (2.24), we obtain

$$x_n(\theta) = A^{-(k+2)}x_{n-1}^{(k+2)}(\theta - \theta_{\min} - \theta_{k+1}) - A^{-(k+2)}Bx_{n-1}^{(k+1)}(\theta - \theta_{k+1}) - \sum_{j=0}^{k} A^{-(j+1)}Bx_{n-1}^{(j)}(\theta - \theta_j) \tag{2.26}$$

which simplifies to

$$x_n(\theta) = A^{-(k+2)}Bx_{n-1}^{(k+2)}(\theta - \theta_{\min} - \theta_{k+1}) - \sum_{j=0}^{k+1} A^{-(j+1)}Bx_{n-1}^{(j)}(\theta - \theta_j) \tag{2.27}$$

$\square$

2.2.1 Construction of Solution to MFDEs with Distributed Delays

We now extend the analysis developed above to the case of models that include a distribution of delayed and advanced terms, representing the situation where the arguments occur in some range of values with some associated distribution. The method is not easily applicable in the case of an equation with distributed delays and we do encounter additional difficulties with neutral MFDEs. If the linear operator L given in $\dot{x}(t) = Lx_t$ is continuous, then by the Riesz representation theorem, there exists an $n \times n$ matrix-valued function $\eta : [\theta_{\min}, \theta_{\max}] \to \mathbb{R}^{n^2}$ whose elements are of bounded variation such that $L\phi = \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\theta)\phi(\theta)$. Examples with discrete arguments are special cases of the more general distributed systems by employing the Dirac distribution.

Consider

$$\dot{x}(t) = \alpha x(t + \theta_{\max}) + \beta x(t + \theta_{\min}). \tag{2.28}$$

Then

$$\begin{aligned}
\dot{x}(t) &= Lx_t \\
&= \int_{\theta_{\min}}^{\theta_{\max}} (\delta_{\theta_{\max}}(\theta)x_t(\theta) + \delta_{\theta_{\min}}(\theta)x_t(\theta)) \\
&= \int_{\theta_{\min}}^{\theta_{\max}} (\delta_{\theta_{\max}}(\theta) + \delta_{\theta_{\min}}(\theta))x_t(\theta) \\
&= \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\theta)x_t(\theta),
\end{aligned}$$

where α and β are constants. Furthermore, let $\eta : [\theta_{\min}, \theta_{\max}] \to \mathbb{R}$ be given such that $\eta(\theta) = 0$ for all $\eta \in (\theta_{\min}, \theta_{\max})$ and $\eta(\theta_{\max}) = \alpha$ and $\eta(\theta_{\min}) = \beta$, then

$$\int_{\theta_{\min}}^{\theta_{\max}} d\eta(\theta)\,\phi(\theta) = \alpha\phi(\theta_{\max}) + \beta\phi(\theta_{\min}) \tag{2.29}$$

To apply the techniques developed above, we extract the leftmost and rightmost values i.e. at $\theta_{\min}$ and $\theta_{\max}$ from the operator L. Suppose that L is atomic at $\theta_{\min}$ and $\theta_{\max}$. Write

$$L\phi = \bar{L}\phi + \alpha\phi(\theta_{\max}) + \beta\phi(\theta_{\min}), \tag{2.30}$$

with $\alpha \cdot \beta \neq 0$. Hence we have

$$\dot{x}(t) = \alpha x_t(\theta_{\max}) + \beta x_t(\theta_{\min}) + \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\theta)\,x_t(\theta) \tag{2.31}$$

which we rearrange to obtain

$$x(t) = \alpha^{-1}\dot{x}(t - \theta_{\max}) - \alpha^{-1}\beta x(t + (\theta_{\min} - \theta_{\max})) - \int_{\theta_{\min}}^{\theta_{\max}} d\widehat{\eta}(\nu)\,x(t + \nu - \theta_{\max}). \tag{2.32}$$

With $\widehat{\eta}$ defined such that $d\widehat{\eta}(\theta_{\max}) = 0$, the modified integral term is not atomic at its boundary points. Define $x_{n(\theta_{\max}-\theta_{\min})}(\theta) := x(n(\theta_{\max} - \theta_{\min}) + \theta)$ and introduce the recurrence

$$x_n(\theta) = \alpha^{-1}\dot{x}_{n-1}(\theta - \theta_{\min}) - \alpha^{-1}\beta x_{n-1}(\theta) - \int_{\theta_{\min}}^{\theta_{\max}} d\widehat{\eta}(\nu)\, x_{n-1}(\theta + \nu - \theta_{\min}). \tag{2.33}$$

For $\dot{x}_{n-1}(\theta - \theta_{\min})$ we have $0 < \theta - \theta_{\min} < \theta_{\max} - \theta_{\min}$, however $\theta - \theta_{\min}$ cannot be greater than $\theta_{\max}$ but $\theta_{\max} - \theta_{\min} > \theta_{\max}$ since $\theta_{\min} < 0$, yielding the threshold $\theta = \theta_{\max} + \theta_{\min}$.

For the integral term, ν varies from $\theta_{\min}$ to $\theta_{\max}$ for a fixed θ. In the interval I_{n-1},we have $\theta < \theta - \theta_{\min} + \nu < \theta + \theta_{\max} - \theta_{\min}$ where $\theta + \theta_{\max} - \theta_{\min}$ lies in I_n. We have $\nu < \theta + \nu - \theta_{\min} < \nu + \theta_{\max} - \theta_{\min}$ with threshold when $\theta + \nu - \theta_{\min} = \theta_{\max}$ that is, when $\theta = \theta_{\max} + \theta_{\min} - \nu$. Note however that ν is variable. Hence to move into I_n, we add $\theta_{\max} - \theta_{\min}$ to θ. The integral would therefore require values close to $x_n(\theta)$ i.e. $x_{n-1}(\theta + \theta_{\max} - \theta_{\min}) = x_n(\theta)$.

With $\theta \in [\theta_{\min}, \theta_{\max}]$, the argument of x lies in the range

$$t - (\theta_{\max} - \theta_{\min}) \leq t - \theta_{\max} + \theta \leq t.$$

Hence to calculate $x(t)$ we require x over the interval $t - (\theta_{\max} - \theta_{\min})$. The presence of distributed delays mean that $x(t)$ requires up to instantaneous values of x for all $t > \theta_{\max}$. To remedy this situation, we employ functions with compact support such that the integral has value 0 in the boundary interval of width ϵ. Therefore ν varies from $\theta_{\min}$ to $\theta_{\max} - \epsilon$ and hence $\theta < \theta - \theta_{\min} + \nu < \theta + \theta_{\max} - \theta_{\min} - \epsilon$. In this case the threshold is $\theta + \theta_{\max} - \theta_{\min} - \epsilon = \theta_{\max}$ or equivalently $\theta = \theta_{\min} + \epsilon$. When $\theta_{\min} < \theta < \theta_{\min} + \epsilon$, we apply the equation to I_{n-1}.

The method of steps needs a clear interval before $t = \theta_{\max}$, therefore $x(t)$ can only be calculated implicitly by re-using the equation $x(t) = \dot{x}(t - \theta_{\max}) - \int_{\theta_{\min}}^{\theta_{\max}} d\widehat{\eta}(\theta)\, x_{t-\theta_{\max}}(\theta)$.

2.3 Existence of Solutions to MFDEs

Theorem 2.5. *The solution x of (2.12) with $\phi \in C^\infty([\theta_{\min}, \theta_{\max}], \mathbb{C}^n)$ exists and is differentiable (C^∞) if and only if*

$$\phi^{(n+1)}(0) = A\phi^{(n)}(\theta_{\max}) + B\phi^{(n)}(\theta_{\min}), \quad n = 0, 1, 2, \cdots$$

If a solution x of (2.6) exists and is differentiable, then the solution is unique.

Proof. Since ϕ belongs to $C^\infty([\theta_{\min}, \theta_{\max}], \mathbb{C}^n)$, for each interval I_n the function $x(t)$ exists and is infinitely many times differentiable. We now investigate the conditions necessary for the existence and continuity of the solution $x(t)$. For the solution to be continuous at $\theta = \theta_{\max} + \theta_{\min}$, we require that

$$x_n^+(\theta) = x_n^-(\theta), \qquad \text{at } \theta = \theta_{\max} + \theta_{\min} \tag{2.34}$$

That is

$$\begin{aligned} A^{-1}x_{n-1}(\theta-\theta_{\min}) - A^{-1}Bx_{n-1}(\theta) &= A^{-2}\ddot{x}_{n-1}(\theta-(\theta_{\max}+\theta_{\min})) \\ &- A^{-2}B\dot{x}_{n-1}(\theta-\theta_{\max}) - A^{-1}Bx_{n-\theta_{\max}}(\theta) \end{aligned} \tag{2.35}$$

and on substituting $\theta = \theta_{\max} + \theta_{\min}$ and simplifying, we obtain

$$\ddot{x}_{n-1}(0) = A\dot{x}_{n-1}(\theta_{\max}) + B\dot{x}_{n-1}(\theta_{\min}) \tag{2.36}$$

and when $n = 1$ we have

$$\ddot{\phi}(0) = A\dot{\phi}(\theta_{\max}) + B\dot{\phi}(\theta_{\min}) \tag{2.37}$$

that is, $\ddot{\phi} = L\dot{\phi}$.

For continuity at $\theta = \theta_{\max}$, we require that

$$x_n(\theta_{\max}) = x_{n+1}(\theta_{\min})$$

$$x_n(\theta_{\max}) = A^{-2}\ddot{x}_{n-1}(-\theta_{\min}) - A^{-2}Bx_{n=1}(0) - A^{-1}Bx_{n-1}(\theta_{\max}). \tag{2.38}$$

At $\theta_{\min}$, we have $\theta \in [\theta_{\min}, \theta_{\max} + \theta_{\min}]$ and

$$x_{n+1}(\theta_{\min}) = A^{-1}\dot{x}_n(0) - A^{-1}Bx_n(\theta_{\min}) \tag{2.39}$$

This should equal $x_n(\theta_{\max})$, implying that

$$x_n(\theta_{\max}) = A^{-1}\dot{x}_n(0) - A^{-1}Bx_n(\theta_{\min}) \tag{2.40}$$

hence

$$\dot{x}_n(0) = Ax_n(\theta_{\max}) + Bx_n(\theta_{\min}) \tag{2.41}$$

that is, $\dot{x}_n(0) = Lx_n$.

□

Corollary 2.6. *There exists a unique continuous solution of (2.12) on $[\theta_{\min}, \infty)$ if $x_0 = \phi$ is smooth on $[\theta_{\min}, \theta_{\max}]$ with*

$$\phi^{(n+1)}(0) = A\phi^{(n)}(\theta_{\max}) + B\phi^{(n)}(\theta_{\min}), \quad n \geq 0. \tag{2.42}$$

Proof. Fix n, then x is continuous on I_n if and only if x is continuous across the points $\theta_k = a_k + \theta_{\min} \in J_k \cap J_{k+1},\ 0 \leq k \leq l-1$, and across $\theta = a$ across I_n and I_{n+1}. We need to have

$$x_n(a_{k+1}) = A^{-(k+1)}x_{n-1}^{(k+1)}(a_{k+1}-\theta_{\min}-a_k) - \sum_{j=0}^{k} A^{-(j+1)}Bx_{n-1}^{(j)}(a_{k+1}-a_j),\ 0 \le k \le l-1,$$

$$x_n(a_{k+1}) = A^{-(k+2)}x_{n-1}^{(k+2)}(a_{k+1}-\theta_{\min}-a_k) - \sum_{j=0}^{k+1} A^{-(j+1)}Bx_{n-1}^{(j)}(a_{k+1}-a_j),\ 0 \le k \le l-1.$$

□

Theorem 2.7. *Let*

$$\dot{x}(t) = \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\theta)\, x_t(\theta). \tag{2.43}$$

The solution ϕ of the MFDE with distributed delays (2.43) exists and is differentiable when ϕ belongs to the set

$$M = \{\ \phi \in C^\infty([\theta_{\min}, \theta_{\max}])\ :\ \phi^{(n+1)}(0) = L\phi^{(n)}\ \}.$$

Proof. Since ϕ belongs to $C^\infty([\theta_{\min}, \theta_{\max}], \mathbb{C}^n)$, for each interval I_n the function $x(t)$ exists and is infinitely many times differentiable.

To show continuity at end points, we make reference to the recurrence given by the equation (2.33) presented again below upon rearrangement:

$$\dot{x}_{n-1}(\theta - \theta_{\min}) = \alpha x_n(\theta) + \beta x_{n-1}(\theta) + \alpha \int_{\theta_{\min}}^{\theta_{\max}} d\widehat{\eta}(\nu)\, x_{n-1}(\theta + \nu - \theta_{\min}).$$

Given an initial function $\phi \in C^\infty([\theta_{\min}, \theta_{\max}], \mathbb{C}^n)$, and setting $\theta = \theta_{\min}$, we obtain

$$\dot{x}_{n-1}(0) = \alpha x_n(\theta_{\max}) + \beta x_{n-1}(\theta_{\min}) + \int_{\theta_{\min}}^{\theta_{\max}} d\widehat{\eta}(\nu)\, x_{n-1}(\nu). \tag{2.44}$$

Differentiating (2.44) n times gives

$$\phi^{(n+1)}(0) = L\phi^{(n)}, \tag{2.45}$$

where we recall from (2.30) that $L\phi = \bar{L}\phi + \alpha\phi(\theta_{\max}) + \beta\phi(\theta_{\min})$. □

2.3.1 Possible Initial Functions $\phi(\theta)$ that satisfy the MFDE

Lemma 2.8. *The function $\phi(\theta) = e^{p\theta}$ where $p \in \mathbb{C}$ such that $p = Ae^{p\theta_{\max}} + Be^{p\theta_{\min}}$, is a characteristic root of and satisfies the MFDE (2.12) .*

Proof. Let $\phi(\theta) = e^{p\theta}$, then $\phi^{(n)}(\theta) = p^n e^{p\theta}$. Hence $\phi^{(n+1)}(0) = A\phi^{(n)}(\theta_{\max}) + B\phi^{(n)}(\theta_{\min})$, giving $p^{n+1} = p^n Ae^{p\theta_{\max}} + p^n Be^{p\theta_{\min}}$ and therefore, $p = Ae^{p\theta_{\max}} + Be^{p\theta_{\min}}$, which can be seen to be the characteristic equation of (2.12) and thus satisfies Theorem 2.5 . □

If we consider polynomials, $\phi(\theta) = \theta^n$, then $\phi^{(n)}(\theta) = n!\theta$ and $\phi^{(n+1)}(\theta) = 0$. Therefore, $0 = n!A\theta + n!B\theta$, giving the restriction $A = -B$. To determine if there exists an initial function $\phi(\theta)$ that satisfies Theorem 2.5 for all A and B, we consider functions of the Gaussian distribution $g(e^{-\theta^2})$, or $h(\sin(\theta))$ such that

$$\phi^{(n)}(0) = \phi^{(n)}(\theta_{\max}) = \phi^{(n)}(\theta_{\min}) = 0. \tag{2.46}$$

2.4 Semigroup Associated to an MFDE

In [36], a spectral analysis of the semigroup associated to certain MFDE is presented. The associated semigroup to the solution of this equation and its corresponding infinitesimal generator are defined on a closed subspace of $C^\infty[\theta_{\min}, \theta_{\max}]$. Expressions for the resolvent associated to the infinitesimal generator and its point spectrum are presented. A particular feature in this work is that the semigroup is defined on a topological space which is not a Banach space.

We apply this to a general autonomous linear MFDE. The idea is that for an autonomous linear MFDE, from $\dot{x}(t) = Lx_t$, we get $x^{(n+1)}(t) = Lx_t^{(n)}$, for any $n \in \mathbb{N}$.

To motivate the ideas developed in this section, we recall some basic facts about semigroup theory. The following Cauchy problem

$$\dot{x}(t) = Ax(t),\, t \geq 0 \tag{2.47}$$

with $x(0) = \phi$ where $A = (a_{ij})$ is an $n \times n$ matrix with $a_{ij} \in \mathbb{C}$ for $i, j = 1, 2, \cdots, n$ has a unique solution $x(t) = x_0 e^{tA}$ with A and e^{tA} linear operators. The family of matrix operators $T(t) = e^{tA}$, $t \geq 0$, is a uniformly continuous semigroup on $\mathbb{C}$. The representation of the solution as $x(t) = T(t)x_0$, $t \geq 0$, yields the derivation of some properties of the solution from the properties of the family $T(t)$. This idea can be extended to a general Banach space. We note that if A is a linear unbounded operator, $A : D(A) \subset X \to X$, with some additional conditions, then we can associate with A a so-called C_0-semigroup of linear operators $T(t) \in L(X)$, $t \geq 0$.

We recall that a semi-norm on a vector space X is a real-valued function p on X such that $p(x + y) \leq p(x) + p(y)$ and $p(\alpha x) = |\alpha| p(x) \,\forall\, x, y \in X$, and scalars α and that p is a norm if $p(x) \neq 0$ if $x \neq 0$. If a system of semi-norms $\mathcal{P}$ is countable, then we may assume that $\mathcal{P} = \{\|\|_k : f \in \mathbb{N}\}$, where $\|\|_1 \leq \|\|_2 \leq \cdots$. This may be obtained by setting $\|f\|_k = \max_{j=1,2,\cdots,k} p_j(f)$ where $\{p_j : j \in \mathbb{N}\}$ is a countable system of semi-norms.

Definition 2.9. *See Rudin [54]. Let $\{p_\alpha\}$ be a family of semi-norms on a vector space X. Then the α^{th} open strip (ball) of radius r centred at $x \in X$ is*

$$B_r^\alpha = \{y : p_\alpha(x - y) < r\}. \tag{2.48}$$

Let $\mathcal{E} = \{B_r^\alpha(x)\}$ be the collections of all open strips in X, then the topology $\mathcal{T}(\mathcal{E})$ generated by $\mathcal{E}$ is called the topology induced by $\{p_\alpha\}$. Since each p_α is a semi-norm, $B_r^\alpha(x)$ is convex.

The set consisting of finite intersections of the balls, $\mathcal{B} = \left\{\bigcap_{j=1}^n B_r^{\alpha_j}\right\}$ forms a base for the induced topology.

Consider the topological space $C^\infty[-\theta_{\min}, \theta_{\max}]$ endowed with the induced topology by the countable (implying the existence of a surjection from $\mathbb{N}$ to $C^\infty[\theta_{\min}, \theta_{\max}]$) system of semi-norms:

$$p_k(f) = \max_{x\in[\theta_{\min},\theta_{\max}]} |f^{(k)}(x)|$$

with k a non-negative integer such that each k gives a different semi-norm. Convergence in this topology implies the uniform convergence of the function and each of its derivatives of any order.

Definition 2.10. *A sequence of functions $f_n(x)$ defined on an set S is said to converge uniformly to $f(x)$ on S if $\|f_n - f\|_S \to 0$ as $n \to \infty$, i.e. $\lim_{n\to\infty} f_n = f$ uniformly on S.*

If the sequence of functions $f_n(x)$ converges uniformly to $f(x)$ on the S, then $f_n(x)$ converges pointwise to $f(x)$.

Denote

$$\|f\|_k = \sum_{i=0}^{k} \|f^{(i)}\|_\infty = \sum_{i=0}^{k} \max|f^{(i)}| = \sum_{i=0}^{k} p_i(f), \qquad \text{for } k = 0, 1, \cdots .$$

Consider the following closed subspace of $C^\infty[\theta_{\min}, \theta_{\max}]$:

$$M = \{\ \phi \in C^\infty[\theta_{\min}, \theta_{\max}]\ :\ \phi^{(n)}(0) = B\phi^{(n-1)}(\theta_{\min}) + A\phi^{(n-1)}(\theta_{\max}), \quad n = 1, 2, \dots\ \}, \tag{2.49}$$

and each $t \geq 0$, a subspace of $C^\infty[\theta_{\min}, \theta_{\max}]$ such that when $\phi \in M$, we have a unique solution ${}^\phi x$.

Definition 2.11. *Let X be a real Banach space. A strongly continuous semigroup of linear operators, (C_0-semigroup), is a one-parameter family $T(t) : X \to X$, $t \geq 0$ of linear operators that satisfy*

1. $T(0) = I$;
2. $T(t_1 + t_2) = T(t_1)T(t_2), \qquad t_1,\ t_2 \geq 0$;
3. $\lim_{t\downarrow 0} \frac{T(t)\phi - \phi}{t} = 0, \qquad \phi \in X$

We consider the operator T_t on the solution $x(t)$ of (2.5) defined by

$$(T_t\phi)(\theta) = {}^\phi x(t + \theta), \qquad \theta \in [\theta_{\min}, \theta_{\max}].$$

Theorem 2.12. *The family of operators $\{ T_t \ : \ t \geq 0 \}$ defined by $T_t : M \to M$ on the solutions $x(t)$ of equation (2.5) defines a strongly continuous semi-group in the space M. Furthermore, T_t is unique. The domain of T_t follows from Theorem 2.5.*

Proof. We define function $y_t(\theta) := T_t x(\theta) = {}^{\phi}x_t(\theta)$ and require that $y_t^{(n)}(0) = Ay_t^{(n-1)}(\theta_{\max}) + By_t^{(n-1)}(\theta_{\min})$. We also have $\dot{y}_t(0) = \dot{x}_t(0) = Ax_t(\theta_{\max}) + Bx_t(-\theta_{\min})$. Upon differentiating $n-1$ times, we have $y_t^{(n)} = x_t^{(n)}(0) = Ax_t^{(n-1)}(\theta_{\max}) + Bx_t^{(n-1)}(\theta_{\min})$. Hence $y_t(\theta) = (T_t\phi)(\theta) \in M$ for each fixed $t \geq 0$ and the restriction of $T_t x$ to $[\theta_{\min}, \theta_{\max}]$, is $\phi \in M$, so the domain of T_t is M.

We check (the semigroup property) if $T_{t+s} = T(t)T(s)$.

We have that

$$T_{t+s}(\theta) = {}^{\phi}x_{t+s}(\theta), \tag{2.50}$$

which is the solution at time $t + s + \theta$ with the restriction $\theta \in [\theta_{\min}, \theta_{\max}]$.

We also have that

$$T_t(T_s)(\theta) = T_t({}^{\phi}x(s+\theta)) = (T_t\psi)(\theta) = {}^{\psi}x_t(\theta), \tag{2.51}$$

the solution at time $t+\theta$ with restriction $\psi(\theta) = {}^{\psi}x_s(\theta)$. We obtain equality when ${}^{\phi}x_{t+s}(\theta) = {}^{\psi}x_t(\theta)$. However, ${}^{\phi}x_{t+s}(\theta)$ is a solution of

$$\begin{aligned} \dot{x}_t(0) &= Ax_t(\theta_{\max}) + Bx_t(\theta_{\min}) \\ {}^{\phi}\dot{x}_{t+s}(0) &= A^{\phi}x_{t+s}(\theta_{\max}) + B^{\phi}x_t(\theta_{\min}) \\ \dot{\psi}_t(0) &= A\psi_t(\theta_{\max}) + B\psi_t(\theta_{\min}) \qquad \text{solution with } \psi \text{ restricted to} \qquad [\theta_{\min}, \theta_{\max}]. \end{aligned} \tag{2.52}$$

To show that T_t is continuous for each fixed t, we shall prove that there exists n_k for each k and some constant $c \geq 0$ such that $\|T_t\phi\|_k \leq c\sum_{i=0}^{n_k} \|\phi\|_i$ for each fixed $t \geq 0$.

$$\begin{aligned} \|T_t\phi\|_k &= \sum_{i=0}^{k} p_i T_t\phi \\ &= \sum_{i=0}^{k} p_i {}^{\phi}x_t(\theta) \\ &= \sum_{i=0}^{k} \max |{}^{\phi}x_t^{(i)}(\theta)|. \end{aligned} \tag{2.53}$$

We recall the solutions constructed via the method of steps. When $\theta_{\max} > |\theta_{\min}|$, we write (2.43) in terms of the initial function ϕ as

$$x_n(\theta) = A^{-(k+2)}B\phi^{(k+2)}(\theta + \theta_{\min} - a_{k+1}) - \sum_{j=0}^{k+1} A^{-(j+1)}B\phi^{(j)}(\theta - a_j) \tag{2.54}$$

When $|\theta_{\min}| > \theta_{\max}$, we write (2.24) in terms of ϕ

$$x_n(\theta) = A^{-(k+1)}\phi^{(k+1)}(\theta + \theta_{\min} - a_k) - \sum_{j=0}^{k} A^{-j}B\phi^{(j)}(\theta - a_j), \quad \theta \in J_k,\ 0 \le k \le l. \tag{2.55}$$

To establish a bound for $x_t(\theta)$ which has width $\theta_{\max} - \theta_{\min}$, we consider contiguous intervals I_{n-1}, I_n or I_n, I_{n+1} that encompass $x_t(\theta)$. The value of n is calculated by subtracting multiples of $\theta_{\max} + \theta_{\min}$ from t and found to be $n = \lfloor \frac{t+\theta_{\max}}{\theta_{\max}+\theta_{\min}} \rfloor$. Since $\theta_{\max}$, $\theta_{\min} \in \mathbb{R}$, we can only estimate the order of the derivatives of ϕ in the interval I_n as $k = \lfloor \frac{n\theta_{\max}}{\theta_{\min}} \rfloor$ when $\theta_{\max} > \theta_{\min}$ and $k = \lfloor \frac{n\theta_{\min}}{\theta_{\max}} \rfloor$ when $\theta_{\min} > \theta_{\max}$. We may therefore use the estimate

$$p_k(T_t x(\theta)) \le \max_{I_{n-1}} |x^{(k)}(\theta)| + \max_{I_n} |x^{(k)}(\theta)|,$$

and the equations (2.54) and (2.55) and find that

$$\max_{[\theta_{\min},\theta_{\max}]} |x_t^{(k)}| \le c \sum_{i=1}^{k} \|\phi\|_i.$$

Finally, $\lim_{t\to t_0} T_t x = T_{t_0} x$ if and only if $\lim_{t\to 0} \|T_{t_0+t}x(\theta) - T_{t_0}x(\theta)\|_k = 0$ for $t_0 \ge 0$ and $k = 0, 1, \ldots$.

Hence by assuming that $\theta_{\min} \le \theta \le \theta_{\max}$ and taking into account the uniform continuity of $x^{(k)}$ in the closed interval $[t_0 + \theta_{\min}, t_0 + \theta_{\max}]$ ($t, t' \in [\theta_{\min}, \theta_{\max}]$), it follows that

$$\max_{[\theta_{\min},\theta_{\max}]} |x^{(k)}(\tau + t_0 + \theta) - x^{(k)}(t_0 + \theta)| \le \max_{|t-t'|\le|\tau|} |x^{(k)}(t) - x^{(k)}(t')| \to 0$$

as $\tau \to 0$.

The uniqueness of T_t follows from Theorem 2.5.

From the definition of T_t and with $T_0 = I$, the identity, and $T_{t+s} = T_t T_s$ for each $t, s \ge 0$. $\square$

2.4.1 Infinitesimal Generator

The solution $x(t)$, of (2.12) , can be extended to the left and with the additional condition of the existence of an inverse, the operator set (semigroup) T_t can be made to constitute a group. Consider the space $D(A) = \{\ \phi \in C^\infty[\theta_{\min}, \theta_{\max}]\ :\ \lim_{t\to 0} \frac{(T_t\phi)(\theta) - \phi(\theta)}{t}\}$ and define the operator $A : D(A) \to C^\infty[\theta_{\min}, \theta_{\max}]$ as

$$A\phi(\theta) = \lim_{t\to 0} \frac{(T_t\phi)(\theta) - \phi(\theta)}{t} = \lim_{t\to 0} \frac{\phi(t+\theta) - \phi(\theta)}{t},$$

with the limit taken in the topology of $C^\infty[\theta_{\min}, \theta_{\max}]$. The operator A is called the infinitesimal generator associated to the semigroup T_t.

Theorem 2.13. *The infinitesimal generator A maps M in M and $Ax = \dot{x}$, where M is given by 2.49.*

Proof. Firstly we show that $Ax = \dot{x}$. Assuming $x \in D(A)$, then

$$\lim_{t \to 0} \max_{\theta \in [\theta_{\min}, \theta_{\max}]} \left| \frac{\phi(t+\theta) - \phi(\theta)}{t} - A\phi(\theta) \right| = 0.$$

This implies that, for each $\theta \in [\theta_{\min}, \theta_{\max}]$, there is the pointwise limit

$$A\phi(\theta) = \lim_{t \to 0} \frac{\phi(t+\theta) - \phi(\theta)}{t} = \dot{\phi}(\theta).$$

Next, we show that the domain of the operator A is M where M is the subspace defined in (2.49). This follows since from Theorem 2.12 $T_t : M \to M$.

By the Mean Value Theorem, for each $n = 1, 2, \ldots$, there exists a number $0 < \xi_t < t$ such that

$$\lim_{t \to 0} \left| \frac{x^{(n-1)}(t+\theta) - x^{(n-1)}(\theta)}{t} - x^{(n)}(\theta) \right| = |x^{(n)}(\theta + \xi_t) - x^{(n)}(\theta)|.$$

Since $x^{(n)}$ is continuous in the closed interval $[\theta_{\min}, \theta_{\max}]$, and therefore uniformly continuous in this interval, for each $\epsilon > 0$ there exists $\delta > 0$ such that if $|\xi_t| < \delta$

$$\lim_{\xi \to 0} \max_{\theta \in [\theta_{\min}, \theta_{\max}]} |x^{(n)}(\theta + \xi_t) - x^{(n)}(\theta)| = 0.$$

Since $\lim_{t \to 0} \xi_t = 0$, and applying the limit of composition of functions,

$$\lim_{t \to 0} \max_{\theta \in [\theta_{\min}, \theta_{\max}]} \left| \frac{x^{n-1}(t+\theta) - x^{(n-1)}(\theta)}{t} - x^{(n)}(\theta) \right| = 0$$

for each $n \in \mathbb{N}$.

Next we verify that the range of A is M. We take $\phi \in M = \mathrm{dom}(A)$, and define

$$\psi(\theta) = A\phi(\theta) = \dot{\phi}(\theta).$$

We then need to verify that $\psi^n(0) = A\psi^{n-1}(\theta_{\max}) + B\psi^{n-1}(\theta_{\min})$. We have

$$\dot{\psi}(0) = \ddot{\phi}(0) = A\dot{\phi}(\theta_{\max}) + B\dot{\phi}(\theta_{\min}) \tag{2.56}$$

which we differentiate $n-1$ times to obtain

$$\begin{aligned} \psi^n(0) = \phi^{n+1}(0) &= A\phi^n(\theta_{\max}) + B\phi^n(-\theta_{\min}) \\ &= A\psi^{n-1}(\theta_{\max}) + B\psi^{n-1}(\theta_{\min}) \end{aligned} \tag{2.57}$$

Hence we have $\psi(\theta) \in M$ since we took $\psi(\theta) = \dot{\phi}(\theta)$, with $\dot{\phi}(\theta) \in M$, where M is defined by 2.49 .

□

2.5 Spectral Analysis of the Infinitesimal Generator

On the set M, the MFDE becomes an ODE because of the generator of the semigroup. In order to determine the resolvent associated to the infinitesimal generator, we consider the equation

$$Ax(\theta) = \lambda x(\theta) + f(\theta) \qquad \text{where } f, x \in M.$$

Since $Ax = x'$, the former equation can thus be rewritten as

$$\dot{x} = \lambda x + f, \tag{2.58}$$

which we consider with the initial conditions given by 2.6

$$x^{(n+1)}(0) = Bx^{(n)}(\theta_{\min}) + Ax^{(n)}(\theta_{\max}), \quad n = 0, 1, \ldots. \tag{2.59}$$

The solution of the ODE in equation 2.58 is given by

$$x(\theta) = ce^{\lambda\theta} + \int_0^{\theta} e^{\lambda(\theta-s)} f(s)\, ds, \tag{2.60}$$

where s is a dummy variable and where the derivative is given by

$$\dot{x}(\theta) = \lambda ce^{\lambda\theta} + \lambda \int_0^{\theta} e^{\lambda(\theta-s)} f(s)\, ds + f(\theta). \tag{2.61}$$

We determine the constant c such that the solution satisfies the conditions (2.59). Since $x \in M$, it satisfies the condition $\dot{x}(0) = Bx(\theta_{\min}) + Ax(\theta_{\max})$. Substituting this condition into the expressions coming from (2.61) for $\theta = 0$, and from (2.60) for $\theta = -\theta_{\min}, \theta_{\max}$, we obtain

$$c = \frac{B \int_0^{-\theta_{\min}} e^{-\lambda(\theta_{\min}-s)} f(s)\, ds + A \int_0^{\theta_{\max}} e^{\lambda(\theta_{\max}-s)} f(s)\, ds - f(0)}{\lambda - Be^{-\lambda\theta_{\min}} - Ae^{\lambda\theta_{\max}}} \tag{2.62}$$

and note that the denominator is the characteristic equation for (2.59).

For the general continuous linear operator Lx_t, with x and f vectors, using $\dot{x}(0) = Lx_0 = \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\theta) x_0(\theta)$, we have

$$\dot{x}(0) = \lambda c + f(0) \tag{2.63}$$

and

$$Lx_0 = \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\theta) \left\{ ce^{\lambda\theta} + \int_0^{\theta} e^{\lambda(\theta-s)} f(s) ds \right\} \tag{2.64}$$

and equating the two expressions, we obtain

$$\lambda c + f(0) = \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\theta) c e^{\lambda\theta} + \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\theta) \int_0^{\theta} e^{\lambda(\theta - s)} f(s) ds \tag{2.65}$$

which upon solving for c we have

$$c = \left[\lambda I - \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\theta) e^{\lambda\theta}\right]^{-1} \left\{ \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\theta) \int_0^{\theta} e^{\lambda(\theta - s)} f(s) ds - f(0), \right\} \tag{2.66}$$

with c a vector and the expression in square brackets, a matrix.

Since $x \in C^{\infty}([\theta_{\min}, \theta_{\max}], \mathbb{R})$ and, using induction, we verify that $x \in M$, if $f \in M$. Finally, in view of $(A - \lambda I)x = f$, we have $x = R(\lambda, A)f$ where $R(\lambda, A)$ denotes the resolvent associated to A. From (2.62), we get the expression for the resolvent associated to A to be

$$R(\lambda, A)f(\theta) = c e^{\lambda\theta} + \int_0^{\theta} e^{\lambda(\theta - s)} f(s)\, ds. \tag{2.67}$$

We observe that the operator R is defined only for λ such that $g(\lambda) = \lambda - Be^{-\lambda\theta_{\min}} - Ae^{\lambda\theta_{\max}}$ is different from zero since it appears in the denominator .

Theorem 2.14. *The eigenvalues of A satisfy the characteristic equation $g(\lambda) = \lambda - Be^{-\lambda\theta_{\min}} - Ae^{\lambda\theta_{\max}} = 0$ and there is no other spectrum.*

Proof. There exists $x \neq 0$ such that $Ax = \lambda x$ if and only if $x(\theta) = ce^{\lambda\theta}$. We will prove that $x \in M$. To do it, we replace $t = 0$ in (2.6) obtaining $\dot{x}(0) = Bx(\theta_{\min}) + Ax(\theta_{\max})$ which implies, taking out c, $g(\lambda) = 0$. Similarly, we obtain for the n-th derivative $x^{(n)}(0) = Bx^{(n-1)}(-\theta_{\min}) + Ax^{(n-1)}(\theta_{\max})$, that is, $g(\lambda) = 0$. Therefore, the point spectrum of A is the set of $\lambda \in \mathbb{C}$ such that $g(\lambda) = 0$. We now show that there is no other spectrum implying that if $g(\lambda) \neq 0$ then λ belongs to the resolvent of A. It is sufficient to show that the resolvent is a continuous operator for such a λ. We make use of the following result:

Theorem 2.15 (Folland,[23])**.** *Let X and Y be vector spaces with the topologies defined by the families of semi-norms $\{p_{\alpha}\}_{\alpha \in A}$ and $\{g_{\beta}\}_{\beta \in B}$, respectively, where A and B are certain sets of indexes. Let $R : X \to Y$ be linear. Then, R is continuous if and only if for all $\beta \in B$ there exist $\alpha_1, \cdots, \alpha_k \in A$ and $c > 0$ such that $g_{\beta}(Rx) \leq c \sum_{j=1}^{k} p_{\alpha_j}(x)$.*

We rewrite the resolvent given by (2.67) and using the relation for c, in the form

$$\begin{aligned} R(\lambda, A)f(\theta) &= cB \int_0^{\theta_{\min}} e^{-\lambda(\theta_{\min} + s)} f(s) ds\, .e^{\lambda\theta} + cA \int_0^{\theta_{\max}} e^{\lambda(\theta_{\max} - s)} f(s) ds\, .e^{\lambda\theta} \\ &- cf(0)e^{\lambda\theta} + \int_0^{\theta} e^{\lambda(\theta - s)} f(s) ds \end{aligned} \tag{2.68}$$

or simply as

$$R(\lambda, A)f = c_\lambda R_1 f + c_\lambda R_2 f + c_\lambda R_3 f + R_4 f,$$

where $c_\lambda = (\lambda - e^{-\lambda\theta_{\min}} - e^{\lambda\theta_{\max}})^{-1}$, and prove that the $R_i f$'s are continuous for $i = 1, 2, 3, 4$. $\square$

Theorem 2.16. *For initial functions $\phi(\theta) = e^{\lambda\theta}$, $\theta \in [\theta_{\min}, \theta_{\max}]$, and $\lambda \in \sigma_p(A)$, we have $T_t\phi(\theta) = e^{\lambda(\theta+t)}$ for $t \in \mathbb{R}$.*

Proof. Clearly the function $e^{\lambda\tau}$, for $\tau \in \mathbb{R}$, represents the smooth solution of equation $\dot{x}(t) = x(t - \theta_{\min}) + x(t + \theta_{\max})$ if and only if $\lambda - e^{-\lambda\theta_{\min}} - e^{\lambda\theta_{\max}} = 0$, that is, if and only if $\lambda \in \sigma_p(A)$. In view of $T_t x(\theta) = x(t+\theta)$, where $x(t+\theta)$ is the solution of equation $\dot{x}(t) = Bx(t-\theta_{\min}) + Ax(t+\theta_{\max})$ with the initial function $\phi(\theta)$, $\theta \in [-\theta_{\min}, \theta_{\max}]$, we have $T_t e^{\lambda\theta} = e^{\lambda(\theta+t)}$ for $t \in \mathbb{R}$. $\square$

Chapter 3

Symmetries and Reversibility of Neutral MFDEs

We know that for ODEs, the Γ-equivariance of the vector field with respect to the action of a compact group Γ is equivalent to the Γ-symmetry of the equation in the sense that the action of Γ leaves globally invariant the set of solutions of the ODE (sends a solution into another one). Clearly, the equivariance of the vector field induces the symmetry of the equations. The result follows from the fact that the converse is true when we can always uniquely solve the Cauchy problem for an ODE.

Definition 3.1. *A Neutral Functional Differential equation (NMFDE) is an equation the form*

$$\dot{x}(t) = f(x_t, \dot{x}_t) \tag{3.1}$$

that is, $\dot{x}(t)$ depends not only upon the past history of x but also on the past history of $\dot{x}(t)$.

Lemma 3.2. *Let Γ be a compact group. If an ODE is Γ-symmetric, then its vector field f is Γ-equivariant.*

Proof. Let $x(t)$ be a solution of the ODE $\dot{x}(t) = f(t, x(t))$ with initial condition $x(t_0) = x_0$. For any $\gamma \in \Gamma$, denote by $y(t) = \gamma\, x(t)$ the γ-symmetric solution. Therefore $\dot{y}(t) = f(t, y(t))$ and so

$$\dot{y}(t_0) = \gamma\, \dot{x}(t_0) = \gamma f(t_0, x_0) = f(t_0, y(t_0)) = f(t_0, \gamma x_0),$$

valid for any t, x_0 and $\gamma \in \Gamma$. $\square$

In this chapter we explore those ideas in the more general context of NMFDEs, extending the notion of reversibility to these cases.

3.1 Symmetry and Equivariance of Vector Fields

We study the reversibility and equivariance MFDEs given by the following equation;

$$\dot{x}(t) = f(t, x_t), \tag{3.2}$$

with phase space defined by $X = C([\theta_{\min}, \theta_{\max}], \mathbb{C}^n)$, which when equipped with the sup-norm, is a Banach space. We have $x_t \in X$ and assume that $0 \in [\theta_{\min}, \theta_{\max}]$. Let $f : \mathcal{U} \subset (\mathbb{R}\times X \to \mathbb{C}^n)$ be a smooth enough function. If f is Frechet differentiable at 0, we identify by the Riesz representation theorem, its Frechet derivative at zero by $f'(0)$, with the regular measure induced by a function of bounded variation, the $n \times n$-matrix function η on $[\theta_{\min}, \theta_{\max}]$ such that the linear operator $L(\lambda)\phi = f'(0)\phi$ is given by

$$L(\lambda)\phi = \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda, \theta)\, \phi(\theta),$$

where the integration variable is $\theta \in [\theta_{\min}, \theta_{\max}]$. For such an η, we always regard it as extended to $\mathbb{R}$ so that

$$\begin{aligned} \eta(\lambda, \theta) &= \eta(\lambda, \theta_{\min}), \quad \theta \leq \theta_{\min}, \\ \eta(\lambda, \theta) &= \eta(\lambda, \theta_{\max}), \quad \theta \geq \theta_{\max}. \end{aligned}$$

Let Γ be a compact group acting on $\mathbb{R}^n$ via a representation $\rho : \Gamma \to \mathbb{O}(n)$. Recall that if Γ is a compact group we can always choose co-ordinates so that the action of Γ is orthogonal (hence $\Gamma \subset \mathbb{O}(n)$). Given an interval $I \subset \mathbb{R}$, the action of Γ induces an action on functions $x : I \to \mathbb{R}^n$ in a straightforward way: for $\gamma \in \Gamma$, $(\gamma x)(t) = \gamma x(t)$. Such actions also respect the regularity of a function, that is, if a function f belongs to one of the function spaces we consider in this thesis, then γf belongs to the same function space. Given a NMFDE, an element $\gamma \in \Gamma$ is called a symmetry of the NMFDE if $y(t) = \gamma x(t)$ is a solution whenever x is a solution. Closely linked, the non linearity of an NMDE is Γ-equivariant when it commutes with a group action of Γ. Clearly, Γ-equivariance implies Γ-symmetry.

Define $L : I \times \mathbb{R}^k \times X \to \mathbb{R}^n$ as

$$(L(t, \lambda)z)(\theta) = \int_{\theta_{\min}}^{\theta_{\max}} d\eta(t, \lambda, \theta)\, z(\theta). \tag{3.3}$$

Lemma 3.3. *Given a compact group Γ acting on $\mathbb{R}^n$, the linear map L (3.3) is Γ-equivariant if*

$$\gamma d\eta(t, \lambda, \theta) = d\eta(t, \lambda, \theta)\gamma, \quad \forall \gamma \in \Gamma. \tag{3.4}$$

3.2 Reversibility

To introduce reversibility, we assume that there is a group Γ acting on $\mathbb{R}^n$ with an homomorphism $\chi : \Gamma \to \mathbb{Z}_2$ where $\mathbb{Z}_2 = \{1, -1\}$. Such homomorphism is an example of a character of Γ over $\mathbb{Z}_2$, the cyclic group of order 2. One example of such map could be $\chi(\gamma) = \det(\rho(\gamma))$ where the representation ρ of the group Γ on V is a linear group homomorphism $\rho : \Gamma \to GL(V)$, where $GL(V)$ is the vector space of invertible linear mappings $V \to V$. Given $r > 0$, the existence of χ induces another action of Γ on $C[-r, r]$ defined by

$$(\gamma^\sharp z)(\theta) = \gamma z(\chi(\gamma)\theta), \quad \forall \theta \in [-r, r]. \tag{3.5}$$

For reversible FDEs we consider $\theta_{\max} = -\theta_{\min} = r > 0$. Our FDE will be reversible if $t \mapsto x(t)$ is a solution if and only if $t \mapsto \rho(\gamma)x(\chi(\gamma)t)$ is also a solution. Define $\Xi = \{ \gamma \in \Gamma : \chi(\gamma) = 1 \}$, the subgroup of Γ of spatial symmetries. Then our FDE is Ξ-equivariant. An element in the complement of Ξ in Γ is called a reversing symmetry of the FDE. We do not require that there exists a reversing symmetry, that is, an involution (that is, $\gamma^2 = I$). In general, $\chi(\gamma^2) = (\chi(\gamma))^2 = 1$, and so the only thing we can say is that the composition of two reversing symmetries is a spatial symmetry (see Lemma 3.4). If the FDE is reversible, but does not possess any nontrivial symmetry, i.e. $\Gamma = \mathbb{Z}_2$ and $\Xi = \{\mathbf{1}\}$, we call it purely reversible.

The representation ρ_χ of Γ defined as $\rho_\chi(\gamma) = \chi(\gamma)\rho(\gamma)$ is called the (χ-) dual (representation) of Γ. Following [44], we call the representation of Γ self-dual if it is isomorphic to its dual, that is, if there exists a linear Γ-equivariant $A : V \to V$ such that

$$A\rho(\gamma) = \rho_\chi(\gamma)A, \quad \forall \gamma \in \Gamma. \tag{3.6}$$

Lemma 3.4. *1.* $\chi(\gamma^{-1}) = \chi(\gamma)$.

2. The composition of two reversing symmetries is a spatial symmetry.

3. Let Γ be a compact group acting orthogonally on $\mathbb{R}^n$ with reversing symmetries determined by the map $\chi : \Gamma \to \mathbb{Z}_2$, $\chi(\gamma) = \det(\gamma)$. If there is a reversing symmetry and the representation is self-dual, then n must be even.

4. The symmetries and reversing symmetries of a system form a group Γ and the symmetries form a normal subgroup $H \trianglelefteq \Gamma$. When $H \neq \Gamma$, then H is a subgroup of index 2, so $\Gamma/H \simeq \mathbb{Z}_2$. It is noted that Γ can be written as the semi-direct product $\Gamma \simeq H \rtimes \mathbb{Z}_2$ if and only if Γ/H contains an involution.

Proof. 1. Because $1 = \chi(I) = \chi(\gamma \cdot \gamma^{-1}) = \chi(\gamma)\chi(\gamma^{-1})$.

2. Let γ_i, $i = 1, 2$, such that $\chi(\gamma_i) = -1$, $i = 1, 2$. Then $\chi(\gamma_1\gamma_2) = \chi(\gamma_1)\chi(\gamma_2) = 1$, and so $\gamma_1\gamma_2$ is a spatial symmetry.

3. When the representation is self-dual, (3.6) holds. Taking the determinant on both sides, we find that $\det(A) = (\chi(\gamma))^n \det(A)$, $\forall \gamma \in \Gamma$, and so n must be even.

$\square$

3.3 Reversible Equivariant NMFDEs (REN MFDEs)

A Reversible Equivariant Neutral MFDE is a neutral mixed functional differential equation that possess symmetry transformations of the state variable (equivariance) and time inversion, $t \mapsto -t$, reversibility. Consider the NMFDE

$$\frac{d}{dt}(h(t, x_t)) = f(t, \lambda, x_t), \tag{3.7}$$

where the delays are bounded in $[\theta_{\min}, \theta_{\max}]$ and f, h are given continuous functions with h atomic at zero. We seek conditions on h and f such that (3.7) is reversible-equivariant.

Lemma 3.5. *When the maximum delay and absolute value of the minimum delay are equal, i.e. $\theta_{\min} = -\theta_{\max} = -r$, and $I = [-b, b]$, $b > 0$, the reversibility of equation (3.7) follows from*

$$\begin{aligned} h(\chi(\gamma)t, \lambda, \gamma^\sharp(z))) &= \gamma h(t, \lambda, z), \quad \forall \gamma \in \Gamma,\ \forall t \in I,\ \forall z \in X, &(3.8)\\ f(\chi(\gamma)t, \lambda, \gamma^\sharp(z)) &= \rho_\chi(\gamma) f(t, \lambda, z), \quad \forall \gamma \in \Gamma,\ \forall t \in I,\ \forall z \in X. &(3.9)\end{aligned}$$

Proof. Let $x(t)$ be a solution of (3.7). For reversibility, we require $y(t) = \gamma x(\chi(\gamma)t)$ to be also a solution of (3.7), that is,

$$\frac{d}{dt}(h(t, y_t)) = f(t, \lambda, y_t).$$

We have $y_t(\theta) = y(t + \theta)$. Let $y(t)$ be a solution of the equation 3.7. Then

$$\begin{aligned} y_t(\theta) &= \gamma x(\chi(\gamma)(t + \theta)) \\ &= \gamma x(\chi(\gamma)t + \chi(\gamma)\theta) \\ &= (\gamma^\sharp(x))_{\chi(\gamma)t}. \end{aligned}$$

And so, setting $s = \chi(\gamma)t$,

$$\begin{aligned} \frac{d}{dt}(h(t, y_t)) &= \frac{d}{dt}\left(h(t, (\gamma^\sharp(x))_{\chi(\gamma)t})\right) \\ &= \frac{d}{dt}\left(h(\chi(g)s, (\gamma^\sharp(x))_s\right), \end{aligned}$$

we get

$$
\begin{aligned}
\frac{d}{dt}\left(h(t,y_t)\right) &= \chi(\gamma)\frac{d}{ds}\left(h(\chi(g)s,(\gamma^\sharp(x))_s\right) \\
&= \chi(\gamma)\frac{d}{ds}\left(h(\chi(g)s,(\gamma^\sharp(x))_s\right).
\end{aligned}
$$

For the vector field, we obtain

$$
\begin{aligned}
f(t,y_t) &= f(t,(\gamma^\sharp(x))_{\chi(\gamma)t}) \\
&= f(\chi(\gamma)s,(\gamma^\sharp(x))_{\chi(\gamma)t}) \\
&= \chi(g)\gamma f(s,x_s).
\end{aligned}
$$

From the hypothesis of the lemma, we can conclude. $\square$

3.3.1 Linear Reversible Equivariant Neutral (REN) MFDEs

Here we explore the conditions needed to make a neutral MFDE reversible equivariant. A reversible equivariant neutral MFDE is a neutral MFDE possessing symmetry transformations of the state variable and inversion of the time variable.

Define $L : X \to \mathbb{R}^n$ as

$$
(L(t,\lambda)z)(\theta) = \int_{-r}^{r} d\eta(t,\lambda,\theta)\, z(\theta), \quad \forall t \in I = [-b,b],\ b > 0,\ \forall \theta \in [-r,r]. \tag{3.10}
$$

Lemma 3.6. *The linear map (3.10) is* Γ*-reversible equivariant if*

$$
d\eta(\chi(\gamma)t,\lambda,\chi(\gamma)\theta)\gamma = \chi(\gamma)\gamma\, d\eta(t,\lambda,\theta), \quad \forall t \in I, \theta \in [-r,r], \forall \gamma \in \Gamma. \tag{3.11}
$$

Proof. We need to show that

$$
L(\chi(\gamma)t,\lambda)\gamma^\sharp z = \rho_\chi(\gamma)L(t,\lambda)z, \quad \forall t \in I, \forall \gamma \in \Gamma, \forall z \in X.
$$

We have

$$
\begin{aligned}
L(\chi(\gamma)t,\lambda)\gamma^\sharp z &= \int_{-r}^{r} d\eta(\chi(\gamma)t,\lambda,\theta)(\gamma^\sharp z)(\theta) \\
&= \int_{-r}^{r} d\eta(\chi(\gamma)t,\lambda,\theta)\gamma z(\chi(\gamma)\theta) \\
&= \chi(\gamma)\int_{-\chi(\gamma)r}^{\chi(\gamma)r} d\eta(\chi(\gamma)t,\lambda,\chi(\gamma)\theta)\gamma z(\theta),
\end{aligned}
$$

and

$$\rho_\chi(\gamma)L(t,\lambda)z = \chi(\gamma)\gamma \int_{-r}^{r} d\eta(t,\lambda,\theta)\, z(\theta).$$

Therefore we need

$$\int_{-\chi(\gamma)r}^{\chi(\gamma)r} d\eta(\chi(\gamma)t,\lambda,\chi(\gamma)\theta)\gamma z(\theta) = \int_{-r}^{r} \gamma d\eta(t,\lambda,\theta)\, z(\theta).$$

This relation is satisfied if (3.11) holds true. □

3.4 Group Actions in One Dimension

The group $\Gamma = \mathbb{Z}_2 = \{\kappa, 1\}$, where $\kappa = -1$, had two actions on $\mathbb{R}$: $\kappa_1 x = -x$ or $\kappa_2 x = x$. Moreover, there are two homomorphisms $\chi_j : \mathbb{Z}_2 \to \mathbb{Z}_2$, $j = 1, 2$, namely, $\chi_j(\kappa_i) = (-1)^j$. This gives four possibilities to consider with the following actions on $C([-r, r])$:

$$\begin{aligned} (\kappa_{i,j} z)(\theta) &= (-1)^i z(\theta), \quad \forall \theta \in [-r, r],\ i, j = 1, 2, && (3.12)\\ (\kappa^{\sharp}_{i,j} z)(\theta) &= \kappa_{i,j} z(\chi_j(\kappa_i)\theta) = (-1)^i z((-1)^j \theta), \quad \forall \theta \in [-r, r],\ i = 1, 2. && (3.13) \end{aligned}$$

This means that we have four possibilities for REN MFDEs in one dimension. Namely, reversibility follows from

$$\begin{aligned} h((-1)^j t, \kappa^{\sharp}_{i,j} z) &= (-1)^j h(t, z), \quad \forall t \in I,\ i, j = 1, 2,\\ f((-1)^j t, \kappa^{\sharp}_{i,j} z) &= (-1)^{i+j} f(t, z), \quad \forall t \in I,\ i, j = 1, 2. \end{aligned}$$

They correspond to the following symmetries:

1a $\kappa_{1,2}$ with $\chi(\kappa_1) = 1$, $\mathbb{Z}_2$-equivariant, not reversible,

1b $\kappa_{1,1}$ with $\chi(\kappa_1) = -1$, reversible, $\mathbb{Z}_2$-equivariant,

2a $\kappa_{2,2}$ with $\chi(\kappa_2) = 1$, no equivariance, nor reversibility,

2b $\kappa_{2,1}$ with $\chi(\kappa_2) = -1$, purely reversible.

With the reversibility conditions on h and f, the previous cases become:

1a $h(t, z) = h(t, -z)$ and $-f(t, z) = f(t, -z)$, so $y(t) = -x(t)$ are both solutions,

1b $h(t, z(\theta)) = -h(-t, -z(-\theta))$ and $f(-t, z(\theta)) = f(t, -z(-\theta))$, so $y(t) = -x(-t)$ are both solutions,

2a no conditions,

2b $h(-t, z(-\theta)) = -h(t, z(\theta))$ and $-f(-t, z(\theta)) = f(t, z(-\theta))$, so $y(t) = x(-t)$ are both solutions.

Lemma 3.7. *The linear operator $L : I \times X \to \mathbb{C}$, $I = [-b, b]$, defined by*

$$(L(t)z) = \int_{-r}^{r} d\eta(t, \theta)\, z(\theta),$$

is reversible if and only if

$$d\eta(-t, \theta) = -d\eta(t, \theta), \quad \forall \theta \in [-r, r],\ t \in [-b, b].$$

Any linear one dimensional map is $\mathbb{Z}_2$-symmetric.

Proof. From (3.11), reversibility follows from

$$d\eta(-t, -\theta)\gamma = -\gamma d\eta(t, \theta),$$

with $\gamma = \pm 1$. And so, $d\eta(-t, -\theta) = -d\eta(t, \theta)$. □

Note that in Case 1a any linear operator L be sufficient because the $\mathbb{Z}_2$-symmetry is a condition on the non-linear part of f, namely, f must be odd in z.

As an example of a $\mathbb{Z}_2$ reversibile system, it can be seen that the linear operator

$$L(t)z = a(t)z(r) + b(t)z(0) + c(t)z(-r) \tag{3.14}$$

where $z \in X$, $r > 0$ and $a, b, c : \mathbb{R} \to \mathbb{R}$ is reversible if and only if b is an odd function and $c(t) = -a(-t)$ for all $t \in \mathbb{R}$.

3.5 Group Actions in Two Dimensions

We require two dimensional group actions for the second order equations and will consider the Lie groups $\Gamma : \mathbb{O}(2), \mathbb{SO}(2), \mathbb{D}_n, \mathbb{Z}_n$, $n \geq 2$.

For reversibility, we need to consider the different possibilities for the groups Γ and homomorphisms $\chi : \Gamma \to \mathbb{Z}_2$.

We note that $\mathbb{SO}(2)$ is connected and continuous, implying that χ must be constant and so we can only choose $\chi(\gamma) = 1$.

The presentation of $\mathbb{Z}_n$ is $< \varrho \ : \ \varrho^n = 1 >$. To define χ on $\mathbb{Z}_n$, we only need to fix the value of $\chi(\varrho)$. There are only two possibilities:

1. $\chi(\varrho) = +1$. In this case, $\mathbb{Z}_n$ acts as a pure symmetry, there are no reversors.

2. $\chi(\varrho) = -1$. This is a group homomorphism only when $n = 2m$ is even. In that case the symmetry group Ξ is generated by ϱ^2 and is isomorphic to $Z_{n/2} = \mathbb{Z}_m$. The group $\mathbb{Z}_n = \Xi\{1, \varrho\}$.

The presentation set $\mathbb{D}_n$ is

$$< \varrho, \kappa \ : \ \varrho^n = \kappa^2 = 1, \ \kappa\varrho = \varrho^{-1}\kappa > .$$

There are four possibilities for χ. We examine them in turn.

1. When $\chi(\varrho) = \chi(\kappa) = 1$, there is no reversibility, only pure symmetries . The whole group $\Xi = \mathbb{D}_n$.

2. When $\chi(\varrho) = 1$ and $\chi(\kappa) = -1$, clearly $\chi(\varrho^s) = 1$ and $\chi(\varrho^s\kappa) = -1$, $0 \leq s \leq n-1$. This defines an homomorphism because

$$\chi(\varrho^{s_1} \cdot \varrho^{s_2}\kappa) = \chi(\varrho^{s_1+s_2}\kappa) = -1 = \chi(\varrho^{s_1}) \cdot \chi\varrho^{s_2}\kappa),$$

   and
$$\chi(\varrho^{s_1}\kappa \cdot \varrho^{s_2}\kappa) = \chi(\varrho^{s_1-s_2}\kappa^2) = \chi(\varrho^{s_2-s_1}) = 1 = (-1)^2 = \chi(\varrho^{s_1}\kappa) \cdot \chi(\varrho^{s_2}\kappa).$$

   The symmetries are $\Xi = \mathbb{Z}_n$ and $\mathbb{D}_n$ is the semi-direct product $\mathbb{Z}_2 \rtimes \mathbb{Z}_n$ where $\mathbb{Z}_2 =< \kappa >$.

3. When $\chi(\varrho) = -1$ and $\chi(\kappa) = 1$, $\chi(\varrho^s) = (-1)^s$, and so $n = 2m$ must be even. The character χ cannot be the determinant. The group of pure symmetries is isomorphic to

$$\Xi = \{\varrho^{2s}, \varrho^{2s}\kappa \ : \ 0 \leq s \leq m-1 \ \} = \mathbb{D}_m = \mathbb{Z}_2 \rtimes \mathbb{Z}_m,$$

   where $\mathbb{Z}_m$ correspond to the rotations generated by ϱ^2 and $\mathbb{Z}_2 =< \kappa >$.

4. Finally, $\chi(\rho) = -1$ and $\chi(\kappa) = -1$. Again, $n = 2m$ must be even and the character χ cannot be the determinant. The group of pure symmetries is isomorphic to

$$\Xi = \{\varrho^{2s}, \varrho^{2s+1}\kappa \ : \ 0 \leq s \leq m-1 \ \} = \mathbb{D}_m = \mathbb{Z}_2 \rtimes \mathbb{Z}_m,$$

   where $\mathbb{Z}_m$ correspond to the rotations generated by ϱ^2 and $\mathbb{Z}_2 =< \varrho\kappa >$.

3.6 Group Actions in n Dimensions

Here we study the actions of $\mathbb{Z}_n$ and $\mathbb{Z}_2$ on $\mathbb{R}^n$ by

$$\rho(x_1, \cdots, x_n) = (x_2, x_3, \cdots, x_1), \text{and}\, \kappa(x_1, \cdots, x_n) = (x_n, \cdots, x_1),$$

where ρ, κ together generate the dihedral group $\mathbb{D}_n$ and ρ generates the cyclic group $\mathbf{Z}_n$.

The generator ρ of $\mathbb{Z}_n$ acts on $\mathbb{R}^n$ by $(\rho x)_i = x_{i+1}$. A flip of order 2 or reflection is denoted by κ with action on $\mathbb{R}^n$ given by either of $(\kappa x)_i = x_{n+2-i}$, when the line of reflection passes between the n^{th} and the first component or $(\kappa x)_i = x_{n+2-i}$ when the line of reflection passes through the first component. The dihedral group $\mathbb{D}_n = \mathbb{Z}_n \otimes \mathbb{Z}_2$ of order $2n$ is generated by ρ and κ.

The representation of ρ be given by the cyclic forward shift matrix and that of the flip κ by the permutation matrix of vector indices, by which simple calculations yield the following:

$$\rho = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{3.15}$$

such that the entries $\rho_{i\,j} = \delta_{i\,j-1}$ and consequently we have $(\rho^2 x)_i = x_{i+2}$ with

$$\rho^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & 1 & \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{3.16}$$

with entries given by $\rho^2_{i\,j} = \delta_{i\,j-2}$ and

$$\kappa = \begin{bmatrix} 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & 1 & \vdots & \vdots & 0 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{3.17}$$

with entries given by $\kappa_{i\,j} = \delta_{i+j\,n+1}$.

We define $d\eta$ by

$$d\eta = \begin{bmatrix} \eta_{11}(t,\theta) & \eta_{12}(t,\theta) & \eta_{13}(t,\theta) & \dots & \eta_{1n}(t,\theta) \\ \eta_{21}(t,\theta) & \eta_{22}(t,\theta) & \eta_{23}(t,\theta) & \dots & \eta_{2n}(t,\theta) \\ \dots & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \eta_{n1}(t,\theta) & \eta_{n2}(t,\theta) & \eta_{n3}(t,\theta) & \dots & \eta_{nn}(t,\theta) \end{bmatrix}.$$

3.7 Reversible Equivariance

3.7.1 $\mathbb{Z}_{\frac{n}{2}}$ Action

We explore $\mathbb{Z}_n$ actions and also those of $\mathbb{Z}_{\frac{n}{2}}$ which will be required to determine the matrix structures for $\mathbb{Z}_n$ and $\mathbb{D}_n$ reversibility. We note that ρ^2 rotates pairs of adjacent coordinate points onto subsequent pairs and lets ρ act between the pairs.

Lemma 3.8. *The matrix $d\eta^{\rho^2}$ is generated by two row vectors d_k for the odd numbered rows and b_k for the even numbered rows given by the relations*

$$d_k = \eta_{2i-1\,k+2i-2}, \tag{3.18}$$

that is, $\eta_{ij} = d_{j-2i+2}$ and

$$b_k = \eta_{2i\,k+2i-2}, \tag{3.19}$$

from which $\eta_{ij} = b_{j-2i+2}$

Proof. Note that $\chi(\rho^2) = 1$, with ρ^2 acting as a symmetry.

Given that $\rho_{ij} = \delta_{i\,j-1}$, we have $\rho^2_{ij} = (\rho\cdot\rho)_{ij}$, hence

$$\begin{aligned}(\rho\cdot\rho)_{ij} &= \sum_{k=1}^{n}\rho_{ik}\rho_{kj}\\ &= \sum_{k=1}^{n}\delta_{i\,k-1}\delta_{k\,j-1}\\ &= \delta_{i+1\,j-1}\\ &= \delta_{i\,j-2}.\end{aligned} \tag{3.20}$$

For the $\mathbb{Z}_{\frac{n}{2}}$ action, we require that $[\rho^2 d\eta]_{ij} = [d\eta\rho^2]_{ij}$.

$$\begin{aligned}[\rho^2 d\eta]_{ij} &= \sum_{k=1}^{2}(\rho^2)_{ik}\eta_{kj}\\ &= \sum_{k=1}^{2}\delta_{i\,k-2}\eta_{kj}\\ &= \eta_{i+2\,j}\end{aligned} \tag{3.21}$$

and

$$
\begin{aligned}
[d\eta\rho^2]_{ij} &= \sum_{k=1}^{2} \eta_{ik}(\rho^2)_{kj} \\
&= \sum_{k=1}^{2} \eta_{ik}\delta_{k\,j-2} \\
&= \eta_{i\,j-2}.
\end{aligned} \tag{3.22}
$$

Equating (3.21) to (3.22) we obtain $\eta_{i+2\,j} = \eta_{i\,j-2}$ or $\eta_{ij} = \eta_{i+2\,j+2}$.

$\rho^2 \in \mathbb{Z}_r$ acts as a symmetry on $\mathbb{R}^{2r} = \mathbb{R}^n$ when $n = 2r$ i.e. when n is even. The result is that a row moves up two steps and across two steps, giving families (orbits) of size r. Hence with n^2 entries in the matrix $d\eta$, we have $\frac{n^2}{r} = \frac{(2r)^2}{r} = 4r$ orbits. The action therefore gives pairs of rows, d_k and b_k.

To generate the matrix entries for $d\eta^{\rho^2}$, we consider two row vectors $d_k = [d_1, d_2 \cdots d_n]$ and $b_k = [b_1, b_2 \cdots b_n]$.

For the odd numbered rows, we have $d_k = \eta_{2i-1\,j}$. We let $k = j - (2i-2)$, and $i \mapsto i+1$ whilst $j \mapsto j+2$. Then $k = (j+2) - 2(i+1-1) = j - 2i + 2$. Hence $d_k = \eta_{2i-1\,k+2i-2}$. Setting $j = k + 2i - 2$ gives $k = j - 2i + 2$ and we obtain

$$
\eta_{ij} = d_{j-2i+2}. \tag{3.23}
$$

Similarly, for the even rows, we have $b_k = \eta_{2i\,j}$, and with the arguments used in the odd numbered case, we find that $b_k = \eta_{2i\,k+2i-2}$.

$\square$

Using the result obtained above, we can construct the matrix $d\eta^{\rho^2}$ required for the $\mathbb{Z}_{\frac{n}{2}}$ equivariant action thus:

$$d\eta^{\rho^2} = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 & \dots & d_{n-2} & d_{n-1} & d_n \\ b_1 & b_2 & b_3 & b_4 & \dots & b_{n-2} & b_{n-1} & b_n \\ d_{n-1} & d_n & d_1 & d_2 & \dots & d_{n-4} & d_{n-3} & d_{n-2} \\ b_{n-1} & b_n & b_1 & b_2 & \dots & b_{n-4} & b_{n-3} & b_{n-2} \\ d_{n-3} & d_{n-2} & d_{n-1} & d_n & \dots & d_{n-6} & d_{n-5} & d_{n-4} \\ b_{n-3} & b_{n-2} & b_{n-1} & b_n & \dots & b_{n-6} & b_{n-5} & b_{n-4} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \dots \\ d_7 & d_8 & d_9 & d_{10} & \dots & d_4 & d_5 & d_6 \\ b_7 & b_8 & b_9 & b_{10} & \dots & b_4 & b_5 & b_6 \\ d_5 & d_6 & d_7 & d_8 & \dots & d_2 & d_3 & d_4 \\ b_5 & b_6 & b_7 & b_8 & \dots & b_2 & b_3 & b_4 \\ d_3 & d_4 & d_5 & d_6 & \dots & d_n & d_1 & d_2 \\ b_3 & b_4 & b_5 & b_6 & \dots & b_n & b_1 & b_2 \end{bmatrix}. \tag{3.24}$$

3.7.2 $\mathbb{Z}_n$ Equivariance

We consider the action of the rotation group ρ with character $\chi(\rho) = 1$. Then

Lemma 3.9. *For $\mathbb{Z}_n-$equivariance, the n^2 entries of the matrix $d\eta$ are permutations of an $n-$vector $d = [d_k]$ such that for $i, k = 1, 2, \cdots, n$ we have $d_k = \eta_{i\,i+k}$ i.e. $\eta_{ij} = d_{j-i}$. This yields $\frac{n}{2} + 1\,\rho-$orbits, each with n elements which are pairs of diagonals with distinct entries and a unique main diagonal.*

Proof. We require that $[\chi(\rho)\rho d\eta]_{ij} = [d\eta\rho]_{ij}$. We recall that $\rho_{ij} = \delta_{i\,j-1}$.

$$\begin{aligned} [\chi(\rho)\rho d\eta]_{ij} &= \sum_{k=1}^{n} \rho_{ik}\eta_{kj} \\ &= \sum_{k=1}^{n} \delta_{ik-1}\eta_{kj} \\ &= \eta_{i+1j} \end{aligned} \tag{3.25}$$

and

$$
\begin{aligned}
[d\eta\rho]_{ij} &= \sum_{k=1}^{n} \eta_{ik}\rho_{kj} \\
&= \sum_{k=1}^{n} \eta_{ik}\delta_{kj-1} \\
&= \eta_{ij-1}
\end{aligned} \tag{3.26}
$$

Equating (3.25) to (3.26), we obtain $\eta_{i+1j} = \eta_{ij-1}$, i.e. $\eta_{ij} = \eta_{i-1\,j-1}$. □

This yields $n-$distinct entries in the vector $[d_1, d_2, \cdots, d_n]$ for the matrix $d\eta$ such that $d_k = \eta_{i\,i+k}$, for $k = 1, 2, \cdots, n$, i.e.

$$
d\eta^{\rho} = \begin{bmatrix}
d_n & d_1 & d_2 & \ldots & d_{n-2} & d_{n-1} \\
d_{n-1} & d_n & d_1 & \ldots & d_{n-3} & d_{n-2} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
d_3 & d_4 & d_5 & \ldots & d_1 & d_2 \\
d_2 & d_3 & d_4 & \ldots & d_n & d_1 \\
d_1 & d_2 & d_3 & \ldots & d_{n-1} & d_n
\end{bmatrix}. \tag{3.27}
$$

We note that the relation $\eta_{ij} = d_{j-i}$ makes the first entry to d_η as d_n : however, to ensure that the first entry should be d_1, the modified relation is $\eta_{ij} = d_{j-i+1}$, the usual circulant definition.

3.7.3 $\mathbb{Z}_n$ Reversibility

We consider action of the rotation group $\mathbb{Z}_n$ with element ρ whose character $\chi(\rho) = -1$ and with ρ^2 acting as the symmetry group, whereby $\eta_{ij} = d_{j-2i+2}$. This leads to :

Lemma 3.10. *The matrix $d\eta^{\rho-}$ is generated by the relations $-d_k(t,\theta) = d_{k+1}(-t,-\theta)$ for the odd numbered rows and $-b_k(t,\theta) = b_{k+1}(-t,-\theta)$ for the even numbered rows. Thus for the odd numbered rows, we have the alternating sequence $\eta_{2i-1\,2i-j} = d_1$ and $\eta_{2i-1\,2j} = d_2$, where $d_2(-t,-\theta) = -d_1(t,\theta)$. The same is obtained for the even numbered rows.*

Proof. We require that $[\chi(\rho)\rho d\eta]_{ij}(t,\theta) = [d\eta\rho]_{ij}(-t,-\theta)$. We recall that $\rho_{ij} = \delta_{i\,j-1}$.

For the odd numbered rows, we obtain

$$
\begin{aligned}
[\chi(\rho)\rho d\eta]_{ij}(t,\theta) &= \sum_{k=1}^{n} -\rho_{ik}\eta_{kj}(t,\theta) \\
&= \sum_{k=1}^{n} -\delta_{ik-1}\eta_{kj}(t,\theta) \\
&= \sum_{k=1}^{n} -\delta_{ik-1}d_{j-2k+2}(t,\theta) \\
&= -d_{j-2(i+1)+2}(t,\theta) \\
&= -d_{j-2i}(t,\theta)
\end{aligned} \tag{3.28}
$$

and

$$
\begin{aligned}
[d\eta\rho]_{ij}(-t,-\theta) &= \sum_{k=1}^{n} \eta_{ik}\rho_{kj}(-t,-\theta) \\
&= \sum_{k=1}^{n} \eta_{ik}\delta_{kj-1}(-t,-\theta) \\
&= \sum_{k=1}^{n} d_{k-2i+2}\delta_{kj-1}(-t,-\theta) \\
&= d_{j-1-2i+2}(-t,-\theta) \\
&= d_{j-2i+1}(-t,-\theta)
\end{aligned} \tag{3.29}
$$

Equating (3.28) to (3.29), we obtain $-d_{j-2i}(t,\theta) = d_{j-2i+1}(-t,-\theta)$, that is, $-d_k = d_{k+1}$. $\square$

This yields $2-$distinct entries in the vector $[d_1, d_2, \cdots, d_n]$ for the matrix $d\eta$ such that $d_k = \eta_{i\,i+k}$, for $k = 1, 2, \cdots, n$, i.e.

$$
d\eta^{\rho-} = \begin{bmatrix}
d_1 & -d_1 & d_1 & -d_1 & \ldots & d_1 & -d_1 \\
b_1 & -b_1 & b_1 & -b_1 & \ldots & b_1 & -b_1 \\
d_1 & -d_1 & d_1 & -d_1 & \ldots & d_1 & -d_1 \\
b_1 & -b_1 & b_1 & -b_1 & \ldots & b_1 & -b_1 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
d_1 & -d_1 & d_1 & -d_1 & \ldots & d_1 & -d_1 \\
b_1 & -b_1 & b_1 & -b_1 & \ldots & b_1 & -b_1
\end{bmatrix}. \tag{3.30}
$$

3.7.4 $\mathbb{Z}_2$ Equivariance

We consider the action of the rotation group $\mathbb{Z}_2$ with character $\chi(\kappa) = 1$.

Lemma 3.11. *When $\chi(\kappa) = 1$, the matrix entries for $d\eta$ satisfy $\eta_{i\,j} = \eta_{n+1-i\,n+1-j}$.*

Proof.

$$\begin{aligned}
[\kappa \cdot d\eta]_{i\,j} &= \sum_{k=1}^{n} \kappa_{i\,k}\,\eta_{k\,j} \\
&= \sum_{k=1}^{n} \delta_{i+k\,n+1}\,\eta_{k\,j} \\
&= \eta_{n+1-i\,j} \\
&= \eta_{1-i\,j} \quad \bmod n
\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
[d\eta \cdot \kappa]_{i\,j} &= \sum_{k=1}^{n} \eta_{i\,k}\,\kappa_{k\,j} \\
&= \sum_{k=1}^{n} \eta_{ik}\delta_{j+k\,n+1} \\
&= \eta_{i\,n+1-j} \\
&= \eta_{i\,1-j}
\end{aligned} \tag{3.32}$$

Equating (3.31) to (3.32) we obtain the relation $\eta_{1-i\,j} = \eta_{i\,1-j}$.

□

We therefore obtain the following matrix

$$d\eta^{\kappa} = \begin{bmatrix}
b_{11} & b_{12} & b_{13} & \dots & b_{1(n-1)} & b_{1n} \\
b_{21} & b_{22} & b_{23} & \dots & b_{2(n-1)} & b_{2n} \\
\dots & \dots & \dots & \dots & \dots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
b_{\frac{n}{2}1} & b_{\frac{n}{2}2} & b_{\frac{n}{2}3} & \dots & b_{\frac{n}{2}(n-1)} & b_{\frac{n}{2}n} \\
b_{(\frac{n}{2}+1)n} & b_{(\frac{n}{2}+1)(n-1)} & b_{(\frac{n}{2}+1)(n-2)} & \dots & b_{(\frac{n}{2}+1)2} & b_{(\frac{n}{2}+1)1} \\
\dots & \dots & \dots & \dots & \dots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
b_{2n} & b_{2(n-1)} & b_{2(n-2)} & \dots & b_{22} & b_{21} \\
b_{1n} & b_{1(n-1)} & b_{1(n-2)} & \dots & b_{12} & b_{11}
\end{bmatrix}. \tag{3.33}$$

3.7.5 $\mathbb{Z}_2$ Reversibility

Lemma 3.12. *The matrix entries for $d\eta$ satisfy $-\eta_{i\,j}(t,\theta) = \eta_{n+1-i\,n+1-j}(-t,-\theta)$.*

Proof.

$$\begin{aligned}[\chi(\kappa)\kappa \cdot d\eta]_{i\,j}(t,\theta) &= \sum_{k=1}^{n} \kappa_{i\,k}\, \eta_{k\,j}(t,\theta) \\ &= \sum_{k=1}^{n} -\delta_{i+k\,n+1}\, \eta_{k\,j}(t,\theta) \\ &= -\eta_{n+1-i\,j}(t,\theta) \\ &= -\eta_{1-i\,j}(t,\theta) \quad \text{mod } n \end{aligned} \tag{3.34}$$

and

$$\begin{aligned}[d\eta \cdot \kappa]_{i\,j}(-t,-\theta) &= \sum_{k=1}^{n} \eta_{i\,k}\, \kappa_{k\,j}(-t,-\theta) \\ &= \sum_{k=1}^{n} \eta_{ik} \delta_{j+k\,n+1}(-t,-\theta) \\ &= \eta_{i\,n+1-j}(-t,-\theta) \\ &= \eta_{i\,1-j}(-t,-\theta) \end{aligned} \tag{3.35}$$

Equating (3.34) to (3.35) we obtain the relation $-\eta_{1-i\,j}(t,\theta) = \eta_{i\,1-j}(-t,-\theta)$.

□

3.8 $\mathbb{D}_n-$Reversible Equivariance

3.8.1 The case when $\chi(\rho) = 1$ and $\chi(\kappa) = 1$

Lemma 3.13. *The matrix entries for $d\eta$ satisfy $d_k = d_{n-k+2}$. When n is odd, we may choose $\frac{n+1}{2}$ values and when n is even, we may choose $\frac{n+2}{2}$ values such that the symmetries give the remaining values.*

Proof. From $\mathbb{Z}_n$ equivariance, $d_k = \eta_{i\,i+k}$ from which $\eta_{i\,j} = d_{j-i+1}$. Also, we have $\kappa_{i\,j} = \delta_{i+j\,n+1}$.

For $\mathbb{D}_n$ equivariance, we require $[\kappa \cdot d\eta^\rho]_{i\,j} = [d\eta^\rho \cdot \kappa]_{i\,j}$.

$$\begin{aligned}
[\kappa \cdot d\eta]_{i\,j} &= \sum_{k=1}^{n} \kappa_{i\,k}\,\eta_{k\,j} \\
&= \sum_{k=1}^{n} \delta_{i+k\,n+1}\,\eta_{k\,j} \\
&= \sum_{k=1}^{n} d_{j-k+1} \\
&= d_{j-(n-1-i)+1} \\
&= d_{i+j} \quad \mod n
\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
[d\eta \cdot \kappa]_{i\,j} &= \sum_{k=1}^{n} \eta_{i\,k}\,\kappa_{k\,j} \\
&= \sum_{k=1}^{n} d_{k-i+1}\delta_{j+k\,n+1} \\
&= d_{n+1-j-i+1} \\
&= d_{2-(i+j)}.
\end{aligned} \tag{3.37}$$

Let $k = i + j$, then $2 - (i + j) = 2 - k$. Equating (3.36) to (3.37) we obtain the relation $d_k = d_{n-k+2}$. □

Hence for $n = 4$ or 5, we have 3 distinct elements and 4 for $n = 6$ or 7, and 5 distinct entries for $n = 8$ or 9 etc.

For example, when $n = 6$, we have d_1,fixed, d_4, fixed and $d_2 = d_6, d_3 = d_5$, viz:

$$d\eta = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 & d_3 & d_2 \\ d_2 & d_1 & d_2 & d_3 & d_4 & d_3 \\ d_3 & d_2 & d_1 & d_2 & d_3 & d_4 \\ d_4 & d_3 & d_2 & d_1 & d_2 & d_3 \\ d_3 & d_4 & d_3 & d_2 & d_1 & d_2 \\ d_2 & d_3 & d_4 & d_3 & d_2 & d_1 \end{bmatrix}. \tag{3.38}$$

3.8.2 The case when $\chi(\rho) = 1$ and $\chi(\kappa) = -1$

We consider the action of κ on $d\eta^\rho$ with ρ acting as a symmetry.

From $\mathbb{Z}_n$ equivariance, $d_k = \eta_{i\,i+k}$ from which $\eta_{i\,j} = d_{j-i+1}$. Also, we have $\kappa_{i\,j} = \delta_{i+j\,n+1}$.

Lemma 3.14. *When $\chi(\rho) = 1$ and $\chi(\kappa) = -1$, the matrix entries satisfy the relation $-d_k(t, \theta) = d_{n-k+2}(-t, -\theta)$. When n is odd, we may choose $\frac{n-1}{2}$ values, with the middle value being an odd* *function, whilst we may choose $\frac{n}{2}$ values when n is even, with anti-symmetry relations giving the remaining values.*

Proof. For $\mathbb{D}_n$ reversibility with reversor, κ, we require $[\chi(\kappa)\kappa \cdot d\eta^\rho]_{i\,j}(t,\theta) = [d\eta^\rho \cdot \kappa]_{i\,j}(-t,-\theta)$.

$$\begin{aligned}
[\chi(\kappa)\kappa \cdot d\eta^\rho]_{i\,j}(t,\theta) &= \sum_{k=1}^{n} -\kappa_{i\,k}\,\eta_{k\,j}(t,\theta) \\
&= \sum_{k=1}^{n} -\delta_{i+k\,n+1}\,\eta_{k\,j}(t,\theta) \\
&= \sum_{k=1}^{n} -\delta_{i+k\,n+1}\,d_{j-k+1}(t,\theta) \\
&= -d_{j-(n+1-i)+1}(t,\theta) \\
&= -d_{i+j}(t,\theta) \quad \mod n
\end{aligned} \tag{3.39}$$

and

$$\begin{aligned}
[d\eta \cdot \kappa]_{i\,j}(-t,-\theta) &= \sum_{k=1}^{n} \eta_{i\,k}\,\kappa_{k\,j}(-t,-\theta) \\
&= \sum_{k=1}^{n} d_{k-i+1}\delta_{j+k\,n+1}(-t,-\theta) \\
&= d_{n+1-j-i+1}(-t,-\theta) \\
&= d_{n-(i+j)+2}(-t,-\theta)
\end{aligned} \tag{3.40}$$

Let $k = i + j$, then $n - (i + j) + 2 = n - (i + j - 1) = n - k + 2$. Equating (3.39) to (3.40) we obtain the relation $-d_k(t,\theta) = d_{n-k+2}(-t,-\theta)$.

$\square$

3.8.3 The case when $\chi(\rho) = -1$ and $\chi(\kappa) = 1$

Here, we have κ and ρ^2 acting as symmetries with ρ as the reversor acting on $d\eta^{\rho^2}$.

From $\mathbb{Z}_n$ equivariance, $d_k = \eta_{i\,i+k}$ from which $\eta_{i\,j} = d_{j-k}$. Also, we have $\kappa_{i\,j} = \delta_{i+j\,n+1}$. For $\mathbb{D}_n$ reversibility, we require $(\kappa \cdot d\eta_\rho)_{i\,j} = (d\eta_\rho \cdot \kappa)_{i\,j}$.

Lemma 3.15. *The matrix $d\eta$ has entries of two distinct values satisfying $-d_1(t,\theta) = -d_2(-t,-\theta)$, and given any d_1, the matrix is fully determined. When n is odd, the middle value is an odd function.*

Proof. For $\mathbb{D}_n$ reversibility, we require $[\kappa \cdot d\eta_\rho]_{i\,j} = [d\eta_\rho \cdot \kappa]_{i\,j}$.

$$
\begin{aligned}
[\kappa \cdot d\eta]_{i\,j}(t,\theta) &= \sum_{k=1}^{n} \kappa_{i\,k}\,\eta_{k\,j}(t,\theta) \\
&= \sum_{k=1}^{n} \delta_{i+k\,n+1}\,\eta_{k\,j}(t,\theta) \\
&= \eta_{n+1-1\,j}(t,\theta) \quad \bmod n
\end{aligned} \tag{3.41}
$$

and

$$
\begin{aligned}
[d\eta \cdot \kappa]_{i\,j}(-t,-\theta) &= \sum_{k=1}^{n} \eta_{i\,k}\,\kappa_{k\,j}(-t,-\theta) \\
&= \sum_{k=1}^{n} \eta_{i\,k}\delta_{j+k\,n+1}(-t,-\theta) \\
&= \eta_{i\,n+1-j}(-t,-\theta)
\end{aligned} \tag{3.42}
$$

Let $k = j + 2i - 2$, then $4 - (j + 2i) = 2 - k$. Equating (3.41) to (3.42) we obtain the relation $\eta_{1-i\,j} = \eta_{i\,1-j}$, linking the odd to the even rows.

Let $I = 1 - i$, then

$$
\begin{aligned}
\eta_{I\,j} &= \eta_{1-i\,j} \\
&= \eta_{i\,1-j} \\
&= \eta_{1-I\,1-j}.
\end{aligned} \tag{3.43}
$$

Equating corresponding subscripts from $\eta_{1-i\,j} = \eta_{i\,1-j}$, we find that $d_1 = b_2$ and $d_2 = b_1$. When n is odd, the middle value is an odd function since $-d_{\frac{n+1}{2}}(t,\theta) = d_{\frac{n+1}{2}}(-t,-\theta)$. $\square$

This yields $2-$distinct entries in the vector $[d_1, d_2, \cdots, d_n]$ for the matrix $d\eta$ such that

$$
d\eta = \begin{bmatrix}
d_1 & -d_1 & d_1 & -d_1 & \dots & d_1 & -d_1 \\
-d_1 & d_1 & -d_1 & d_1 & \dots & -d_1 & d_1 \\
d_1 & -d_1 & d_1 & -d_1 & \dots & d_1 & -d_1 \\
-d_1 & d_1 & -d_1 & d_1 & \dots & -d_1 & d_1 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
d_1 & -d_1 & d_1 & -d_1 & \dots & d_1 & -d_1 \\
-d_1 & d_1 & -d_1 & d_1 & \dots & -d_1 & d_1
\end{bmatrix}. \tag{3.44}
$$

3.8.4 The case when $\chi(\rho) = -1$ and $\chi(\kappa) = -1$

In this case, ρ^2 is the symmetry and either ρ or κ is the reversor, acting on $d\eta^{\rho^2}$.

Lemma 3.16. *When $\chi(\rho) = -1$ and $\chi(\kappa) = -1$, the matrix entries satisfy the relation $-\eta_{1-i\,j}(t,\theta) = \eta_{i\,1-j}(-t,-\theta)$, linking odd to even rows such that $-d_1(t,\theta) = b_2(-t,-\theta)$ and $-d_2(t,\theta) = b_1(-t,-\theta)$. Note also that since $-d_1(t,\theta) = d_2(-t,-\theta)$, we have $b_1(t,\theta) = d_1(t,\theta)$ so that $b_2(t,\theta) = -d_1(-t,-\theta)$. This gives two distinct matrix elements.*

Proof. For $\mathbb{D}_n$ reversibility, we require $[\chi(\kappa)\kappa \cdot d\eta^{\rho-}]_{i\,j}(t,\theta) = [d\eta^{\rho-} \cdot \kappa]_{i\,j}(-t,-\theta)$.

$$\begin{aligned}
[\chi(\kappa)\kappa \cdot d\eta]_{i\,j}(t,\theta) &= \sum_{k=1}^{n} -\kappa_{i\,k}\,\eta_{k\,j}(t,\theta) \\
&= \sum_{k=1}^{n} -\delta_{i+k\,n+1}\,\eta_{k\,j}(t,\theta) \\
&= -\eta_{n+1-i\,j}(t,\theta)
\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}
[d\eta \cdot \kappa]_{i\,j}(-t,-\theta) &= \sum_{k=1}^{n} \eta_{i\,k}\,\kappa_{k\,j}(-t,-\theta) \\
&= \sum_{k=1}^{n} \eta_{i\,k}\delta_{j+k\,n+1}(-t,-\theta) \\
&= \eta_{i\,n+1-j)}(-t,-\theta)
\end{aligned} \tag{3.46}$$

Equating (3.45) to (3.46) we obtain the relation $-\eta_{1-i\,j}(t,\theta) = \eta_{i\,1-j}(-t,-\theta)$. $\square$

This yields $2-$distinct entries in the vector $[d_1, d_2, \cdots, d_n]$ for the matrix $d\eta$ such that

$$d\eta = \begin{bmatrix}
d_1 & -d_1 & d_1 & -d_1 & \dots & d_1 & -d_1 \\
d_1 & -d_1 & d_1 & -d_1 & \dots & d_1 & -d_1 \\
d_1 & -d_1 & d_1 & -d_1 & \dots & d_1 & -d_1 \\
d_1 & -d_1 & d_1 & -d_1 & \dots & d_1 & -d_1 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
d_1 & -d_1 & d_1 & -d_1 & \dots & d_1 & -d_1 \\
d_1 & -d_1 & d_1 & -d_1 & \dots & d_1 & -d_1
\end{bmatrix}. \tag{3.47}$$

3.9 The Spectral Analysis of the matrix, $d\eta$

We have found the matricial structure of $d\eta$ to be circulant or block circulant. We require the following results of circulant and block circulant matrices to analyse the various $d\eta$ matrices. A circulant matrix and block circulant matrix can be represented as a sum of Kronecker products with powers of the cyclic forward shift matrix R.

Lemma 3.17. *Let $A = Circ(a)$, where $a = [a_1, a_2, \cdots a_n]$ is the circulant vector. Then*

$$\begin{aligned} A &= a_1 R^0 + a_2 R^1 + a_3 R^2 + \cdots + a_n R^{n-1} \\ &= \sum_{k=0} n - 1 a_{k+1} R^k. \end{aligned} \tag{3.48}$$

Similarly, if $B \in \mathcal{B}Circ(.)$, is block circulant, then

$$\begin{aligned} B &= R^0 \otimes B_1 + R^1 \otimes B_2 + R^2 \otimes B_3 + \cdots + R^{n-1} \otimes B_n \\ &= \sum_{k=1}^{k-1} R^{k-1} \otimes B_k. \end{aligned} \tag{3.49}$$

where $\otimes$ is the Kronecker product.

Proof. We note that the matrix R is a permutation matrix with a single 1 in each row and column, with the other entries zeroes. Raising the circulant matrix R to powers up to n yields new permutation matrices with the ones shifted to the right by one position each time. Also, $R^n = I$. It is easy to observe that the circulant matrix A can then be written as a linear combination of the powers of R. The a_1 terms are generated by multiplying the identity matrix, $R^n = I$, by a_1, and fall on the leading diagonal. The a_2 terms are generated by multiplying R by a_2, with the result that the 1's in R correspond to the a_2 positions in A. This process is repeated n times and the resulting matrices can then be summed to give A in its entirety.

To prove the block circulant relation we define the Kronecker product of two matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$ as the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & \dots & a_{mn}B & & & & \end{bmatrix} \in \mathbb{R}^{mp\times nq}. \tag{3.50}$$

Following the process in the circulant case and replacing the a_k's with R^k, the result for block circulant matrices follows. $\square$

Theorem 3.18. *Let $A \in \mathbb{R}^{n\times n}$ have eigenvalues λ_i, $i = 1, 2, \cdots, n$ and let $B \in \mathbb{R}^{m\times m}$ have eigenvalues μ_j, $j = 1, 2, \cdots, m$. Then the mn eigenvalues of $A \otimes B$ are given by $\lambda_i \otimes \mu_j =$*

$\lambda_1\mu_1, \lambda_1\mu_2, \cdots, \lambda_1\mu_m, \lambda_2\mu_1, \cdots \lambda_2\mu_m, \cdots, \lambda_n\mu_m$.

Furthermore, let $x_1, x_2, \cdots, x_p$ be the linearly independent right eigenvectors of A corresponding to the eigenvalues $\lambda_1, \cdots, \lambda_p\,(p \leq n)$ and let $y_1, y_2, \cdots, y_q$ be the linearly independent right eigenvectors of B corresponding to the eigenvalues $\mu_1, \cdots, \mu_q\,(q \leq m)$, then $x_i \otimes y_j \in \mathbb{R}^{mn}$ are the linearly independent right eigenvectors of $A \otimes B$ corresponding to $\lambda_i\mu_j$.

Proof.

$$\begin{aligned}(A \otimes B)(x \otimes y) &= Ax \otimes By \\ &= \lambda x \otimes \mu y \\ &= \lambda\mu(x \otimes y). \end{aligned} \tag{3.51}$$

□

When $d\eta = \text{Circ(d)}$, the circulant operator, the eigenvalues are given by

$$\lambda_k = d_1 + \rho_n^k d_2 + \cdots + \rho_n^{(n-1)k} d_n \tag{3.52}$$

with eigenvectors

$$\underline{v}_k = \left(\rho_n^k, \rho_n^{2k}, \ldots, \rho_n^{k(n-1)}, 1\right), \quad 1 \leq k \leq n.$$

where $\lambda_k \in \mathbb{C}$, where $\rho = e^{\frac{2\pi i}{n}}$ is the n^{th} root of unity and λ_k exists if and only if $d_i \in \mathbb{C}$.

We observe that the $\mathbb{Z}_2$ actions, either on their own or in conjunction of $\mathbb{Z}_n$ give rise to a partitioning of the matrix $d\eta$ such that

Lemma 3.19. *When $\chi(\kappa) = 1$, then*

$$d\eta = \begin{bmatrix} A & P(B) \\ B & P(A) \end{bmatrix} \tag{3.53}$$

and when $\chi(\kappa) = -1$, then

$$d\eta = \begin{bmatrix} A & -P(B) \\ B & -P(A) \end{bmatrix}. \tag{3.54}$$

where

$$\begin{aligned} P(A) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} [A] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \kappa A \kappa \end{aligned} \tag{3.55}$$

and similarly for $P(B)$.

Proof. Denote the representation of $\mathbb{Z}_2$ by $\begin{bmatrix} 0 & \kappa \\ \kappa & 0 \end{bmatrix}$. Then a matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ commutes with $\mathbb{Z}_2$ if

$$\begin{bmatrix} 0 & \kappa \\ \kappa & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 & \kappa \\ \kappa & 0 \end{bmatrix} \tag{3.56}$$

i.e. $\begin{bmatrix} \kappa C & \kappa D \\ \kappa A & \kappa B \end{bmatrix} = \begin{bmatrix} B\kappa & A\kappa \\ D\kappa & C\kappa \end{bmatrix}$. Comparing corresponding entries, we see that $\kappa B = C\kappa$, i.e. $C = \kappa B\kappa$. Similarly, $D = \kappa A\kappa$. □

Lemma 3.20. *If a matrix M is $\mathbb{Z}_n$ equivariant, then M is $\mathbb{D}_n$ equivariant.*

Proof. Recall the presentation of the non-abelian group $\mathbb{D} = \{<\rho, \kappa>: \ \rho^n = \kappa^2 = I, \ \rho\kappa = \kappa\rho^{-1}\}$. Let $\gamma \in \mathbb{D}_n = \rho^r\kappa^s$ where $r = 0, 1, \cdots, n$ and $s = 0, 1$. Then $M\gamma = \gamma M \Rightarrow M(\rho^r\kappa^s) = (\rho^r\kappa^s)M$. Since M is $\mathbb{Z}_n$ equivariant when $s = 0$, we take $s = 1$. Then

$$M\rho^r\kappa = \rho^r\kappa M. \tag{3.57}$$

From the LHS of (3.57), $M\rho^r\kappa = \rho^r M\kappa$. Equating to the RHS, $\rho^r M\kappa = \rho^r\kappa M \Rightarrow M\kappa = \kappa M$. □

Theorem 3.21. *Matrices that commute with $\mathbb{D}_n$ are of the form* $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$.

Proof. Matrices that commute with $\mathbb{Z}_n$ are circulant. We therefore take $\kappa = 1$ in the $\mathbb{Z}_2$ representation of (3.19) and the result follows. □

We discuss the possiblity of the occurrence or not of Hopf bifurcation (birth of periodic orbits) relating to the matrix $d\eta$ resulting from the $\mathbb{D}_n$, $\mathbb{Z}_n$ and $\mathbb{Z}_2$ actions, and further explore this later, when we analyse some cyclically repetitive structures, ring networks. It is well-known that Hopf bifurcations occur when the Jacobian matrix of a nonlinear dynamical system has a pair of purely imaginary eigenvalues, $i\omega$. The normal form is given in two dimensions by the matrix

$$\begin{bmatrix} 0 & -d \\ d & 0 \end{bmatrix}. \tag{3.58}$$

We now examine the circulant matricial structure of $d\eta$ arising from the $\mathbb{Z}_n$ and $\mathbb{Z}_2$ actions.

Lemma 3.22. *The matrix $d\eta$ is circulant when $\chi(\rho) = 1$.*

Proof. The proof follows from lemma (B.2), given that a circulant matrix is $\rho-$equivariant. □

The following cases outline the matricial structure when $d\eta$ is circulant :

We explore $\mathbb{Z}_n$ equivariance i.e. when $\chi(\rho) = 1$. Recall that $d\eta = \text{Circ(d)}$ where $d = [d_1, d_2, \cdots, d_n] \in \mathbb{R}^{n\times n}$.

The eigenvalues are given by

$$\begin{aligned}\lambda_k &= d_1 + \rho_n^k d_2 + \cdots + \rho_n^{(n-1)k} d_n \\ &= d_1 + d_2 e^{i\theta} + d_3 e^{2i\theta} + \cdots + d_n e^{(n-1)i\theta} \\ &= \sum_{k=1}^{n} d_k[\cos(k-1)\theta + i\sin(k-1)\theta] \end{aligned} \tag{3.59}$$

Lemma 3.23. *Hopf bifurcation can occur from the $\mathbb{Z}_n$ equivariant action on $d\eta$.*

Proof. For Hopf bifurcation to occur, we require that $\lambda_k = i\omega$, i.e. $\sum_{k=1}^{n} d_k \cos(k-1)\theta = 0$ and that $\sum_{k=1}^{n} d_k \sin(k-1)\theta = \omega$. $\square$

Next, we examine the possibility for the occurrence of Hopf bifurcation witn $\mathbb{D}_n$ equivariance i.e. when $\chi(\rho) = 1$ and $\chi(\kappa) = 1$. The matrix is given by $d\eta = \mathrm{Circ(d)}$ where $d = \{d_k : d_k = d_{n-k}\}$.

Lemma 3.24. *Hopf bifurcation cannot occur from the $\mathbb{D}_n$ equivariant action on $d\eta$, when $\chi(\rho) = 1$ and $\chi(\kappa) = 1$.*

Proof. Using the relation $d_k = d_{n-k}$, we see that the values in the eigenvalue relation occur in pairs, and the eigenvalues are seen to be given by

$$\begin{aligned}\lambda_k &= d_n + d_1\rho + d_2\rho^2 + d_3\rho^3 + \cdots + d_3\rho^{n-3} + d_2\rho^{n-2} + d_1\rho^{n-1} \\ &= d_n + d_1(\rho + \bar{\rho}^{1}) + d_2(\rho + \bar{\rho}^{2}) + \cdots + d_{r-1}(\rho^{r-1} + \bar{\rho}^{r-1}) + \begin{cases} 0 & \text{if } n = 2r-1 \\ d_r(-1)^k & \text{if } n = 2r \end{cases} \\ &= d_n + 2\sum_{k=1}^{r-1} d_k \cos(k-1)\theta + \begin{cases} 0 & \text{if } n = 2r-1 \\ d_r(-1)^k & \text{if } n = 2r \end{cases}. \end{aligned}$$

Note that we have made use of $\rho_k = e^{\frac{i2\pi k}{n}} = e^{ik\theta}$, where $\theta := \frac{2\pi}{n}$ and that $\rho^m + \bar{\rho}^m = e^{im\theta} + e^{-im\theta} = 2\cos(m\theta)$ and $\rho_{n-k}^r = e^{\frac{i2\pi(n-k)}{n}} = e^{\frac{-i2\pi kr}{n}} = \bar{\rho}_k^r$.

Hence the eigenvalues $\lambda_k \in \mathbb{R}$, implying that the Hopf bifurcation cannot occur. $\square$

Here we examine the possibility of Hopf bifurcation occurring with $\mathbb{D}_n$ reversibility when $\chi(\rho) = 1$ and $\chi(\kappa) = -1$. Note that here, $d\eta = \mathrm{Circ(d)}$ where $d = \{d_k : -d_k(t,\theta) = d_{n-k}(-t,-\theta)\}$.

Lemma 3.25. *Hopf bifurcation can occur from the $\mathbb{D}_n$ action on $d\eta$, when $\chi(\rho) = 1$ and $\chi(\kappa) = -1$.*

Proof. Using the relation $d_k = -d_{n-k}$, we see that the values in the eigenvalue relation occur in pairs, and the eigenvalues are seen to be given by

$$\begin{aligned}
\lambda_k &= d_n + d_1\rho + d_2\rho^2 + d_3\rho^3 + \cdots - d_3\rho^{n-3} - d_2\rho^{n-2} - d_1\rho^{n-1} \\
&= d_n + d_1(\rho - \rho^{\bar{-}1}) + d_2(\rho - \rho^{\bar{-}2}) + \cdots + d_{r-1}(\rho^{r-1} - \rho^{r\bar{-}1}) - \begin{cases} 0 & \text{if } n = 2r-1 \\ d_r(-1)^k & \text{if } n = 2r \end{cases} \\
&= d_n + 2\sum_{k=1}^{r-1} i\, d_k \sin(k-1)\theta - \begin{cases} 0 & \text{if } n = 2r-1 \\ d_r(-1)^k & \text{if } n = 2r \end{cases} .
\end{aligned}$$

In this case, since $\lambda_k \in i\mathbb{R}$, we may have the Hopf bifurcation.

□

The following cases outline the matricial structure when $d\eta \in \mathbb{R}^{n\times n}$ is block circulant of size $\frac{n}{2} \times \frac{n}{2}$. We observe that $d\eta$ is circulant over $M_2(\mathbb{R})$. Hence $d\eta$ can be partitioned to create a block matrix with 2×2 sub-matrix blocks, which need not be circulant.

Lemma 3.26. *The matrix $d\eta$ is block circulant when $\chi(\rho) = -1$.*

When $d\eta$ is $\mathbb{Z}_n$ reversible with $\chi(\rho) = -1$, the matrix $d\eta^{\rho-}$ may be partitioned into blocks, giving a block circulant structure with each block given by

$$D_1 = \begin{bmatrix} d_1 & -d_1 \\ b_1 & -b_1 \end{bmatrix} \in \mathbb{R}^{2\times 2}. \tag{3.60}$$

Let R be the cyclic forward shift matrix and such that $R^0 = I$, the identity matrix, and $\otimes$ be the Kronecker product. Then $d\eta^{\rho-}$ can be represented by the matrix sum

$$\begin{aligned}
d\eta^{\rho-} &= R^0 \otimes D_1 + R^2 \otimes D_1 + R^3 \otimes D_1 + \cdots + R^{n-1} \otimes D_1 \\
&= \sum_{k=1}^{k-1} R^{k-1} \otimes D_1.
\end{aligned} \tag{3.61}$$

This gives

$$d\eta^{\rho-} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \otimes D_1 = \mathrm{Circ}(1) \otimes D_1, \tag{3.62}$$

with $R, \mathrm{Circ}(1) \in \mathbb{R}^{\frac{n}{2}\times\frac{n}{2}}$.

The eigenvalues and eigenvectors of $\mathrm{Circ}(1)$ can be found from $\mathrm{Circ}(1)x = \lambda x$ as follows:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + \cdots + x_n \\ x_1 + x_2 + \cdots + x_n \\ \vdots \\ x_1 + x_2 + \cdots + x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{bmatrix}. \tag{3.63}$$

We note that Circ(1) has rank 1 and at most one non-zero eigenvalue $\lambda = n$ and $n-1$ eigenvalues of 0. When $\lambda = 0$, one obvious solution is $W := \{[x_1, x_2, \cdots, x_n]^T : x_1 + x_2 + \cdots + x_n = 0\}$, with $\dim W = n - 1$. When $\lambda = n$, we may choose $U := \{[x_1, x_2, x_3, \cdots, x_n]^T : x_1 = x_2 = \cdots = x_n\}$ and $\dim U = 1$. Since $W \cap U = \{0\}$, we have $\dim(W + U) = n$.

The eigenvalues of D_1 are $\mu_1 = 0$ and $\mu_2 = d_1 - b_1$ with corresponding eigenvectors $y_1 = [1, 1]^T$ and $y_2 = [d_1, b_1]^T$.

Lemma 3.27. *Hopf bifurcation cannot occur from the $\mathbb{Z}_n$ reversible action on $d\eta$, i.e. when $\chi(\rho) = -1$.*

Proof. The eigenvalues of D_1 are real i.e. $\mu_1 = 0$ and $\mu_2 = d_1 - b_1 \in \mathbb{R}$. Note that D_1 is not in Hopf normal form. Therefore, the Hopf bifurcation cannot occur in this case. □

If $d\eta$ is $\mathbb{D}_n$ reversible with $\chi(\rho) = -1$ and $\chi(\kappa) = 1$, then the matrix $d\eta^{\rho-,\kappa}$ may be partitioned into blocks, giving a block circulant structure with each block given by

$$D_2 = \begin{bmatrix} d_1 & -d_1 \\ -d_1 & d_1 \end{bmatrix} \in \mathbb{R}^{2\times 2}. \tag{3.64}$$

This gives

$$d\eta^{\rho-,\kappa} = \mathrm{Circ}(1) \otimes D_2. \tag{3.65}$$

The eigenvalues of D_2 are $\mu_1 = 0$ and $\mu_2 = 2d_1$ with corresponding eigenvectors $y_1 = [1, 1]^T$ and $y_2 = [1, -1]^T$.

Lemma 3.28. *Hopf bifurcation cannot occur from the $\mathbb{D}_n$ reversible action on $d\eta$, i.e. when $\chi(\rho) = -1$ and $\chi(\kappa) = 1$.*

Proof. The eigenvalues of D_1 are real i.e. $\mu_1 = 0$ and $\mu_2 = 2d_1 \in \mathbb{R}$. Alternatively, since D_2 is not in Hopf normal form, the Hopf bifurcation cannot occur in this case. □

When $d\eta$ is $\mathbb{D}_n$ reversible with $\chi(\rho) = -1$ and $\chi(\kappa) = -1$, the matrix $d\eta^{\rho-,\kappa-}$ may be partitioned into blocks, giving a block circulant structure with each block given by

$$D_3 = \begin{bmatrix} d_1 & -d_1 \\ d_1 & -d_1 \end{bmatrix} \in \mathbb{R}^{2\times 2}. \tag{3.66}$$

We have

$$d\eta^{\rho-,\kappa-} = \mathrm{Circ}(1) \otimes D_3. \tag{3.67}$$

The characteristic polynomial of D_3 is $\lambda^2 = 0$ giving $\mu_1 = 0$ and $y_1 = [1,\, 1]^T$. The eigenspace span$\{[1,\, 1]^T\}$ is not enough to span $\mathbb{R}^2$.

Lemma 3.29. *Hopf bifurcation cannot occur from the $\mathbb{D}_n$ reversible action on $d\eta$, i.e. when $\chi(\rho) = -1$ and $\chi(\kappa) = -1$.*

Proof. Since D_3 is not in Hopf normal form, the Hopf bifurcation cannot occur in this case. □

The $\mathbb{Z}_2$ equivariant action on $d\eta$ yields $\frac{n^2}{2}$ distinct matrix entries.

Lemma 3.30. *Hopf bifurcation cannot occur from the $\mathbb{Z}_2$ equivariant action on $d\eta$, i.e. when $\chi(\kappa) = 1$.*

Proof. When $n = 2$, we have

$$d\eta^\kappa = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{11} \end{bmatrix} \tag{3.68}$$

with eigenvalues $\mu_i = b_{11} \pm b_{12} \in \mathbb{R}$. Furthermore, we note that since the resulting matrix $d\eta$ has $\frac{n^2}{2}$ distinct entries and therefore not in Hopf normal form, the $\mathbb{Z}_2$ equivariant action does not generate the Hopf bifurcation. □

The $\mathbb{Z}_2$ equivariant action on $d\eta$ yields $\frac{n^2}{2}$ distinct matrix entries and satisfy $\eta_{ij} = -\eta_{1-i\,1-j}$

Lemma 3.31. *Hopf bifurcation can occur from the $\mathbb{Z}_2$ reversible action on $d\eta$, i.e. when $\chi(\kappa) = -1$.*

Proof. When $n = 2$, we have

$$d\eta^\kappa = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{11} \end{bmatrix} \tag{3.69}$$

with eigenvalues $\mu_i = \pm\sqrt{b_{11}^2 - b_{12}^2} \in i\mathbb{R}$ when $b_{12} > b_{11}$. The $\mathbb{Z}_2$ equivariant action may generate the Hopf bifurcation. □

3.10 Ring Networks

There is considerable interest in networks of nonlinear differential equations. Systems with symmetry can lead to interesting oscillatory patterns which can be investigated using the theory of equivariant bifurcations. Consider a ring of n identical elements with forwards and backwards nearest neighbour coupling. We assign to each individual element a linear decay term, a nonlinear forwards-backwards self-connection (feedback) term and nonlinear element to element mixed connection terms. Such a generalised system could obviously be further complicated by making the delay arguments different from each other or distributed.

We investigate and classify ring network systems of NMFDEs, extending previous work on network systems of DDEs and MFDEs. A cell is a finite dimensional system of functional differential equations on a phase space $\mathbb{R}^n$. A coupled cell network $\mathcal{C}$ consists of N cells with equations that are coupled. A network consists of nodes, cells, linked by edges which specify the couplings.

3.10.1 Neutral MFDE in a Ring Network

In this section we study reversibility and equivariance in ring networks resulting from the actions of the dihedral group $\mathbb{D}_n$. The dynamics of coupled cell networks (symmetric networks of coupled identical oscillators) with nearest-neighbour coupling have been studied by authors such as Buono et al. in [9], Campbell et al. in [12] and Benoit et al. in [8]. The equations used by these authors contain some delay terms and are generally of the form

$$\dot{x}_j(t) = f(x_j(t)) + \sum_{k=1}^{n} c_{j,k} h(x_j(t) - x_k(t-\tau)),$$

where f is the internal dynamics function and h the coupling function and $j = 1, \cdots, n$. Another example can be found in Wu [67] i.e. the delayed Hopfield-Cohen-Grossberg model of neural networks given by

$$\dot{u}_i(t) = -u_i(t) + \sum_{j=1}^{n} J_{ij} f(u_j(t-\tau)), \qquad 1 \leq i \leq n,$$

where f is a sigmoidal function normalized so that $f(0) = 0$ and $J = J_{ij}$ is a symmetric circulant matrix with all the diagonal elements identical to zero. We extend and generalise these examples to a neutral MFDE network with (maximum) all-to-all coupling given by the general equation The dynamics of a cell network with (maximum) all-to-all coupling can be written by the neutral equation

$$[H_j(u_j)_t]' = \alpha_j f_j(T_0(u_j)_t) + \sum_{k=1, k\neq j}^{n} \beta_{j\,k} f_{j,k}(T_1(u_j)_t, T_2(u_k)_t), \tag{3.70}$$

where $[.]'$ and $f_j' = D_u f$ denote Frechet derivatives and where the operators T_i are given explicitly by

$$T_0 u_t = u(t),\; T_1 u_t = u(t+\tau),\; T_2 u_t = u(t-\tau) \tag{3.71}$$

and the linearisation at 0 is given by

$$[H'(0)(u_j)_t]' = \alpha_j f_j'(0)(T_0(u_j)_t) + \sum_{k=1\, k\neq j}^{n} \beta_{jk} f_{j,k}'(0)(T_1(u_j)_t, T_2(u_k)_t) \tag{3.72}$$

with $1 \leq j \leq n$, where the states of the cells are characterised by a vector $u_j = [u_{j1}, \ldots, u_{jm}]$, of size nm i.e. each with m components with $u_j : C([\theta_{min}\theta_{max}]) \to \mathbb{R}^m$ and with $H : C([\theta_{min}\theta_{max}]) \to \mathbb{R}^m$. The difference operator H is atomic at a if H is continuous together with its first and second Frechet derivatives with respect to ϕ, and the derivative H_ϕ with respect to ϕ is atomic at a. We may write $H(t)\phi = \int_{\theta_{min}}^{\theta_{max}} d[\mu(t,\theta)]\phi(\theta)$. The linear operators $T_i : X \to \mathbb{R}^l$, $i = 0, 1, 2$, represent how the distributed time effects enter the internal dynamics of a particular cell, for f_j, and the dynamics of the connection or interaction for $f_{j,k}$. We note that $T_i : C([\theta_{min}\theta_{max}]) \to \mathbb{R}^m$, whilst $f_j : \mathbb{R}^m \to \mathbb{R}^m$ and $f_{jk} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$. Furthermore, $T_i = \int_{\theta_{min}}^{\theta_{max}} d\zeta_i(\theta)(u_j)_t$.

1. The Frechet derivative of the difference operator at 0 of the LHS of (3.70) given by $H_j(u_j)_t$ is

$$H_j'(0)(u_j)_t = \int_{\theta_{\min}}^{\theta_{\max}} d\mu_j(\theta)(u_j)_t(\theta). \tag{3.73}$$

   The matrix $d\mu$ has dimensions $nm \times nm$ and is block diagonal with each block $d\mu_j$ of dimension $m \times m$.

2. Consider the equation (3.70). The Frechet derivative gives

$$\begin{aligned}
[\alpha_j f_j(T_0(u_j)_t)]' &= \alpha_j f_j'(T_0(u_j)_t) \cdot T_0'(u_t)_j \\
&= \alpha_j f_j'(0) \cdot T_0(u_j)_t \\
&= \int_{\theta_{\min}}^{\theta_{\max}} \alpha_j f_j'(0) d\zeta_0(\theta)(u_j)_t(\theta)
\end{aligned} \tag{3.74}$$

   which is block diagonal with $m \times m$ blocks.

3. The Frechet derivative of the last term of the RHS is given by

$$\begin{aligned}
\sum_{k=1,k\neq j}^{n} [\beta_{j\,k} f_{j,k}(T_1(u_j)_t, T_2(u_k)_t)]' &= \sum_{k=1,k\neq j}^{n} \beta_{j\,k}[f_{j,k}'(T_1(u_j)_t) \cdot T_1'(u_j)_t + f_{j,k}'(T_2(u_k)_t) \cdot T_2'(u_j)_t] \\
&= \sum_{k=1,k\neq j}^{n} \beta_{jk} f_{jk}'(0) \begin{bmatrix} T_1(u_j)_t \\ T_2(u_k)_t \end{bmatrix},
\end{aligned} \tag{3.75}$$

   with the vector having dimension $2nm$. We split the $m \times 2m$ matrix $f_{jk}'(0)$ into two $m \times m$ sub-matrices to yield

$$\begin{aligned}
\sum_{k=1,k\neq j}^{n} \beta_{jk}[f_{jk1}'(0)|f_{jk2}'(0)] \begin{bmatrix} T_1(u_j)_t \\ T_2(u_k)_t \end{bmatrix} = \int_{\theta_{\min}}^{\theta_{\max}} \sum_{k=1,k\neq j}^{n} [\beta_{jk} f_{jk1}'(0)] d\zeta_1(\theta)(u_j)_t(\theta) + \\
\int_{\theta_{\min}}^{\theta_{\max}} \sum_{k=1,k\neq j}^{n} \beta_{jk} f_{jk2}'(0) d\zeta_2(\theta)(u_k)_t(\theta)
\end{aligned} \tag{3.76}$$

We may therefore rewrite the linearisation of (3.70) from (3.73) , (3.74) and (3.76) as

$$\begin{aligned}
\int_{\theta_{\min}}^{\theta_{\max}} d\mu_j(\theta)(u_j)_t(\theta) = \int_{\theta_{\min}}^{\theta_{\max}} \left[\alpha_j f_j'(0) d\zeta_0(\theta) + [\sum_{k=1,k\neq j}^{n} \beta_{jk} f_{jk1}'(0)] d\zeta_1(\theta)\right] (u_j)_t(\theta) \\
+ \int_{\theta_{\min}}^{\theta_{\max}} \sum_{k=1,k\neq j}^{n} \beta_{jk} f_{jk2}'(0) d\zeta_2(\theta)(u_k)_t(\theta)
\end{aligned} \tag{3.77}$$

The equivariance and reversibility of (3.70) and thus of (3.77) can be determined by employing lemma (3.5) of section (3.3).

Here is a matrix schematic of (3.77)

$$\begin{bmatrix} [d\mu]_1 & & & \\ & [d\mu]_2 & & \\ & & \ddots & \\ & & & [d\mu]_n \end{bmatrix} = \begin{bmatrix} [d\eta]_{11} & [d\eta]_{12} & \cdots & [d\eta]_{1n} \\ [d\eta]_{21} & [d\eta]_{22} & \cdots & [d\eta]_{2n} \\ & \cdots & \ddots & \\ [d\eta]_{n1} & [d\eta]_{n2} & \cdots & [d\eta]_{nn} \end{bmatrix}. \tag{3.78}$$

The $m \times m$ sub-matrices (blocks) of (3.77) are given by

$$[d\eta]_{ij} = \begin{cases} \alpha_j f_j'(0) d\zeta_0(\theta) + [\sum_{k=1,k\neq j}^{n} \beta_{jk} f_{jk1}'(0)] d\zeta_1(\theta) & \text{for} \quad i = j \\ \sum_{k=1,k\neq j}^{n} \beta_{jk} f_{jk2}'(0) d\zeta_2(\theta) \quad \text{for} & i \neq j \end{cases}. \tag{3.79}$$

The diagonal entries are equal, hence independent of j. The coupling values β_{jk} and f_{jk} determine how cell k influences cell j. For $\mathbb{Z}_n$ equivariance, $d\eta$ is circulant, $[\eta]_{ij} = d_{j-i+1}$, and therefore, $f_j'(0) = f'(0)$, $\alpha_j = \alpha$, $\beta_{jk} = \beta_j$and $f_{jk1}'(0) = f_k'(0)$ so that the coupling terms do not depend on the j cell but only on the k cell (e.g. $f_{13}'(0) = f_{23}'(0) = \cdots = f_{n3}'(0)$). As a consequence, all cells would have the same internal dynamics.

For $\mathbb{D}_n$ equivariance, $d\eta$ is block-circulant i.e. $[d\eta]_{ij} = [d\eta]_{j-i+1}$ and furthermore, $[d\eta]_k = [d\eta]_{n-k+2}$. For example, when $n = 6$, the circulant vector (of sub-matrices) of $\frac{n+2}{2}$ distinct entries is $\{[d\eta]_1, [d\eta]_2, [d\eta]_3, [d\eta]_4, [d\eta]_3, [d\eta]_2\}$. We note that the coupling terms also depend only on the k cell i.e. $f_{jk2}'(0) = f_{k2}'(0)$.

3.10.2 The $\mathbb{D}_n$ Actions on the Operators T_i

We examine the $\mathbb{D}_n$ actions on the operators T_i defined in (3.71) .

Lemma 3.32. *The operators T_i are $\mathbb{Z}_n$ and $\mathbb{Z}_2-$equivariant.*

Proof. Since T is linear, we let

$$T\phi \in \{\phi(\tau),\, \phi(-\tau),\, \phi(\tau) + \phi(-\tau),\, \phi(\tau) - \phi(-\tau)\} \tag{3.80}$$

Since $\rho\phi_i = \phi_{i+1}$,

$$\begin{aligned} T(\rho\phi_i) &\in \{\phi_{i+1}(\tau),\, \phi_{i+1}(-\tau),\, \phi_{i+1}(\tau) + \phi_{i+1}(-\tau),\, \phi_{i+1}(\tau) - \phi_{i+1}(-\tau)\} \\ &= \rho T\phi_i \end{aligned} \tag{3.81}$$

Also, since $\kappa\phi_i = \phi_{n+2-i}$,

$$\begin{aligned} T(\kappa\phi_i) &\in \{\phi_{n+2-i}(\tau),\, \phi_{n+2-i}(-\tau),\, \phi_{n+2-i}(\tau)+\phi_{n+2-i}(-\tau),\, \phi_{n+2-i}(\tau)-\phi_{n+2-i}(-\tau)\} \\ &= \kappa T\phi_i \end{aligned} \tag{3.82}$$

□

To set the scene for reversibility, we consider the various possible definitions of of the operator $T\phi$ and utilise $\gamma^\sharp : x(\theta) \to x(-\theta)$ to obtain the following:

Theorem 3.33. *T_i is $\mathbb{Z}_n$ and $\mathbb{Z}_2$ reversible if and only if $T\phi = \phi(\tau) - \phi(-\tau)$.*

Proof. 1. When $T\phi = \phi(\tau)$, then

$$\begin{aligned} T(\rho\phi_i(-\theta)) &= T(\phi_{i+1}(-\theta)) \\ &\neq -\rho T(\phi_i(-\theta)) \end{aligned} \tag{3.83}$$

and similarly

$$\begin{aligned} T(\kappa\phi_i(-\theta)) &= T(\phi_{n+2-i}(-\theta)) \\ &\neq -\kappa T(\phi_i(-\theta)) \end{aligned} \tag{3.84}$$

2. When $T\phi = \phi(-\tau)$, then

$$\begin{aligned} T(\rho\phi_i(\theta)) &= T(\phi_{i+1}(\theta)) \\ &\neq -\rho T(\phi_i(\theta)) \end{aligned} \tag{3.85}$$

and also

$$\begin{aligned} T(\kappa\phi_i(\theta)) &= T(\phi_{n+2-i}(\theta)) \\ &\neq -\kappa T(\phi_i(\theta)) \end{aligned} \tag{3.86}$$

3. When $T\phi = (\phi(\tau) + \phi(-\tau))$, then

$$\begin{aligned} T(\rho\phi_i(-\theta)) &= T(\phi_{i+1}(-\theta) + \phi_{i+1}(\theta)) \\ &\neq -\rho T(\phi_i(-\theta)) \end{aligned} \tag{3.87}$$

with

$$\begin{aligned} T(\kappa\phi_i(-\theta)) &= T(\phi_{n+2-i}(-\theta)) + \phi_{n+2-i}(\theta)) \\ &\neq -\kappa T(\phi_i(\theta)) \end{aligned} \tag{3.88}$$

4. When $T\phi = (\phi(\tau) - \phi(-\tau))$, then

$$
\begin{aligned}
T(\rho\phi_i(-\theta)) &= T(\phi_{i+1}(-\theta) - \phi_{i+1}(\theta)) \\
&= -\rho T(\phi_i(-\theta)) \qquad (3.89)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
T(\kappa\phi_i(-\theta)) &= T(\phi_{n+2-i}(-\theta)) - \phi_{n+2-i}(\theta)) \\
&= -\kappa T(\phi_i(\theta)) \qquad (3.90)
\end{aligned}
$$

□

3.10.3 Another look at $\mathbb{D}_n$ actions on the ring network neutral MFDE

We now study $\mathbb{D}_n-$symmetries of (3.70).

For $\mathbb{Z}_n$ equivariance, we require $(\rho u_j) = u_{j+1}$ and $f(T(\rho u)) = \rho f(Tu)$.

Lemma 3.34. *The NMFDE (3.70)*

$$[H_j(u_j)_t]' = f_j(T_0 u_j) + \sum_{k=1}^{n} f_{j,k}(T_1 u_j, T_2 u_k)$$

is $\rho-$equivariant when

$$
\begin{aligned}
f_{j,k}(\phi,\psi) &= f_{j+1,k+2}(\phi,\psi) \\
f_j(\phi,\psi) &= f_{j+1}(\phi,\psi). \qquad (3.91)
\end{aligned}
$$

Proof. For $f(T(\rho u))$,

$$
\begin{aligned}
f_j(T_0(\rho u_j)_t) + \sum_{k=1}^{n} f_{j,k}(T_1(\rho u_j)_t, T_2(\rho u_k)_t) &= \\
f_j(T_0(u_{j+1})_t) + \sum_{k=1}^{n} f_{j,k}(T_1(u_{j+1})_t, T_2(u_{k+1})_t), \quad (j \mod n) \qquad (3.92)
\end{aligned}
$$

For $\rho f(Tu)$,

$$
\begin{aligned}
\rho f_j(T_0(u_j)_t) + \rho \sum_{k=1}^{n} f_{j,k}(T_1(u_j)_t, T_2(u_k)_t) &= \\
f_{j+1}(T_0(u_{j+1})_t) + \sum_{k=1}^{n} f_{j+1,k+2}(T_1(u_{j+1})_t, T_2(u_{k+1})_t). \qquad (3.93)
\end{aligned}
$$

□

We note that for an $n-$node network, we obtain n conditions on the coefficients corresponding to the order of rotational symmetry.

Similarly, for $\mathbb{Z}_2$ equivariance,

Lemma 3.35. *The NMFDE (3.70)*

$$[H_j(u_j)_t]' = f_j(T_0 u_j) + \sum_{k=1}^{n} f_{j,k}(T_1 u_j, T_2 u_k)$$

is $\mathbb{Z}_2$ equivariant when

$$\begin{aligned} f_{j,k}(\phi,\psi) &= f_{n+2-j,n+2-k}(\phi,\psi) \\ f_j(\phi,\psi) &= f_{n+2-j}(\phi,\psi). \end{aligned} \tag{3.94}$$

Proof. For the left hand side,

$$\begin{aligned} f_j(T_0(\kappa u_j)) + \sum_{k=1}^{n} f_{j,k}(T_1(\kappa u_j)_t, T_2(\kappa u_k)_t) &= \\ f_j(T_0(u_{n+2-j})_t) + \sum_{k=1}^{n} f_{j,k}(T_1(u_{n+2-j})_t, T_2(u_{n+2-j})_t), \quad (j \mod n) \end{aligned} \tag{3.95}$$

whilst for the right hand side,

$$\begin{aligned} \kappa f_j(T_0 u_j) + \kappa \sum_{k=1}^{n} f_{j,k}(T_1(u_j)t, T_2(u_k)_t) &= \\ f_{n+2-j}(T_0(u_{n+2-j})_t) + \sum_{k=1}^{n} f_{n+2-j,n+2-k}(T_1(u_{n+2-j})_t, T_2(u_{n+2-k})_t). \end{aligned} \tag{3.96}$$

□

3.10.4 Reversibility

We now study the possibility of reversibility resulting from $\mathbb{D}_n$ actions.

Case 1: $\chi(\rho) = +1$, $\chi(\kappa) = +1$

Lemma 3.36. *With $\chi(\rho) = +1$, $\chi(\kappa) = +1$, the NMFDE (3.70) is $\mathbb{D}_n$ equivariant given the conditions in (3.91) and (3.94).*

Proof. See above. □

Note that since we have already explored the actions of ρ and κ above, we only need to apply ρ_χ and κ_χ in the following arguments for reversibility.

Case 2: $\chi(\rho) = +1,\ \chi(\kappa) = -1$

In this case, the system (6.5) is $\rho-$equivariant as shown above.

Lemma 3.37. *The equation (3.70) is $\kappa-$reversible if and only if*

$$f_{j,k}(\phi, \psi) = -f_{n+2-j,n+2-k}(\psi, \phi) \tag{3.97}$$

Proof.

$$\begin{aligned}
f_j(T_0(\kappa u_j(-\theta))) + \sum_{k=1}^{n} f_{j,k}(T_1(\kappa u_j(-\theta)), T_2(\kappa u_k(-\theta))) &= \\
f_j(T_0 u_{n+2-j}(-\theta)) + \sum_{k=1}^{n} f_{j,k}(T_1 u_{n+2-j}(-\theta), T_2 u_{n+2-k}(-\theta)) &= \\
f_j(T_0 u_{n+2-j}(-\theta)) + \sum_{k=1}^{n} f_{j,k}(T_2 u_{n+2-j}(\theta), T_1 u_{n+2-k}(\theta)) & \qquad (3.98)
\end{aligned}$$

and

$$\begin{aligned}
-\kappa f_j(T_0 u_j(\theta)) - \kappa \sum_{k=1}^{n} f_{j,k}(T_1 u_j(\theta), T_2 u_k(\theta)) &= \\
-f_{n+2-j}(T_0 u_{n+2-j}(\theta)) - \sum_{k=1}^{n} f_{n+2-j,k}(T_1 u_{n+2-j}(\theta), T_2 u_{n+2-k}(\theta)). & \qquad (3.99)
\end{aligned}$$

□

Note the switch between T_1 and T_2 in (3.98) .

Case 3: $\chi(\rho) = -1,\ \chi(\kappa) = +1$

In this case, the system is $\kappa-$equivariant if condition (3.94) is satisfied.

When $\chi(\rho) = -1$ and $\chi(\kappa) = +1$, then $\chi(\rho^s) = (-1)^s$, which requires n to be even i.e. $n = 2m$,

$$\mathbb{D}_m = \mathbb{Z}_m \ltimes \mathbb{Z}_2 = \left\{\rho^{2s}, \rho^{2s}\kappa : 0 \leq s \leq m-1\right\}.$$

In this case, $\mathbb{Z}_m$ corresponds to rotations by $\rho^2 u_j = u_{j+2}$ and $\mathbb{Z}_2 =< \kappa >$.

Lemma 3.38. *The equation (3.70) is $\rho-$reversible if and only if*

$$\begin{aligned} f_j(-\phi) &= -f_{j+2}(\phi), \\ f_{j,k}(\phi,\psi) &= -f_{j+2,k+2}(\psi,\phi) \end{aligned} \tag{3.100}$$

Note the switch in the arguments.

Proof. For the left hand side (LHS)

$$\begin{aligned} f_j(T_0\rho^2 u_j(-\theta)) + \sum_{k=1}^{n} f_{j,k}(T_1\rho^2 u_j(-\theta), T_2\rho^2 u_k(-\theta)) &= \\ f_j(T_0 u_{j+2}(-\theta)) + \sum_{k=1}^{n} f_{j,k}(T_1 u_{j+2}(-\theta), T_2 u_{k+2}(-\theta)) &= \\ f_j(T_0 u_{j+2}(-\theta)) + \sum_{k=1}^{n} f_{j,k}(T_2(u_{j+2}(\theta), T_1 u_{k+2}(\theta)) & \end{aligned} \tag{3.101}$$

Note the switch in arguments.

For the right hand side (RHS),

$$\begin{aligned} -\rho^2 f_j(T_0 u_j(\theta)) - \rho^2 \sum_{k=1}^{n} f_{j,k}(T_1 u_j(\theta), T_2 u_k(\theta)) &= \\ -f_{j+2}(T_0 u_{j+1}(\theta)) - \sum_{k=1}^{n} f_{j+2,k+2}(T_1 u_{j+2}(\theta), T_2 u_{k+2}(\theta)). & \end{aligned} \tag{3.102}$$

Upon comparing coefficients and terms with similar arguments, (3.100) follows. □

Case 4: $\chi(\rho) = -1$, $\chi(\kappa) = -1$

When $\chi(\rho) = -1$ and $\chi(\kappa) = -1$, then $\chi(\rho^s) = (-1)^s$, which requires n to be even i.e. $n = 2m$,

$$\mathbb{D}_m = \mathbb{Z}_m \ltimes \mathbb{Z}_2 = \left\{\rho^{2s}, \rho^{2s+1}\kappa : 0 \le s \le m-1\right\}$$

In this case, $\mathbb{Z}_m$ corresponds to rotations by $\rho^2 u_j = u_{j+2}$ and $\mathbb{Z}_2 =< \kappa >$.

Lemma 3.39. *The equation (3.70) is $\rho-$ and $\kappa-$reversible if and only if*

$$\begin{aligned} f_j(-\phi) &= -f_{j+2}(\phi), \\ f_{j,k}(\phi,\psi) &= -f_{j+2,k+2}(\psi,\phi) \end{aligned} \tag{3.103}$$

3.11 Group Actions in a Three-cell Network

Consider a 3-ring network with nearest neighbour coupling, equivariant with respect to the finite group: $\Gamma = \mathbb{D}_3 = \mathbb{Z}_3 \otimes \mathbb{Z}_2$ generated by a rotation ϱ and a reflection κ. The map $\chi : \mathbb{D}_3 \to \mathbb{Z}_2$ is determined from $\chi(\varrho) = 1$ and $\chi(\kappa) = -1$, so κ is the reversor. Consider the action of $\mathbb{Z}_3$ on $X = C([-r, r], \mathbb{R}^3)$ defined by the shift to the right $\varrho(u_1, u_2, u_3) = (u_2, u_3, u_1)$ and $\kappa(u_1, u_2, u_3) = (u_1, u_3, u_2)$.

For $\mathbb{Z}_3$-equivariance, the nonlinearity satisfies $f(\varrho(u_1, u_2, u_3)) = \varrho f(u_1, u_2, u_3)$. The linearisation Df of f satisfies

$$Df(u_1, u_2, u_3) = \begin{pmatrix} L_1u_1 + L_2u_2 + L_3u_3 \\ L_3u_1 + L_1u_2 + L_2u_3 \\ L_2u_1 + L_3u_2 + L_1u_3 \end{pmatrix} \tag{3.104}$$

where the L_i's are linear MFDEs

$$L_1u = a_1u(r) + a_0u(0) + a_2u(-r), \tag{3.105}$$

$$L_2u = b_1u(r) + b_0u(0) + b_2u(-r), \tag{3.106}$$

$$L_3u = c_1u(r) + c_0u(0) + c_2u(-r). \tag{3.107}$$

For reversibility, we require

$$f(\kappa^\sharp(u_1, u_2, u_3)) = -\kappa f(u_1, u_2, u_3), \tag{3.108}$$

where $\kappa(u_1, u_2, u_3) = (u_1, u_3, u_2)$. Note that in $\kappa^\sharp u$, time is reversed, i.e. $u(t + r) \mapsto u(t - r)$. When $L_iu^\sharp = L_iu$ and $L_{i+1}u^\sharp = -L_iu$, $i = 1, 2, 3$, the linear operator is $\mathbb{D}_3$-reversible. We show that L_1 is $\mathbb{Z}_2$-reversible if $a_2 = -a_1$ and $a_0 = 0$. Hence,

$$L_1u = a(u(r) - u(-r)).$$

Notice the forward and backward terms with symmetric delays.

Chapter 4

Centre Manifold Theory of Nonlinear MFDEs

In this chapter, we briefly summarise the centre manifold theory, one of the techniques employed to simplify a complex n-dimensional dynamical system by reducing the dimension of the state (phase) space. The purpose is to reduce the dimension of a system near a local bifurcation. We then carry out a versal unfolding of a DDE and examine the versal unfolding of an NMFDE under the Bogdanov-Takens bifurcation .

A broader coverage of the centre manifold theory can be found in the work of A. Vanderbauwhede and G. Iooss ([64]) and Kuznetsov [41] . Centre manifold reduction in the case of functional differential equations are provided in the works of Faria et al. [21], Babram et al. [5] and Guo et al. in [31] .

Recall that a manifold is a subspace of dimension $m < n$ of $\mathbb{R}^n$ that may be required to satisfy continuity and differentiability conditions. If a solution to a differential equation starts and remains on a curve or surface (manifold), then the manifold is said to be invariant. It is known that the classification of the equilibria of a linear(ised) system depends on the eigenvalues of the Jacobian matrix.

An equilibrium point of a nonlinear system whose linearization has eigenvalues all with negative real parts is stable. If the linearization has any eigenvalues with positive real part then the equilibrium point is not stable. If the nonlinear dynamical system with an equilibrium point at the origin and with the linearization there having no eigenvalues with positive real part, then by a suitable linear transformation we can always rewrite the dynamics in terms of stable coordinates $x \in \mathbb{R}^c$ and $y \in \mathbb{R}^s$ (where c is the dimension of E^c, the centre eigenspace and s is the dimension of E^s, the stable eigenspace at the origin) such that $\mathbb{R} = E^s \oplus E^u \oplus E^c$.. For example, an autonomous system can be transformed into block diagonal form with the linearised terms separated as

$$\dot{x} = Bx + f(x,y), \quad \dot{y} = Cy + g(x,y), \quad x \in \mathbb{R}^{n-m}, \quad y \in \mathbb{R}^m, \tag{4.1}$$

where n is the dimension of the system with f and g smooth. Let the origin be an isolated equilibrium point i.e. $f(0,0) = g(0,0) = 0$ and that the eigenvalues of B have zero real parts but non-zero imaginary parts, and the eigenvalues of C have negative real part.

Consider the system

$$\dot{x} = f(x) = Ax + g(x) \tag{4.2}$$

where $f(0) = 0$ and $A = D_x^0 f(0)$. The spectrum $\sigma(A)$ can be divided into the disjoint union of the stable spectrum σ_s, the unstable spectrum σ_u, and the centre spectrum σ_c . To those spectra correspond three eigenspaces, namely: the stable subspace E^s, the span of the stable eigenvectors, the unstable subspace E^u, the span of the unstable eigenvectors, the centre subspace E^c.

From the eigenspaces above, there exist the following invariant manifolds: a stable invariant manifold W^s tangent to E^s at 0, a stable invariant manifold W^u tangent to E^u at 0, a stable invariant manifold W^c tangent to E^c at 0. Note that stable component W^s is bounded forward in time, the unstable component W^u is bounded backward in time whilst the center component W^c is bounded in both directions. The invariant manifolds may be given in the form of the graph of a function $\psi_c : (E^c, 0) \to E^s \oplus E^u$, invariant under the flow. Their linear approximation is zero at 0 and therefore the graph is tangent to the subspace E^c. The graph of ψ_c denoted by W^c and tangent at the generalised eigenspace of $A = D_x^0 f(0)$ with purely imaginary eigenvalues is called the *center manifold* of the system at 0.

4.1 Centre Manifold Theory

We present the center manifold theory of A. Vanderbauwhede and G. Iooss [64]. We note that since this theory is developed for Banach spaces, it is applicable to the state space $C([\theta_{\min}, \theta_{\max}], \mathbb{C}^c)$ that we have employed for functional differential equations. Proofs of the theorems mentioned below may be found in [64].

We let X, Y and Z be Banach spaces, and let $A \in \mathcal{L}(X, Z)$ be a continuous linear operator and with $g \in C^k(X, Y)$, the Banach space of $k-$times continuously differentiable functions for some $k \geq 1$ and furthermore, let E and H be Banach spaces, $V \subset E$ an open subset, $k \in \mathbb{N}$ and $\eta \geq 0$. Then we define

$$C_b^k(V, H) = \{ \ v \in C^k(V, H) \ : \ |v|_{j,V} = \sup_{x \in V} \|D_j v(x)\| < \infty, \ 0 \leq j \leq k \ \},$$

equipped with the sup norm on all derivatives up to order k, and

$$C_b^{0,1}(E, H) = \{ \ v \in C^{0,1}(E, H) \ : \ |v|_{\text{Lip}} = \sup_{x,y \in E, x \neq y} \frac{\|v(x) - v(y)\|}{\|x - y\|} < \infty \},$$

where $|v|_{\text{Lip}}$ denotes the Lipschitz constant. When $V = E$ we may write $|v|_j$ for $|v|_{j,E}$. We also define

$$BC^{\eta}(\mathbb{R},E) = \{ \ v \in C^0(\mathbb{R},E) \ : \ \|v\|_\eta = \sup_{t\in\mathbb{R}} e^{-\eta t}\|v(t)\|_E < \infty \ \}, \tag{4.3}$$

the space of exponentially growing functions. We note that $BC^{\eta}(\mathbb{R},E) \subset BC^{\zeta}(\mathbb{R},E)$ if $0 \leq \eta < \zeta$ and that $\|v\|_\zeta \leq \|v\|_\eta$, $\forall v \in BC^{\eta}(\mathbb{R},E)$, that is, $(BC^{\eta}(\mathbb{R},E))_{\eta\geq 0}$ forms a scale of Banach spaces.

The following hypothesis is placed on the operator A :

1. (H1) There exists a continuous projection $P_c \in \mathcal{L}(Z,X)$ onto a finite-dimensional subspace $X_c \subset X$ such that $AP_cx = P_cAx$, $\forall x \in X$, and such that, taking

$$Z_h = (I-P_c)(Z), \ X_h = (I-P_c)(X), \ Y_h = (I-P_c)(Y),$$

where the subscript c refers to the centre and h, to the hyperbolic components, and

$$A_c = A \mid_{X_c} \in \mathcal{L}(X_c), \ A_h = A \mid_{X_h} \in \mathcal{L}(X_h, Z_h),$$

then the following hold

   (a) $\sigma(A_c) \subset i\mathbb{R}$ (where $\sigma(A)$ denotes the spectrum of the operator A);

   (b) there exists some $\beta > 0$ such that for each $\eta \in [0,\beta)$ and, for each $f \in BC^{\eta}(\mathbb{R},Y_h)$, the linear problem

$$\dot{x}_h = A_hx_h + f(t), \quad x_h \in BC^{\eta}(\mathbb{R},X_h),$$

   as a unique solution $x_h = K_hf$, where $K_h \in \mathcal{L}(BC^{\eta}(\mathbb{R},Y_h), BC^{\eta}(\mathbb{R},X_h)$ for each $\eta \in [0,\beta)$ and $\|K_h\|_\eta \leq \gamma(\eta)$, $\forall \eta \in [0,\beta)$, for some continuous function $\gamma : [0,\beta) \to \mathbb{R}_+$.

4.1.1 Some Centre Manifold Theorems

The aim here is to find solutions of (4.2) which belong to $BC^{\eta}(\mathbb{R},X)$ for some $\eta \in (0,\beta)$, under the hypothesis (H1). We use the notation $P_h = I_Z - P_c$.

Lemma 4.1 (Vanderbauwhede and Iooss)**.** *Assumming that (H1) holds true and $g \in C^0_b(X,Y)$ and let $x(t) : \mathbb{R} \to X$ be a solution of (4.2) and let $\eta \in (0,\beta)$. Then the following statements are equivalent:*

1. $x(t) \in BC^{\eta}(\mathbb{R},X)$,

2. $x(t) \in BC^{\zeta}(\mathbb{R},X)$, $\forall \zeta > 0$,

3. $P_hx(t) \in C^0_b(\mathbb{R},X_h)$.

Proof. Let $x(t) = P_cx(t)$ and $x_h(t) = P_hx(t)$. Then $x(t)$ is a solution of the ordinary differential equation

$$\dot{x}_c = A_cx_c + P_cg(x(t)), \tag{4.4}$$

which we may obtain via the variation of constants formula

$$x_c(t) = e^{A_c t} x_c(0) + \int_0^t e^{A_c(t-s)} P_c g(x(s))\, ds, \quad \forall t \in \mathbb{R}. \tag{4.5}$$

Because $\sigma(A_c) \subset i\mathbb{R}$ and g is globally bounded, $x(t)_c \in BC^\zeta(\mathbb{R}, X_c)$ for all $\zeta > 0$. In the same manner, we see that $x_h(t)$ is a solution of the equation

$$\dot{x}_h = A_h x_h + P_h g(x(t)). \tag{4.6}$$

Now, $P_h g(x(t)) \in C_b^0(\mathbb{R}, Y_h)$, and hence, by (H1), the equation (4.6) has a unique solution in $C_b^0(\mathbb{R}, X_h)$ given by $K_h(P_h g(x(t)))$. Moreover, this solution is also the unique solution of (4.6) in $BC^\eta(\mathbb{R}, X_h)$ for each $\eta \in (0, \beta)$. Then since $x(t) \in BC^\eta(\mathbb{R}, X_h)$ and $\eta \in (0, \beta)$, the foregoing argument shows that

$$x_h(t) = K_h(P_h g(x(t))). \tag{4.7}$$

Since $K_h(P_h g(x(t)))$ belongs to $BC^0(\mathbb{R}, X_h) = C_b^0(\mathbb{R}, X_h)$, it follows that condition (1) implies the condition (3).

Now assume that the condition (3) holds and since $C_b^0(\mathbb{R}, X_h) \subset BC^\zeta(\mathbb{R}, X_h)$ for each $\zeta > 0$, it follows that $x_h(t) \in BC^\zeta(\mathbb{R}, X_h)$ for all $\zeta > 0$. Furthermore, because $x_c(t) \in BC^\zeta(\mathbb{R}, X_c)$ for each $\zeta > 0$, it follows that $x(t) = x_c(t) + x_h(t) \in BC^\zeta(\mathbb{R}, X)$ for all $\zeta > 0$. Therefore the condition (3) implies condition (2) which in turn implies (1), the result follows. □

Lemma 4.2 (Vanderbauwhede and Iooss)**.** *Assume that the hypothesis (H1) holds and $g \in C_b^0(X, Y)$. Let $x(t) \in BC^\eta(\mathbb{R}, X)$ for some $\eta \in (0, \beta)$. Then $x(t)$ is a solution of (4.2) if and only if*

$$x(t) = e^{A_c t} P_c x(0) + \int_0^t e^{A_c(t-s)} P_c g(x(s))\, ds + K_h(P_h g(x))(t), \quad \forall t \in \mathbb{R}. \tag{4.8}$$

Proof. If $x(t)$ is a solution of (4.2) then using (4.5) and (4.7) shows that $x(t)$ satisfies (4.8). Conversely, if $x(t)$ satisfies (4.8) then projecting with P_c shows that $x_c(t) = P_c x(t)$ is a solution of (4.4), while projecting with P_h gives (4.7), and hence, by hypothesis (H1)(ii), $x_h(t) = P_h x(t)$ is a solution of (4.6). Thus using (4.4) and (4.6) we can see that that $x(t)$ is a solution of (4.2). □

Theorem 4.3 (Vanderbauwhede and Iooss)**.** *Assume (H1). Then there exists a $\delta_0 > 0$ such that, for all $g \in C_b^{0,1}(X, Y)$ satisfying*

$$|g|_{\mathrm{Lip}} < \delta_0, \tag{4.9}$$

there exists a unique $\psi \in C_b^{0,1}(X_c, X_h)$ such that for all $x(t) : \mathbb{R} \to X$, we have the following equivalent statements :

1. *$x(t)$ is a solution of (4.2) and $x(t)$ belongs to $BC^\eta(\mathbb{R}, X)$ for some $\eta \in (0, \beta)$;*

2. $P_h(x(t)) = \psi(P_c x(t))$, for all $t \in \mathbb{R}$, and $P_c x(t) : \mathbb{R} \to X_c$ is a solution of the ODE

$$\dot{x}_c = A_c x_c + P_c g(x_c + \psi(x_c)). \tag{4.10}$$

The following result follows as a consequence of the above theorem:

Corollary 4.4 (Vanderbauwhede and Iooss)**.** *Assuming that (H1) is true and let $g \in C_b^{0,1}(X,Y)$ be such that (4.9) holds. Then the problem*

$$\begin{cases} \dot{x} = Ax + g(x), \\ P_c x(0) = x_c, \quad x \in BC^\eta(\mathbb{R}, X), \end{cases} \tag{4.11}$$

has for each $x_c \in X_c$ and each $\eta \in (0, \beta)$ a unique solution given by

$$x(t, x_c) = x_c(t, x_c) + \psi(x_c(t, x_c)), \tag{4.12}$$

where $x_c(t, x_c)$ is the unique solution of (4.10) satisfying $x_c(0) = x_c$.

Following the prior hypotheses and results,

$$W^c = \{ \ x_c + \psi(x_c) \ : \ x_c \in X_c \ \} \subset X \tag{4.13}$$

is called the unique global centre manifold of (4.2).

The next problem is to examine the smoothness of this centre manifold.

Theorem 4.5 (Vanderbauwhede and Iooss)**.** *Assume that the hypothesis (H1) holds. Then there exists for each $k \geq 1$ a number $\delta_k > 0$ such that, if $g \in C_b^{0,1}(X,Y) \cap C_b^k(V_\rho, Y)$, with $V_\rho = \{ \ x \in X \ : \ \|P_h x\| < \rho \ \}$ and $\rho > \|K_h\|_0 |P_h g|_0$, and if moreover*

$$|g|_{\mathrm{Lip}} < \delta_k, \tag{4.14}$$

then the mapping ψ given by Theorem 4.3 belongs to the space $C_b^k(X_c, X_h)$. Moreover, if $g(0) = 0$ and $Dg(0) = 0$, then $\psi(0) = 0$ and $D\psi(0) = 0$, also.

Lemma 4.6 (Vanderbauwhede and Iooss)**.** *Let E be a Banach space, $\rho > 0$ and $w \in C_b^1(V_\rho, E)$ where $V_\rho = \{ \ x \in X \ : \ \|P_h x\| < \rho \ \}$. Let $\eta \geq 0$ and $V_\rho^\eta = \{ \ \tilde{u} \in BC^\eta(\mathbb{R}, X) \ : \ \tilde{u}(t) \in V_\rho, \forall t \in \mathbb{R} \ \}$. Define $W : V_\rho^\eta \to BC^\eta(\mathbb{R}, E)$ and $W^{(1)} : V_\rho^\eta \to \mathcal{L}(BC^\eta(\mathbb{R}, X), BC^\eta(\mathbb{R}, E))$ by*

$$W(\tilde{u})(t) = w(\tilde{u}(t))$$

and

$$\left(W^{(1)}(\tilde{u})(\tilde{v})\right)(t) = Dw(\tilde{u}(t))\tilde{v}(t),$$

for all $t \in \mathbb{R}$, $\tilde{u} \in V_\rho^\eta$ *and* $\tilde{v} \in BC^\eta(\mathbb{R}, X)$.

Let $\Phi \in C^0(BC^\eta(\mathbb{R}, X_c), V_\rho^\eta)$ *be such that*

1. Φ *is of class* C^1 *from* $BC^\eta(\mathbb{R}, X_c)$ *into* $BC^{\eta+\mu}(\mathbb{R}, X)$ *for each* $\mu > 0$*;*

2. *its derivative takes the form*

$$D\Phi(\tilde{u})(\tilde{v}) = \Phi^{(1)}(\tilde{u})\tilde{v}, \quad \forall \tilde{u}, \tilde{v} \in BC^\eta(\mathbb{R}, X_c),$$

   for some globally bounded $\Phi^{(1)} : BC^\eta(\mathbb{R}, X_c) \to \mathcal{L}(BC^\eta(\mathbb{R}, X_c), BC^\eta(\mathbb{R}, X)$.

Then $W \circ \Phi \in C_b^0(BC^\eta(\mathbb{R}, X_c), BC^\eta(\mathbb{R}, E))$*. Moreover,* $W \circ \Phi$ *is of class* C^1 *from* $BC^\eta(\mathbb{R}, X_c)$ *into* $BC^{\eta+\mu}(\mathbb{R}, E)$ *for each* $\mu > 0$ *with*

$$D(W \circ \Phi)(\tilde{u})\tilde{v} = W^{(1)}(\Phi(\tilde{u}))\Phi^{(1)}(\tilde{u})\tilde{v}, \quad \forall \tilde{u}, \tilde{v} \in BC^\eta(\mathbb{R}, X_c).$$

Proof. The proof of this lemma uses the same arguments as used in the proof of Lemma 3.7 of [63]. □

4.1.2 Local Centre Manifold

Using the theorems 4.3 and 4.5 of the global centre manifolds leads to the following theorem on the existence of a local centre manifold for (4.2).

Theorem 4.7 (Vanderbauwhede and Iooss)**.** *Assume that (H1) holds and let* $g \in C^k(X, Y)$ *for some* $k \geq 1$ *with* $g(0) = 0$ *and* $Dg(0) = 0$*. Then there exist a neighborhood* Ω *of the origin in* X *and a mapping* $\psi \in C_b^k(X_c, X_h)$ *with* $\psi(0) = 0$ *and* $D\psi(0) = 0$ *and such that the following properties hold:*

1. *if* $x_c(t) : I \to X_c$ *is a solution of (4.10) such that* $x(t) = x_c(t) + \psi(x_c(t)) \in \Omega$ *for all* $t \in \mathbb{R}$*, then* $x(t) : I \to X$ *is a solution of the system (4.2);*

2. *if* $x(t) : \mathbb{R} \to X$ *is a solution of (4.2) such that* $x(t) \in \Omega$ *for all* $t \in \mathbb{R}$*, then*

$$P_h x(t) = \psi(P_c x(t)), \quad \forall t \in \mathbb{R},$$

   and $P_c x(t) : \mathbb{R} \to X_c$ *is a solution of (4.10).*

Corollary 4.8 (Vanderbauwhede and Iooss)**.** *By the conditions of Theorem 4.7, one sees that there exists a neighborhood* Ω *of the origin in* X *such that all solutions* $x(t) : \mathbb{R} \to X$ *of (4.2) which satisfy* $x(t) \in \Omega$ *for all* $t \in \mathbb{R}$ *are of class* C^k *as a mapping from* $\mathbb{R}$ *into* X*.*

4.2 Center Manifolds for MFDEs

The center manifold reduction permits the study of a dynamical system near a non-hyperbolic equilibrium point as of an ordinary differential equation.

A complicated dynamical system can be put into a simpler form by some change of coordinates, an example being the the use of Jordan form for square matrices. The process of calculating the normal form of an ordinary differential equation involves projecting the system onto the center manifold and obtaining an approximate expression of normal form. Faria and Magalhaes [21] obtained the normal forms for FDEs by recursive changes of variables without computing beforehand the center manifold of the singularity. The resulting equation is an abstract form of ODE in an enlarged phase space.

Consider the MFDE

$$\dot{x}(t) = L(\lambda)x_t + F(\lambda, x_t), \tag{4.15}$$

where the delays are bounded in $[\theta_{\min}, \theta_{\max}]$, $\lambda \in \mathbb{R}^m$ are bifurcation parameters and $L(\lambda) : X \to \mathbb{R}^n$, be a linear operator and where X is the Banach space of continuous functions $C([\theta_{\min}, \theta_{\max}] : \mathbb{R}^n)$ equipped with the supremum norm

$$\|\phi\| = \sup_{\theta_{\min}<\theta<\theta_{\max}} |\phi(\theta)|, \qquad \phi \in X. \tag{4.16}$$

By the Riesz representation theorem we may write

$$L(\lambda)z = \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda, \theta)\, z(\theta)$$

where $\eta : [\theta_{\min}, \theta_{\max}] \to \mathbb{R}^n$ is a function of bounded variation. We may rewrite (4.15) using $L_0 = L(0)$ as

$$\dot{x}(t) = L_0 x_t + [L(\lambda) - L_0]x_t + F(\lambda, x_t). \tag{4.17}$$

We let $A(\lambda)$ be the infinitesimal generator of the linearised system

$$\dot{x}(t) = L(\lambda)x_t \tag{4.18}$$

with spectrum $\sigma(A(\lambda))$ and let Λ_λ be the set of purely imaginary eigenvalues. When $\lambda = 0$, the following bilinear form is defined

$$(\psi, \phi) = \psi(0)\phi(0) - \int_{\theta_{\min}}^{0} \int_{0}^{\theta_{\max}} \psi(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi \tag{4.19}$$

and used to decompose the state space X as $X = E^c \oplus Q$ with E^c being the generalised eigenspace of Λ_0 and Q is the infinite dimensional complementary subspace. A basis for the subspace E^c is $\Phi_{\Lambda_0} = \{\Phi_{\Lambda_1}, \cdots, \Phi_{\Lambda_m}\}$. We let B denote the finite dimensional matrix representation of $A(\lambda)$

that is restricted to Φ_{Λ_0} such that $A\Phi_{\Lambda_0} = \Phi_{\Lambda_0} B$. The basis of the dual space * in X^* is given by $\Psi = col\,\{\Psi_1, \cdots, \Psi_m\}$ such that $(\Psi, \Phi) = I$, the identity matrix.

It is shown by Faria et al in [21] that the equation (4.17) can be written as an ODE

$$\dot{x}_t = Ax_t + X_0 F(x_t)$$

in the Banach space BC of continuous functions from $[\theta_{\min}, \theta_{\max}]$ to $\mathbb{R}^n$ bounded and continuous on $[\theta_{\min}, \theta_{\max}]$ with a possible jump discontinuity at $\theta_{\max}$. The elements of BC are given in the form $\phi + X_0\delta$, where $\phi \in X$, $\delta \in \mathbb{R}^n$ with $X_0(\theta) = 0$ for $\theta \in [\theta_{\min}, \theta_{\max}]$ and $X_0(\theta_{\max}) = I$.

Let $\Pi : BC \to E^c$ be a continuous projection defined by

$$\Pi(\phi + X_0\delta) = \Phi[(\Psi, \phi) + \Psi(0)\delta].$$

We may write $BC = E^c \oplus \ker \Pi$ such that $Q = E^s \oplus E^u \subset \ker \Pi$. Decompose $x_t = \Phi y_t + z_t$ where $y_t \in \mathbb{R}^m$ and $z_t \in \ker \Pi \cap D(A) \equiv Q^1$ where $D(A)$ is the domain of A. The equation (4.17) is therefore equivalent to the following system

$$\begin{aligned} \dot{y} &= By + \Psi(0)\left\{[L(\lambda) - L_0](\Phi y + z) + F(\Phi y + z)\right\} \\ \dot{z} &= A_{Q^1} z + (I - \Pi)X(0)\left\{[L(\lambda) - L_0](\Phi y + z) + F(\Phi y + z)\right\} \end{aligned} \tag{4.20}$$

where $A_{Q^1} : Q^1 \to \ker \Pi$ such that $A_{Q^1}\phi = \dot{\phi} + X_0[L(\phi) - \dot{\phi}(0)]$. The system (4.20) can be transformed to the system

$$\begin{aligned} \dot{y} &= By + \sum_{j \geq 2} \frac{f_j^1(y, z)}{j!} \\ \dot{z} &= A_{Q^1} z + \sum_{j \geq 2} \frac{f_j^2(y, z)}{j!} \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} f_j^1(y, z) &= \Psi(0) F_j(\Phi y + z) \\ f_j^2(y, z) &= (I - \Pi)X_0 F_j(\Phi y + z) \end{aligned}$$

where F_j is the j^{th} Frechet derivative of F.

4.2.1 The Effects of Symmetry on the Center Manifold

We now briefly explore the effects of symmetry and reversibility on the center manifold. Golubitsky et al. [28] and Cicogna et al. in [16] amongst others, explore the effects of the action of a Lie group Γ on the center subspace. Cicogna et al. show that the center manifold inherits the symmetry properties of the original system.

Consider the situation when the equation (4.2) commutes with a group representation i.e. there exists a group Γ representing the symmetries of (4.2), such that

$$\gamma A x = A\gamma x \quad \text{and} \quad \gamma g(x) = g(\gamma x), \quad \forall x \in X, \gamma \in \Gamma.$$

Thus the group Γ leaves then the subspace X invariant.

It follows from the uniqueness of the global centre manifold that

$$\gamma\psi(x_c) = \psi(\gamma x_c), \quad \forall x_c \in X_c, \gamma \in \Gamma. \tag{4.22}$$

Hence the centre manifold is invariant under the action of Γ and that the reduced ODE on this centre manifold is equivariant under the action of Γ in X_c.

Aspects of group actions can be found in appendix (D). We now highlight some results from [28] .

Proposition 4.9. *The action of Γ allows the Jacobian matrix $(dg)_{0,0}$ to have purely imaginary eigenvalues when there is a $\Gamma-$invariant subspace of $\mathbb{R}^n$ that is either of the form*

- *$V \oplus V$ where V is absolutely irreducible,*
- *irreducible but not absolutely irreducible.*

Recall that a representation of Γ is absolutely irreducible if the only linear maps that commute with Γ are real multiples of the identity. The next problem is to consider reversibility, i.e. when a system anti-commutes with a symmetry γ.

$$\gamma A x = -A\gamma x, \quad \gamma g(x) = -g(\gamma x), \quad \forall x \in X.$$

We need to show then that $\gamma\psi(x_c) = \psi(\gamma x_c)$ and that the reduced vector field on the centre manifold anti-commutes with γ_c, the restriction of γ to X^c.

Since on X_h we have $A_h(\lambda - A_h)^{-1} = (\lambda - A_h)^{-1}A_h$, it follows that

$$A_h\gamma_+(t)|_{X_h} = \gamma_+(t)A_h,\ t > 0, \quad \text{and} \quad A_h\gamma_-(t)|_{X_h} = \gamma_-(t)A_h,\ t < 0. \tag{4.23}$$

4.3 Versal Unfoldings

We introduce the notion of (uni)versal unfolding first in the context of ODEs and vector fields and then to some FDEs.

Definition 4.10. *A family $f(x, \alpha)$ of vector fields is an unfolding of $f_0(x)$ if $f(x, 0) = f_0(x)$ for parameters $\alpha \in \mathbb{C}^p$ where p is an integer.*

Definition 4.11. *An unfolding $f(x, \alpha)$ of $f_0(x)$ is versal if it contains all possible qualitative dynamics that can occur near to $f_0(x)$.*

This means that every other unfolding in some neighbourhood of $f_0(x)$ will have the same dynamics as some family induced by $f(x, \alpha)$, through the addition of small parameters. Transversality guarantees that a parameter, α perturbs a nonhyperbolic equilibrium point transversely i.e. $D_\alpha f(0, 0) \neq 0$. An equilibrium point is nondegenerate, it cannot be removed by sufficiently small perturbations, such as by a small change in the value of α.

Recall that a dynamical system is a manifold $\mathcal{M}$ called the state space endowed with a family of smooth evolution functions Φ_t that for any element of $t \in T$, the time, map a point of the phase space back into the phase space. The system is called a flow when $t \in \mathbb{R}$. An unfolding is essentially a smoothly embedded submanifold that is transverse to $\mathcal{M}$. The following diagram depicts the conjugacy between two flows:

$$\begin{array}{ccc} x & \xrightarrow{\varphi_t} & \varphi_t(x, \alpha) \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ y & \xrightarrow{\psi_t} & \psi_t(y, \alpha) \end{array}$$

implying that

$$\psi_t(h(x, \alpha), \alpha) = h(\varphi_t(x, \alpha), \alpha) \tag{4.24}$$

in which $y = h(x, \alpha)$ is a homeomorphism.

Let $x_1, \ldots, x_n$ be local coordinates on $\mathcal{M}$, and let $\mathcal{R}(x_1, \ldots, x_n)$ denote the ring of smooth functions. We define the Jacobian ideal of f, denoted by J_f as follows:

$$J_f := \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle. \tag{4.25}$$

Then a basis for the versal unfolding of f is given by the quotient, known as the local algebra of f :

$$\frac{\mathcal{R}(x_1, \ldots, x_n)}{J_f} \tag{4.26}$$

The dimension of the local algebra is called the Milnor number of f. The minimum number of unfolding parameters for a versal unfolding is equal to the Milnor number.

4.3.1 Versal Unfolding of a DDE

Buono et al. study the versal unfolding of linear RFDEs, considering the projection of such families onto finite dimensional invariant manifolds and address the versality of the resulting parameterised family of linear ODEs. The approach extends the versal unfoldings of matrices to the situation of parameterised linear RFDEs.

Beginning with the RFDE

$$\dot{x}(t) = \mathcal{L}_0(x_t) \tag{4.27}$$

whose semiflow (i.e. with $t > 0$), is restricted to a finite dimensional subspace defined by the matrix B, construct a parameterised family $\mathcal{L}(\alpha)$ with semiflow defined by a versal unfolding $\mathcal{B}(\alpha)$ of the matrix B. If B be a $c \times c$ matrix with complex entries, then a p-parameter unfolding of B is a p-parameter analytic family of matrices $\mathcal{B}(\alpha)$ such that $\mathcal{B}(\alpha_0) = B$ for some $\alpha_0 \in \mathbb{C}^p$.

A center manifold reduction the FDE $\dot{x}(t) = F(x_t, \alpha)$ yields $\dot{z}(t) = Bz + G(z, \alpha)$. We apply a change of coordinate transformation $z = Cy + h(y)$ to obtain $C\dot{y} + h_y\dot{y} = B(Cy + h(y)) + G(Cy + h(y), \alpha)$. Extracting the linear parts, we obtain $C\dot{y} = BCy$ that is $\dot{y} = C^{-1}BCy$ from which yields the similarity class of B, and a versal unfolding is transverse to the similarity class manifold. It therefore follows that $\mathcal{B}(\alpha)$ is a versal unfolding of B if for all q-parameter unfoldings $A(\beta)$ with $A(\beta_0) = B$, there exists an analytic mapping $\phi : \mathbb{C}^q \to \mathbb{C}^p$ and an analytic family of invertible matrices $C(\beta)$ satisfying the conditions

$$A(\beta) = C(\beta)\mathcal{B}(\phi(\beta))(C(\beta))^{-1}, \qquad \phi(\beta_0) = \alpha_0, \qquad C(\beta_0) = I, \tag{4.28}$$

providing an orbit, Σ, for B under similarity, which implies a one-to-one correspondence between the solutions. A sufficient condition of versality is

$$\mathbf{Mat}_{c\times c} = T\Sigma_{\mathcal{B}(\alpha_0)} + D_\alpha\mathcal{B}(\alpha_0) \cdot \mathbb{C}^p \tag{4.29}$$

summing the tangent space $T\Sigma_{\mathcal{B}(\alpha_0)}$ and normal components to Σ at $\mathcal{B}(\alpha_0)$. Versal unfolding describes all perturbations near to the orbit of B that break the dynamical behaviour of the system.

4.3.2 Bogdanov-Takens Bifurcation

Consider the NMFDE system

$$\frac{d}{dt}Hx_t = \mathcal{L}_0(x_t) \tag{4.30}$$

with a unique solution given by $x(., \phi)$ with initial function ϕ at zero, then (4.30) determines a C_0-semigroup of bounded linear operators given by $T(t)\phi = x_t(\phi)$, for $t \geq 0$, where $x_t(\phi)$ is the solution of (4.30) with $x_0(\phi) = \phi$. Recall the infinitesimal generator A of $\{T(t)\}_{t\geq 0}$

$$D(A) = \left\{\phi \in C_n : \frac{d\phi}{d\theta} \in C_n,\ H\frac{d\phi}{d\theta} = \mathcal{L}_0(\alpha)\phi\right\}, \qquad A\phi = \frac{d\phi}{d\theta}. \tag{4.31}$$

Let $C_n = C([-\tau, 0], \mathbb{C}^n)$ be the Banach space of continuous functions from the interval $[-\tau, 0]$ into $\mathbb{C}^n$. Let A_0 be the infinitesimal generator of the semiflow generated by (4.27) and let $\Lambda \in \mathbb{C}$ a non-empty finite set of the eigenvalues of A_0 and P the corresponding c-dimensional generalised eigenspace. Using adjoint theory, we may split C_n as

$$C_n = P \oplus Q \tag{4.32}$$

where Q is invariant under (4.27). Define $C_n^* = C([-\tau, 0], \mathbb{C}^{n*})$, where $\mathbb{C}^{n*}$ is the n-dimensional space of row vectors. Furthermore, we let $\Phi = (\varphi_1, \cdots, \varphi_c)$ be a basis for P and $\Psi = \mathrm{col}(\psi 1, \cdots, \psi_c)$ be a basis for the dual space P^* in C_n^*, chosen such that $< \Psi, \Phi >_n = \mathbf{I}_c$ is the $c \times c$ identity matrix. This means that $Q = \{\varphi \in C_n \; :< \Psi, \varphi >_n = 0\}$. If we denote by B the $c \times c$ constant matrix such that $\dot{\Phi} = \Phi B$, then the spectrum of B coincides with Λ. Furthermore, we choose the bases Φ and Ψ such that $\dot{\Psi} = -B\Psi$. Using the decomposition (4.32), any $z \in C_n$ can be written as $z = \Phi x + y$ where $x \in \mathbb{C}^c$ and $y \in Q$ is a C^1 function.

The conditions for Bogdanov-Takens (BT) bifurcation are

(BT 1): $\lambda = 0$ is a characteristic value of A_0 with algebraic multiplicity 2 and geometric multiplicity 1, when $\alpha = 0$;

(BT 2): all other eigenvalues of A_0 have non-zero real parts.

When *(BT 1) - (BT 2)* hold, we call $(x, \alpha) = (0, 0)$ a Bogdanov-Takens (BT) point of the system.

We obtain the generalised eigenspace as follows:

By *(BT 1)*, there exist linearly independent functions $\phi_1, \phi_2 \in C_n$ such that

$$A_0\phi_1 = 0, \; A_0\phi_2 = \phi_1 \tag{4.33}$$

and the following equation

$$A_0\phi = \phi_2 \tag{4.34}$$

has no solution $\phi \in C_n$.

We note that $A_0\phi_1 = 0$ is equivalent to

$$\begin{aligned} \dot{\phi}_1(\theta) &= 0, \qquad \theta_{min} \le \theta < \theta_{max} \\ H\dot{\phi}_1 &= \mathcal{L}_0\phi_1, \qquad \theta = 0 \end{aligned} \tag{4.35}$$

and $\mathcal{A}_0\phi_2 = \phi_1$ is equivalent to

$$\begin{aligned} \dot{\phi}_2(\theta) &= \phi_1^0, \qquad \theta_{min} \le \theta < \theta_{max} \\ H\dot{\phi}_2 &= \mathcal{L}_0\phi_2, \qquad \theta = 0. \end{aligned} \tag{4.36}$$

Hence, to obtain explicit expressions for Φ and Ψ we use

$$\Phi = [\phi_1, \phi_2] = [v_1, v_2 + v_1\theta] \tag{4.37}$$

and

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -w_1 s + w_2 \\ w_1 \end{pmatrix} \tag{4.38}$$

where $v_1, v_2, w_1, w_2 \in \mathbb{R}^3$ satisfy

$$\Delta(0)v_1 = 0,\ D_\lambda \Delta(0)v_1 + \Delta(0)v_2 = 0 \tag{4.39}$$

and

$$w_1\Delta(0) = 0,\ \ w_1 D_\lambda \Delta(0) + w_2 \Delta(0) = 0 \tag{4.40}$$

The Orbit Σ of the matrix B

We now study the orbit Σ of $B(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, the Jordan matrix associated with the double zero eigenvalue with geometric multiplicity 1, defined in (4.28) and (4.29) where $\Sigma : \mathbb{R}^4 \to \mathbb{R}^4$, and we let

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \qquad \text{with} \qquad C^{-1} = \frac{1}{c_{11}c_{22} - c_{12}c_{21}} \begin{bmatrix} c_{22} & -c_{21} \\ -c_{12} & c_{11} \end{bmatrix}. \tag{4.41}$$

We have up to similarity,

$$\Sigma = CBC^{-1} = \frac{1}{c_{11}c_{22} - c_{12}c_{21}} \begin{bmatrix} -c_{11}c_{21} & c_{11}^2 \\ -c_{21}^2 & c_{11}c_{21} \end{bmatrix} \tag{4.42}$$

To determine the tangent space $T\Sigma_{\mathcal{B}(\alpha_0)}$ for $\mathcal{B}(0)$, we require $C = I$. We define the components of $\Sigma(c_{11}, c_{12}, c_{21}, c_{22})$ by the maps $\Sigma_1 = \frac{-c_{11}c_{21}}{D}$, $\Sigma_2 = \frac{-c_{11}^2}{D}$, $\Sigma_3 = \frac{-c_{21}^2}{D}$, and $\Sigma_4 = \frac{c_{11}c_{21}}{D}$, where $D = c_{11}c_{22} - c_{12}c_{21}$.

Evaluating the Jacobian $\left[\frac{\partial \Sigma_i}{\partial c_{ij}}\right]|_{B(0)}$, we obtain

$$T\Sigma_{\mathcal{B}(\alpha_0)} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{bmatrix} = \begin{bmatrix} -c_{21} \\ c_{11} - c_{22} \\ 0 \\ c_{21} \end{bmatrix} \tag{4.43}$$

$$= Span \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \tag{4.44}$$

The tangent space $T\Sigma_{\mathcal{B}(\alpha_0)}$, where we recall that $\alpha_i \in \mathbb{C}^p$, is two-dimensional hence the normal space

$$D_\alpha \mathcal{B}(\alpha_0) \cdot \mathbb{C}^p = Span \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \tag{4.45}$$

is two-dimensional, yielding the following matrices $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, corresponding to α_1, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, corresponding to α_2. We therefore have

$$B(\alpha) = \begin{bmatrix} 0 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix}. \tag{4.46}$$

We note that (4.46) has a double zero eigenvalue when $\alpha_1 = \alpha_2 = 0$. We have $\Delta_{B(\alpha)}(\lambda) = -\lambda\alpha_2 + \lambda^2 - \alpha_1$ and $D_\lambda \Delta_{B(\alpha)}(\lambda) = -\alpha_2 + 2\lambda$. Applying the double zero conditions, we find that $\lambda = \frac{\alpha_2}{2}$ and $\alpha_1 = -\frac{\alpha_2^2}{4}$

4.3.3 Unfolding a Neutral Mixed Functional Differential Equation

We now study the versal unfolding of the following Neutral MFDE with Bogdanov-Takens bifurcation (double zero eigenvalue with geometric multiplicity one):

$$\frac{d}{dt}[x(t) + x(t+1)] = H_0 x_t = x(t+1) - x(t-1) \tag{4.47}$$

We determine the Bogdanov-Takens singularity and the generalised eigenspace associated with the zero eigenvalue by using a center manifold reduction and normal form theory. If $\mathcal{B}$ is a versal unfolding which depends on the least number of parameters, then $\mathcal{B}$ is called a mini-versal unfolding. Thus, one may view a versal unfolding of B as the most general C^∞ perturbation of B up to similarity and change of parameters.

Lemma 4.12. *The characteristic equation of (4.47) has a double zero eigenvalue, with a two dimensional center eigenspace P and a real Λ-mini-versal unfolding of (4.47) is given by*

$$\frac{d}{dt}[x(t) + x(t+1)] = H(\alpha)x_t = x(t+1) - x(t-1) + \alpha_1 x(t+1) + \alpha_2 x(t-1). \tag{4.48}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$.

Proof. We construct the proof of the lemma using aspects of normal form theory and numerous applicable theorems.

In order to test the multiplicity of λ_1 as a root of $\det \Delta(\lambda) = 0$, we need to evaluate at λ_1.

The characteristic equation of (4.47) is given by

$$\det \Delta(\lambda) = -\lambda - \lambda e^{\lambda} + e^{\lambda} - e^{-\lambda} = 0, \tag{4.49}$$

giving $\lambda_1 = 0$ as an eigenvalue. The derivatives are given by

$$\frac{d}{d\lambda}\Delta(\lambda) = \Delta'(\lambda) = -1 - e^{\lambda} - \lambda e^{\lambda} + e^{\lambda} + e^{-\lambda}$$

and the second derivative

$$\Delta''(\lambda) = -e^{\lambda} - \lambda e^{\lambda} - e^{\lambda} + e^{\lambda} - e^{-\lambda}.$$

We obtain $\Delta'(0) = -1 - e^0 - 0 + e^0 + e^0 = 0$ but that $\Delta''(0) \neq 0$. This means that λ_1 is a double eigenvalue by the Bogdanov-Takens conditions (BT 1) and (BT 2).

Let $I \subset \mathbb{R}$ and for each $\theta \in I$, define $\Phi_c(\theta) = (\phi_1(\theta), \cdots, \phi_c(\theta))$. Note that from (4.47), we have $\mathcal{L}_0\phi = \phi(1) - \phi(-1) - \dot{\phi}(0) - \dot{\phi}(1)$ giving $\mathcal{L}_0(1) = 1 - 1 - 0 - 0 = 0$. For the condition $\mathcal{L}_0\phi_1 = 1$, we obtain ϕ_2 thus:

$$\begin{aligned} \mathcal{L}_0\phi_1 &= 0 \Rightarrow v_1 = 1 \\ \mathcal{L}_0\phi_2 &= \phi_1 = 1 \Rightarrow \phi_2 = \theta \end{aligned} \tag{4.50}$$

We obtain the basis $\Phi = (1, \theta)$ for P from $\Phi(1) = \{\phi_1(1), \phi_2(1)\} = \{1, 1\}$ and $\Phi(-1) = \{\phi_1(-1), \phi_2(-1)\} = \{1, -1\}$, hence

$$\Phi = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{4.51}$$

with rank $c = 2$. This satisfies lemma 6.6 of Buono et al. in [11], thus $\varphi_1(0) = 1$ and $\varphi_2(0) = -1$, neither of which equals 0 and

$$\varphi_2(0)\Delta(\lambda_j) = 0 \tag{4.52}$$

i.e. $(1, -1)0 = 0$ since in this $1-$D case, $\Delta(\lambda_j) = 0$.

Consider the smoothly parameterised family of linear RFDEs

$$\dot{z}(t) = \mathcal{L}(\alpha)(z_t) \tag{4.53}$$

which can be rewritten as

$$\begin{aligned} \dot{z}(t) &= \mathcal{L}_0(z_t) + [\mathcal{L}(\alpha) - \mathcal{L}_0](z_t) \\ \dot{\alpha}(t) &= 0. \end{aligned} \tag{4.54}$$

It can be shown that the dynamics of (4.53) near the equilibrium solution (z, α) reduces to the c-dimensional parameterised linear system

$$\dot{x} = \mathcal{B}(\alpha)x, \tag{4.55}$$

where

$$\mathcal{B}(\alpha) = B(0) + \Psi(0)[\mathcal{L}(\alpha) - \mathcal{L}_0](\Phi + h(\alpha)) \tag{4.56}$$

with $\mathcal{B}(0) = B$ and $h(0) = 0$, we have

$$D_\alpha \mathcal{B}(0) = \Psi(0) D_\alpha[\mathcal{L}(\alpha)(\Phi)]|_{\alpha=0}. \tag{4.57}$$

The parameterised family (4.53) is said to be a Λ-versal unfolding for (4.27) if the matrix $\mathcal{B}(\alpha)$ is a versal unfolding for B.

Theorem 4.13 (Buono et al.)**.** *Let $\mathcal{L}(\alpha)$ be a δ-parameter family of bounded linear operators from $C([-\tau, 0], \mathbb{C}^n)$ into $\mathbb{C}^n$ defined by*

$$\mathcal{L}(\alpha) = \mathcal{L}_0 + \sum_{m=1}^{\delta} \alpha_m L_m \tag{4.58}$$

where the α_m are complex parameters, $\mathcal{L}_0$ is as in (4.27) and L_m is given by

$$L_m(z) = \sum_{j=0}^{c-q} A_j^m z(\tau_j) \tag{4.59}$$

where A_j^m are $n \times n$ matrices. Then (4.53) is a Λ-mini-versal unfolding of (4.27).

Although the theory has been carried out in complex spaces, $\mathcal{L}_0$ and $\mathcal{L}(\alpha)$ are usually real i.e. bounded operators from $C([-\tau, 0], \mathbb{R}^n)$ into $\mathbb{R}^n$, real versal unfoldings can be constructed by decomplexification.

Theorem 4.14 (Buono et al.)**.** *Suppose that $\Lambda = \{\Lambda_0, \Lambda_h, \bar{\Lambda_h}\}$ where Λ_0 is the subset of the real eigenvalues and Λ_h the subset of non-real eigenvalues, then a real Λ-mini versal unfolding of (4.27) is given by*

$$\mathcal{L}(\alpha) = \mathcal{L}_0 + \sum_{p=1}^{\delta_0} \alpha_p L_p + \sum_{s=\delta_0+1}^{s=\delta_0+\delta_h} (\beta_s \Re(L_s) + \beta_{s+\delta_h} \Im(L_s)) \tag{4.60}$$

where $\alpha_p \in \mathbb{R}$ for $p = 1, \cdots, \delta_0$, and $\beta_s, \beta_{s+\delta_h} \in \mathbb{R}$ for $s = \delta_0 + 1, \cdots, \delta_0 + \delta_h$ whilst L_p is a bounded linear operator from $C([-\tau, 0], \mathbb{R}^n)$ into $\mathbb{R}^n$ and L_s is a bounded linear operator from $C([-\tau, 0], \mathbb{R}^n)$ into $\mathbb{C}^n$.

Corollary 4.15. *If $\Lambda = \Lambda_0$, then a real Λ-versal unfolding of (4.27) is given by (4.60) with*

$\beta_s = \beta_{s+\delta_h} = 0$ *for all* $s = \delta_0 + 1, \cdots, \delta_0 + \delta_h$.

$\square$

Chapter 5

Hopf Bifurcation via Lyapounov-Schmidt Reduction

Bifurcation theory studies the qualitative changes in the behaviour or solutions of a dynamical system when the system's parameter (bifurcation parameters) values are varied. Local bifurcations occur in the neighbourhood of an equilibrium point, where solutions are constant in time. Hopf bifurcation theorems are used to prove the existence of periodic solutions of a non-linear system near an equilibrium point when a conjugate pair of distinct eigenvalues of the linearised system crosses the imaginary axes. As an example, if we consider the equation

$$\dot{x}(t) = f(x_t, \lambda), \tag{5.1}$$

where $f : \mathcal{C} \to \mathbb{R}^n$, and where $\mathcal{C} = C([\theta_{\min}, \theta_{\max}], \mathbb{R}^n)$ is the Banach space of continuous functions from the interval $[\theta_{\min}, \theta_{\max}]$ to $\mathbb{R}^n$, then the point (x^*, λ^*) which satisfies $f(x^*, \lambda^*) = 0$, is an equilibrium point. Here $x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m$ and the bifurcation is determined by the linearisation of the vector field $L = D_x f(x^*, \lambda^*)$. Hopf showed when the linearised operator L has simple eigenvalues $\pm i$ and has no other eigenvalues on the imaginary axis, a one-parameter family of periodic solutions to (5.1) could be found. The Hopf bifurcation starts with the formation of a limit cycle (an isolated closed trajectory), from a stable focus which is analogous to a fixed point. A stable limit cycle (supercritical Hopf bifurcation) attracts trajectories from both its inside and outside whilst an unstable limit cycle (subcritical Hopf bifurcation) repels trajectories on both sides. The bifurcation occurs as a pair of complex conjugate eigenvalues crosses the imaginary axis thereby switching from a stationary state of the system at an equilibrium to oscillatory behaviour on a limit cycle.

The presence of symmetry may cause purely imaginary eigenvalues to arise with higher multiplicities which cause the bifurcation problem to become more complicated. The most common approach to study bifurcation problems in FDEs involves the computation of (normal forms of) reduced bifurcation equations on centre manifolds. However, as stated before, major difficulties that need to be overcome in the construction of centre manifolds for MFDE are the absence of a semi-flow and the ill-posedness of the natural initial value problem. This precludes the direct application of the ideas developed by Faria and Magalhaes [21] for retarded (delay) differential equations (RFDEs).

Rustichini [56, 55] applied the Hopf bifurcation theorems to MFDEs using the center manifold theorem of chapter 4 and the Lyapunov-Schmidt (L-S) reduction, see appendix C. Here, the proof of the Hopf bifurcation theorem does not involve a solution operator since a semigroup cannot be easily obtained, but utilises functional analytic arguments, setting the problem in the space of periodic functions of fixed period. The linearisation of the MFDE defines a linear operator acting on this space, and in fact, it can be identified by an operator of the delay type, for which a continuous semigroup (solution operator) can be defined. The task is then reduced to the study of the zeros of the bifurcation problem. To utilise the centre manifold theorem for (5.1), we identify (by using the Riesz representation theorem) its Frechet derivative at zero by $f'(0)$, with the regular measure induced by a function of bounded variation η, by

$$f'(0)\phi = \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\theta)\phi(\theta) \qquad \forall \phi \in \mathcal{C}. \tag{5.2}$$

We write $L\phi = f'(0)\phi$ and associate with this linear operator the characteristic matrix $\Delta(\alpha) \equiv \alpha I - \int_{\theta_{\min}}^{\theta_{\max}} e^{\alpha\theta} d\eta(\theta)$. The strategy involved here is to work with functions defined on the entire real line, the space of exponentially bounded functions defined in (4.3) of chapter 4 and construct the centre manifold using the implicit function theorem. The spectrum of the characteristic equation can be divided into the disjoint union of the stable spectrum σ_s, the unstable spectrum σ_u, and the centre spectrum σ_c.

Guo and Lamb [30] study equivariant Hopf bifurcation by applying a Lyapunov-Schmidt (LS) reduction to neutral functional differential equations (NFDEs) and Guo [29] applies the LS reduction to MFDEs. To deal with the problem caused by the presence of eigenvalues with high multiplicities resulting from the action of a symmetry group Γ, Golubitsky et al. in [28] describe a procedure that reduces the dimension of $\ker L$ in the Lyapunov-Schmidt reduction. Solutions are sought which lie in a fixed point subspace of a subgroup Σ of Γ defined by

$$\text{Fix}(\Sigma) = \{y \in \ker L : \sigma y = y, \forall \sigma \in \Sigma\}.$$

The system is then shown to have a bifurcation of periodic solutions whose spatio-temporal symmetry can be completely characterised by Σ.

An alternative approach is in [58] where Sieber finds periodic orbits in state dependent delay differential equations as roots of algebraic equations. The caricature example is

$$\dot{x}(t) = \lambda - x(t - x(t) = f(x_t, \lambda),$$

with the maximal delay limited to a value τ, and furthermore, the function f is only as smooth as its argument x. Sieber constructs the algebraic system for periodic orbits of functional differential equations (FDEs) using the notion of periodic boundary-value problems for FDEs on the interval $[-\pi, \pi]$ with periodic boundary conditions identified with the unit circle i.e. such that functions on the interval satisfy $x^{(j)}(-\pi) = x^{(j)}(\pi)$ for some integer $j \geq 0$. The functions x can then be extended to arguments in $\mathbb{R}$ by defining $x(t) = x(t - 2k\pi)$ where k is an integer chosen such that $-\pi \leq t - 2k\pi < \pi$. The norm in the space is given by

$$||x||_j = \max_{t \in [-\pi,\pi]} \{|x(t)|, |\dot{x}(t)|, \cdots, |x^{(j)}|\}.$$

5.1 Setting up the Scene

We present a treatment of Hopf bifurcation for equivariant MFDEs on the basis of equivariant Lyapunov-Schmidt reduction. In the process, we obtain explicit expressions in terms of the original system that determine the monotonicity of the period and Hopf bifurcation direction of branches of bifurcating symmetric periodic solutions. With these expressions at our disposal, the study of equivariant Hopf bifurcation in explicit examples can be performed without having to resort to lengthy computations associated to centre manifold reduction.

We consider a general system of the following parameterized MFDE, considering a structural form, that is, using matrices and vectors. A general parameterized MFDE is given by

$$\dot{x}(t) = L(\lambda)x_t + f(\lambda, x_t), \tag{5.3}$$

where $\lambda \in \mathbb{R}^k$ and $x_t \in X = C([\theta_{\min}, \theta_{\max}], \mathbb{C}^n)$ is a continuous for any $t \in \mathbb{R}$. Furthermore, we assume that $L(\lambda) : X \to \mathbb{R}^n$ is a linear operator and $f : X \to \mathbb{R}^n$ is a smooth enough nonlinear operator satisfying $f(0, 0) = 0$ and $D_x f(0, 0) = 0$. Those last conditions imply that the origin is a steady state of (5.3). We assume that (5.3) is Γ-equivariant where Γ is a compact group. The linearisation of (5.3) around the equilibrium 0 is

$$\dot{x}(t) = L(\lambda)x_t. \tag{5.4}$$

In some sense, X is indeed a state space for the homogeneous equation (5.4), even though one cannot view this equation as an initial value problem.

The parameterised system of Neutral MFDE is of the form

$$\frac{d}{dt}h(\lambda, x_t) = f(\lambda, x_t), \tag{5.5}$$

where $h, f : \mathbb{R} \times C([\theta_{\min}, \theta_{\max}], \mathbb{R}^n) \to \mathbb{R}^n$ are two continuously differentiable mappings which satisfy $f(0, 0) = 0$ for all $\lambda \in \mathbb{R}$. We consider the generalised Neutral MFDE system in the operator form (5.5) and note that linear operators in the system of equations can help identify symmetries. Furthermore, we seek conditions that may be imposed on the operators to obtain reversibility and symmetry.

Let $D(\lambda), L(\lambda) : C([\theta_{\min}, \theta_{\max}], \mathbb{R}^n) \to \mathbb{R}^n$ be the linearised operators of $h(\lambda, .)$ and $f(\lambda, .)$ respectively and further, assuming that $D(\lambda)$ is atomic at 0, then by the Riesz representation theorem there exists $n \times n$ matrix-valued functions $\mu, \eta : [\theta_{\min}, \theta_{\max}] \to \mathbb{R}^{n^2}$ whose components each are of bounded variation, such that for $\phi \in C([\theta_{\min}, \theta_{\max}], \mathbb{R}^n)$

$$D(\lambda)\,\phi = \phi(0) - \int_{\theta_{\min}}^{\theta_{\max}} d\mu(\lambda,\theta)\,\phi(\theta), \qquad L(\lambda)\,\phi = \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda,\theta)\,\phi(\theta).$$

For each λ the linear system of equations, under suitable conditions,

$$\frac{d}{dt} D(\lambda) x_t = L(\lambda) x_t \tag{5.6}$$

generates a strongly continuous semigroup of linear operators with infinitesimal generator A_λ defined as follows:

Let $X^1 = C^1([\theta_{\min}, \theta_{\max}], \mathbb{C}^n)$, define the following operator $A_\lambda : \mathrm{dom}(A_\lambda) \subset X \to X$, via $\mathrm{dom}(A_\lambda) = \{\ \phi \in X^1 :\ \frac{d}{d\theta}\phi(0) = L(\lambda)\phi\ \}$,

$$A_\lambda \phi = \frac{d}{d\theta}\phi. \tag{5.7}$$

The following result summarises information about A_λ.

Lemma 5.1. *1. The operator A_λ defined in (5.7)is closed and densely defined and Γ-equivariant. The domain $\mathrm{dom}(A_\lambda)$ is invariant under $\lambda^\sharp$ whilst A_λ and $\lambda^\sharp$ anti-commute when λ is a reversing symmetry.*

2. The spectrum of A_λ is the point spectrum with $\alpha \in \sigma(A_\lambda)$ if and only if α satisfies

$$\Delta_{A_\lambda}(\alpha) = 0, \tag{5.8}$$

where the holomorphic characteristic matrix $\Delta_{A_\lambda} : \mathbb{C}^{k+1} \to \mathrm{M}(n, \mathbb{C})$ is given by

$$\Delta_{A_\lambda}(\alpha) = \alpha I - L(\lambda)\exp(\alpha(\cdot)). \tag{5.9}$$

Moreover, $\phi \in X$ is an eigenfunction of A_λ associated with the eigenvalue $\bar{\alpha}$ if and only if $\phi(\theta) = e^{\bar{\alpha}\theta}\underline{a}$ for $\theta \in [\theta_{\min}, \theta_{\max}]$ and some vector $\underline{a} \in \mathbb{C}^n$ such that $\Delta_{A_\lambda}(\bar{\alpha})\underline{a} = 0$.

Proof. The operator A_λ is closed because differentiation is a closed operation and $L(\lambda)$ is bounded. The density of the domain $\mathrm{dom}(A_\lambda)$ follows from the density of C^1-functions in X together with the fact that for any $\epsilon > 0$ and any neighbourhood of zero, one can modify an arbitrary C^1-function ϕ in such a way that $\frac{d}{d\theta}\phi(0)$ can be set at will, while $\phi(0)$ remains unchanged and $\|\phi\|$ changes by at most ϵ.

We now explore the equivariance of

$$A_\lambda \phi = \frac{d\phi}{d\theta}.$$

For equivariance we require $A_\lambda \gamma = \gamma A_\lambda$, and we have

$$A_\lambda \gamma\phi(\theta) = \frac{d}{d\theta}(\gamma\phi)(\theta) = \gamma\frac{d}{d\theta}\phi(\theta) = \gamma A_\lambda \phi(\theta) \tag{5.10}$$

provided the domain is $\gamma-$invariant. Furthermore, $\gamma\phi \in \operatorname{dom}(A_\lambda)$ if

$$(\dot{\gamma\phi})(0) = L(\lambda)(\gamma\phi). \tag{5.11}$$

Note that $\frac{d}{d\theta}\phi(0) = L(\lambda)\phi$ and therefore

$$\gamma\frac{d}{d\theta}\phi(0) = \gamma L(\lambda)\phi = L(\lambda)\gamma\phi, \tag{5.12}$$

since L is $\gamma-$equivariant.

To explore the reversibility of the operator A_λ, we recall some basic definitions and requirements for reversibility.

Definition 5.2. *The action of $\gamma^\sharp$ is defined by $(\gamma^\sharp x)(\theta) = \gamma x(\chi(\gamma)\theta), \quad \forall\theta \in [\theta_{\min}, \theta_{\max}]$ where $\gamma^\sharp : x(\theta) \to x(-\theta)$ and $\rho_\chi(\gamma) = \chi(\gamma)\rho(\gamma)$ and for the autonomous case we require that $F(\gamma^\sharp x) = \rho_\chi(\gamma)F(x)$.*

We show that $\operatorname{dom}(A_\lambda)$ is $\gamma^\sharp$-invariant. Recall that $\gamma\phi \in \operatorname{dom}(A_\lambda)$ if $\frac{d}{d\theta}(\gamma\phi)(0) = L(\lambda)(\gamma\phi)$. By the reversibility of $L(\lambda)\phi$, we have

$$L(\lambda)\gamma^\sharp\phi(\theta) = \gamma\chi(\gamma)L(\lambda)\phi(\theta) \tag{5.13}$$

$$\begin{aligned} L(\lambda)[\gamma\phi(\chi(\gamma)\theta)] &= \gamma\chi(\gamma)L(\lambda)\phi(\theta) \\ &= \gamma\chi(\gamma)\frac{d}{d\theta}\phi(0). \end{aligned} \tag{5.14}$$

Define $\psi(\theta) := \gamma\phi(\chi(\theta)) = \gamma^\sharp\phi(\theta)$ so that $\frac{d}{d\theta}\psi(\theta) = \gamma\frac{d}{d\theta}\phi(\chi(\theta))$, then

$$L(\lambda)\psi(\theta) = \frac{d}{d\theta}\psi(0) \tag{5.15}$$

showing that $\psi \in \mathrm{dom}(A_\lambda)$.

For the reversibility of A_λ , we require

$$A_\lambda \gamma^\sharp \phi(\theta) = \chi(\gamma)\gamma A_\lambda \phi(\theta) \tag{5.16}$$

We have

$$\begin{aligned} A_\lambda \gamma^\sharp \phi(\theta) &= (\gamma^\sharp \dot{\phi})(\theta) \\ &= \chi(\gamma)\gamma \frac{d}{d\theta}\phi(\chi(\gamma)\theta) \\ &= \chi(\gamma)\gamma A_\lambda \phi(\chi(\gamma)\theta). \end{aligned} \tag{5.17}$$

Therefore (by the $\gamma^\sharp$ action),

$$\gamma^\sharp A_\lambda \phi(\theta) = \gamma A_\lambda \phi(\chi(\gamma)\theta). \tag{5.18}$$

Hence

$$A_\lambda \gamma^\sharp \phi(\theta) = \chi(\gamma)\gamma^\sharp A_\lambda \phi(\theta). \tag{5.19}$$

Definition 5.3. *A scalar α is called an eigenvalue of the operator A if there is a nontrivial solution v of $Av = \alpha v$. Such an v is called an eigenvector corresponding to the eigenvalue α. The set of eigenvalues of A_λ is also called the point spectrum of A_λ, denoted by $\sigma(A_\lambda)$.*

Let α be an eigenvalue of A_λ. Then

$$A_\lambda v = \dot{v} = \alpha v$$

giving $v = \underline{a} e^{\alpha\theta}$. It follows that

$$\begin{aligned} \dot{v}(0) &= L(\lambda) v \\ \alpha \underline{a} &= \underline{a} L(\lambda) e^{\alpha\theta} \\ \underline{a}(\alpha I - L(\lambda) e^{\alpha\theta}) &= 0 \end{aligned} \tag{5.20}$$

and

$$\Delta_{A_\lambda}(\alpha) = \alpha I - L(\lambda)\exp(\alpha(\cdot)). \tag{5.21}$$

□

Suppose that a pair of roots of the characteristic equation (5.8) crosses the imaginary axis at $\alpha_\pm = \pm i\alpha_0$ for a certain parameter value λ_0, say $\lambda_0 = 0$. Under suitable conditions, the Hopf bifurcation theorem can be lifted to the infinite dimensional setting of (5.3) and, hence, one may conclude the existence of a branch of periodic solutions to (5.3) bifurcating from the trivial equilibrium $x = 0$ for $\lambda \approx \lambda_0$. The symmetry group Γ often causes purely imaginary roots to be multiple. When a compact Lie group Γ acts on a vector space V , the space can be decomposed into a finite number m of Γ-irreducible subspaces $\bigoplus_1^m U_i$, an isotypic decomposition in which Γ acts absolutely irreducibly, and distinctly, on each of the U_i. Note that a subspace $U \subset V$ is Γ-irreducible if it is Γ-invariant (i.e $\gamma \cdot u \in U,\ \forall\, \gamma \in \Gamma,\ u \in U$) with the only Γ-invariant subspaces being $\{0_U\}$ and U. So, we always assume that:

(H1): The characteristic equation (5.8) has a pair of purely imaginary roots at $\pm i\alpha_0$, each of multiplicity m, and no other root belongs to $i\alpha_0\mathbb{Z}$.

In studying the bifurcation problem we wish to consider how the eigenvalues of A_λ cross the imaginary axis and to describe the structure of the associated eigenspace $\mathcal{E}_\lambda(\alpha)$. We consider the following nontrivial restrictions on the corresponding imaginary eigenspace of A_0.

(H2): $\mathcal{E}_0(\pm i\alpha_0)$ is Γ-simple. This means that

- the eigenspace $\mathcal{E}_0 = \mathcal{E}_1 \bigoplus \mathcal{E}_2$, where $\mathcal{E}_j$, for $j = 1, 2$ is absolutely irreducible for Γ by which the only linear mappings that commute with Γ on the eigenspace are scalar multiples of the identity, which may occur in problems involving orthogonal groups or their subgroups which contain reflections

  or

- the eigenspace $\mathcal{E}_0$ is irreducible but not absolutely irreducible for Γ, which may arise when the group Γ contains only rotations, for example $\Gamma = \mathbb{SO}(2)$.

Thus, the linear structure around $(0, \pm i\alpha_0)$ is given by the following:

Proposition 5.4. *Under conditions (H1, H2), for sufficiently small λ, the infinitesimal generator A_λ has one pair of complex conjugate eigenvalues $\hat{\alpha}_\pm(\lambda) = r(\lambda) \pm is(\lambda)$, each of multiplicity m. Moreover, r and s are smooth functions of λ and satisfy that $r(0) = 0$ and $s(0) = \alpha_0$. The corresponding right and left eigenvectors are smooth functions $\underline{a}(\lambda)$, respectively $\underline{b}(\lambda)$, of λ satisfying*

$$\Delta_{A_\lambda}(\hat{\alpha}(\lambda))\,\underline{a}(\lambda) = 0, \tag{5.22}$$

$$\Delta_{A_\lambda}(\hat{\alpha}(\lambda))^T \underline{b}(\lambda) = 0. \tag{5.23}$$

Proof. We make use of the IFT and Lemma 1.5 in Page 265 of [28] (matrices with real entries) refer to Hale et al. [32] and Wu [67] (FDEs) to obtain the following results about the multiplicity of this eigenvalue and its associated eigenvectors of A_λ.

Consider the characteristic matrix

$$\Delta(\lambda, \alpha)\underline{a}(\lambda) = \alpha I - L(\lambda)(e^{\alpha \cdot} I). \tag{5.24}$$

For Hopf bifurcation, the relevant action is that of $\Gamma \times S^1$, where $\nu \in S^1$ acts by multiplication by $e^{i\nu}$. A subgroup $\Sigma \subset \Gamma$ is called axial if it is an isotropy subgroup having a one-dimensional fixed-point subspace. We call a subgroup $\Sigma \subset \Gamma \times S^1$ **C**-axial if it is an isotropy subgroup that has a two-dimensional fixed-point subspace.

If Γ acts absolutely irreducibly, then the Jacobian will be a real-valued multiple of the identity (and diagonal) i.e. $L(\lambda) = c(\lambda)I$ and $L(\lambda)$ will possess only real eigenvalues. In this case $\Gamma \cong SO(2)$, possesses only rotations. Therefore, for Hopf bifurcation to occur, we require one or other of the conditions of proposition (5.4) should hold.

We note that $\Delta(\lambda, \alpha)$ is a matrix with nonlinear entries and we employ the Lyapunov-Schmidt reduction, reducing the problem to mappings of the kernel of the linearisation. In the first step of the reduction, we consider the linear part $\Delta(\lambda_0, i\omega_0)a(\lambda)$ with $\ker \Delta(\lambda_0, i\omega_0)$ being the generalised eigenspace with eigenvalues of the form $\pm i\omega$.

In the second step, we consider the splittings

$$C^1_{2\pi} = \ker \Delta(\lambda_0, i\omega_0) \oplus (\ker\Delta(\lambda_0, i\omega_0))^\perp \quad \text{and} \quad C_{2\pi} = (\mathrm{ran}\Delta(\lambda_0, i\omega_0))^\perp \oplus \mathrm{ran}\Delta(\lambda_0, i\omega_0). \tag{5.25}$$

We therefore decompose $\mathbb{C}^n$ into

$$\mathbb{C}^n = \ker \Delta(\lambda_0, i\omega_0) \oplus \mathrm{ran}\Delta(\lambda_0, i\omega_0) \tag{5.26}$$

with associated projection operators P and $(I - P)$ onto the kernel and range, which commute with the action of Γ since $\ker \Delta(\lambda_0, i\omega_0)$ is Γ-invariant. We rewrite $\Delta_{A_\lambda}(\hat{\alpha}(\lambda))\, a(\lambda) = 0$ as

$$\begin{aligned} \Delta(\lambda_0, i\omega_0)\nu &= (I - P)[\Delta(\lambda_0, i\omega_0) - \Delta(\alpha, \lambda)](a_0 + \nu) \\ P[\Delta(\lambda_0, i\omega_0) - \Delta(\alpha, \lambda)](a_0 + \nu) &= 0 \end{aligned} \tag{5.27}$$

with $\alpha(\lambda_0) = \pm i\omega_0$, $a(\lambda_0) = a_0$ and where $a = a_0 + \nu$ is the unique decomposition such that

$$a_0 \in \ker \Delta(\lambda_0, i\omega_0), \quad \nu \in \mathrm{ran}\Delta(\lambda_0, i\omega_0).$$

Note that $i\omega$ has multiplicity m where $2m = \dim \ker \Delta(\alpha, \lambda)$ since the eigenvalues come in pairs. Also, $\Delta(\alpha, \lambda)a_0 = 0$ whilst $\Delta(\alpha, \lambda)\nu \neq 0$.

From the first equation of (5.27), we see that

$$\begin{aligned} \Delta(\lambda_0, i\omega_0)\nu &= (I - P)[\Delta(\lambda_0, i\omega_0) - \Delta(\alpha, \lambda)](a_0 + \nu) \\ &= (I - P)[\Delta(\lambda_0, i\omega_0)(a_0 + \nu)] - (I - P)[\Delta(\alpha, \lambda)(a_0 + \nu)] \end{aligned} \tag{5.28}$$

is true *if and only if*

$$(I-P)[\Delta(\alpha,\lambda)(a_0+\nu)]=0. \tag{5.29}$$

Similarly, the second equation of (5.27) is true *if and only if*

$$P[\Delta(\alpha,\lambda)(a_0+\nu)]=0. \tag{5.30}$$

We seek solutions for all λ close to λ_0. We define

$$\Phi(\alpha,\lambda,a_0,\nu):=\Delta(\lambda_0,i\omega_0)\nu-(I-P)[\Delta(\lambda_0,i\omega_0)-\Delta(\alpha,\lambda)](a_0+\nu)=0. \tag{5.31}$$

Applying the Implicit Function Theorem, we obtain

$$\begin{aligned}\frac{\partial\Phi}{\partial\nu} &= \Delta(\lambda_0,i\omega_0)-(I-P)[\Delta(\lambda_0,i\omega_0)-\Delta(\alpha,\lambda)], \quad \text{evaluated at } (\lambda_0,i\omega_0)\\ &= \Delta(\lambda_0,i\omega_0),\end{aligned} \tag{5.32}$$

which is non-singular on the range, hence we can solve for ν by

$$g(\alpha,\lambda,a_0)=G(\alpha,\lambda)a_0=\nu \tag{5.33}$$

where $G(\alpha,\lambda)$ is a continuously differentiable $n\times n$ matrix which commutes with the action of Γ on $\ker\Delta(\lambda_0,i\omega_0)$ and is such that $G(\lambda_0,i\omega_0)=0$. The existence of an eigenvalue α near $i\omega_0$ for the parameter λ near λ_0 is equivalent to the existence of a solution to the following equation

$$\Psi(\alpha,\lambda)a_0:=P[\Delta(\lambda_0,i\omega_0)-\Delta(\alpha,\lambda)][I_m+G(\alpha,\lambda)]a_0 \tag{5.34}$$

which is obtained by substituting $G(\alpha,\lambda)$ into the second equation of (5.27). By appropriately choosing a basis for $\mathbb{C}^n$, we may write

$$\Delta(\alpha,\lambda)=\begin{bmatrix}\Delta_{11}(\alpha,\lambda) & \Delta_{12}(\alpha,\lambda)\\ \Delta_{21}(\alpha,\lambda) & \Delta_{22}(\alpha,\lambda)\end{bmatrix} \tag{5.35}$$

where $\Delta_{11}(\alpha,\lambda)$ is an $n\times n$ matrix, $\Delta_{22}(\alpha,\lambda)$ of order $(n-m)\times(n-m)$ with

$$\Delta(\lambda_0,i\omega_0)=\begin{bmatrix}0 & 0\\ 0 & \Delta_{22}(\lambda_0,i\omega_0)\end{bmatrix}, \quad \text{with } \det\Delta_{22}(\lambda_0,i\omega_0)\neq 0.$$

We also have the projection matrix on the kernel as

$$P=\begin{bmatrix}I_m & 0\\ 0 & 0\end{bmatrix}$$

It follows that

$$
\begin{aligned}
\Psi(\alpha,\lambda)a_0 &= \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} \left[\begin{bmatrix} 0 & 0 \\ 0 & \Delta_{22}(\lambda_0, i\omega_0) \end{bmatrix} - \begin{bmatrix} \Delta_{11}(\alpha,\lambda) & \Delta_{12}(\alpha,\lambda) \\ \Delta_{21}(\alpha,\lambda) & \Delta_{22}(\alpha,\lambda) \end{bmatrix} \right] \begin{bmatrix} a_0 \\ G(\alpha,\lambda)a_0 \end{bmatrix} \\
&= -\begin{bmatrix} \Delta_{11}(\alpha,\lambda) & \Delta_{12}(\alpha,\lambda) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ G(\alpha,\lambda)a_0 \end{bmatrix} \\
&= -\Delta_{11}(\alpha,\lambda)a_0 - \Delta_{12}(\alpha,\lambda)G(\alpha,\lambda)a_0. \qquad (5.36)
\end{aligned}
$$

Furthermore, we substitute ν by $G(\alpha,\lambda)a_0$ in the first equation of (5.27) to obtain

$$
\Delta(\lambda_0, i\omega_0)G(\alpha,\lambda)a_0 = (I-P)[\Delta(\lambda_0, i\omega_0) - \Delta(\alpha,\lambda)][I_m + G(\alpha,\lambda)]a_0,
$$

that is,

$$
\begin{aligned}
\begin{bmatrix} 0 & 0 \\ 0 & \Delta_{22}(\lambda_0, i\omega_0) \end{bmatrix} G(\alpha,\lambda)a_0 &= (I-P) \left[\begin{bmatrix} 0 & 0 \\ 0 & \Delta_{22}(\lambda_0, i\omega_0) \end{bmatrix} - \begin{bmatrix} \Delta_{11}(\alpha,\lambda) & \Delta_{12}(\alpha,\lambda) \\ \Delta_{21}(\alpha,\lambda) & \Delta_{22}(\alpha,\lambda) \end{bmatrix} \right] \begin{bmatrix} a_0 \\ G(\alpha,\lambda)a_0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \left[\begin{bmatrix} 0 & 0 \\ 0 & \Delta_{22}(\lambda_0, i\omega_0) \end{bmatrix} - \begin{bmatrix} \Delta_{11}(\alpha,\lambda) & \Delta_{12}(\alpha,\lambda) \\ \Delta_{21}(\alpha,\lambda) & \Delta_{22}(\alpha,\lambda) \end{bmatrix} \right] \begin{bmatrix} a_0 \\ G(\alpha,\lambda)a_0 \end{bmatrix} \\
&= \left[\begin{bmatrix} 0 & 0 \\ 0 & \Delta_{22}(\lambda_0, i\omega_0) \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \Delta_{21}(\alpha,\lambda) & \Delta_{22}(\alpha,\lambda) \end{bmatrix} \right] \begin{bmatrix} a_0 \\ G(\alpha,\lambda)a_0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ -\Delta_{21}(\alpha,\lambda) & \Delta_{22}(\lambda_0, i\omega_0) - \Delta_{22}(\alpha,\lambda) \end{bmatrix} \begin{bmatrix} a_0 \\ G(\alpha,\lambda)a_0 \end{bmatrix} \qquad (5.37)
\end{aligned}
$$

which upon simplifying gives

$$
\begin{bmatrix} 0 \\ \Delta_{22}(\lambda_0, i\omega_0)G(\alpha,\lambda)a_0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\Delta_{21}(\alpha,\lambda)a_0 + (\Delta_{22}(\lambda_0, i\omega_0) - \Delta_{22}(\alpha,\lambda))G(\alpha,\lambda)a_0 \end{bmatrix} \qquad (5.38)
$$

yielding

$$
\Delta_{21}(\alpha,\lambda) = -\Delta_{22}(\alpha,\lambda)G(\alpha,\lambda) \qquad (5.39)
$$

and upon substituting into $\Delta(\alpha,\lambda)$, we have

$$
\begin{aligned}
\Delta(\alpha,\lambda) &= \begin{bmatrix} \Delta_{11}(\alpha,\lambda) & \Delta_{12}(\alpha,\lambda) \\ -\Delta_{22}(\alpha,\lambda)G(\alpha,\lambda) & \Delta_{22}(\alpha,\lambda) \end{bmatrix} \\
&= \begin{bmatrix} I_m & 0 \\ 0 & \Delta_{22}(\alpha,\lambda) \end{bmatrix} \begin{bmatrix} \Delta_{11}(\alpha,\lambda) & \Delta_{12}(\alpha,\lambda) \\ -G(\alpha,\lambda) & I_m \end{bmatrix} \qquad (5.40)
\end{aligned}
$$

Therefore, from (7.9) it follows that

$$\det \Delta(\alpha,\lambda) = (-1)^m \det \Delta_{22}(\alpha,\lambda) \det \Psi(\alpha,\lambda). \tag{5.41}$$

We recall that that $\dim \mathcal{E} = 2m$. We also have that

$$\begin{cases} \frac{\partial^k}{\partial \alpha^k} \det \Delta(\alpha,\lambda_0)|_{\alpha=i\omega_0} & = \quad 0 \quad \text{for} \quad 0 \le k \le m-1 \\ \frac{\partial^m}{\partial \alpha^m} \det \Delta(\alpha,\lambda_0)|_{\alpha=i\omega_0} & \neq 0 \end{cases} \tag{5.42}$$

Since $\det \Delta_{22}(\lambda_0, i\omega_0) \neq 0$, we see from (5.42) that

$$\begin{cases} \frac{\partial^k}{\partial \alpha^k} \det \Psi(\alpha,\lambda) & = \quad 0 \quad \text{for} \quad 0 \le k \le m-1 \\ \frac{\partial^m}{\partial \alpha^m} \det \Psi(\alpha,\lambda) & \neq 0 \end{cases} \tag{5.43}$$

Further, under the assumption (H3) on the existence of an m-dimensional absolutely irreducible Γ-representation, we can assume that

$$\Psi(\alpha,\lambda) \quad = \quad \begin{bmatrix} \psi_{11}(\alpha,\lambda) & \psi_{12}(\alpha,\lambda) \\ \psi_{21}(\alpha,\lambda) & \psi_{22}(\alpha,\lambda) \end{bmatrix} \tag{5.44}$$

for some 2×2 matrices with real entries. The Γ-equivariance of $G(\alpha,\lambda)$ implies that $\Psi(\alpha,\lambda)$ commutes with the diagonal action of Γ on $\mathcal{E} \oplus \mathcal{E}$ and hence ψ_{ij} commutes with the action of Γ on $\mathcal{E}$. By the absolute irreducibility of $\mathcal{E}$, we have

$$\psi_{ij}(\alpha,\lambda) = \Psi_{ij}(\alpha,\lambda) I_m \tag{5.45}$$

for some scalar functions $\Psi_{ij}(\alpha,\lambda)$. Hence

$$\det \Psi(\alpha,\lambda) = p^m(\alpha,\lambda)$$

where

$$p(\alpha,\lambda) = \Psi_{11}(\alpha,\lambda)\Psi_{22}(\alpha,\lambda) - \Psi_{12}(\alpha,\lambda)\Psi_{21}(\alpha,\lambda) \tag{5.46}$$

By (5.43) we obtain

$$p(\alpha,\lambda) = 0, \quad \text{and} \quad \frac{\partial}{\partial \alpha} p(\lambda_0,\alpha)|_{\alpha=i\omega_0} \neq 0. \tag{5.47}$$

Therefore, by the implicit function theorem it follows that there exists a continuous function $\alpha(\lambda)$ such that $\alpha(\lambda_0) = i\omega_0$ and $p(\alpha,\lambda) = 0$. Hence $\alpha(\lambda)$ is an eigenvalue of A_λ with multiplicity $2m$

and corresponding eigenvector

$$a(\lambda) = [I + G(\alpha(\lambda), \lambda))]a, \quad a \in \ker \Delta(\lambda_0, i\omega_0). \tag{5.48}$$

Furthermore, if Γ is a reversing symmetry, then

$$\Delta(\alpha, \lambda)\gamma^{\sharp}\phi = \chi(\gamma)\gamma\Delta(\alpha, \lambda)\phi \tag{5.49}$$

Again, we may write

$$\Delta(\alpha, \lambda) = \begin{bmatrix} \Delta_{11}(\alpha, \lambda) & \Delta_{12}(\alpha, \lambda) \\ \Delta_{21}(\alpha, \lambda) & \Delta_{22}(\alpha, \lambda) \end{bmatrix}$$

in which each block is an $(n \times n)$ matrix. We take the reversor to be the $\mathbb{Z}_2$ representation,

$$\gamma = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

Thus (5.57) implies that

$$\begin{bmatrix} \Delta_{11}(\alpha, \lambda) & \Delta_{12}(\alpha, \lambda) \\ \Delta_{21}(\alpha, \lambda) & \Delta_{22}(\alpha, \lambda) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = - \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \Delta_{11}(\alpha, \lambda) & \Delta_{12}(\alpha, \lambda) \\ \Delta_{21}(\alpha, \lambda) & \Delta_{22}(\alpha, \lambda) \end{bmatrix} \tag{5.50}$$

Hence

$$\begin{bmatrix} \Delta_{11}(\alpha, \lambda) & -\Delta_{12}(\alpha, \lambda) \\ \Delta_{21}(\alpha, \lambda) & -\Delta_{22}(\alpha, \lambda) \end{bmatrix} = \begin{bmatrix} -\Delta_{11}(\alpha, \lambda) & -\Delta_{12}(\alpha, \lambda) \\ \Delta_{21}(\alpha, \lambda) & \Delta_{22}(\alpha, \lambda) \end{bmatrix}$$

which yields for reversibility, the matrix representation

$$\Delta(\alpha, \lambda) = \begin{bmatrix} 0 & \Delta_{12}(\alpha, \lambda) \\ \Delta_{21}(\alpha, \lambda) & 0 \end{bmatrix} \tag{5.51}$$

Therefore, the system (5.3) with infinitesimal generator A_λ possessing complex eigenvalues with high multiplicity satisfying the characteristic equations (5.22) and (5.22) has a bifurcation of periodic solutions whose spatio-temporal symmetry can be characterised by the group Γ. Furthermore, we see that the system is $\mathbb{Z}_2$ reversible when the representation (5.51) is obtained. $\square$

5.2 Hopf Bifurcation on Loop Spaces

The purely imaginary eigenvalues of A_0 have high multiplicity, so the standard Hopf bifurcation theorem cannot be applied directly. So, we first develop the equivariant Lyapunov-Schmidt reduction for (5.3) to consider the existence of periodic solutions. Let $\omega_0 = 2\pi/\alpha_0$ and C_{ω_0} (respectively, $C^1_{\omega_0}$) be the set of continuous (respectively, differentiable) n-dimensional ω_0-periodic mappings with range in $\mathbb{K}^n$. If we denote

$$\|u\|_{\infty,0} = \max\{\ |u_i(\theta)|\ :\ 1 \le i \le n,\ \theta \in [0, \omega_0]\ \}$$

for $u = (u_1, \ldots, u_n) \in C_{\omega_0}$ and $\|u\|_{\infty,1} = \max\{\|u\|_{\infty,0}, \|\dot{u}\|_{\infty,0}\}$ for $u \in C^1_{\omega_0}$, then C_{ω_0} and $C^1_{\omega_0}$ are Banach spaces when they are endowed with the norms $\|\ \|_{\infty,0}$ and $\|\ \|_{\infty,1}$, respectively. It is easy to see that they are Banach representations of the group $\Gamma \times S^1$ with the action given by

$$(\gamma, \xi)\, u(t) = \gamma u(t + \xi/\alpha_0), \quad (\gamma, \theta) \in \Gamma \times S^1.$$

We introduce the inner product for $u, v \in C_{\omega_0}$

$$\langle u, v \rangle = \frac{1}{\omega_0} \int_0^{\omega_0} \overline{u(t)} \cdot v(t)\, dt.$$

Let $\beta \in (-1, 1)$, define $u(t) = x(t/(1+\beta))$. By varying the newly introduced small variable β, one keeps track not only of solutions of (5.3) with period ω_0 but also of solutions with nearby period because x is of period $\omega_0/(1+\beta)$.

Lemma 5.5. *For $u \in C_{\omega_0}$ and $\beta \in (-1, 1)$, let*

$$u_{t,\beta}(\theta) = u(t + (1+\beta)\theta) \quad \theta \in [\theta_{\min}, \theta_{\max}].$$

1. *The equation (5.3) can be rewritten as*

$$F(u, \lambda, \beta) = -(1+\beta)\dot{u}(t) + L_\lambda u_{t,\beta} + f(\lambda, u_{t,\beta}) = 0 \tag{5.52}$$

   where $F : C^1_{\omega_0} \times \mathbb{R}^{k+1} \to C_{\omega_0}$. As a matter of fact, solutions of (5.52) correspond to $\omega_0/(1+\beta)$-periodic solutions of (5.3).

2. *The function F is $\Gamma \times S^1$-equivariant (and reversible if necessary) with its linear part $D_u F(0, \lambda, \beta)$ written as $\mathcal{L}(\lambda, \beta)$.*

3. (a) *The adjoint $L^*(\lambda)u_t$ of $L(\lambda)u_t$ with respect to the scalar product $\langle \cdot, \cdot \rangle$ is given by*

$$L^*(\lambda)u_t = \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda, \theta)^T u_t(-\theta) = \int_{-\theta_{\max}}^{-\theta_{\min}} d\eta(\lambda, -\theta)^T u_t(\theta). \tag{5.53}$$

   (b) *As a corollary, the adjoint $\mathcal{L}^*$ of $\mathcal{L}$ with respect to the scalar product $\langle \cdot, \cdot \rangle$ is given by*

$$\mathcal{L}^*(\lambda, \beta)u = -(1+\beta)\dot{u} + L^*(\lambda)u_{t,\beta}. \tag{5.54}$$

4. There exist an $R > 0$ such that, for any $u \in C^1_{\omega_0}$,

$$\mathcal{L}(\lambda, \beta)u = -(1+\beta)\dot{u} + \int_{-R}^{0} d\tilde{\eta}(\lambda, \theta)\, u_{t,\beta}(\theta),$$

where

$$\begin{aligned}\int_{-R}^{0} d\tilde{\eta}(\lambda, \theta)\, u_t(\theta) &= \int_{\theta_{\min}}^{0} d\eta(\lambda, \theta)\, u_t(\theta) + \big(\eta(\lambda, 0^+) - \eta(\lambda, 0^-)\big)\, u_t(0) \\ &\quad + \int_{-R}^{-R+\theta_{\max}} d\eta(\lambda, R+\theta)\, u_t(\theta)\end{aligned}$$

5. ([29]) The operator $\mathcal{L}_0 u = -\dot{u} + L(0)u_{t,\beta}$ is the linearisation of F at the origin. The spaces $\ker\mathcal{L}_0$, $\mathrm{ran}\mathcal{L}_0$ and $\mathcal{D} = (\ker\mathcal{L}_0)^\perp \cap C^1_{\omega_0}$ are $\Gamma \times S^1$-equivariant subspaces of C_{ω_0}. Moreover, $C_{\omega_0} = \ker\mathcal{L}_0 \oplus \mathrm{ran}\mathcal{L}_0$ and $C^1_{\omega_0} = \ker\mathcal{L}_0 \oplus \mathcal{D}$.

Proof. 1. Note that if $u(t) = x(t/(1+\beta))$, then $(1+\beta)\dot{u}(t) = \dot{x}(t/(1+\beta))$. Then, if $\xi = t/(1+\beta)$, for $\theta \in [\theta_{\min}, \theta_{\max}]$,

$$x_\xi(\theta) = x(\xi + \theta) = x(\frac{t}{(1+\beta)} + \theta) = x(\frac{t + (1+\beta)\theta}{(1+\beta)}) = u(t + (1+\beta)\theta) = u_t((1+\beta)\theta).$$

To check that ω_0-periodic functions are sent to ω_0-periodic functions by F. In particular, for $\theta \in [\theta_{\min}, \theta_{\max}]$,

$$u_{t+\omega_0,\beta}(\theta) = u(t + \omega_0 + (1+\beta)\theta) = u(t + (1+\beta)\theta) = u_{t,\beta}(\theta).$$

Moreover,

$$\begin{aligned}(L(\lambda)u)(t+\omega_0) &= \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda, \theta)\, u_{t+\omega_0}(\theta) \\ &= \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda, \theta)\, u_t(\theta) = (L(\lambda)u)(t).\end{aligned}$$

Therefore, $\mathcal{L}u$ is ω_0-periodic if $u \in C^1_{\omega_0}$.

2. Equivariance and Reversibility:

Define

$$L(\lambda, \beta) = \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda, \theta)u(\theta) \tag{5.55}$$

For γ-equivariance, we require that

$$d\eta(\lambda, \theta)\gamma = \gamma d\eta(\lambda, \theta) \tag{5.56}$$

We need to show that

$$\mathcal{L}(\lambda)\gamma u = \gamma\mathcal{L}(\lambda)u \tag{5.57}$$

$$\begin{aligned} L(\lambda)\gamma u &= \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda,\theta)(\gamma u)\theta \\ &= \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda,\theta)\gamma u(\theta) \\ &= \gamma\int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda,\theta)u(\theta) \\ &= \gamma L(\lambda)u \end{aligned} \tag{5.58}$$

For reversibility, we require that

$$d\eta(\lambda,\chi(\gamma)\theta)\gamma = \chi(\gamma)\gamma d\eta(\lambda,\theta) \tag{5.59}$$

We need to show that

$$L(\lambda)\gamma^{\sharp}u = \chi(\gamma)\gamma L(\lambda)u \tag{5.60}$$

From (5.60)

$$\begin{aligned} L(\lambda)\gamma^{\sharp}u &= \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda,\theta)(\gamma^{\sharp}u)\theta \\ &= \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda,\theta)\gamma u(\chi(\gamma)\theta) \\ &= \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda,\theta)\chi(\gamma)\gamma u(\theta) \\ &= \chi(\gamma)\int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda,\theta)\gamma u(\theta). \end{aligned} \tag{5.61}$$

Also, (5.60) gives

$$\chi(\gamma)\gamma L(\lambda)u = \chi(\gamma)\gamma\int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda,\theta)u(\theta) \tag{5.62}$$

Since (5.60) is satisfied, reversibility follows.

3. (a) Given u, v in $C^1_{\omega_0}$, we would like to show that $\langle v, \mathcal{L}u\rangle = \langle \mathcal{L}^* v, u\rangle$. And so,

$$\begin{aligned}
\omega_0\langle v, Lu_t\rangle &= \int_0^{\omega_0} \overline{v(t)} \cdot Lu_t dt \\
&= \int_0^{\omega_0}\int_{\theta_{\min}}^{\theta_{\max}} \Big(\overline{v(s)} \cdot d\eta(\lambda,\theta)\, u_s(\theta)\Big)\, ds, \quad s+\theta = t, \\
&= \int_0^{\omega_0}\int_{\theta_{\min}}^{\theta_{\max}} \Big(d\eta(\lambda,\theta)^T\overline{v(t-\theta)} \cdot u(t)\Big)\, dt \\
&= \int_0^{\omega_0} \left(\int_{\theta_{\min}}^{\theta_{\max}} \overline{d\eta(\lambda,\theta)^T\, v_t(-\theta)}\right) \cdot u(t)\, dt \\
&= \int_0^{\omega_0} \overline{L^*(\lambda)v_t} \cdot u(t)\, dt \\
&= \langle L^*(\lambda)v_t, u\rangle.
\end{aligned}$$

(b) We need to deal with the $\dot{u}$ part of $\mathcal{L}$. We have

$$\begin{aligned}
\langle v, -\dot{u}\rangle &= \int_0^{\omega_0} -\overline{v(t)} \cdot \dot{u}(t)\, dt \\
&= \Big[\overline{v(t)} \cdot -u(t)\Big]_0^{\omega_0} + \int_0^{\omega_0} \overline{\dot{v}(t)} \cdot u(t)\, dt \\
&= \langle dv, u\rangle.
\end{aligned}$$

4. Let $K = \min\{\ k \in \mathbb{N}\ :\ k\omega_0 > (\theta_{\max} - \theta_{\min})\ \}$, and define $R = K\omega_0$. then,

$$\begin{aligned}
\mathcal{L}(\lambda,\beta)u &= -(1+\beta)\dot{u} + \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda,\theta)\, u_{t,\beta}(\theta) \\
&= -(1+\beta)\dot{u} + \int_{\theta_{\min}}^{0} d\eta(\lambda,\theta)\, u_{t,\beta}(\theta) + \int_0^{\theta_{\max}} d\eta(\lambda,\theta)\, u_{t,\beta}(\theta) \\
&\quad + \big(\eta(\lambda,0^+) - \eta(\lambda,0^-)\big)\, u_{t,\beta}(0) \\
&= -(1+\beta)\dot{u} + \int_{\theta_{\min}}^{0} d\eta(\lambda,\theta)\, u_{t,\beta}(\theta) + \big(\eta(\lambda,0^+) - \eta(\lambda,0^-)\big)\, u_{t,\beta}(0) \\
&\quad + \int_{-R}^{-R+\theta_{\max}} d\eta(\lambda, R+\theta)\, u_{t,\beta}(R+\theta) \\
&= -(1+\beta)\dot{u} + \int_{\theta_{\min}}^{0} d\eta(\lambda,\theta)\, u_{t,\beta}(\theta) + \big(\eta(\lambda,0^+) - \eta(\lambda,0^-)\big)\, u_{t,\beta}(0) \\
&\quad + \int_{-R}^{-R+\theta_{\max}} d\eta(\lambda, R+\theta)\, u_{t,\beta}(\theta) \\
&= -(1+\beta)\dot{u} + \int_{-R}^{0} d\tilde{\eta}(\lambda,\theta)\, u_{t,\beta}(\theta).
\end{aligned}$$

5. Obviously, the elements of $\ker\mathcal{L}_0$ correspond to solutions of the linear system $\dot{u} = L(0)u_t$

satisfying $u(t) = u(t+)$. Let $\mathcal{L}_0^*$ be the adjoint operator of $\mathcal{L}_0$, satisfying

$$\langle v, \mathcal{L}_0 u\rangle = \langle \mathcal{L}_0^* v, u\rangle, \quad \forall u, v \in C^1_{\omega_0}.$$

It follows from (H1) that $\ker\mathcal{L}_0 \cong \ker\Delta_{A_0}(\pm i\alpha_0)$ and $\ker\mathcal{L}_0^* = \ker\Delta^*_{A_0}(\pm i\alpha_0)$, both of which are $2m$-dimensional.

□

5.3 Lyapounov-Schmidt Reduction

Let P and $I-P$ denote the projection operators defined by $P : C_{\omega_0} \to \mathrm{ran}\mathcal{L}_0$ and $I - P : C_{\omega_0} \to \ker\mathcal{L}_0$. Obviously, P and $I-P$ are $\Gamma\times S^1$-equivariant. Thus $F(u,\lambda,\beta) = 0$ is equivalent to the following system:

$$PF(v+w,\lambda,\beta) = 0, \tag{5.63}$$

$$(I-P)F(v+w,\lambda,\beta) = 0. \tag{5.64}$$

Here we have written $u \in C_{\omega_0}$ in the form $u = v + w$, with $v = (I-P)u \in \ker\mathcal{L}_0$ and $w = Pu \in \mathcal{D}$. Near the critical point $(u,\lambda,\beta) = (0,0,0)$, the IFT implies that (5.63) can be solved for $w = W(v,\lambda,\beta)$, where $W : \ker\mathcal{L}_0 \times \mathbb{R}^{1+l} \to \mathcal{D}$ is a continuously differentiable S^1-equivariant map satisfying $W(0,0,0) = 0$. Substituting $w = W(v,\lambda,\beta)$ into (5.64), we have

$$B(v,\lambda,\beta) \equiv (I-P)F(v + W(v,\lambda,\beta),\lambda,\beta) = 0. \tag{5.65}$$

Thus, we reduce our Hopf bifurcation problem to the problem of finding zeros of the map $B : \ker\mathcal{L}_0 \times \mathbb{R}^{1+l} \to \ker\mathcal{L}_0$. We refer to B as the bifurcation map of the system (5.3). It follows from the $\Gamma\times S^1$-equivariance of F and W that the bifurcation map B is also $\Gamma\times S^1$-equivariant. Moreover, $B(0,0,0) = 0$ and $B_v(0,0,0) = 0$. Finding periodic solutions to (5.3) rests on prescribing in advance the symmetry of the solution we seek. This can often be used to select a subspace on which the eigenvalues are simple. In addition, we should take temporal phase-shift symmetries in terms of the circle group S^1 into account as well as spatial symmetries. Here, we place emphasis on two-dimensional fixed-point subspaces and assume that

(H3): $\dim\mathrm{Fix}(\Sigma, \mathcal{E}_0 \pm i\alpha_0) = 2$ for some subgroup $\Sigma \subset \Gamma\times S^1$.

(H4): $\frac{d}{d\lambda} r(0) \neq 0$, where $r(0)$ is the real part of the eigenvalue with the differentiation done with respect to the bifurcation parameter.

Assumption (H4) is the transversality condition analogous to those of the standard Hopf bifurcation theorem. Now, we can present our main results about equivariant Hopf bifurcation.

Theorem 5.6. *Under conditions (H1 − 4), in every neighbourhood of the origin, the system (5.63) has a bifurcation of periodic solutions whose spatio-temporal symmetry can be completely characterized by* Σ.

Proof. We consider the restriction mapping $\tilde{B} : \mathrm{Fix}(\Sigma, \ker \mathcal{L}_0) \times \mathbb{R}^{1+l} \to \ker \mathcal{L}_0$ of $B : \ker \mathcal{L}_0 \times \mathbb{R}^2 \to \ker \mathcal{L}_0$ on $\mathrm{Fix}(\Sigma, \ker \mathcal{L}_0) \times \mathbb{R}^2$, that is,

$$\tilde{B}(v, \lambda, \beta) = (I - P)F(v + W(v, \lambda, \beta), \lambda, \beta)$$

for $v \in \mathrm{Fix}(\Sigma, \ker \mathcal{L}_0)$, $\lambda \in \mathbb{R}$ and $\beta \in \mathbb{R}$. Clearly, $\tilde{B}$ is also $\Gamma \times S^1$-equivariant, and satisfies that $\tilde{B}(0, 0, 0) = 0$ and $D_v\tilde{B}(0, 0, 0) = 0$. Moreover, it is easy to see that $\mathrm{ran}\tilde{B} \subset \mathrm{Fix}(\Sigma, \ker \mathcal{L}_0)$. Namely, $\tilde{B}$ maps $\mathrm{Fix}(\Sigma, \ker \mathcal{L}_0) \times \mathbb{R}^2$ to $\mathrm{Fix}(\Sigma, \ker \mathcal{L}_0)$. Therefore, we only need to consider the existence of nontrivial zeroes of $\tilde{B}$. Without loss of generality, assume that $\mathrm{Fix}(\Sigma, \ker \mathcal{L}_0) = \mathrm{span}\{\phi, \bar{\phi}\}$ where $\phi(\theta) = e^{i\alpha_0\theta}\underline{a}(0)$. As stated in Proposition 5.4, for sufficiently small λ, there is a C^1-function $\underline{a}(\lambda) \in \mathbb{C}^n$ satisfying $\Delta_{A_\lambda}(\hat{\alpha})\underline{a}(\lambda) = 0$. We differentiate it with respect to λ at $\lambda = 0$ and obtain

$$\left[D_\lambda(\Delta_{A_0})(i\alpha_0) + \frac{d}{d\lambda}\hat{\alpha}(0)D_\alpha(\Delta_{A_0})(i\alpha_0)\right]\underline{a}(0) + \Delta_{A_0}(i\alpha_0)\frac{d}{d\lambda}\underline{a}(0) = 0. \tag{5.66}$$

In addition, there exists $\underline{b} \in \mathbb{C}^n$ such that $\overline{\underline{b}}^T\Delta_{A_0}(i\alpha_0) = 0$ and $\psi(\theta) = e^{i\alpha_0\theta}\underline{b} \in \mathrm{Fix}(\Sigma, \ker \mathcal{L}_0^*) = \mathrm{Fix}(\Sigma, \ker \mathcal{L}_0)^*$. Thus, multiplying both sides of (5.66) by $\overline{\underline{b}}^T$ gives us

$$\overline{\underline{b}}^T D_\lambda(\Delta_{A_0})(i\alpha_0)\underline{a}(0) + \frac{d}{d\lambda}\hat{\alpha}'(0)\overline{\underline{b}}^T D_\alpha(\Delta_{A_0})(i\alpha_0)\underline{a}(0) = 0. \tag{5.67}$$

In fact, we can normalize $\overline{\underline{b}}^T$ such that $\overline{\underline{b}}^T D_\alpha(\Delta_{A_0})(i\alpha_0)\underline{a}(0) = 1$. Thus, it follows from (5.67) that $\frac{d}{d\lambda}\hat{\alpha}(0) = -\overline{\underline{b}}^T D_\alpha(\Delta_{A_0})(i\alpha_0)\underline{a}(0)$. For each $\Phi \in \mathrm{Fix}(\Sigma, \ker \mathcal{L}_0)$, $\Phi = z\phi + \bar{z}\bar{\phi}$ where $z = \langle \psi, \Phi \rangle \in \mathbb{C}$. Let

$$g(z, \lambda, \beta) = \langle \psi, \tilde{B}(z\phi + \bar{z}\bar{\phi}, \lambda\beta) \rangle \tag{5.68}$$

This inner product is taken since $P\tilde{B}$ is orthogonal to ψ. Thus, we only need to consider the existence of nontrivial solutions to $g(z, \lambda, \beta) = 0$. It follows that $g_z(0, 0, 0) = g_{\bar{z}}(0, 0, 0) = 0$.

Proposition 5.7. *g is S^1-equivariant.*

Proof. Let $\rho \in S^1$, the circle group of phase shifts. We let $\rho \in S^1$ act via multiplication by $e^{i\nu}$ for $\nu \in [0, 2\pi]$ or alternatively by $(\nu.x)(t) = x(t - \nu)$. Given that $\tilde{B} : \mathrm{Fix}(\Sigma, \ker \mathcal{L}_0) \times \mathbb{R}^{1+l} \to \ker \mathcal{L}_0$ where $\mathrm{Fix}(\Sigma, \ker \mathcal{L}_0) = \mathrm{span}\{\phi, \bar{\phi}\}$ and where $\phi(\theta) = e^{i\alpha_0\theta}\underline{a}(0)$.

We have

$$
\begin{aligned}
g(\rho\cdot z,\lambda,\beta) &= \langle\psi, \tilde{B}(\rho\cdot z\phi+\rho\cdot\bar{z}\bar{\phi},\lambda\beta)\rangle \\
&= \langle\psi, e^{i\nu} z e^{i\alpha_0\theta}\underline{a}_1(0)+e^{i\nu}\bar{z}e^{-i\alpha_0\theta}\underline{a}_2(0)\rangle \\
&= \langle\psi, z e^{i(\nu+\alpha_0\theta)}\underline{a}_1(0)+\bar{z}e^{-i(\nu+\alpha_0\theta)}\underline{a}_2(0)\rangle \\
&= e^{i\nu}\langle\psi, z e^{i\alpha_0\theta}\underline{a}_1(0)+\bar{z}e^{-i\alpha_0\theta}\underline{a}_2(0)\rangle \\
&= \rho g(z,\lambda,\beta), \qquad (5.69)
\end{aligned}
$$

which follows from a change of variable, $s=\nu+\alpha_0\theta$. □

Using similar arguments to that in [28], we can find two functions $p, q:\mathbb{R}^3\to\mathbb{R}$ such that

$$
g(z,\lambda,\beta)=p(|z|^2,\lambda,\beta)\,z+q(|z|^2,\lambda,\beta)\,iz. \qquad (5.70)
$$

It follows from $g_z(0,0,0)=0$ that $p(0,0,0)=q(0,0,0)=0$. Then solving $g=0$ is equivalent to either $z=0$ or solving $p(r^2,\lambda,\beta)=q(r^2,\lambda,\beta)=0$. In view of the implicitly defined function $W(v,\lambda,\beta)$, which vanishes through first order in $v=z\phi+\bar{z}\bar{\phi}$, we have

$$
F(v+W(v,\lambda,\beta),\lambda,\beta)=-(1+\beta)\dot{v}(t)+L(\lambda)v_{t,\beta}+O(|z|^2).
$$

Therefore, with $v=\Phi$,

$$
\begin{aligned}
g_\lambda(z,0,0) &= \langle\psi, F_\lambda(\Phi,0,0)\rangle=\langle\psi,\dot{L}(0)\Phi_t\rangle+O(|z|^2) \\
&= \langle\psi,\dot{L}(0)\phi_t\rangle z+\langle\psi,\dot{L}(0)\bar{\phi}_t\rangle\bar{z}+O(|z|^2) \\
&= z\underline{\bar{b}}^T\dot{L}(0)(e^{i\alpha_0\theta}\underline{a}(0)+O(|z|^2)=z\hat{\alpha}'(0)+O(|z|^2).
\end{aligned}
$$

In addition,

$$
\begin{aligned}
g_\beta(z,0,0) &= \langle\psi, F_\beta(\Phi,0,0)\rangle=\langle\psi,-\dot{\Phi}(t)+i\alpha_0L(0)(\theta\Phi_t(\theta))\rangle+O(|z|^2) \\
&= \langle\psi,-i\alpha_0\phi(t)+i\alpha_0L(0)(\theta\phi_t(\theta))\rangle z+\langle\psi,-i\alpha_0\bar{\phi}(t)+i\alpha_0L(0)(\theta\bar{\phi}_t(\theta))\rangle\bar{z}+O(|z|^2) \\
&= -i\alpha_0 z+i\alpha_0L(0)(\theta e^{i\alpha_0\theta}\underline{a}(0))z+O(|z|^2)=-i\alpha_0 z+O(|z|^2).
\end{aligned}
$$

Therefore, $G_\lambda(0,0,0)=\frac{d}{d\lambda}\hat{\alpha}(0)$ and $G_\beta(0,0,0)=-\alpha_0$. So, the Jacobian determinant of the real and imaginary part of function g with respect to λ and β is

$$
\begin{vmatrix}
\mathrm{re}(g_\lambda(0,0,0)) & \mathrm{re}(g_\beta(0,0,0)) \\
\mathrm{im}(g_\lambda(0,0,0)) & \mathrm{im}(g_\beta(0,0,0))
\end{vmatrix}
=-\alpha_0\Re(\frac{d}{d\lambda}\hat{\alpha}(0))=-\alpha_0\frac{d}{d\lambda}r(0).
$$

Thus, under condition (H4), the above Jacobian determinant is nonzero. The IFT implies that there exists a unique function $\lambda = \bar{\lambda}(r^2)$ and $\beta = \bar{\beta}(r^2)$ satisfying $\lambda(\bar{0}) = \bar{\beta}(0) = 0$ such that

$$p(r^2, \bar{\lambda}(r^2), \bar{\beta}(r^2)) = q(r^2, \bar{\lambda}(r^2), \bar{\beta}(r^2)) = 0 \tag{5.71}$$

for all sufficient small r. Therefore, $g(z, \bar{\lambda}(|z|^2), \bar{\beta}(|z|^2)) = 0$ for z sufficiently near 0, and so the system (5.3) has a bifurcation of periodic solutions whose spatio-temporal symmetry can be completely characterized by Σ. □

Theorem 5.6 implies that a Hopf bifurcation for (5.3) occurs at $\lambda = 0$. In every neighbourhood of the origin there is a branch of Σ-symmetric periodic solutions $x(t, \lambda)$ with $x(t, \lambda) \to 0$ as $\lambda \to 0$. The period ω_λ of $x(t, \lambda)$ satisfies that $\lim_{\lambda \to 0} \omega_\lambda = \omega_0$. Moreover, Σ-equivariance implies that there are $|\Gamma \times S^1/\Sigma|$ different periodic solutions, which have isotopy subgroups conjugate to Σ in $\Gamma \times S^1$.

5.4 Bifurcation Direction

We recall the following basic notions in bifurcation theory:

Consider $\dot{x}(t) = f(\lambda, x_t)$. Assume that when $\lambda = \hat{\lambda}$ there is an equilibrium $\hat{x}$ for which

1. $\frac{\partial f(\hat{\lambda}, \hat{x})}{\partial x} = 0$, then $f(\lambda, x_t)$ has a stationary point with respect to x at $(\hat{\lambda}, \hat{x})$.

2. $\frac{\partial^2 f(\hat{\lambda}, \hat{x})}{\partial x^2} \neq 0$, then $f(\lambda, x_t)$ is an extremum.

3. $\frac{\partial f(\hat{\lambda}, \hat{x})}{\partial \lambda} \neq 0$, then $f(\lambda, x_t)$ has no stationary point with respect to λ at $(\hat{\lambda}, \hat{x})$.

4. $\frac{\partial^2 f(\hat{\lambda}, \hat{x})}{\partial \lambda \partial x} \neq 0$, then varying λ shifts the phase curve.

The direction of bifurcation is determined by the sign of the second order derivatives.

In what follows, we consider the bifurcation direction.

Theorem 5.8. *In addition to conditions ($H1-4$), assume that $\mathcal{L}(\alpha)$ and $f(\alpha, \cdot)$ are sufficiently smooth. Then there exists a branch of Σ-symmetric periodic solutions, parameterized by α, bifurcating from the trivial solution $x = 0$ of (5.3). Moreover,*

1. $\Re(\frac{d}{d\lambda}\hat{\alpha}(0))\Re(g_{21})$ *determines the direction of the bifurcation: the bifurcation is supercritical (respectively, subcritical), i.e. the bifurcating periodic solutions exist for $\lambda > 0$ (respectively, $\lambda < 0$), if $\Re(\frac{d}{d\lambda}\hat{\alpha}(0)\Re(g_{21}) < 0$ (respectively, > 0), and*

2. $\Re(\frac{d}{d\lambda}\hat{\alpha}(0))\Im(\frac{d}{d\lambda}\hat{\alpha}(0)g_{21})$ *determines the period of the bifurcating periodic solutions along the branch: the period is greater than (respectively, smaller than) ω_0 if it is positive (respectively, negative).*

Proof. Assuming sufficient smoothness of f, we write

$$f(0,v)=\frac{1}{2}A(v,v)+\frac{1}{6}B(v,v,v)+o(\|v\|^3),$$

where $A=D_{xx}f(0,0)$ and $B=D_{xxx}f(0,0)$. Write $W(zq+\bar{z}\bar{q},0,0)$ and $g(z,0,0)$ as

$$W(z\phi+\bar{z}\bar{\phi},0,0)=\sum_{s+l\geq 2}\frac{1}{s!l!}W_{sl}z^s\bar{z}^l,\quad g(z,0,0)=\sum_{s+l\geq 2}\frac{1}{s!l!}g_{sl}z^s\bar{z}^l.$$

It follows from (5.70) that $g_{21}=p_u(0,0,0)+iq_u(0,0,0)$. From (5.71), we can calculate the derivatives of $\bar{\lambda}(r^2)$ and $\bar{\beta}(r^2)$ and evaluate at $r=0$:

$$\frac{d}{d\lambda}\bar{\lambda}(0)=-\frac{\Re(g_{21})}{\Re(\frac{d}{d\lambda}\alpha(0))},\quad \frac{d}{d\lambda}\bar{\beta}(0)=-\frac{\Im(\frac{d}{d\lambda}\alpha(0)\overline{g_{21}})}{\Re(\frac{d}{d\lambda}\alpha(0))}.$$

The bifurcation direction is determined by the sign of $\frac{d}{d\lambda}\bar{\lambda}(0)$, and the monotonicity of the period of bifurcating closed invariant curve depends on the sign $\bar{\beta}'(0)$.

Using a similar argument as that in [27], we have

$$g_{21}=\langle\psi,B(\phi,\phi,\bar{\phi})\rangle+2\langle\psi,A(\phi,W_{11})\rangle+\langle\psi,A(\phi,W_{20})\rangle.$$

We still need to compute W_{11} and W_{20}. In fact, it follows that

$$W_{20}=-\mathcal{L}_0^{-1}PA(\phi,\phi),\quad W_{11}=-\mathcal{L}_0^{-1}PA(\phi,\bar{\phi}).$$

In order to evaluate function W_{20}, we must solve the following differential equations

$$\dot{W}_{20}-L(0)W_{20}=PA(\phi,\phi). \tag{5.72}$$

Note that $A(\phi,\phi)=A(e^{i\alpha_0(\cdot)}\underline{a}(0),e^{i\alpha_0(\cdot)}\underline{a}(0))e^{2i\alpha_0 t}$. So, $g_{20}=\langle\psi,A(\phi,\phi)\rangle=0$. Namely, $A(\phi,\phi)\in\mathrm{ran}\mathcal{L}_0$. Hence, the projection P acts on $A(\phi,\phi)$ as the identity, and (5.72) is an inhomogeneous differential equation with constant coefficients. Thus, there is a particular solution of (5.72) of the form $W_{20}^*=D_2e^{2i\alpha_0\theta}$. Substituting W_{20}^* into (5.72) and comparing the coefficients, we obtain

$$D_2=\Delta^{-1}(0,2i\alpha_0)A(Ae^{i\alpha_0(\cdot)}\underline{a}(0),e^{i\alpha_0(\cdot)}\underline{a}(0)). \tag{5.73}$$

In addition, W_{20}^* is orthogonal to ψ, so it belongs to $\mathrm{ran}\mathcal{L}_0$. Thus $W_{20}(0,0,0)$ is equal to W_{20}^* with D_2 determined by (5.73). Similarly, we have $W_{02}=\overline{D}_2e^{-2i\alpha_0 t}$ and $W_{11}=D_0$ where

$$D_0=\Delta_{A_0}^{-1}(0)A(e^{i\alpha_0(\cdot)}\underline{a}(0),e^{-i\beta_0(\cdot)}\overline{\underline{a}(0)}).$$

Therefore,

$$g_{21} = \underline{b}^T B(e^{i\alpha_0(\cdot)}\underline{a}(0), e^{i\alpha_0(\cdot)}\underline{a}(0), e^{-i\alpha_0(\cdot)}\overline{\underline{a}(0)}) + 2\underline{b}^T A(e^{i\alpha_0(\cdot)}\underline{a}(0), D_0) + \underline{b}^T A(e^{-i\alpha_0(\cdot)}\overline{\underline{a}(0)}, D_2).$$

□

5.5 One Dimensional Lyapounov-Schmidt Reduction for Neutral MFDEs

At a point where the linear equation has m eigenvalues with zero real parts and all other eigenvalues having negative real parts, we split the state space thus: $X = X_0 \oplus X_s \oplus X_u$, where X_0 is an $m-$dimensional subspace spanned by solutions corresponding to the m purely imaginary eigenvalues, the stable subspace X_s and the unstable subspace X_u.

The Hopf bifurcation requires at least a $2-$dimensional system and an MFDE is infinite dimensional. Recall the general parameterised autonomous NMFDE

$$\frac{d}{dt}h(\lambda, x_t) = f(\lambda, x_t), \tag{5.74}$$

where $h, f : \mathbb{R} \times C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ are two continuously differentiable mappings which satisfy $f(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Recall that when a zero set (equilibrium) $(\hat{\lambda}, \hat{x})$ varies, we have a bifurcation. Introducing a perturbation $(\hat{\lambda} + \xi, \hat{x} + z)$ of $(\hat{\lambda}, \hat{x})$ which we may take to equal $(0, 0)$, and substituting into (5.74), we obtain

$$\frac{d}{dt}h(\hat{\lambda} + \xi, \hat{x}_t + z_t) = f(\hat{\lambda} + \xi, \hat{x}_t + z_t). \tag{5.75}$$

If $\hat{x}$ is a steady state, we obtain an autonomous equation of the form (5.74) otherwise we obtain a non-autonomous equation. Therefore, depending on the properties of $\hat{x}$, we may have a problem that is autonomous, non-autonomous or periodic.

Let $D(\lambda), L(\lambda) : C([\theta_{\min}, \theta_{\max}], \mathbb{R}^n) \to \mathbb{R}^n$ be the linearised operators of $h(\lambda, .)$ and $f(\lambda, .)$ respectively and further, assuming that $D(\lambda)$ is atomic at 0, then by the Riesz representation theorem there exist $n \times n$ matrix-valued functions $\mu, \eta : [\theta_{\min}, \theta_{\max}] \to \mathbb{R}^{n^2}$ whose components each are of bounded variation, such that for $\phi \in C([-\tau, 0], \mathbb{R}^n)$

$$D(\lambda)\phi = \phi(0) - \int_{\theta_{\min}}^{\theta_{\max}} d\mu(\lambda, \theta)\phi(\theta), \qquad L(\lambda)\phi = \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda, \theta)\phi(\theta).$$

To investigate the behaviour of (5.75) we cast it in the linear operator form

$$A(\lambda)x = \frac{d}{dt}\left[x_t(0) - \int_{\theta_{\min}}^{\theta_{\max}} d\mu(\lambda, \theta)x_t(\theta)\right] - \int_{\theta_{\min}}^{\theta_{\max}} d\eta(\lambda, \theta)x_t(\theta). \tag{5.76}$$

If $\hat{x}$ is not constant (i.e. not an equilibrium), then the matrix-valued function becomes $d\eta(\lambda, \theta, t)$. On $\ker A(\lambda)$, we seek values of λ that yield solutions of $A(\lambda)$. Let α be the eigenvalue(s) of $A(\lambda)$. When $\alpha = \pm i\omega$, Hopf bifurcation follows.

5.5.1 Lyapounov-Schmidt Reduction of $1-$D NMFDEs

We consider a general linear autonomous NMFDE of the form

$$\frac{d}{dt}[x(t) + b(\mu x(t+r) + \eta x(t-r))] = a(\epsilon x(t+r) + \delta x(t-r)), \tag{5.77}$$

where $a, b : \mathcal{U} \to \mathbb{R}$ represent the bifurcation parameters and with $\mu, \eta, \epsilon, \delta \in \mathbb{R}$ being perturbations.

We analyse the spectrum of (5.77).

Lemma 5.9. *Given $a, b, \mu, \eta, \epsilon, \delta \in \mathbb{R}^*$, the spectrum of (5.77) is given by the following.*

1. *The purely imaginary roots $\pm iy$ are given by the solutions of*

$$\cos(2ry) = P\cos(ry - \theta) - 1 \tag{5.78}$$

where

$$P = -\frac{2R}{ab(\mu+\eta)(\epsilon+\delta)},$$

$$R = [(ab(\mu-\eta)(\epsilon-\delta))^2 + (a(\epsilon+\delta))^2]^{\frac{1}{2}},$$

and

$$\theta = \tan^{-1}\frac{b(\mu-\eta)(\epsilon-\delta)}{\epsilon+\delta}.$$

2. *The neutral equation has an infinite number of purely imaginary roots when $P \geq 2$ whilst the non- neutral equation has at most a finite number of imaginary roots.*

3. *There is a unique pair of real roots $\pm x$ given by*

$$x = \frac{a(\epsilon e^{rx} + \delta e^{-rx})}{1 + b(\mu e^{rx} + \eta e^{-rx})}. \tag{5.79}$$

The number of real roots is finite.

4. *In general, roots come as quadruples α_0, $\overline{\alpha_0}$, $-\alpha_0$ and $-\overline{\alpha_0}$.*

Proof. The characteristic equation of (5.77) is given by

$$\Delta_{A_0}(\alpha) = \alpha + b\mu\alpha e^{\alpha r} + b\eta\alpha e^{-\alpha r} - a\epsilon e^{\alpha r} - a\delta e^{-\alpha r}$$

and corresponding eigenfunctions $e^{\hat{\alpha}t}$ where $\Delta_A(\hat{\alpha}) = 0$. Letting $\alpha = x + iy$ and substituting into the characteristic equation and separating the imaginary and real parts we find

$$x = \frac{by\sin(ry)(\mu e^{rx} - \eta e^{-rx}) + a\cos(ry)(\epsilon e^{rx} + \delta e^{-rx})}{1 + b\cos(ry)(\mu e^{rx} + \eta e^{-rx})}, \tag{5.80}$$

$$y = \frac{-bx\sin(ry)(\mu e^{rx} - \eta e^{-rx}) + a\sin(ry)(\epsilon e^{rx} - \delta e^{-rx})}{1 + b\cos(ry)(\mu e^{rx} + \eta e^{-rx})}. \tag{5.81}$$

1. To find imaginary roots, we set $x = 0$ in (5.80,5.81), yielding imaginary roots are given by $\pm iy$ where y's are roots of the following equations

   $$y = \frac{-a\cos(ry)(\epsilon + \delta)}{b(\mu - \eta)} \tag{5.82}$$

   and

   $$y = \frac{a\sin(ry)(\epsilon - \delta)}{1 + b\cos(ry)(\mu + \eta)} \tag{5.83}$$

   Equating, re-arranging and using, $(a\cos\theta + b\sin\theta = R\cos(\theta - \alpha), \quad \cos^2(ry) = \frac{1}{2}(1 + \cos(2ry)))$, we obtain

   $$-a(\epsilon + \delta)\cos(ry) - ab(\epsilon + \delta)(\mu + \eta)\cos^2(ry) = ab(\mu - \eta)(\epsilon - \delta)\sin(ry) \tag{5.84}$$

   from which

   $$ab(\epsilon + \delta)(\mu + \eta)\cos(2ry) = -2R\cos(ry - \theta) - ab(\mu + \eta)(\epsilon + \delta)$$

   where $R = [(ab(\mu - \eta)(\epsilon - \delta))^2 + (a(\epsilon + \delta))^2]^{\frac{1}{2}}$

   and after some simplification, we have

   $$\cos(2ry) = P\cos(ry - \theta) - 1 \tag{5.85}$$

   where

   $$P = -\frac{2R}{ab(\mu + \eta)(\epsilon + \delta)},$$

   and

   $$\theta = \tan^{-1}\frac{b(\mu - \eta)(\epsilon - \delta)}{\epsilon + \delta}.$$

   We note that,

   Lemma 5.10. *The graphs of the of (5.85) intersect for $P > 0$ if and only if $P \geq 2$.*

Proof. The graphs will not intersect if $\max P\cos(ry-\theta)-1 < \max\cos(2ry)$ i.e.

$$P - 1 < 1$$

hence if

$$P < 2.$$

$\square$

2. To find real roots, we set $y = 0$ in (5.90,5.91). Real roots are solutions of (5.89). We obtain the single equation

$$x = \frac{a(\epsilon e^{rx} + \delta e^{-rx})}{1 + b(\mu e^{rx} + \eta e^{-rx})} \tag{5.86}$$

$\square$

5.5.2 Some Degenerate Cases of the 1-D NMFDE

We now examine some degenerate cases that may arise from setting some of the parameters and/or perturbations in (5.77) equal to zero or to unity and obtain various simpler forms of (5.82, 5.83) or alternatively (5.85) and (5.86), as follows:

1. **Mixed equation**

   Setting $b = \mu = \eta = 0$ and $\epsilon = 1$, $\delta = -1$, we obtain the $1-D$, linear MFDE with one delay are of the form

   $$\dot{x}(t) = a(x(t+r) - x(t-r)), \tag{5.87}$$

   where $a : \mathcal{U} \to \mathbb{R}$ represent the parameters.

   We analyse the spectrum of (5.87)as follows:

   Lemma 5.11. *Given $a \in \mathbb{R}^*$, the spectrum of (5.87) is the following.*

   (a) When $a \in [-3\pi/4r, 0]$ or $a \geq 1/2r$, there is a finite number of imaginary roots $\pm iy$ given by

   $$y = 2a\sin(ry). \tag{5.88}$$

   (b) When $a \in (0, 1/2r]$, there is a unique pair of real roots $\pm x$ given by

   $$x = 2a\sinh(rx) \tag{5.89}$$

Proof. Substituting the values $b = \mu = \eta = 0$ and $\epsilon = 1$, $\delta = -1$, into (5.82, 5.83) and (5.86), we find that

$$x = 2a\cos(ry)\sinh(rx), \tag{5.90}$$

$$y = 2a\sin(ry)\cosh(rx). \tag{5.91}$$

(a) To find imaginary roots, we set $x = 0$ in (5.90,5.91). Then we find that imaginary roots are given by $\pm iy$ where y's are roots of (5.88). Describing h_1, we see that it is even and has zeroes on $k\pi/r$, $k \in \mathbb{Z}^*$. we describe it on $\mathbb{R}_+$. Its limit as $y \to 0$ is r. It is monotonically decreasing to 0 on $[0, \pi/r]$. The function h_1 is negative on $((2k-1)\pi/r, 2k\pi/r)$, $k \in \mathbb{N}$, with minimum at $y = (4k-1)\pi/2r$ of value $-2r/(4k-1)\pi$. It is positive on $(2k\pi/r, (2k+1)\pi/r)$ with maximum at $y = (4k+1)\pi/2r$ of value $2r/(4k+1)\pi$.

(b) To find real roots, we set $y = 0$ in (5.90,5.91). Real roots are solutions of (5.89). The function h_2 is even and monotonically increasing from r to $+\infty$.

□

Because of the symmetries of the spectrum, it is enough to look at roots in the closed positive quadrant of $\mathbb{C}$. For the general case, (5.91) can be solved for x if

$$u = \frac{1}{2a}\frac{y}{\sin(ry)} \geq 1.$$

This means that solutions exists if and only if $a\sin(ry) \geq 0$. There are roots when $h_1(y) \leq 1/2a$, $a > 0$, and when $1/2a \leq h_1(y)$, $a < 0$. They are given by

$$x = \frac{1}{r}\ln\left(u + \sqrt{u^2 - 1}\right).$$

And so,

$$\sqrt{u^2 - 1} = \frac{\sqrt{y^2 - 4a^2\sin^2(ry)}}{2a\sin(ry)}.$$

Therefore, we need to solve

$$\ln\left(\frac{y + \sqrt{y^2 - 4a^2\sin^2(ry)}}{2a\sin(ry)}\right) = r\cot(ry)\sqrt{y^2 - 4a^2\sin^2(ry)}. \tag{5.92}$$

The equation (5.92) may be solved graphically or numerically.

2. **Delay equation**

   Here we set $b = 0$, $\epsilon = 0$ and $\delta = \pm 1$

$$\dot{x}(t) = \pm ax(t - r) \tag{5.93}$$

   and obtain purely imaginary roots from the solution of (5.82, 5.83) and (5.86)

$$y = \pm a \sin(ry) \tag{5.94}$$

   and real roots from

$$x = \pm ae^{-rx} \tag{5.95}$$

3. **Neutral delay equation**

   In this case, we let $\mu = \epsilon = 0$, $\eta = \pm 1$ and $\delta = \pm 1$ giving

$$[x(t) \pm bx(t - r)]' = \pm a\delta x(t - r) \tag{5.96}$$

   which yields a slight modification to (5.85) as

$$\cos(2ry) = Q\cos(ry - \theta) - 1 \tag{5.97}$$

   whilst (5.86) takes the form

$$x = \frac{ae^{-rx}}{1 + be^{-rx}}$$

4. **Simple advanced equation**

   Setting $b = \delta = 0$ and $\epsilon = \pm 1$, we obtain

$$\dot{x}(t) = \pm ax(t + r) \tag{5.98}$$

   and (5.82, 5.83) and (5.86)yield

$$y = \pm a \sin(ry)$$

   and

$$x = \pm ae^{rx}.$$

5. **Neutral advanced equation**

   Here, we let $\eta = \delta = 0$, $\mu = \pm 1$ and $\epsilon = \pm 1$ giving

$$[x(t) \pm bx(t + r)]' = \pm ax(t + r) \tag{5.99}$$

which also yields a slight modification to (5.85) as

$$\cos(2ry) = Q\cos(ry - \theta) - 1$$

whilst (5.86) takes the from

$$x = \frac{ae^{rx}}{1 + be^{rx}}$$

The five cases analysed above are of the general one-dimensional neutral mixed functional differential equation (5.77) obtained by setting some of its parameters to zero or unity.

Chapter 6

Bifurcation of Neutral MFDE in a Ring Network

In this chapter, we study the Hopf bifurcation of a cell network with maximum coupling. The dynamics of coupled cell networks (symmetric networks of coupled identical oscillators) with nearest-neighbour coupling have been studied by authors such as Buono et al. in [9], Campbell et al. in [12] and Benoit et al. in [8]. The equations used by these authors contain some delay terms and are generally of the form

$$\dot{x}_j(t) = f(x_j(t)) + \sum_{k=1}^{n} c_{j,k} h(x_j(t) - x_k(t-\tau)),$$

where f is the internal dynamics function and h the coupling function and $j = 1, \cdots, n$. Wu [67] studies the delayed Hopfield-Cohen-Grossberg model of neural networks given by

$$\dot{u}_i(t) = -u_i(t) + \sum_{j=1}^{n} J_{ij} f(u_j(t-\tau)), \qquad 1 \leq i \leq n,$$

where f is a sigmoidal function normalized so that $f(0) = 0$ and $J = J_{ij}$ is a symmetric circulant matrix with all the diagonal elements identical to zero. Most systems considered hitherto are those with nearest neighbour coupling. If each pair of cells is coupled, we have a complete graph (clique). We extend and generalise these to a neutral MFDE network with (maximum) all-to-all coupling given by the general equation

$$[H_j(u_j)_t]' = f_j(T_0(u_j)_t) + \sum_{k=1, k\neq j}^{n} f_{j,k}(T_1(u_j)_t, T_2(u_k)_t) \tag{6.1}$$

with $1 \leq j \leq n$, where the states of the cells are characterised by a vector $u = (u_1, \ldots, u_n)$, each with possibly multiple components.

The linear operators $T_i : X \to \mathbb{R}^l$, $i = 0, 1, 2$, represent how the distributed time effects enter the internal dynamics of a particular cell, for f_j, and the dynamics of the connection or interaction for $f_{j,k}$. We define the operators explicitly by the following :

$$T_0 x_t = x(t);\ T_0\phi = \phi(0) \tag{6.2}$$

$$T_1 x_t = x(t+\theta);\ T_1\phi = \phi(\theta) \tag{6.3}$$

$$T_2 x_t = x(t-\theta);\ T_2\phi = \phi(-\theta) \tag{6.4}$$

The dynamics of a cell network with (maximum) all-to-all coupling can be written by the general equation

$$[H_j(u_j)_t]' = f_j(T_0(u_j)_t) + \sum_{k=1, k\neq j}^{n} f_{j,k}(T_1(u_j)_t, T_2(u_k)_t) \tag{6.5}$$

with $1 \leq j \leq n$, where the states of the cells are characterised by a vector $u = (u_1, \ldots, u_n)$, each with possibly multiple components. The linearisation of the neutral MFDE (6.5) at 0 is given by

$$[H'(0)(u_j)_t]' = f_j'(0)(T_0(u_j)_t) + \sum_{k=1}^{n} f_{j,k}'(0)(T_1(u_j)_t, T_2(u_k)_t). \tag{6.6}$$

In vector form, we may write this equation as

$$[H'(0)(u_j)_t]' = f_j'(0)\mathbf{I}(T_0(u_j)_t) + \mathbf{M}(T_1(u_j)_t, T_2(u_k)_t) \tag{6.7}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix and $\mathbf{M}$ the $n \times n$ matrix

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \ldots & m_{1n} \\ m_{21} & m_{22} & m_{23} & \ldots & m_{2n} \\ m_{31} & m_{32} & m_{33} & \ldots & m_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{n1} & m_{n2} & m_{n3} & \ldots & m_{nn} \end{bmatrix} \tag{6.8}$$

in which $m_{jk} = f_{j,k}'(0)$.

Definition 6.1. *Let* $\mathcal{I} = (0, 1, \ldots, n-1)$. *The matrix* M *is circulant if there exists a function* $\mathcal{M} : \mathcal{I} \to \mathbb{C}$ *such that*

$$m_{i,j} = m_{i-j(\mod n),0} = \mathcal{M}(i - j(\mod n)) \quad \forall i, j \in \mathcal{I}.$$

M is a circulant matrix, in which each row is generated from a vector by a cyclic shift of the row above it. Here the entries of M are vectors $v \in \mathbb{C}^n$ and we denote$M$ by

$$M = \text{circ}\{m\} = \text{circ}\{m_0, m_1, \ldots, m_{n-1}\}.$$

Note that $m_{i,j} = m_{i-j(\mod n)}$. We define the shift operator $R : \mathbb{C}^n \to \mathbb{C}^n$, a rotation, by

$$R(m_0, m_1, \ldots, m_{n-1}) = (m_{n-1}, m_0, \ldots, m_{n-2}).$$

To obtain the characteristic matrix equation, we substitute the ansatz $u = e^{\lambda t}$ into (6.6)and obtain

$$\begin{aligned}
[H'(0)e^{\lambda(t+\theta)}]' &= f_j'(0)e^{\lambda t} + \sum_{k=1}^{n} f_{j,k}(0)(e^{\lambda(t+\theta)}, e^{\lambda(t-\theta)}) \\
\lambda H''(0)e^{\lambda(t+\theta)} &= f_j'(0)e^{\lambda t} + \sum_{k=1}^{n} f_{j,k}(0)(e^{\lambda(t+\theta)}, e^{\lambda(t-\theta)}) \\
\lambda H''(0)e^{\lambda\theta} &= f_j'(0) + \sum_{k=1}^{n} f_{j,k}(0)(e^{\lambda\theta}, e^{-\lambda\theta})) \qquad (6.9)
\end{aligned}$$

since $H'(0)$ is a linear operator. The last line follows from dividing by $e^{\lambda t}$.

6.0.1 Examples of Networks with Nearest-neighbour Coupling

The following are examples in the literature, of networks with n identical elements and nearest-neighbour coupling, which are derivations of this form.

1. An example of a network with nearest-neighbour coupling is the neutral DDE studied by Lamb and Guo [30]:

$$\frac{d}{dt}[u_j(t) - cu_j(t-1)] = -3g(u_j(t-1)) + g(u_{j+1}(t-1)) + g(u_{j-1}(t-1)). \qquad (6.10)$$

   The adjacency matrix for nearest-neighbour coupling is

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ & \ddots & & & \ddots & & \\ 1 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \qquad (6.11)$$

   We linearise (6.10) and rewrite it in vector form as

$$\frac{d}{dt}[u(t) - cu(t-1)] = -3I\dot{g}(0)u(t-1) + B\dot{g}(0)u(t-1). \qquad (6.12)$$

The matrices $\mu(\theta)$ and $\eta(\theta)$ are therefore given by

$$(\delta_0 - c\delta_{-1})I = -3\dot{g}(0)\delta_{-1}I + \dot{g}(0)\delta_{-1}B. \tag{6.13}$$

2. A particular practical example is the symmetric ring of delay-coupled lasers investigated by numerous researchers including Buono and Collera [18]. The equations are symmetric with respect to rotations of the electric fields, the symmetry groups are $\mathbb{Z}^n \times S^1$ and $\mathbb{D}^n \times S^1$. Typically, the system links the electric field E and the inversion N. The DDE system with bi-directional coupling is

$$\begin{aligned}\dot{E}_j(t) &= (1+i\alpha)(N_j(t)E_j(t)) + \kappa e^{-iC_p}(E_{j-1}(t-\tau) + E_{j+1}(t-\tau)),\\ \dot{N}_j(t) &= \frac{1}{T}(P - N_j(t) - (1+2N_j(t))|E_j(t)|^2),\end{aligned} \tag{6.14}$$

for $j = 1, \ldots, n \pmod n$, with five parameters α, κ, C_p, P and T.

We take $H_j(t) = [E_j(t), N_j(t)]^T \in \mathbb{C} \times \mathbb{R}$ for $j = 1, 2, \cdots, n$ cast the DDE (6.14) in the form (3.70) as follows:

$$\begin{bmatrix}\dot{E}_j(t)\\ \dot{N}_j(t)\end{bmatrix} = \begin{bmatrix}(1+i\alpha)PE_j(t)\\ \frac{1}{T}(-\frac{N(t)}{P}|E_j(t)|^2)\end{bmatrix} + \begin{bmatrix}\kappa e^{-iC_p}(E_{j-1}(t-\tau) + E_{j+1}(t-\tau))\\ 0\end{bmatrix} \tag{6.15}$$

We linearise and write (6.15) in vector form making use of the nearest neighbour coupling matrix B, as

$$\begin{bmatrix}\dot{\phi}(0)\\ \dot{\psi}(0)\end{bmatrix} = \begin{bmatrix}(1+i\alpha)\psi(0)\phi(0)\\ \frac{1}{T}(P - \psi(0) - (1+2\psi(0))|\phi(0)|^2)\end{bmatrix} I + \begin{bmatrix}\kappa e^{-iC_p}\phi(-\tau)\\ 0\end{bmatrix} B \tag{6.16}$$

giving the matrix

$$d\eta = \begin{bmatrix}(1+i\alpha)P\delta_0\\ \frac{1}{T}(-\frac{1}{P}\delta_0|\delta_0|^2)\end{bmatrix} I + \begin{bmatrix}\kappa e^{-iC_p}\delta_{-\tau}\\ 0\end{bmatrix} B \tag{6.17}$$

6.1 A Ring Network Example

We now apply the results to a system of neutral equations with mixed arguments by considering a ring of n identical elements with forwards and backwards nearest neighbour coupling. We assign to each individual element a linear decay term, a nonlinear forwards-backwards self-connection (feedback) term and nonlinear element to element mixed connection terms. This is an example of (6.5) which includes aspects of the general form not considered in the examples given in the literature; combining the neutral term and mixed terms. Since it considers nearest-neighbour coupling, there would be many zero terms i.e. $f_{j,k} = 0$ for $k > j+1$ or $k < j-1$.

The example NMFDEs takes the form

$$\frac{d}{dt}[Hu_j] = \alpha_0 T_0 u_j + \alpha_1 f_1(T_1 u_j - T_2 u_i) \quad + \\ \alpha_2 f_2(T_1 u_{j+1} - T_2 u_{i+1}) + \alpha_2 f_2(T_1 u_{j-1} - T_2 u_{j-1}) \tag{6.18}$$

The equation for the system is given in component form by

$$\begin{aligned} \frac{d}{dt}[u_i(t) + u_i(t+\tau_s) - u_i(t-\tau_s)] &= -u_i(t) + \alpha[f(u_i(t+\tau_s) - u_i(t-\tau_s))] \\ &+ \beta[g(u_{i+1}(t+\tau) - u_{i+1}(t-\tau))] \\ &+ \beta[g(u_{i-1}(t+\tau) - u_{i-1}(t-\tau))]. \qquad i(\mod n) \end{aligned} \tag{6.19}$$

Lemma 6.2. *The linearisation of (6.19) around the equilibrium point x^* is*

$$\begin{aligned} \frac{d}{dt}[Hu_t] &= -u_i(t) + \alpha \dot{f}(0)(u_i(t+\tau_s) - u_i(t-\tau_s)) \\ &+ \beta \dot{g}(0)(u_{i+1}(t+\tau) - u_{i+1}(t-\tau)) \\ &+ \beta \dot{g}(0)(u_{i-1}(t+\tau) - u_{i-1}(t-\tau)), \qquad i(\mod n) \end{aligned} \tag{6.20}$$

Note that the subscripts 0, +, − on T denote the instantaneous, advanced and delayed terms respectively. If we define

$$\begin{aligned} T_0\phi_i &= (-\phi_i(0), \phi_i(\tau_s) - \phi_i(-\tau_s)) \\ T_+\phi_{i+1} &= (\phi_{i+1}(\tau) - \phi_{i+1}(-\tau)) \\ T_-\phi_{i-1} &= (\phi_{i-1}(\tau) - \phi_{i-1}(-\tau)), \end{aligned} \tag{6.21}$$

then equation (6.20) can be written in vector form as

$$\frac{d}{dt}[H\mathbf{u}_t] = \alpha \dot{f}(0)\mathbf{I}(T_0\mathbf{u}_t) + \beta \mathbf{M}\dot{g}(0)(T_+\mathbf{u}_t - T_-\mathbf{u}_t) \tag{6.22}$$

where I is the $n \times n$ identity matrix, $H\phi = \phi(0) + \phi(\tau_s) - \phi(-\tau_s)$ and $T\phi = M(\phi(\tau) - \phi(-\tau))$ with M given by the $n \times n$ adjacency matrix,

$$M = \begin{bmatrix} 0 & \beta\dot{g}(0) & 0 & \dots & \beta\dot{g}(0) \\ \beta\dot{g}(0) & 0 & \beta\dot{g}(0) & \dots & 0 \\ 0 & \beta\dot{g}(0) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta\dot{g}(0) & 0 & \dots & \beta\dot{g}(0) & 0 \end{bmatrix} \tag{6.23}$$

and $\phi(\tau) = [\phi_1(\tau), \phi_2(\tau), \dots, \phi_n(\tau)]^T$.

The matrix M in (6.23) can be decomposed into

$$M = \beta\dot{g}(0)(C + C^T) \tag{6.24}$$

where

$$C = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{6.25}$$

$C = \mathrm{circ}_n(0, 1, 0, \dots, 0)$.

Equivariance and Reversibility

We deduced the necessary conditions for $\mathbb{D}_n-$ equivariance and reversibility for the general equation (6.5) and since (6.18) is a derivation of that form, its $\mathbb{D}_n-$equinariance and reversibility follow as corollaries.

6.2 Equivariant Hopf Bifurcation

We recall that a bifurcation from a steady-state can be caused by the loss of the stability of the trivial solution $\lambda = \lambda_0$ of the characteristic equation. The loss of stability occurs when a pair of complex conjugate eigenvalues leaving the left complex half-plane through complex conjugate points on the imaginary axis at the critical value α_0. If 0 is not an eigenvalue of the characteristic equation, then the Implicit Function Theorem implies that stationary solutions of (6.19) cannot bifurcate from the trivial solution at $(0, \lambda_0)$.

Proposition 6.3. *The system (6.19) has a unique and uniform steady state whose components are given by*

$$x^* = \alpha f(0) + 2\beta g(0) \tag{6.26}$$

.

Proof. At an equilibrium, x^*, a steady state, the right hand side of (6.19) equals 0, we have $u' = 0$ implying that $u_j(t + \tau) = u_j(t - \tau)$, hence $u = k$, a constant. Hence the right hand side gives

$$k + \alpha f(0) + 2\beta g(0) = 0$$

Let $w = u_j(t + \tau) - u_j(t - \tau)$, then $w = 0$. □

Lemma 6.4. *The characteristic equation of the linearised equation (6.19) obtained using the ansatz $u(t) = e^{\lambda t}v$ is found to be*

$$\begin{aligned} \lambda e^{\lambda t}(1 + e^{\lambda\tau} - e^{-\lambda\tau})vI &= -e^{\lambda t}v + \alpha f'(0)(e^{\lambda(t+\tau_s)} - e^{\lambda(t-\tau_s)})vI \\ &+ \beta g'(0)(e^{\lambda(t+\tau)} - e^{\lambda(t-\tau)})Cv \\ &+ \beta g'(0)(e^{\lambda(t+\tau)} - e^{\lambda(t-\tau)})C^T v. \end{aligned} \tag{6.27}$$

Upon dividing through by $e^{\lambda t}v$ we have,

$$\begin{aligned} [\lambda(1 + e^{\lambda\tau_s} - e^{-\lambda\tau_s})]I &= [-1 + \alpha f'(0)(e^{\lambda\tau_s} - e^{-\lambda\tau_s})]I \\ &+ \beta g'(0)(e^{\lambda\tau} - e^{-\lambda\tau})C \\ &+ \beta g'(0)(e^{\lambda\tau} - e^{-\lambda\tau})C^T. \end{aligned}$$

Therefore the characteristic matrix of the linearisation about the trivial solution is

$$\begin{aligned} \mathcal{M}_n(\lambda) &= [\lambda(1 + e^{\lambda\tau_s} - e^{-\lambda\tau_s})]I - [-1 + \alpha f'(0)(e^{\lambda\tau_s} - e^{-\lambda\tau_s})]I \\ &- \beta g'(0)(e^{\lambda\tau} - e^{-\lambda\tau})C - \beta g'(0)(e^{\lambda\tau} - e^{-\lambda\tau})C^T. \end{aligned} \tag{6.28}$$

The matrices on the right, I, C and C^T together have the structure of the matrix M in equation (6.23), i.e.

$$M = \begin{bmatrix} m_{11} & m_{12} & 0 & \dots & m_{12} \\ m_{12} & m_{11} & m_{12} & \dots & 0 \\ 0 & m_{12} & m_{11} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{12} & 0 & \dots & m_{12} & m_{11} \end{bmatrix} \tag{6.29}$$

where $m_{11} = \lambda(1 + e^{\lambda\tau_s} - e^{-\lambda\tau_s}) + 1 + \alpha f'(0)(e^{-\lambda\tau_s} - e^{\lambda\tau_s})$ and
$m_{12} = \beta g'(0)(e^{-\lambda\tau} - e^{\lambda\tau})$.

By theorem (B.3), and using $\rho_1 = 1$, we have the first eigenvalue

$$\begin{aligned} \lambda_1 &= \lambda(1 + e^{\lambda\tau_s} - e^{-\lambda\tau_s}) + 1 + \alpha f'(0)(e^{-\lambda\tau_s} - e^{\lambda\tau_s}) \\ &+ 2\beta g'(0)(e^{-\lambda\tau} - e^{\lambda\tau}) \end{aligned} \tag{6.30}$$

Recalling that the determinant of a matrix equals the product of its eigenvalues (counting multiplicities), it can be shown that

Lemma 6.5. *The characteristic equation is given by*

$$\begin{aligned}
\det \mathcal{M}(0,\lambda) &= \prod_{j=0}^{n-1} \lambda(1+e^{\lambda\tau_s}-e^{-\lambda\tau_s})+1+\alpha f'(0)(e^{-\lambda\tau_s}-e^{\lambda\tau_s})+2\beta g'(0)(e^{-\lambda\tau}-e^{\lambda\tau})\cos\frac{2\pi j}{n} \\
&= \prod_{j=0}^{n-1} \Delta_j(\lambda) \\
&= \Delta_0(\lambda)\prod_{j=1}^{n-1} \Delta_j(\lambda) \\
&= (a+2b)\prod_{j=1}^{n-1}(a+2b\cos\frac{2\pi j}{n}) = 0 \qquad (6.31)
\end{aligned}$$

where $a = \lambda(1+e^{\lambda\tau_s}-e^{-\lambda\tau_s})+1+\alpha f'(0)(e^{-\lambda\tau_s}-e^{\lambda\tau_s})$, and $b = \beta g'(0)(e^{-\lambda\tau}-e^{\lambda\tau})$.

6.2.1 Hopf Bifurcation

We now examine conditions which lead to the Hopf bifurcation theorem for the system (6.18). We denote the characteristic polynomial by $E(\lambda)$ (6.31) i.e.

$$E(\lambda) = \Delta_0(\lambda)\prod_{j=1}^{n-1} \Delta_j(\lambda). \qquad (6.32)$$

From (6.31) for each $\Delta_j(\lambda)$ we have

$$\Delta_j(\lambda) = \lambda(1+e^{\lambda\tau_s}-e^{-\lambda\tau_s})+1+\alpha\dot{f}(0)(e^{-\lambda\tau_s}-e^{\lambda\tau_s})+2\beta\dot{g}(0)(e^{-\lambda\tau}-e^{\lambda\tau})\cos\frac{2\pi j}{n} \qquad (6.33)$$

Let $\lambda = \mu + i\omega, \quad \mu,\omega \in \mathbb{R}$ and $\Delta_j(\lambda) = \Re_j(\mu,\omega) + \Im_j(\mu,\omega)$, then upon substituting $\lambda = \mu + i\omega$ into (6.33) we obtain

$$\begin{aligned}
\Re_j(\mu,\omega) &= \mu+\mu(e^{\mu\tau_s}-e^{-\mu\tau_s})\cos\omega\tau_s-\omega(e^{\mu\tau_s}+e^{-\mu\tau_s})\sin\omega\tau_s+1 \\
&+ \alpha\dot{f}(0)(e^{-\mu\tau_s}-e^{\mu\tau_s})\cos\omega\tau_s+2\beta\dot{g}(0)(e^{-\mu\tau}-e^{\mu\tau})\cos\omega\tau\cdot\cos\frac{2\pi j}{n}, \qquad (6.34)
\end{aligned}$$

and

$$\begin{aligned}
\Im_j(\mu,\omega) &= \omega+\omega(e^{\mu\tau_s}-e^{-\mu\tau_s})\cos\omega\tau_s-\alpha\dot{f}(0)(e^{-\mu\tau_s}+e^{\mu\tau_s})\sin\omega\tau_s \\
&- 2\beta\dot{g}(0)(e^{-\mu\tau}+e^{\mu\tau})\sin\omega\tau\cdot\cos\frac{2\pi j}{n}. \qquad (6.35)
\end{aligned}$$

It is clear that the characteristic equation has a simple pair of pure imaginary roots $\lambda = \pm i\omega$ for parameters such that $\Delta_0(\pm i\omega) = 0$, when, upon substituting $i\omega$ into (6.31),

$$i\omega(1+2i\sin(\omega\tau_s))+1+\alpha\dot{f}(0)(-2i\sin(\omega\tau_s))+2\beta\dot{g}(0)(-2i\sin(\omega\tau))\cos\frac{2\pi j}{n}=0 \tag{6.36}$$

giving

$$\sin(\omega\tau_s)=\frac{1}{2\omega} \tag{6.37}$$

Taking $|\sin(\omega\tau_s)|\leq 1$, gives $|\frac{1}{2\omega}|\leq 1$ and hence $\omega\geq\frac{1}{2}$.

We find that

$$\tau_s=\frac{1}{\omega}[\sin^{-1}(\frac{1}{2\omega})+2k\pi]$$

with $\frac{-\pi}{2}<\omega\tau_s<\frac{\pi}{2}$.

Also, we have

$$\sin(\omega\tau)=\frac{\omega-2\alpha\dot{f}(0)\sin(\omega\tau_s)}{4\beta\dot{g}(0)\cos\frac{2\pi j}{n}}. \tag{6.38}$$

Upon substituting (6.37) into (6.38), we obtain

$$\sin(\omega\tau)=\frac{\omega^2-\alpha\dot{f}(0)}{4\omega\beta\dot{g}(0)\cos\frac{2\pi j}{n}}, \tag{6.39}$$

yielding

$$\tau=\frac{1}{\omega}\left[\sin^{-1}\left(\frac{\omega^2-\alpha\dot{f}(0)}{4\omega\beta\dot{g}(0)\cos\frac{2\pi j}{n}}\right)+2k\pi,\right] \tag{6.40}$$

with

$$\left|\frac{\omega^2-\alpha\dot{f}(0)}{4\omega\beta\dot{g}(0)\cos\frac{2\pi j}{n}}\right|\leq 1.$$

Fixing the parameters $\alpha,\beta,\omega,\dot{f}(0)$ and $\dot{g}(0)$, we can determine τ_s and τ such that Hopf bifurcation occurs.

Lemma 6.6 (The eigenvalue conditions). *Let α,β,τ_s,τ be such that there is a solution of (6.37) and (6.38)for some $j\in 1,2,\ldots$. Then*

- *The characteristic matrix $\mathcal{M}_n(\lambda)$ is continuously differentiable with respect to β.*
- *The infinitesimal generator, $A(\beta)$, of the linear operator (6.22) has a repeated pair of eigenvalues $\pm i\omega$.*
- *The generalised eigenspace, P, of $A(\beta)$ for $i\omega$ is spanned by the eigenvectors*

$$\{e^{i\omega\theta}v_j,e^{i\omega\theta}\bar{v}_j,e^{-i\omega\theta}v_j,e^{-i\omega\theta}\bar{v}_j\}$$

Proof. The proof is organised in parts.

- Differentiability

  The differentiability of $\mathcal{M}_n(\lambda)$ follows from its definition, (6.28).

- Eigenvalues

  The eigenvalues of $A(\beta)$ correspond to the roots of the characteristic equation. From (6.31), $A(\beta)$ has a repeated pair of eigenvalues.

- Eigenspace

  From the properties of v_j in (B.3), we have

$$\begin{aligned}\mathcal{M}_N(\lambda)v_j &= [\lambda(1+e^{\lambda\tau_s}-e^{-\lambda\tau_s})+1 \\ &+ \alpha\dot{f}(0)(e^{-\lambda\tau_s}-e^{\lambda\tau_s})]Iv_j+[\beta\dot{g}(0)(e^{-\lambda\tau}-e^{\lambda\tau})](C+C^T)v_j\end{aligned}$$

□

The transversality condition requires that the eigenvalue should cross the imaginary axis with non-vanishing speed $\Re\alpha'(\tau_c)$.

Lemma 6.7 (Transversality condition)**.** *Let $a, \tau_s, \tau, \omega, \dot{f}(0)$ and $\dot{g}(0)$, be fixed such that there is a solution (τ_c, τ_{sc}) of (6.38) and (6.37). If $\dot{g}(0) \neq 0$, then*

$$\Re(\frac{d\lambda}{d\beta})|_{\lambda=i\omega} \neq 0. \tag{6.41}$$

Proof. To check that the roots of the characteristic equation are simple, it is enough to verify that

$$\frac{\partial\Delta_0(i\omega)}{\partial\lambda}|_{\lambda=i\omega} \neq 0.$$

Upon substituting $\lambda = i\omega$ into

$$\Delta_0(\lambda) = \lambda(1+e^{\lambda\tau_s}-e^{-\lambda\tau_s})+1+a\dot{f}(0)(e^{-\lambda\tau_s}-e^{\lambda\tau_s})+2\beta\dot{g}(0)(e^{-\lambda\tau}-e^{\lambda\tau}), \tag{6.42}$$

we obtain the following conditions,

$$\frac{\partial\Delta_0(i\omega)}{d\lambda} = k_1 + ik_2 \tag{6.43}$$

where

$$
\begin{aligned}
k_1 &= 1 - 2a\tau_s \dot{f}(0)\cos(\omega\tau_s) - 4\beta\tau\dot{g}(0)\cos(\omega\tau) \neq 0 \\
k_2 &= 2\sin(\omega\tau_s) + 2\omega\tau_s\cos(\omega\tau_s) \neq 0 \qquad (6.44)
\end{aligned}
$$

We also find that

$$
\frac{\partial\Delta_0(i\omega)}{d\beta} = -4\dot{g}(0)i\sin(\omega\tau) \qquad (6.45)
$$

By the implicit function theorem,

$$
\frac{\partial\Delta_0(i\omega)}{\partial\lambda} \cdot \frac{d\lambda}{d\beta} + \frac{d\Delta_0(i\omega)}{d\beta} = 0
$$

giving

$$
\frac{d\lambda}{d\beta} = -\frac{\partial\Delta_0(i\omega)}{d\beta} / \frac{\partial\Delta_0(i\omega)}{\partial\lambda}. \qquad (6.46)
$$

□

Using (6.43), (6.44), (6.45) and (6.46), we obtain

$$
\Re(\frac{d\lambda}{d\beta}|_{\lambda=i\omega}) = \Re\frac{4i\dot{g}(0)\sin(\omega\tau)}{k_1 + ik_2} \qquad (6.47)
$$

and upon multiplying the numerator and denominator by $k_1 - ik_2$ yields

$$
\Re(\frac{d\lambda}{d\beta}|_{\lambda=i\omega}) = \frac{4k_2\dot{g}(0)\sin(\omega\tau)}{k_1^2 + k_2^2} \qquad (6.48)
$$

Hence the transversality condition is

$$
8\dot{g}(0)\sin(\omega\tau)[\sin(\omega\tau_s) + \omega\tau_s\cos(\omega\tau_s)] \neq 0 \qquad (6.49)
$$

and k_1, k_2 cannot simultaneously be zero.

From lemmas (6.6) and (6.7), conditions (H1), (H2), (H4) of the Hopf bifurcation theorem are satisfied. We now arrive at the following theorem.

Theorem 6.8 (Equivariant Hopf bifurcation)**.** *Let $\tau, \tau_s, \alpha, a, \beta, \dot{f}(0)$ and $\dot{g}(0)$ be fixed and such that there is a solution (ω_c, β_c) of (6.37) and (6.38). If condition (6.47) holds, then the system (6.19) undergoes an equivariant Hopf bifurcation as β varies through β_c i.e. there exists a periodic orbit of frequency ω bifurcating from the steady state $x^* = \alpha f(0) + 2\beta g(0)$.*

Chapter 7

REN Optimisation with Delays

One of the main historical motivation for the study of MFDEs comes from the Euler-Lagrange equations for optimisation problems with delays. In this chapter we extend the theory to our reversible/symmetric framework. Kolesnikova *et al.* [39] determine the necessary and sufficient conditions for a given ODE or PDE to admit a variational formulation .

We consider symmetric functionals with delayed argument extending the work of [33] and the series [3, 1, 2]. The study is mainly concerned with the necessary conditions on a function x to be a critical point of the functional

$$J(x) = \int_a^b g(t, L_1(t)x_t, L_2(t)\dot{x}_t)\, dt, \tag{7.1}$$

where $a < b - r < b$ where r is the maximum delay and g is called the Lagrangian. The phase spaces for the problem are

$$X^- = \mathrm{PWS}([-r, 0], \mathbb{R}^n) \subset Y^- = \mathrm{PWC}([-r, 0], \mathbb{R}^n).$$

We recall that a function is said to be piecewise continuous (PWC) on an interval if it is of class C^0 on the interval except possibly for a finite set of simple (jump) discontinuities. Piecewise smooth (PWS) refers to the equivalent case where a function is of class C^∞. The function $x \in X^-$ intervene in J via two delayed linear operators

$$L_i : [a, b] \times Y^- \to \mathbb{R}^{k_i}, \quad L_i(t)\phi = \int_{-r}^0 d\eta_i(t, \theta)\, \phi(\theta),$$

for some $k_i \in \mathbb{N}$, $i = 1, 2$, and then via the continuous nonlinearity

$$g : [a, b] \times \mathbb{R}^{k_1+k_2} \to \mathbb{R}$$

with continuous partial derivatives of the first two orders with respect to the last two variables.

Moreover, we assume that there is a compact group Γ acting on $\mathbb{R}^n$ and $\mathbb{R}^k_i$, $i = 1, 2$, and that the linear operators L_i's, $i = 1, 2$, are Γ-equivariant, that is,

$$d\eta_i(t, \theta)\, \gamma\phi = \gamma d\eta_i(t, \theta)\, \phi, \quad i = 1, 2,$$

for all $\gamma \in \Gamma$ and $\phi \in Y^-$ and the function g is Γ-invariant, that is,

$$g(t, \gamma u, \gamma v) = g(t, u, v)$$

for all $u, v \in \mathbb{R}^{k_i}$. Therefore $J : \mathrm{PWS}([a - \tau, b], \mathbb{R}^n) \to \mathbb{R}$ is a smooth invariant functional.

7.0.1 Examples

1. The classical functional

$$J(x) = \int_a^b f(t, x(t), x(t - r), \dot{x}(t), \dot{x}(t - r))\, dt$$

is written in the form (7.1) with $k_1 = k_2 = 2$, using $L_1 = L_2 = L : X^- \to \mathbb{R}^2$, $L_2 : Y^- \to \mathbb{R}^2$ and $g : \mathbb{R}^5 \to \mathbb{R}$ defined as

$$L\phi = (\phi(0), \phi(-r)), \quad g(t, u, v) = f(t, u_1, u_2, v_1, v_2).$$

2. If we have many discrete delays r_i, $1 \leq i \leq k$, on the state variable, the functional

$$J(x) = \int_a^b f(t, x(t), x(t - r_1), \ldots, x(t - r_k), \dot{x}(t))\, dt$$

is written in the form (7.1) with $r = \max_{1 \leq i \leq k}\{r_i\}$, $k_1 = k + 1$ and $k_2 = 1$, using $L_1 : Y^- \to \mathbb{R}^{k+1}$, $L_2 : Y^- \to \mathbb{R}$ and $g : \mathbb{R}^{2+k+1} \to \mathbb{R}$ defined as

$$L_1\phi = (\phi(0), \phi(-r_1), \ldots, \phi(-r_k)), \quad L_2\phi = \phi(0), \quad g(t, u, v) = f(t, u_1, \ldots, u_{k+1}, v).$$

3. If we have a distributed delay on the state variable, for instance

$$J(x) = \int_a^b f(t, (Lx)(t), \dot{x}(t))\, dt,$$

where

$$(Lx)(t) = C(t) \int_{t-r}^t e^{-s} x(s)\, ds$$

with the normalisation constant $C(t) = \dfrac{e^t}{e^r - 1}$, the functional J is written in the form (7.1) with $k_1 = k_2 = 1$ using $L_1 : X^- \to \mathbb{R}^{k+1}$, $L_2 : Y^- \to \mathbb{R}$ and $g : \mathbb{R}^{2+k+1} \to \mathbb{R}$ defined as

$$L_1\phi = \frac{1}{e^r - 1}\int_{-r}^{0} e^{-\theta}\phi(\theta)\,d\theta$$

and $L_2\phi = \phi(0)$ and $g(t,u,v) = f(t,u,v)$. The shape of L_1 follows from the following calculations

$$\begin{aligned}(Lx)(t) &= \frac{e^t}{e^r - 1}\int_{t-r}^{t} e^{-s}x(s)\,ds \\ &= \frac{e^t}{e^r - 1}\int_{-r}^{0} e^{-(t+\theta)}x(t+\theta)\,d\theta = \frac{1}{e^r - 1}\int_{-r}^{0} e^{-\theta}x_t(\theta)\,d\theta = L_1x_t.\end{aligned}$$

4. We can also deal with variable delays. Take $d\eta(t,\theta) = \delta_{-\alpha(t)}(\theta)$, then

$$L(t)y_t = y(t - \alpha(t)).$$

7.0.2 Critical Points

To discuss critical points of the functional J we set boundary conditions for the admissible functions. Classically, there are many boundary conditions possible :

1. asymmetric conditions, where $x_a(\theta) = \phi(\theta)$, $\theta \in [-r, 0]$ and $x(b)$ is fixed;
2. symmetric conditions, where $x_a(\theta) = \phi(\theta)$, $\theta \in [-r, 0]$, and $x_b(\theta) = \psi(\theta)$, $\theta \in [-r, 0]$;
3. periodic conditions where we choose $\phi = \psi$ in the symmetric conditions.

To calculate necessary conditions for critical points, we need to define admissible variations that correspond to the differences between two admissible functions. In each of the previous case they correspond to the following PWS functions $\xi : [a - r, b] \to \mathbb{R}$ satisfying

1. for asymmetric conditions, $\xi(a + \theta) = 0$, $\forall\theta \in [-r, 0]$ and $\xi(b) = 0$;
2. for symmetric conditions, $\xi(a + \theta) = \xi(b + \theta) = 0$, $\forall\theta \in [-r, 0]$;
3. for periodic conditions, $\xi_a = \xi_b$.

To be able to determine the Euler-Lagrange equations for the critical points of J, we need the following result.

Theorem 7.1. *If x is a critical point of J, then there exists a constant c such that*

$$\Phi(t) + c = \int_{-r}^{0} g_v(t-\theta, L_1(t-\theta)x_{t-\theta}, L_2(t-\theta)\dot{x}_{t-\theta})\,d\eta_2(t-\theta,\theta), \qquad (7.2)$$

for $a \le t \le b-r$, and

$$\Psi(t) + c = \int_{t-b}^{0} g_v(t-\theta, L_1(t-\theta)x_{t-\theta}, L_2(t-\theta)\dot{x}_{t-\theta})\, d\eta_2(t-\theta, \theta), \tag{7.3}$$

for $b-r \le t \le b$, where Φ and Ψ are defined later in equations (7.8) and (7.9), respectively.

Proof. If x is a critical point of J, the directional derivative of J at x in the direction of all admissible variations ξ must be 0. Explicitly, for small $\epsilon \in \mathbb{R}$, let $F(\epsilon) = J(x + \epsilon\xi)$, then, taking the derivative at $\epsilon = 0$, we see that

$$\frac{d}{dt}F(0) = D_x J(x)\xi \;=\; \int_a^b g_u(t, L_1(t)x_t, L_2(t)\dot{x}_t)\, L_1(t)\xi_t\, dt \tag{7.4}$$

$$+ \int_a^b g_v(t, L_1(t)x_t, L_2(t)\dot{x}_t)\, L_2(t)\dot{\xi}_t\, dt. \tag{7.5}$$

We need now to introduce the explicit form of the linear operators in (7.4,7.5). We get

$$\begin{aligned} D_x J(x)\xi \;=\; & \int_a^b \int_{-r}^0 g_u(t, L_1(t)x_t, L_2(t)\dot{x}_t)\, d\eta_1(t,\theta)\, \xi(t+\theta)\, dt \\ & + \int_a^b \int_{-r}^0 g_v(t, L_1(t)x_t, L_2(t)\dot{x}_t)\, d\eta_2(t,\theta)\, \dot{\xi}(t+\theta)\, dt. \end{aligned}$$

Interchanging the integrals in t and θ, we can change co-ordinates to $s = t+\theta$, keeping $\theta \in [-r, 0]$. We get

$$\begin{aligned} D_x J(x)\xi \;=\; & \int_{-r}^0 \int_{a+\theta}^{b+\theta} g_u(s-\theta, L_1(s-\theta)x_{s-\theta}, L_2(s-\theta)\dot{x}_{s-\theta})\, d\eta_1(s-\theta,\theta)\, \xi(s)\, ds \\ & + \int_{-r}^0 \int_{a+\theta}^{b+\theta} g_v(s-\theta, L_1(s-\theta)x_{s-\theta}, L_2(s-\theta)\dot{x}_{s-\theta})\, d\eta_2(s-\theta,\theta)\, \dot{\xi}(s)\, ds. \end{aligned}$$

Recalling that in general

$$\int_a^{b+\theta} f(t)\, dt = \int_a^{b-r} f(t)\, dt + \int_{b-r}^{b+\theta} f(t)\, dt,$$

we can use the properties of the admissible variations, ξ and $\dot{\xi}$ are 0 on $[a-r, a]$, to say that

$$D_x J(x)\xi \;=\; \int_{-r}^0 \int_a^{b+\theta} g_u(s-\theta, L_1(s-\theta)x_{s-\theta}, L_2(s-\theta)\dot{x}_{s-\theta})\, d\eta_1(s-\theta,\theta)\, \xi(s)\, ds \tag{7.6}$$

$$+ \int_{-r}^0 \int_a^{b+\theta} g_v(s-\theta, L_1(s-\theta)x_{s-\theta}, L_2(s-\theta)\dot{x}_{s-\theta})\, d\eta_2(s-\theta,\theta)\, \dot{\xi}(s)\, ds. \tag{7.7}$$

To apply the Fundamental Lemma of the Calculus of Variations (du Bois-Reymond), we need to express (7.6) in term of $\dot{\xi}$ using the integration by part formula,

$$\int_a^b f(t)\,\xi(t)\,dt = [F(t)\,\xi(t)]_a^b - \int_a^b F(t)\,\dot{\xi}(t)\,dt$$

where $\frac{d}{dt}F = f$ with $F(b-r) = 0$. We also need to interchange the range of integration between θ and s, from $\theta \in [-r, 0]$, $s \in [a, b+\theta]$ to $s \in [a, b-r]$, $\theta \in [-r, 0]$ and $s \in [b-r, b]$, $\theta \in [s-b, 0]$. The first part (7.6) of $D_x J(x)\xi$ becomes (we introduce $g_u(\dots)$ to simplify notations, the dots replace the arguments of (7.6)) or (7.7), respectively)

$$\begin{aligned}\int_{-r}^0 \int_a^{b+\theta} g_u(\dots)\,d\eta_1(s-\theta,\theta)\,\xi(s)\,ds \;&=\; \int_a^{b-r}\left[\int_{-r}^0 g_u(\dots)\,d\eta_1(s-\theta,\theta)\right]\xi(s)\,ds \\ &\quad + \int_{b-r}^b \left[\int_{s-b}^0 g_u(\dots)\,d\eta_1(s-\theta,\theta)\right]\xi(s)\,ds.\end{aligned}$$

We can now integrate by parts to get

$$\int_a^{b-r}\left[\int_{-r}^0 g_u(\dots)\,d\eta_1(s-\theta,\theta)\right]\xi(s)\,ds = [\Phi(s)\,\xi(s)]_a^{b-r} - \int_a^{b-r} \Phi(s)\,\dot{\xi}(s)\,ds$$

where

$$\Phi(s) = \int_{b-r}^s \int_{-r}^0 g_u(z-\theta, L_1(z-\theta)x_{z-\theta}, L_2(z-\theta)\dot{x}_{z-\theta})\,d\eta_1(z-\theta,\theta)\,dz, \tag{7.8}$$

and

$$\int_{b-r}^b \left[\int_{s-b}^0 g_u(\dots)\,d\eta_1(s-\theta,\theta)\right]\xi(s)\,ds = [\Psi(s)\,\xi(s)]_{b-r}^b - \int_{b-r}^b \Psi(s)\,\dot{\xi}(s)\,ds$$

where

$$\Psi(s) = \int_{b-r}^s \int_{s-r}^0 g_u(z-\theta, L_1(z-\theta)x_{z-\theta}, L_2(z-\theta)\dot{x}_{z-\theta})\,d\eta_1(z-\theta,\theta)\,dz. \tag{7.9}$$

We need to split (7.7) to compare the integrals. Because $\Phi(b-r) = \Psi(b-r) = \xi(a) = \xi(b) = 0$, we get the following

$$\begin{aligned}D_x J(x)\xi \;&=\; \int_a^{b-r}\left[-\Phi(s) + \int_{-r}^0 g_v(\dots)\,d\eta_2(s-\theta,\theta)\right]\dot{\xi}(s)\,ds \\ &\quad + \int_{b-r}^b \left[-\Psi(s) + \int_{s-b}^0 g_v(\dots)\,d\eta_2(s-\theta,\theta)\right]\dot{\xi}(s)\,ds.\end{aligned}$$

Using the fundamental lemma, we can conclude because the integrands must vanish. □

We can then deduce the following corollary.

Corollary 7.2. *If x is a critical point of J, then x must satisfy the following MFDE*

$$\begin{aligned} \int_{-r}^{0} g_u(t-\theta, L_1(t-\theta)x_{t-\theta}, L_2(t-\theta)\dot{x}_{t-\theta})\, d\eta_1(t-\theta,\theta) &= \\ \frac{d}{dt}\left[\int_{-r}^{0} g_v(t-\theta, L_1(t-\theta)x_{t-\theta}, L_2(t-\theta)\dot{x}_{t-\theta})\, d\eta_2(t-\theta,\theta)\right], \end{aligned} \tag{7.10}$$

for $a \le t \le b-r$, and

$$\begin{aligned} \int_{t-r}^{0} g_u(t-\theta, L_1(t-\theta)x_{t-\theta}, L_2(t-\theta)\dot{x}_{t-\theta})\, d\eta_1(t-\theta,\theta) &= \\ \frac{d}{dt}\left[\int_{t-b}^{0} g_v(t-\theta, L_1(t-\theta)x_{t-\theta}, L_2(t-\theta)\dot{x}_{t-\theta})\, d\eta_2(t-\theta,\theta)\right], \end{aligned} \tag{7.11}$$

for $b-\tau \le t \le b$.

Proof. The result follows from differentiating (7.2) and (7.3). □

7.0.3 Example

When we have a single delay, we basically recover the results of Hughes ([33]). Let

$$J(y) = \int_a^b f(t, y(t-\tau), y(t), \dot{y}(t-\tau), \dot{y}(t))\, dt, \tag{7.12}$$

be a nonlinear functional. We shall examine quadratic forms, that is homogeneous quadratic polynomials f in the variables x, y, q, r where $x(t) = y(t-\tau)$, $q(t) = \dot{x} = \dot{y}(t-\tau)$ and $r(t) = \dot{y}(t)$.

Corollary 7.3. *If y is a critical point of J, then y must satisfy the following MFDEs*

$$\begin{aligned} D_3 f(t, y_t(-\tau), y_t(0), \dot{y}_t(-\tau), \dot{y}_t(0)) + D_2 f(t+\tau, y_t(0), y_t(\tau), \dot{y}_t(0), \dot{y}_t(\tau)) &= \\ \frac{d}{dt}\left[D_5 f(t, y_t(-\tau), y_t(0), \dot{y}_t(-\tau), \dot{y}_t(0)) + D_4 f(t+\tau, y_t(0), y_t(\tau), \dot{y}_t(0), \dot{y}_t(\tau))\right], \end{aligned} \tag{7.13}$$

for $a \le t \le b-\tau$, and

$$D_3 f(t, y_t(-\tau), y_t(0), \dot{y}_t(-\tau), \dot{y}_t(0)) = \frac{d}{dt}\left(D_5 f(t, y_t(-\tau), y_t(0), \dot{y}_t(-\tau), \dot{y}_t(0))\right), \tag{7.14}$$

for $b-\tau \le t \le b$.

If y minimises J on the set of admissible functions, then the following relation holds at $t = b-\tau$:

$$\begin{aligned} D_5 f(b-\tau, y_b(-2\tau), y_b(-\tau), \dot{y}_b(-2\tau^-), \dot{y}_b(-\tau^-)) + D_4 f(b, y_b(-\tau), y_b(0), \dot{y}_b(-\tau^-), \dot{y}_b(0^-)) &= \\ D_5 f(b-\tau, y_b(-2\tau), y_b(-\tau), \dot{y}_b(-2\tau^+), \dot{y}_b(-\tau^+)). \end{aligned} \tag{7.15}$$

Proof. We cast (7.12) in the form (7.1) by defining $u = (x, y) \in \mathbb{R}^2$ and $v = (q, r) \in \mathbb{R}^2$, $r = \tau$, so that

$$g(t, u, v) = f(t, x, y, q, r)$$

and the operators $L_1 = L_2 = L : Y^- \to \mathbb{R}^2$ by

$$L\phi = \begin{pmatrix} \phi(-\tau) \\ \phi(0) \end{pmatrix} = \int_{-\tau}^{0} d\eta(\theta)\, \phi(\theta)$$

where $d\eta(\theta)$ is a two dimensional vector $(\delta_{-\tau}(\theta), \delta_0(\theta))$ of measures of mass 1 at $\theta = -\tau$ and $\theta = 0$.

We can now use Corollary 7.2. The derivatives g_u and g_v are vectors in $\mathbb{R}^2$ given by $(D_2 f, D_3 f)$ and $(D_4 f, D_5 f)$, respectively. The integrals in θ means that the only non zero term is for $\theta = -\tau$ in the first component and for $\theta = 0$ in the second. The values of the operators are

$$L_1(t-\theta) y_{t-\theta} = \begin{cases} L y_{t+\tau} = (y(t), y(t+\tau))\,, & \theta = -\tau; \\ L y_t = (y(t-\tau), y(t))\,, & \theta = 0, \end{cases}$$

with similar results for $L_2(t-\theta)\dot{y}_{t-\theta}$, replacing y by $\dot{y}$. Therefore the first component of the term in g_u, for $\theta = -\tau$, becomes

$$D_2 f(t+\tau, y(t), y(t+\tau), \dot{y}(t), \dot{y}(t+\tau))$$

and the second, for $\theta = 0$,

$$D_3 f(t, y(t-\tau), y(t), \dot{y}(t-\tau), \dot{y}(t)).$$

For the term in g_v, we exchange D_2 for D_4 and D_3 for D_5.

At $t = b - \tau$ we have compatibility conditions between (7.6) and (7.7). The function y is continuous, but its derivative is not. So we need to take limits on the left and on the right at $t = b - \tau$: the left limit of (7.6) must be equal to the right limit of (7.7). Note that the integral terms vanish because the boundaries are identical and the integrand are continuous, therefore we get (7.15). □

7.0.4 Quadratic Objective Functions

Now consider the general constant coefficient quadratic form

$$f(t, x, y, q, r) = a_1 x^2 + a_2 xy + a_3 xq + a_4 xr + b_1 y^2 + b_2 yq + b_3 yr + c_1 q^2 + c_2 qr + d_1 r^2, \qquad (7.16)$$

where the coefficients are constant of time t. We get the following result.

Lemma 7.4. *Given $\tau > 0$, let L^+, L^- and L^0 be the following operators from $(C[-\tau, \tau], \mathbb{R})$ to $\mathbb{R}$:*

$$L^+\phi = \phi(\tau) + \phi(-\tau), \quad L^-\phi = \phi(\tau) - \phi(-\tau), \quad L^0\phi = \phi(0).$$

Let $p(t) = \dot{y}(t)$ and $x = (y, p)$, then the Euler-Lagrange equation (7.13) can be written as a neutral MFDE

$$\frac{d}{dt}h(x_t) = g(x_t), \tag{7.17}$$

where

$$h(x_t) = \begin{bmatrix} L^0 & 0 \\ 0 & c_2L^+ + 2(c_1 + d_1)L^0 \end{bmatrix} \begin{bmatrix} y_t \\ p_t \end{bmatrix}$$

and

$$g(x_t) = \begin{bmatrix} 0 & L^0 \\ a_2L^+ + 2(a_1 + b_1)L^0 & (a_4 - b_2)L^- \end{bmatrix} \begin{bmatrix} y_t \\ p_t \end{bmatrix}.$$

Proof. The Euler-Lagrange equation (7.13) is given by

$$\begin{aligned} a_2x + 2b_1y + b_2q + b_3r + 2a_1y + a_2z + a_3\dot{y} + a_4\dot{z} &= \\ \frac{d}{dt}\left[a_4x + b_3y + c_2q + 2d_1r + a_3x + b_2y + 2c_1q + c_2r\right], \end{aligned} \tag{7.18}$$

that is

$$\begin{aligned} a_2y(t-\tau) + 2(a_1 + b_1)y(t) + b_2\dot{y}(t-\tau) + a_2y(t+\tau) + a_4\dot{y}(t+\tau) &= \\ a_4\dot{y}(t-\tau) + c_2\ddot{y}(t-\tau) + 2d_1\ddot{y}(t) + b_2\dot{y}(t+\tau) + 2c_1\dot{y}(t-\tau) + c_2\ddot{y}(t+\tau). \end{aligned} \tag{7.19}$$

Revisiting the second order equation (7.19), we reduce it to a system of first order equations by letting $p = \dot{y}$, and so $\dot{p} = \ddot{y}$. Then,

$$\begin{aligned} a_2y(t-\tau) + 2(a_1 + b_1)y(t) + b_2p(t-\tau) + a_2y(t+\tau) + a_4p(t+\tau) \\ = a_4p(t-\tau) + c_2\dot{p}(t-\tau) + 2d_1\dot{p}(t) + b_2p(t+\tau) + 2c_1\dot{p}(t) + c_2\dot{p}(t+\tau), \end{aligned}$$

simplifies in operator form as

$$c_2L^+\dot{p}_t + 2(c_1 + d_1)\dot{p}(t) = (a_4 - b_2)L^-p_t + a_2L^+y_t + 2(a_1 + b_1)y(t).$$

which may wholly be written in operator form as (7.17). $\square$

In order to obtain a first order MFDE from (7.19) we require $c_1 = c_2 = d_1 = 0$, giving the simplified form

$$(a_4 - b_2)[\dot{y}(t+\tau) - \dot{y}(t-\tau)] = -2(a_1 + b_1)y(t) - a_2[y(t-\tau) + y(t+\tau)]. \tag{7.20}$$

Equation (7.19) rearranges to a neutral MFDE.

7.0.5 Optimising One Dimensional DDEs with Harvesting

Here we shall concentrate on optimal harvesting problems. Consider the harvesting model with delay represented by the DDE

$$\frac{d}{dt}N = g(t, N_t) - h(t, N) \tag{7.21}$$

where $N(t)$ is the population density vector, g represents the dynamics of the populations without the harvesting strategy h and τ is a delay. The goal is to maximise the exploitation of resource across a period of time $[t_0, t_1]$. This can be interpreted in many different ways; for example, maximising catches in fisheries or income under some initial and final conditions (such as periodic orbits).

Here we shall consider simple cases corresponding to a Bolza problem. The goal is to maximise catches H over $[t_0, t_1]$ assuming that N is known over $[t_0 - \tau, t_0]$ and at $t = t_1$. Re-writing (7.21), we have

$$h(t, N) = g(t, N_t) - \frac{d}{dt}N(t).$$

Considering the interval from t to $t + dt$,

$$\Delta H = H(t + \Delta t, N) - H(t, N) \approx h(t, N)\, dt.$$

The amount harvested from t_0 to t_1 is thus given by

$$H(N) = \int_{t_0}^{t_1} h(t, N)\, dt = \int_{t_0}^{t_1} [g(t, N_t) - \dot{N}(t)] dt, \tag{7.22}$$

which is the functional we wish to maximise.

In general, as in Theorem 1.2 of [33], we have then

$$H(N) = \int_{t_0}^{t_1} f((t, N(t-\tau), N(t), \dot{N}(t-\tau), \frac{d}{dt}N(t))\, dt. \tag{7.23}$$

A selection of Predator-Prey models with delay and harvesting is provided in [53] and include the following examples. In one dimension, let r be the intrinsic birth rate of a population and K the carrying capacity of its environment (both depending on time), the delayed Verhulst logistic versions of (7.21) are, either,

$$g(t, N_t) = r(t-\tau)N_t \left[1 - \frac{N(t)}{K(t)}\right], \tag{7.24}$$

if there is competition at the current time between the young and the adult population, or

$$g(t, N_t) = r(t)N_t\left[1 - \frac{N_t}{K(t-\tau)}\right], \tag{7.25}$$

if there is competition between adults at reproduction time $t-\tau$.

The critical points satisfy the following condition.

Corollary 7.5 (Hughes, [33])**.** *If x is a minimum of J, then x must satisfy the following MFDE*

$$\begin{aligned} D_3 f(t, x_t(-\tau), x_t(0), \dot{x}_t(-\tau), \dot{x}_t(0)) + D_2 f(t+\tau, x_t(0), x_t(\tau), \dot{x}_t(0), \dot{x}_t(\tau)) &= \\ \frac{d}{dt}\left[D_5 f(s, x_s(-\tau), x_s(0), \dot{x}_s(-\tau), \dot{x}_s(0)) + D_4 f(s+\tau, x_s(0), x_s(\tau), \dot{x}_s(0), \dot{x}_s(\tau))\right]&, \end{aligned} \tag{7.26}$$

for $a \leq t \leq b-\tau$, and

$$D_3 f(t, x_t(-\tau), x_t(0), \dot{x}_t(-\tau), \dot{x}_t(0)) = \frac{d}{dt}\left(D_5 f(t, x_t(-\tau), x_t(0), \dot{x}_t(-\tau), \dot{x}_t(0))\right), \tag{7.27}$$

for $b-\tau \leq t \leq b$.

If x minimises J on the set of admissible functions, then the following relation holds at $t = b-\tau$:

$$\begin{aligned} D_5 f(b-\tau, x_b(-2\tau), x_b(-\tau), \dot{x}_s(-2\tau^-), \dot{x}_s(-\tau^-)) + D_4 f(b, x_b(-\tau), x_b(0), \dot{x}_b(-\tau^-), \dot{x}_b(0^-)) \\ = D_5 f(b-\tau, x_b(-2\tau), x_b(-\tau), \dot{x}_s(-2\tau^+), \dot{x}_s(-\tau^+)). \end{aligned} \tag{7.28}$$

7.0.6 The Logistic Equation

Proposition 7.6. *When the Lagrangian g is given by (7.24), then the equations (7.13,7.14) can be applied to yield the difference equation*

$$N(t+\tau) = K - \frac{r(t)}{r(t+\tau)}N(t-\tau). \tag{7.29}$$

Proof. From (7.22) and (7.14), the right-hand side of the Euler-Lagrange equation for (7.23) is equal to $\frac{d}{dt}(-1+0) = 0$. The two derivatives in (7.13) give

$$-r(t)\frac{N(t-\tau)}{K} + r(t+\tau)\left[1 - \frac{N(t+\tau)}{K}\right] = 0, \tag{7.30}$$

which may be rewritten as (7.29). $\square$

Note that for equation (7.25), we find directly the solutions of $f_{N(t-\tau)} = 0$, that corresponds to the classical constant coefficient result $N(t) = \dfrac{K}{2}$ ([17]). Finally, we can state a few results following from Proposition 7.6.

Theorem 7.7. *Suppose that $r \in C^1(\mathbb{R})$. Then the solution $N : \mathbb{R} \to \mathbb{R}$ of equation (7.29) satisfying $N(t) = \phi(t)$ for $t \in [-2\tau, 0]$, exists, is continuous and differentiable if and only if $\phi \in C^\infty[-2\tau, 0]$ and*

$$\phi(0) + \frac{r(-\tau)}{r(0)}\phi(-2\tau) = K. \tag{7.31}$$

Proof. Note that equation (7.29) yields $N(t)$ which can be differentiated and substituted to give the harvesting strategy, $H = f - \frac{d}{dt}N$. For example, in the case of proportional harvesting, $H(t, N) = eN$, the effort e required would be given by $e = \frac{f - \frac{d}{dt}N}{N}$. To check for continuity of the solution, let $\phi \in [0, 2\pi]$ be the history function, then $\phi(0)$ must equal $N(0)$. Stepping forward, (7.29) can be rewritten as

$$N(t) = K - \frac{r(t-\tau)}{r(t)}\phi(t - 2\tau), \qquad \text{for} \quad 0 \leq t \leq 2\tau. \tag{7.32}$$

Hence for continuity,

$$N(0) = K - \frac{r(-\tau)}{r(0)}\phi(-2\tau) = \phi(0),$$

which gives condition (7.31).

For the next interval $2\tau \leq t \leq 4\tau$, we have

$$\begin{aligned} N(t) &= K - \frac{r(t-\tau)}{r(t)}N(t-2\tau) \\ &= K - \frac{r(t-\tau)}{r(t)}\left[K - \frac{r(t-3\tau)}{r(t-2\tau)}\phi(t-4\tau)\right] \end{aligned}$$

on substitution from (7.32). For continuity at $t = 2\tau$, we require the following

$$N(2\tau) = K - \frac{r(\tau)}{r(2\tau)}\left[K - \frac{r(-\tau)}{r(0)}\phi(-2\tau)\right].$$

But,

$$K - \frac{r(\tau)}{r(2\tau)}\phi(0) = K\left(1 - \frac{r(\tau)}{r(2\tau)}\right) + \frac{r(\tau)r(-\tau)}{r(2\tau)r(0)}\phi(-2\tau).$$

Rearranging, we have

$$K\frac{r(\tau)}{r(2\tau)} = \frac{r(\tau)}{r(2\tau)}\phi(0) + \frac{r(\tau)r(-\tau)}{r(2\tau)r(0)}\phi(-2\tau),$$

which is the same as (7.31). Note that $N(t)$ is differentiable if both $\phi(t)$ and $r(t)$ are differentiable. □

7.0.7 Periodic Solutions

For a solution to be $2\tau-$periodic, we require $N(t) = N(t - 2\tau)$. We obtain the following result

Lemma 7.8. *For the equation*

$$N(t) = K - \frac{r(t-\tau)}{r(t)} N(t-2\tau)$$

to have a periodic solution, we need

$$\phi(0) = K\left[1 + \frac{r(\tau)}{r(0)}\right]^{-1} = K\frac{r(0)}{r(0) + r(-\tau)}$$

Proof. For $0 \leq t \leq 2\tau$,

$$N(t) = K - \frac{r(t-\tau)}{r(t)}\phi(t-2\tau) = \phi(t-2\tau).$$

This gives

$$\phi(t-2\tau) = K\left[1 + \frac{r(t-\tau)}{r(t)}\right]^{-1}. \tag{7.33}$$

Hence,

$$N(t) = K - \frac{r(t-\tau)}{r(t)}.K\left[1 + \frac{r(t-\tau)}{r(t)}\right]^{-1} = K\frac{r(t)}{r(t) + r(t-\tau)}$$

From equation (7.33)

$$\phi(0) = K\left[1 + \frac{r(\tau)}{r(2\tau)}\right]^{-1} = K\frac{r(2\pi)}{r(\tau) + r(2\tau)}$$

. We also require r to be $2\tau-$periodic, which therefore yields

$$\phi(0) = K\left[1 + \frac{r(\tau)}{r(0)}\right]^{-1} = K\frac{r(0)}{r(0) + r(-\tau)}.$$

□

Chapter 8

Conclusions and Future Work

The use of functional differential equations and by extension, MFDEs, has become prevalent in recent years. They are applied to control, biological and economic systems. Some of the systems have an inherent symmetric nature, such as the cyclical arrangement of coupled cells in a network. It is necessary to study the existence and stability of periodic solutions of such systems. The effects of symmetries on such systems lead to many interesting patterns of oscillation. In this work, our focus has been on the question of existence and uniqueness of solutions, the reversibility and equivariance of NMFDEs and bifurcation, using the center manifold and equivariant Lyapunov-Schmidt reduction techniques.

8.1 Results

In this chapter, we give a brief summary of some of the results that we have obtained thus far:

We have proven the existence and uniqueness of solution of an MFDEs with asymmetrical constant deviating arguments, as an initial value problem. We used and extended the analysis to the challenging case of distributed delayed and advanced arguments, showing that the method can be applied if the history function has compact support. We also study the infinitesimal generator of the semi-group associated to the generalised autonomous MFDE and its spectral analysis.

Our other main contribution was the develop a reversible-equivariant theory for NMFDEs, laying emphasis on $\mathbb{D}_n-$reversible-equivariant systems. We obtained the matricial structures and conditions that are necessary for an NMFDE system to be $\mathbb{D}_n$, $\mathbb{Z}_n$ and $\mathbb{Z}_2$ reversible-equivariant. We applied the results to a system of ring networks of cyclically arranged identical cells with forward and backward coupling.

For bifurcation, we explored the occurrence of Hopf bifurcation resulting from the actions of $\mathbb{D}_n$, $\mathbb{Z}_n$ and $\mathbb{Z}_2$. The Hopf bifurcation of NMFDEs was analysed using the centre manifold and Lyapunov-Schmidt reduction processes. We explored the symmetries and reversing symmetries of the MFDE and developed the equivariant Lyapunov-Schmidt reduction to explore the existence of periodic solutions. Furthermore, we carried out a unfolding of an NMFDE under the Bogdanov-Takens bifurcation using the center manifold reduction.

Optimal control systems amongst others in economics motivated the studies of variational problems with delayed arguments. We studied the problem of determining the necessary and sufficient conditions for optimality in variational problems with delayed arguments. We obtained the critical points of symmetric functionals with distributed delays from which the resulting Euler-Lagrange equations yield MFDEs, and thereby extend the work of Hughes. The Euler-Lagrange equations ensuing from the optimisation of the logistic equation yielded a difference equation.

8.2 Future Work

We now list a number of possible extensions to the work undertaken in this thesis:

1. The existence and uniqueness of solutions of the equation (2.12) of Chapter 2 was shown. A possible extension would be to study the effects of imposing symmetries and reversibilities on the matrices A and B and on bifurcation analyses thereof.

2. Buono et al. in [11] establish a general theory for the equivariant versal unfolding of DDEs. We studied the versal unfolding of an NMFDE with Bogdanov-Takens bifurcation. Further work needs to be done to establish the theory for the equivariant versal unfolding of MFDEs.

3. Lattice differential equations (LDEs) are systems of ordinary differential equations with a discrete spatial structure and have inherent symmetry properties. LDEs naturally lead to MFDEs. Georgi [26] studies bifurcations from homoclinic orbits in reversible lattice differential equations whilst Chow et al. [14] study propagation failure and lattice induced anisotropy for traveling wave on LDEs. The reversibility , equivariance and bifurcation analyses of such system would be an important area to explore.

4. The effects of reversibility and symmetries on the Euler-Lagrange equations resulting from the optimisation of functionals with delayed arguments is an area that needs further studying.

Appendix A

Functions of Bounded Variation

This text is adapted from Verduyn Lunel [65].

In line with general usage in the study of delay equations we shall work with kernels of bounded variation. A partition $\underline{\sigma}(x)$ (of length n) of $[0, x]$ is a finite ordered set $(\sigma_0, \ldots, \sigma_n) \in \mathbb{R}^n$ such that $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_n = x$. The width of the partition is $\mu(\sigma(x)) = \max_{1 \leq j \leq n}(\sigma_j - \sigma_{j-1})$. Given x, $P(x)$ is the set of all partitions of any length of $[0, x]$.

Let $f; \mathbb{R}_+ \to \mathbb{R}$ be a given function. The total variation function $V(f)$ is defined by

$$V(f)(x) = \sup_{\underline{\sigma} \in P(x)} \sum_{j=1}^{n} |f(\sigma_j) - f(\sigma_{j-1})|.$$

In general, for $0 \leq x \leq y < \infty$,

$$0 \leq V(f)(x) \leq V(f)(y) < \infty. \tag{A.1}$$

If $V(f)$ is a bounded function, then (A.1) implies that

$$T(f) = \lim_{x \to \infty} V(f)(x)$$

exists and is finite. In that case we say that f is of bounded variation, in short $f \in BV$, and we call $T(f)$ the total variation of f. A complex function f is called of bounded variation if and only if its real and complex parts are in BV. A vector-valued function f is called of bounded variation if and only if all components of f are of bounded variation. If both g and h are non-decreasing bounded functions then $f = g - h$ is of bounded variation. Actually the following result shows that this property can be used to give an equivalent definition.

Theorem A.1 (Titchmarsh [61]). *If $f : R_+ \to \mathbb{R}^n$ is of bounded variation, then f can be expressed in the form*

$$f = g - h,$$

where both g and h are non-decreasing bounded functions.

The next theorem explains the importance of the class $NBV[\mathbb{R}_+]$ and makes it possible to apply abstract integration theory. To formulate the result, recall the definition of a Borel measure. A Borel measure is a measure μ defined on the σ-ring generated by the compact subsets of $\mathbb{R}$ and such that $\mu(K) < \infty$ for every compact subset K of $\mathbb{R}$.

Theorem A.2. *If $f, g \in NBV[\mathbb{R}_+]$ and if μ_f denotes the Borel measure corresponding to f. Then,*

1. *Suppose $\{\phi_j\}_{j=1}^{\infty}$ is a sequence of complex measurable functions on $\mathbb{R}_+$ such that*

$$\phi(x) = \lim_{j\to\infty} \phi_j(x)$$

exists for every $x \in \mathbb{R}_+$. If there exists a function $\chi \in L^1(\mu_f)$ such that for every j

$$|\phi_j(x)| \leq \chi(x), \text{ a.e.}$$

with respect to μ_f, then $\phi \in L^1(\mu_f)$ and

$$\lim_{j\to\infty} \int_{\mathbb{R}_+} |\phi - \phi_j| \, df = 0.$$

2. *Let ϕ be a Borel measurable function on $\mathbb{R}_+ \times \mathbb{R}_+$. Suppose that*

$$\int_{\mathbb{R}_+} |df(x)| \int_{\mathbb{R}_+} |\phi(x,y)| \, dg(y) < \infty,$$

then

$$\int_{\mathbb{R}_+} df(x) \int_{\mathbb{R}_+} \phi(x,y) \, dg(y) = \int_{\mathbb{R}_+} dg(x) \int_{\mathbb{R}_+} \phi(x,y) \, df(y).$$

3. *If ϕ is a continuous bounded function on $\mathbb{R}_+$. Then, for all finite intervals $[a,b]$,*

$$\int_a^b \phi \, df = [\phi(x) f(x)]_a^b - \int_a^b f \, f d\phi.$$

Moreover,

$$\int_a^b \phi \, df \leq \sup_{x\in[a,b]} |\phi(x)| \left(V(f)(b) - V(f)(a)\right).$$

Define the subclass $NBV[a,b]$ of $NBV[\mathbb{R}_+]$ by

$$NBV[a,b] = \{ \ f \in NBV[\mathbb{R}_+] \ : \ f(t) = 0, \ t \leq a, \quad f(t) = f(b), \ t \geq b \ \}$$

and use for $f \in NBV[a,b]$ the following convention

$$\int_a^b \phi \, df = \int_{\mathbb{R}_+} \phi \, d\mu_f.$$

Because of Theorem A.23, for every $f \in NBV[a,b]$, the mapping

$$\phi \mapsto \int_a^b \phi \, df$$

defines a continuous linear functional on $C[a,b]$.

A.1 The Riesz Representation Theorem

Theorem A.3 (Riesz Representation Theorem, [54], 6.19)**.** *Let L be a continuous linear functional on $C[a,b]$. There exists a unique $f \in NBV[\mathbb{R}_+]$ such that, for all $\phi \in C[a,b]$,*

$$L(\phi) = \langle \phi, L \rangle = \int_a^b \phi \, df. \tag{A.2}$$

and $\|L\| = T(f)$.

In order to to be able apply this to delay equations note the following conventions. Write $\int df \phi$ instead of $\int \phi \, df$, and if ϕ is a $\mathbb{C}^n$-valued function with values as column-vectors and those of f as row vectors, then take $\int df \phi$ to denote

$$\sum_{j=1}^{n} \int df_j \phi_j = \sum_{j=1}^{n} \int \phi_j df_j. \tag{A.3}$$

Also every continuous linear mapping from $C([a,b], \mathbb{C}^n)$ into $\mathbb{C}^n$ can be uniquely represented by

$$\phi \mapsto \int_a^b d\zeta \phi, \tag{A.4}$$

where ζ is a $n \times n$-matrix whose elements belong to $NBV[a,b]$.

A.2 Examples

Next consider a linear system of autonomous retarded (delay) RFDEs:

$$\dot{x}(t) = \int_0^h d\zeta(\theta) \, x(t-\theta), \quad t \geq 0, \tag{A.5}$$

satisfying the initial condition

$$x(t) = \phi(t), \quad -h \leq t \leq 0, \tag{A.6}$$

where the matrix-valued function ζ belongs to $NBV[0,h]$ and the initial condition ϕ is a given continuous function, in short $\phi \in C[-h,0]$. In the study of the behaviour of the solution of the above system of RFDEs, it turns out to be useful to rewrite the problem as a Volterra convolution integral equation (or, as it is frequently called, a renewal equation). We split up the integral to separate the part involving the known ϕ from the part involving the unknown x:

$$\begin{aligned}\dot{x}(t) &= \int_0^t d\zeta\, x(t-\theta) + \int_t^h d\zeta(\theta)\, \phi(t-\theta) \\ &= -\int_0^t d_\theta \zeta(t-\theta)\, x(\theta) - \int_{-h}^0 d_\theta \zeta(t-\theta)\, \phi(\theta)\end{aligned}$$

(recall that ζ is defined to be constant on $[h, \infty)$). Next we integrate from 0 to t and obtain

$$x(t) - \phi(0) = -\int_0^t \int_0^\sigma d_\theta \zeta(\sigma-\theta)\, x(\theta)\, d\sigma - \int_0^t \int_{-h}^0 d_\theta \zeta(\sigma-\theta)\, \phi(\theta)\, d\sigma.$$

So, because of Theorem A.22

$$\begin{aligned}x(t) - \phi(0) &= -\int_0^t d_\theta \int_\theta^t \zeta(\sigma-\theta)\, d\sigma x(\theta) - \int_{-h}^0 d_\theta \int_\theta^t \zeta(\sigma-\theta)\, d\sigma \phi(\theta) \\ &= -\int_0^t \zeta(t-\theta)\, x(\theta)\, d\theta + \int_{-h}^0 (\zeta(t-\theta) - \zeta(-\theta))\, \phi(\theta)\, d\theta.\end{aligned}$$

We summarize the end result of our manipulations as follows. The solution x of (A.5) satisfies the renewal equation

$$x - \zeta * x = f,$$

where by definition

$$f(t) = \phi(0) + \int_{-h}^0 (\zeta(t-\theta) - \zeta(-\theta))\, \phi(\theta)\, d\theta. \tag{A.7}$$

A.2.1 Remarks on the Example

1. The function f defined in equation (A.7) is constant for $t \geq h$.

2. The function f defined by (A.7) is absolutely continuous. Actually, for $\phi \in C[-h,0]$,

$$\dot{f}(t) = \int_t^h d\zeta(\theta)\, \phi(t-\theta) \tag{A.8}$$

   is well-defined and even of bounded variation.

3. The formula (A.7) makes perfect sense if $\phi(0)$ is given as an element of $\mathbb{R}^n$ while $\phi(\theta)$ for $\theta \in [-h, 0]$ is given as an integrable function. Moreover, in [20] it is proved that the mapping defined by equation (A.8) has a continuous extension to a mapping from $L^1[-h, 0] \to L^1[-h, 0]$. So f is still absolutely continuous, although there is no explicit formula for $\dot{f}$ anymore.

4. Partial integration shows that the derivative of the solution of the linear autonomous RFDE (A.5) also satisfies a renewal equation of the form

$$\dot{x} - \zeta * \dot{x} = h,$$

where h is defined on $[0, 1)$ and is constant on the interval $[h, 1)$. See Chapter 12 of [65] for detailed results about the close connection between delay and renewal equations.

Appendix B

Circulant Diagonalisation

The action of $\mathbb{Z}_n$ on $\mathbb{K}^n$ is a typical symmetry for a ring network. In this section we state and show known results about such action of $\mathbb{Z}_n$ on $\mathbb{K}^n$.

B.1 Shift and Fourier Matrices

The shift to the right ρ is defined as the permutation on $\mathbb{K}^n$

$$\rho(x_1, \ldots, x_n) = (x_n, x_1, \ldots, x_{n-1}). \tag{B.1}$$

It generates an action of $\mathbb{Z}_n$ on $\mathbb{K}^n$. It acts as the matrix of coefficients $\rho_{ij} = \delta_{1(i-j)}$ where δ is the Kronecker symbol and $(i-j)$ is calculated modulo n..

To analyse the spectral and diagonalisation properties of ρ, we denote by $\rho_n = \exp \frac{2\pi i}{n}$ the primary n^{th}-root of unity. We define the Fourier matrix F_n (of order n), the matrix with entries

$$(F_n)_{ij} = \frac{1}{\sqrt{n}} \rho_n^{i \cdot j}.$$

Lemma B.1. *The matrix ρ is orthogonal of eigenvalues*

$$\lambda_k = \rho_n^{n-k}, \quad 1 \leq k \leq n,$$

with corresponding eigenvectors

$$v_k = \left((F_n)_{1k}, \cdots, (F_n)_{nk}\right), \quad 1 \leq k \leq n.$$

The matrix F_n realises the diagonalisation of ρ,

$$F^* \rho F = \operatorname{diag}(\rho_n^{n-1}, \rho_n^{n-2}, \cdots, 1).$$

Proof. To show that ρ is orthogonal, we need to check that $\rho\rho^T = I_n$. We can calculate that

$$(\rho\rho^T)_{ij} = \sum_{k=1}^{n} \rho_{ik}(\rho^T)_{kj} = \sum_{k=1}^{n} \rho_{ik} \cdot \rho_{jk} = \sum_{k=1}^{n} \delta_{1(i-k)} \cdot \delta_{1(j-k)} = \delta_{ij} = (I)_{ij}$$

To analyse the eigenvalues and eigenvectors of ρ, we note first that when ζ is a n^{th}-root of unity such that $\rho \neq 1$,

$$\sum_{i=0}^{n-1} \zeta^i = \frac{1-\zeta^n}{1-\zeta} = 0.$$

Second, $F_n^{-1} = F_n^*$ because

$$(F_n^* F_n)_{lm} = \sum_{k=1}^{n} (F_n^*)_{lk}(F_n)_{km} = \frac{1}{n}\sum_{k=1}^{n} \bar{\rho_n}^{lk} \rho_n^{km} = \frac{1}{n}\sum_{k=1}^{n} \rho_n^{k(m-l)} = \delta_{lm},$$

because ρ_n^{m-l} is a root of unity when $l \neq m$.

Next, for $1 \leq l, m \leq n$, we calculate

$$\begin{aligned}
(F_n^* \rho F_n)_{lm} &= \sum_{j=1}^{n} (F_n^* \rho)_{lj} (F_n)_{jm} = \sum_{j,k=1}^{n} (F_n^*)_{lk} \rho_{kj} (F_n)_{jm} \\
&= \frac{1}{n} \sum_{j,k=1}^{n} \bar{\rho_n}^{lk} \delta_{1(k-j)} \rho_n^{jm} = \frac{1}{n} \sum_{j=1}^{n} \bar{\rho_n}^{l(j+1)} \rho_n^{jm} = \frac{\bar{\rho_n}^{l}}{n} \left(\sum_{j=1}^{n} \rho_n^{j(m-l)} \right) = \bar{\rho_n}^{l} \delta_{lm},
\end{aligned}$$

therefore

$$F^* \rho F = \operatorname{diag}(\rho_n^{n-1}, \rho_n^{n-2}, \cdots, 1).$$

Let e_k be a unit vector in $\mathbb{K}^n$, the eigenvector corresponding to λ_k is

$$v_k = F_n e_k = \left((F_n)_{1k}, \ldots, (F_n)_{nk}\right), \quad 1 \leq k \leq n.$$

□

B.2 Circulant Matrices

A matrix M is circulant if each row is generated from a vector by a cyclic shift of the row above it. Let $m \in \mathbb{K}^n$, we denote by $M = \operatorname{circ}(m)$ the circulant matrix whose first row is m.

Lemma B.2. *A circulant matrix is ρ-equivariant. Moreover, M is circulant if and only if*

$$M = \sum_{i=1}^{n} M_{1i} \rho^{n+1-i}. \tag{B.2}$$

Proof. Note that $M_{ij} = m_{(j-i+1)}$.

Given $1 \leq i, j \leq n$, the ρ-equivariance of M follows from

$$\begin{aligned}(M\rho)_{ij} &= \sum_{k=1}^{n} M_{ik}\rho_{kj} = \sum_{k=1}^{n} m_{(k-i+1)}\delta_{1(k-j)} = m_{(j-(i-1)+1)} \\ &= \sum_{k=1}^{n} \delta_{1(i-k)}m_{(j-k+1)} = \sum_{k=1}^{n} \rho_{ik}M_{kj} = (\rho M)_{ij}.\end{aligned}$$

Clearly M given by the equation (B.2) is ρ-equivariant and so it is circulant. Note that, if N is any matrix,

$$(\rho N)_{jk} = \sum_{i=1}^{n} \rho_{jl}N_{lk} = \sum_{i=1}^{n} \delta_{1(j-l)}N_{lk} = N_{(j-1)k},$$

and so, by induction, modulo n,

$$(\rho^l)_{jk} = \rho_{(j-l+1)k}. \tag{B.3}$$

Now, if M is circulant, modulo n, and using (B.3),

$$\begin{aligned}M_{jk} &= m_{(k-j+1)} = M_{1(1+k-j)} = \sum_{i=1}^{n} M_{1i}\delta_{1(j+i-k)} = \sum_{i=1}^{n} M_{1i}\rho_{(j+i-k)k} \\ &= \sum_{i=1}^{n} M_{1i}\rho_{[(j-(n+1-i)+1]k} = \sum_{i=1}^{n} M_{1i}\rho_{(j+i-k)k} = \sum_{i=1}^{n} M_{1i}\rho_{[(j-(n+1-i)+1]k} \\ &= \sum_{i=1}^{n} M_{1i}(\rho^{n+1-i})_{jk} = \left(\sum_{i=1}^{n} M_{1i}(\rho^{n+1-i})\right)_{jk},\end{aligned}$$

and the conclusion follows. □

We now state a standard result on circulant matrices.

Theorem B.3 (The Circulant Diagonalisation Theorem)**.** *Let $m \in \mathbb{K}^n$, $M = \mathrm{circ}(m)$. The eigenvalues of M are*

$$\lambda_k = m_1 + \rho_n^k m_2 + \ldots + \rho_n^{(n-1)k} m_n, \quad 1 \leq k \leq n,$$

of eigenvector

$$v_k = \left(\rho_n^k, \rho_n^{2k}, \ldots, \rho_n^{k(n-1)}, 1\right), \quad 1 \leq k \leq n.$$

Proof. From (B.2), $M = \sum_{i=1}^{n} M_{1i}\rho^{n+1-i}$. We know that F_n diagonalise ρ, and so, it diagonalises any power ρ^l and they have the same eigenvectors v_k, $1 \leq k \leq n$. Therefore

$$(F_n)^* M F_n = \sum_{i=1}^{n} m_i (F_n)^* \rho^{n+1-i} F_n = \mathrm{diag}(\lambda_1, \cdots, \lambda_n),$$

where the eigenvalues are

$$\lambda_k = m_1 + \rho_n^k m_2 + \ldots + \rho_n^{(n-1)k} m_n, \quad 1 \leq k \leq n,$$

with the corresponding eigenvectors v_k. $\square$

B.2.1 Commutativity of M and $\mathbb{D}_n$

Consider now the following action on $\mathbb{K}^n$:

$$S(x_1, \ldots, x_n) = (x_n, x_{n-1}, \ldots, x_1),$$

that is, S acts as the matrix of coefficients $S_{ij} = \delta_{i(n-j+1)}$, $1 \leq i, j \leq n$.

Lemma B.4. $\rho S = S\rho^T$.

Proof. Calculate (modulo n), for $1 \leq j, k \leq n$,

$$(\rho S)_{jk} = \sum_{i=1}^{n} \rho_{ji} S_{ik} = \sum_{i=1}^{n} \delta_{1(j-i)} \delta_{i(n-k+1)} = \delta_{1(j-(n-k+1))} = \delta_{1(j+k-1)}.$$

Similarly,

$$(S\rho^T)_{jk} = \sum_{i=1}^{n} S_{ji} (\rho^T)_{ik} = \sum_{i=1}^{n} S_{ji} \rho_{ki} = \sum_{i=1}^{n} \delta_{j(n-i+1)} \delta_{1(k-i)} = \delta_{1(k-(n-j+1))} = \delta_{1(j+k-1)},$$

and the conclusion. $\square$

As a consequence, we can characterise any $\mathbb{D}_n$-equivariant matrix using the previous notation.

Theorem B.5 ($M - \mathbb{D}_n$ Equivariance)**.** *The matrix*

$$M = \sum_{i=1}^{n} m_i \rho^{n-i+1} \tag{B.4}$$

is $\mathbb{D}_n$-equivariant if and only if $(m_2, \ldots, m_n) = (m_n, \ldots, m_2)$.

Proof. We already know that such M in (B.4) is $\mathbb{Z}_n$-equivariant. It remains to impose that $MS = SM$. And so,

$$MS = \sum_{i=1}^{n} m_i \rho^{n-i+1} S = S \left(\sum_{i=1}^{n} m_i \rho^{-n+i-1} \right) = S \left(\sum_{i=1}^{n} m_i \rho^{i-1} \right) = S \left(I + \sum_{i=2}^{n} m_i \rho^{i-1} \right).$$

Hence, the conclusion. $\square$

Appendix C

The Implicit Function Theorem and the Lyapounov-Schmidt Reduction

C.1 Implicit Function Theorem (IFT)

One can use the Contraction Mapping Theorem (CMT) to prove the following general version of the IFT in higher dimensions. Let $(V, \| \|)$ be a normed space, we denote by $B_{v_0}(r)$ the closed ball of centre v_0 and radius r, that is, $B_{v_0}(r) = \{ \ v \in V \ : \ \|v - v_0\| \leq r \ \}$.

Theorem C.1 (Implicit Function Theorem; [15])**.** *Let X, Y, Λ be open subsets of Banach spaces and let $F : X \times \Lambda \to Y$ be a k-times continuously differentiable map on $X \times \Lambda$ such that*

1. *$F(x_0, \lambda_0) = 0$,*
2. *$D_x F(x_0, \lambda_0)$ is invertible.*

Then, there exist neighbourhoods $\lambda_0 \in L \subset \Lambda$, $x_0 \in U \subset X$ and $\bar{x} : L \to X$ such that

$$F(\bar{x}(\lambda), \lambda) = 0, \quad \forall \lambda \in L,$$

and $\bar{x}(x_0) = y_0$. Moreover,

1. *all the solutions of $F = 0$ inside $U \times L$ belong to the curve parametrised by λ: $\lambda \mapsto (\bar{x}(\lambda), \lambda)$,*
2. *the regularity of F determines the regularity of $\bar{x}$, more precisely, $\bar{x}$ has as many derivatives as F has.*

Actually, we can be much more precise when F has locally Lipschitz derivatives (like when F is twice continuously differentiable). Suppose, in addition to 1. and 2., that

1. *$L_0 > 0$ is a constant $0 < L_0$ such that $\|D_x F(x_0, \lambda_0)^{-1}\| \leq L_0$,*

2. *there exist Lipschitz constants for F and $D_x F$, say L_1, L_2, on some neighbourhoods $x_0 \in B_{x_0}$ and $\lambda_0 \in B_{\lambda_0}$, that is,*

$$\begin{aligned} \|F(u,\lambda) - F(v,\mu)\| &\leq L_1(\|u-v\| + \|\lambda - \mu\|), \\ \|D_x F(u,\lambda) - D_x F(v,\mu)\| &\leq L_2(\|u-v\| + \|\lambda - \mu\|), \end{aligned}$$

for all $u, v \in B_{x_0}$ and $\lambda, \mu \in B_{\lambda_0}$.

Then, for all $\theta \in (0,1)$, there exist $r_1(\theta), r_2(\theta) > 0$ and $\bar{x} : B_{\lambda_0}(r_1(\theta)) \to B_{x_0}(r_2(\theta))$ such that

$$F(\bar{x}(\lambda), \lambda) = 0, \quad \forall \lambda \in B_{\lambda_0}(r_1(\theta)),$$

with $\bar{x}(x_0) = y_0$ and

$$\|\bar{x}(\lambda) - \bar{x}(\mu)\| \leq \frac{L_0 L_1}{1-\theta} \|\lambda - \mu\|. \tag{C.1}$$

C.2 Bifurcation and Lyapunov-Schmidt (L-S) Reduction

The L-S reduction applies easily to the class of Fredholm maps. We are going to describe first linear Fredholm operators.

Definition C.2. *Given two Banach spaces X, Y and a linear operator $\mathcal{L} : X \to Y$, we say that $\mathcal{L}$ is Fredholm if it is bounded with finite-dimensional kernel* $\ker \mathcal{L}$ *and cokernel* $\mathrm{coker}\mathcal{L} = Y/\mathrm{im}\mathcal{L}$. *It follows, in particular, that* $\mathrm{im}\mathcal{L}$ *is a closed linear subspace of Y. With each Fredholm operator, we associate its Fredholm index as*

$$\mathrm{ind}\mathcal{L} = \dim\ker \mathcal{L} - \mathrm{dim}\mathrm{coker}\mathcal{L}. \tag{C.2}$$

We can now define general Fredholm maps.

Definition C.3. *We say that a nonlinear map $F : U \subset X \to Y$ of class C^r, $r \geq 1$, is a Fredholm map if $D_x F(x)$ is a Fredholm linear operator at every point $x \in U$. Note that, in that case,* $\mathrm{ind}(D_x F(x))$ *is constant on open component of U (see [59]).*

The notion of a germ (of a function, set etc.) is useful in our context when we are not concerned with the exact neighbourhood of definition of maps, but want to analyse the qualitative properties of maps and their perturbations.

Definition C.4. *Two functions F, G defined on two neighbourhoods of a point x_0 are (germ) equivalent if they coincide in a neighbourhood of x_0. A germ (of function) is an equivalent class under germ equivalence.*

Given two topological spaces V, W, we denote a germ $F : V \to W$ at $x_0 \in V$ by $F : (V, x_0) \to W$, even $F : (V, x_0) \to (W, f(x_0))$ if we want also to look at the germ structure in the target space near $f(x_0)$. We define in a similar manner the germs of set, varieties at a point (or even at a set S) as equivalence classes by the filtration of neighbourhood of the point (or the set S). Note that the zero-set of a germ is a germ of set. In an abstract way, germs at x_0 are equivalence classes of mathematical objects under the filtration by the neighbourhoods of x_0. They identify the properties that remain true whatever close we are from x_0 for every representative of the class. We can now state the L-S reduction.

When the operator $D_x F$ is invertible, using the IFT,

$$F(x, \lambda) = 0$$

has locally unique branches of solutions parametrised by λ. When $D_x F$ is singular, the solution set of $F(x, \lambda) = 0$ can be more complicated. To analyse such situation, we can use the Lyapounov-Schmidt (L-S) procedure to reduce the dimension of the problem by 'factoring out' the invertible part of $D_x F$. We collect here information on the L-S reduction technique to get a finite dimensional bifurcation equation whose solution set is in $1-1$ correspondence with the original bifurcation equation. Typically this technique is used on nonlinear equations in function spaces to obtain a finite dimensional reduced bifurcation equation. The method is basically a consequence of the IFT. As such, the Taylor series expansion of the reduced bifurcation equation is available and singularity theory helps to study it systematically. There are numerous exposition of the technique in the literature. We mention a recent one due to Kielhöfer [38] and classic references due to Vanderbauwhede [62], particularly when the problem is equivariant, and Chow and Hale [15] for a comprehensive use in various cases, using the language of classical nonlinear analysis. Those books have extensive discussions of the issues and references to other important work we are not mentioning here.

Theorem C.5 (Lyapounov-Schmidt Reduction; [15, 38, 62])**.** *Let $X, \widetilde{X}$ and Λ be real Banach spaces such that $X \hookrightarrow \widetilde{X}$ is a continuous imbedding. We assume that $F : (X \times \Lambda, (0,0)) \to (\widetilde{X}, 0)$ is a C^k-Fredholm map, $2 \leq k \leq \infty$, of finite index. Let P be a projector $\widetilde{X} \to \ker(D_x^o F)$.*

1. *There exists a unique C^k-function $\widetilde{w} : (\ker(D_x^o F) \times \Lambda, (0,0)) \to (\mathrm{im}(D_x^o F), 0)$ such that*

$$(I - P)F(v + \widetilde{w}(v, \lambda), \lambda) = 0. \tag{C.3}$$

2. *Define the reduced bifurcation function $f : (\ker D_x^o F \times \Lambda, 0) \to (\ker D_x^o F, 0)$ by*

$$f(v, \lambda) = PF(v + \widetilde{w}(v, \lambda), \lambda). \tag{C.4}$$

Then the germ at $(0,0)$ of the solution set of $F(x, \lambda) = 0$ is diffeomorphic to the germ at

$(0,0)$ *of the solution set of* $f(v,\lambda)=0$. *More precisely,* $(x,\lambda)=(v+\widetilde{w}(v,\lambda),\lambda)$ *is a solution of* $F(x,\lambda)=0$ *if and only if* (v,λ) *is a solution of* $f(v,\lambda)=0$.

Proof. 1. Let $\mathcal{L}=D_x^o F$. Because F is Fredholm of finite index,

(a) $0<\dim(\ker\mathcal{L})=\mathrm{codim}(\mathrm{im}\mathcal{L})<\infty$,

(b) $\widetilde{X}=\ker\mathcal{L}\oplus\mathrm{im}\mathcal{L}$.

Introduce the splitting $x=v+w$ in X where $v\in\ker\mathcal{L}$ and $w\in\mathrm{im}\mathcal{L}$, and define $H:(X\times\Lambda,(0,0))\to(\widetilde{X},0)$ by

$$H(v,w,\lambda)=(I-P)F(v+\widetilde{w}(v,\lambda),\lambda). \tag{C.5}$$

The derivative $D_wH(0,0,0)=(I-P)\mathcal{L}$ is a bijection. Using the IFT, all the solutions of $H(v,w,\lambda)=0$ near the origin are described by a unique C^k-function $\widetilde{w}:(\ker\mathcal{L}\times\Lambda,(0,0))\to(\mathrm{im}\mathcal{L}),0)$ such that $H(v,\widetilde{w}(v,\lambda),\lambda)=0$, that is (C.3).

2. The equation $F(x,\lambda)=0$ splits into

$$\begin{aligned} PF(v+w,\lambda) &= 0, && \text{(C.6)}\\ (I-P)F(v+w,\lambda) &= 0. && \text{(C.7)} \end{aligned}$$

In the first part we solved (C.7) for w as a function of (v,λ). Replacing w by $\widetilde{w}$ into equation (C.6), we obtain an equivalent equation $f(v,\lambda)=0$ where f is given by (C.4).

□

C.2.1 Equivariant L-S Reduction

When F is equivariant under the action of a compact group Γ, that is

$$F(\gamma x,\lambda)=\tilde{\lambda}F(x,\lambda),\quad \forall\gamma\in\Gamma, \tag{C.8}$$

where the actions λ and $\tilde{\lambda}$ of Γ on X and $\widetilde{X}$ are not necessarily the same, one can keep track of the symmetry on the kernel of $\mathcal{L}$ and have a Γ-equivariant f for the actions Γ induced on $\ker\mathcal{L}$ and $\mathrm{coker}\mathcal{L}$ (see [62]).

Theorem C.6 (Equivariant Lyapounov-Schmidt Reduction; [62]). *In addition to the context of Theorem C.5, suppose that (C.8) holds for the actions of a compact group* Γ *on* X *and* $\widetilde{X}$. *Then,*

1. $\ker\mathcal{L}$ *and* $\mathrm{im}\mathcal{L}$ *are globally* Γ*-invariant and we can choose* $\mathrm{coker}\mathcal{L}$ *to be globally* Γ*-invariant,*

2. *the solution* $\widetilde{w}$ *of (C.3) is* Γ*-equivariant, and so*

3. *the reduced bifurcation function f defined in (C.4) is* Γ*-equivariant with respect to the actions* γ *and* $\tilde{g}$*, resp., of* Γ *on* $\ker \mathcal{L}$ *and* $\mathrm{coker}\mathcal{L}$*, resp..*

Proof. Because F is Γ-equivariant, $\mathcal{L} = D_x^o F$ is Γ-equivariant: $\mathcal{L}\gamma v = \tilde{g}\mathcal{L}v$, $\forall v \in X$ and $\gamma \in \Gamma$.

1. If $v \in \ker \mathcal{L}$, $\mathcal{L}\gamma v = \tilde{g}\mathcal{L}v = 0$ and so $\gamma(\ker \mathcal{L}) \subset \ker \mathcal{L}$ for all $\gamma \in \Gamma$. Similarly, if $w \in \mathrm{im}\mathcal{L}$, there exists $v \in X$ such that $w = \mathcal{L}v$. Then, for any $\gamma \in \Gamma$, $\tilde{g}w = \tilde{g}\mathcal{L}v = \mathcal{L}\gamma v \in \mathrm{im}\mathcal{L}$, so $\mathrm{im}\mathcal{L}$ is globally Γ-invariant. Finally, we can Γ-average the projector $P : \widetilde{X} \to \ker \mathcal{L}$ to obtain an Γ-equivariant projector.

2. With the previous choices, the operator H in equation (C.5) is Γ-equivariant, and so, from the uniqueness of the solutions of (C.3), $\widetilde{w}$ is Γ-equivariant.

3. Finally, f in (C.4) is the composition of Γ-equivariant maps, and so is itself Γ-equivariant with respect to the actions g and $\tilde{g}$ of Γ on $\ker \mathcal{L}$ and $\mathrm{coker}\mathcal{L}$.

□

C.2.2 Bifurcation Equivalence and L-S Reduction

Bifurcation equivalence is the correct notion in dealing with the various choices when calculating the L-S reduction of bifurcation problems. This is shown in the following theorem that is an adaptation of the appendix of Vanderbauwhede in [37]. First we need to define what we mean by 'equivalence' of bifurcation maps. Let $f_1, f_2 : (\mathbb{R}^{n+l}, (0,0)) \to (\mathbb{R}^m, 0)$ be two finite dimensional Γ-equivariant bifurcation maps defined around the origin of $\mathbb{R}^{n+l}$. We say that f_1 and f_2 are bifurcation equivalent if there exist

1. a local, orientation preserving, diffeomorphism (X, L) of $(\mathbb{R}^{n+l}, (0,0))$, such that
   (a) $X : (\mathbb{R}^{n+l}, (0,0)) \to (\mathbb{R}^n, 0)$, $L : (\mathbb{R}^l, 0) \to (\mathbb{R}^l, 0)$ such that $D_x X(0,0)$ and $D_\lambda L(0)$ are in their respective components of the identity,
   (b) X is Γ-equivariant, that is, $X(\gamma x, \lambda) = \gamma X(x, \lambda)$ for all $\gamma \in \Gamma$,
2. a local matrix valued map $T : (\mathbb{R}^{n+l}, (0,0)) \to \mathrm{GL}(\mathbb{R}, m)$ such that
   (a) T is in the connected component of the identity,
   (b) T is Γ-equivariant, that is, $T(\gamma v, \lambda)\tilde{g} = \tilde{g}T(v, \lambda)$ for all $\gamma \in \Gamma$.

Two bifurcation equivalent maps have diffeomorphic bifurcation diagrams and behave similarly under perturbation. Moreover, this equivalence relation is well-adapted to the L-S reduction because of the following result.

Theorem C.7. *Let* $F : (X \times \Lambda, (0,0)) \to (\widetilde{X}, 0)$ *be an equivariant bifurcation function satisfying the hypotheses of Theorem C.6 (the equivariant L-S reduction). Assume that we define two reduced bifurcation equations* f_1 *and* f_2 *by*

1. *choosing two* Γ*-invariant complements of* $\ker \mathcal{L}$ *in* X*,*
2. *choosing two* Γ*-invariant complements of* $\ker \mathcal{L}$ *in* $\widetilde{X}$*, and*
3. *choosing two systems of co-ordinates in* $\ker \mathcal{L}$*.*

Then, f_1 *and* f_2 *are* Γ*-equivariant bifurcation equivalent maps.*

Appendix D

Linear Group Actions

In invariant theory, an important question is whether a mathematical object can be obtained from another by a group action or transformation. We consider an invariant to be a function or some mathematical object which takes the same value on the objects on which the group acts. Classical invariant theory considers the intrinsic properties of functions or polynomials i.e. the properties that are not affected by some group action or a change of variables. Such properties include factorisation and the multiplicities of roots. The determinant of a matrix is an invariant under similarity. Equivariance deals with functions that commute with some group action. A function f is said to be equivariant to a transformation T if $f(Tx) = Tf(x)$ and invariant if $f(Tx) = f(x)$. An equivariant function is invariant if the group action is trivial. The symmetries of a dynamical system can simplify the solution process such as in the separation of variables.

The subject is vast and we highlight the parts that are relevant to our work, namely invariant and equivariant theory. For a discussion of equivariant nonlinear mappings, we follow the exposition presented in the books by Golubitsky et al. [27, 28].

D.1 Representations

Recall that the endomorphism algebra, set of linear maps, of a vector space V is given by $\mathrm{End}(V) = \{\ L : V \to V\}$ under addition and composition of maps. We may regard a group representation, informally, as a way of writing it as a group of matrices. Let V be an n-dimensional vector space over a field $\mathbb{K}$ (of characteristic zero e.g. $\mathbb{R}$ or $\mathbb{C}$) and Γ be a group. Then the general linear group $\mathrm{GL}(n, \mathbb{K})$ is the group of invertible $n \times n$ matrices with entries in $\mathbb{K}$ under matrix multiplication, i.e.

$$\mathrm{GL}(n, \mathbb{K}) = \{\ A \in \mathrm{M}(n, \mathbb{K}) : \det A \neq 0\}.$$

Alternatively, we define the group of invertible linear maps $\pi : V \to V$, i.e $\mathrm{Aut}(V)$, then

$$\mathrm{GL}(V) = \mathrm{Aut}(V) \subset \mathrm{M}(n, \mathbb{K}) \cong \mathbb{K}^{n^2}.$$

A representation of Γ over $\mathbb{K}$ is a homomorphism $\rho : \Gamma \to \mathrm{GL}(V)$. The vector space V is called the representation space of Γ.

If we fix a basis $\{v_1, v_2, \cdots, v_n\}$ then each $\rho(\gamma)$ can be written in a matrix form. We define the trivial representation by $\rho : \gamma \mapsto I_n$, $\forall \gamma \in \Gamma$. A representation preserves the linear structure i.e.

$$\rho(\gamma)(v_1 + v_2) = \rho(\gamma)v_1 + \rho(\gamma)v_2, \quad \forall v_1, v_2 \in V, \tag{D.1}$$

$$\rho(\gamma)kv = k\rho(\gamma)v, \quad \forall k \in \mathbb{K},\ \forall v \in V. \tag{D.2}$$

D.2 Group Actions

Let G be a group and X a non-empty set. G is said to act on X if there is a function $\alpha : G \times X \to X$ normally denoted by $(g, x) \mapsto gx$, such that $ex = x$ for all $x \in X$ and that $(gh)x = g(hx)$ for all $g, h \in G$. The orbit of x is the subset $\mathcal{O}(x) = \{\ gx\ :\ g \in G\ \}$ of X obtained by applying elements of G. The stabilizer of x is the subset $\mathcal{S}(x) = \{\ g \in G\ :\ gx = x\ \}$ of G.

Matrices act on polynomials by changing the variables. Given $A \in G \subset \mathrm{GL}(n, \mathbb{K})$ and a vector $x \in V$, then the action or change in variables is $A \cdot f$. For any matrix $A \in G \subset \mathrm{GL}(n, \mathbb{K})$ and a polynomial function $f \in \mathbb{K}[x_1, \cdots, x_n]$, $(A \cdot f)(x)$ is also a polynomial in $\mathbb{K}[x_1, \cdots, x_n]$.

Proposition D.1. *An action of a group G on a set X is a homomorphism $\alpha : G \to \mathrm{Aut}(X)$.*

Let Γ be a group acting on a vector space V via a representation ρ. The pair (V, ρ) is called a Γ-space. For a given $x \in V$, the set of points γx generated by the action of the group Γ on x is called the group orbit of x (or Γ-orbit). Explicitly, the orbit of x is

$$\mathcal{O}(x) = \Gamma x = \{\ \gamma x\ :\ \gamma \in \Gamma\ \}.$$

Lemma D.2. *The action of a group Γ induce a partition of V into orbits.*

Proof. If $y \in \mathcal{O}(x)$, then there is a $\tilde{g} \in \Gamma$ such that $y = \tilde{g}x$. Then, $\gamma y = \gamma \tilde{g} x = \gamma_1 x$ and so $\mathcal{O}(y) \subset \mathcal{O}(x)$. Similarly, $\gamma x = \gamma \tilde{g}^{-1} y$, and so $\mathcal{O}(x) \subset \mathcal{O}(y)$. Therefore, $\mathcal{O}(x) = \mathcal{O}(y)$, and the conclusion. □

D.3 Integration on Compact Groups

Let $f : \Gamma \to \mathbb{R}$ be a continuous real-valued function. The Haar integral $\int_\Gamma f(\gamma)\, d\gamma \in \mathbb{R}$ is defined on a compact Lie group to satisfy the following properties:

1. Linearity: $\int_\Gamma (\lambda f(\gamma) + \mu g(\gamma))\, d\gamma = \lambda \int_\Gamma f(\gamma)\, d\gamma + \mu \int_\Gamma g(\gamma)\, d\gamma$ for all $\lambda, \mu \in \mathbb{R}$;

2. Positivity: If $f(\gamma) \geq 0$ for all $\gamma \in \Gamma$, then $\int_\Gamma f\, d\gamma \geq 0$;

3. Translation Invariance: $\int_\Gamma f(\delta\gamma)\, d\gamma = \int_\Gamma f(\gamma)\, d\gamma$.

Proposition D.3. *Let Γ be a compact Lie group acting on a vector space V and let $\rho(\gamma)$ be the matrix associated with $\gamma \in \Gamma$. Then there exists an inner product on V such that $\rho(\gamma)$ is orthogonal for all $\gamma \in \Gamma$.*

Proof. By Proposition D.3, compact Lie groups which are subgroups of $\mathrm{GL}(n, \mathbb{K})$ can be identified with a subgroup of the orthogonal group $\mathbb{O}(n)$. The orthogonal group preserves the inner product

$$\langle \rho(\delta)v, \rho(\delta)w\rangle_\Gamma = \langle v, w\rangle_\Gamma. \tag{D.3}$$

Let $\langle , \rangle_\Gamma$ be any inner product on V and define

$$\langle v, w\rangle_\Gamma = \langle \rho_\gamma v, \rho_\gamma w\rangle_\Gamma.$$

By the linearity property of the Haar integral, this is also an inner product on V. Furthermore, the translation invariance of the integral implies that the inner product satisfies D.3. □

D.4 Irreducibility

Let Γ be a Lie group acting on a vector space V. We say that a subspace $W \subset V$ is called Γ-invariant if $\gamma w \in W$, for all $\gamma \in \Gamma$, $w \in W$. A representation (ρ, V) of Γ is irreducible if its only invariant subspaces are 0 and V, i.e. if it has no proper invariant subspace.

The direct sum of two representations (ρ_1, V_1) and (ρ_2, V_2) of Γ is defined as $\rho_1 \oplus \rho_2 : \Gamma \to \mathrm{GL}(V_1 \oplus V_2)$ such that

$$(\rho_1 \oplus \rho_2)(\gamma)(v_1, v_2) = (\rho_1(\gamma)(v_1), \rho_2(\gamma)(v_2))$$

for which in matrix notation is

$$\gamma \mapsto \begin{pmatrix} \rho_1(\gamma) & 0 \\ 0 & \rho_2(\gamma) \end{pmatrix}.$$

A representation is completely reducible if it can be written as the direct sum of irreducible representations i.e. if $V = V_1 \oplus \cdots \oplus V_r$ where each V_i is a Γ-invariant irreducible representation.

Proposition D.4 (Complementary Subspace)**.** *Let Γ be a compact Lie group acting on a vector space V and let $W \subset V$ be a Γ-invariant subspace. Then, there exist a Γ-invariant complementary subspace U such that $V = W \oplus U$.*

Proof. By proposition D.3 there exists a Γ-invariant inner product on V. Let $U = W^\perp$ where

$$W^\perp = \{\ v \in V :\ \langle w, v\rangle_\Gamma = 0 \quad \text{for all} \quad w \in W\ \}.$$

Then, from the invariance of the inner product, U is also an invariant subspace. Hence $V = U \oplus W$. $\square$

Corollary D.5 (Complete Reducibility). *Let Γ be a compact Lie group acting on a vector space V. Then there exist Γ-irreducible subspaces $V_1, \cdots, V_k$ such that $V = V_1 \oplus \cdots \oplus V_k$.*

A representation of a group Γ on a vector space V is absolutely irreducible if the only linear mappings on V that commute with Γ are scalar multiples of the identity.

Theorem D.6 (Maschke, [51]). *Let Γ be a finite group, V a G-module and W a G-submodule of V. Then there is a submodule U of V such that $V = W \oplus U$.*

Schur's lemma concerns the properties of matrices which commute with the matrices of an irreducible representation. The first lemma states that a non-zero matrix which commutes with the matrices of an irreducible representation is a constant multiple of the identity matrix.

Theorem D.7 (Schur's Lemma). *Let $\rho : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ be an irreducible representation of Γ, and suppose that δ is an $n \times n$ matrix such that $\rho(\gamma)\delta = \delta\rho(\gamma)$ for all $\gamma \in \Gamma$. Then, $\delta = \lambda I$ for some $\lambda \in \mathbb{C}$.*

Proof. Let λ be an eigenvalue of δ. Since every non-constant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$, the characteristic polynomial of δ has at least one root $\lambda \in \mathbb{C}$. By definition, $\det(\delta - \lambda I) = 0$ and the matrix $\delta - \lambda I$ is not invertible. For all $\gamma \in \Gamma$,

$$(\delta - \lambda I)\rho(\gamma) = \delta\rho(\gamma) - \lambda\rho(\gamma) = \rho(\gamma)\delta - \lambda\rho(\gamma) = \rho(\gamma)(\delta - \lambda I),$$

since δ commutes with every $\rho(\gamma)$. Therefore, $\delta - \lambda I$ commutes with every $\rho(\gamma)$ and, since by the choice of λ it is not invertible, we have $\delta = \lambda I$. $\square$

D.4.1 Isotypic Decomposition

When a compact Lie group Γ acts on a vector space V, the space can be decomposed into a finite number m of Γ-irreducible subspaces U_i, giving

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_m.$$

The decomposition may not be unique and some of the U_i could be Γ-isomorphic to each other, meaning that there is an isomorphism $\mu : U_i \to U_j$ that commutes with the action of Γ given by

$$\mu(\gamma u_i) = \gamma(\mu(u_j)), \quad \text{for all} \quad u_i \in U_i,\ \gamma \in \Gamma$$

The sum of each set of Γ-isomorphic subspaces gives a subspace W_i, called the isotypic components of V. The space V can then be written as the direct sum of isotypic components

$$V = W_1 \oplus \cdots \oplus W_k, \quad k \leq m.$$

This decomposition is unique and the proof can be found in [28].

Examples. 1. A 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be expressed as the sum of two other matrices in the following way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$$

which we write as $V = V_1 \oplus V_2$. If we consider the usual action by matrix multiplication of $\mathbb{SO}(2)$, it is easy to see that the action is isomorphic to its standard irreducible action on $\mathbb{K}^2$ by $\mu_1 : V_1 \to K^2$ and $\mu_2 \to V_2$ defined by

$$\mu_1 \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}, \qquad \mu_2 \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

and hence V_1 and V_2 are $\mathbb{SO}(2)$-isomorphic and the decomposition is unique, consisting of the whole space. However, decomposition merely into irreducible subspaces such as $\begin{pmatrix} a & a \\ c & c \end{pmatrix}$ is not unique.

2. Consider a finite abelian group Γ. If a representation is irreducible, then the only linear mappings that commute with it are scalar multiples of the the identity, hence

$$\rho(\gamma) = c_\gamma I, \quad \forall \gamma \in \Gamma.$$

Such a representation can only be irreducible when it is one-dimensional.

D.5 Invariant and Equivariant Maps

An equivariant map between Γ-spaces is a map which commutes with the group actions. If Γ is a compact Lie group that acts linearly on the spaces V and W, then a map $f : V \to W$ commutes with Γ, or is Γ-equivariant, if

$$f(\gamma_V v) = \gamma_W f(v), \quad \forall v \in V, \ \gamma \in \Gamma.$$

We consider the case where Γ is a matrix subgroup of $GL(n, \mathbb{R})$ acting on the polynomial algebra $\mathbb{K}[x_1, x_2, \cdots, x_n]$ by the substitution of the variables. By this linear action, for each $\gamma \in \Gamma$

we have $\gamma \cdot x_i = \sum_{j=1}^{n} a_{ij} x_i$ for some $a_{ij} \in \mathbb{K}$. When a subalgebra of polynomials is invariant under the action of a group Γ, we denote it by $\mathbb{K}[x_1, \cdots, x_n]^\Gamma$.

D.5.1 Equivariant Linear Mappings

Linear maps do commute with non-irreducible representations and the following result which can be found in [28], follows thereof.

Lemma D.8. *Let Γ be a compact Lie group acting on V, let $A : V \to V$ be a linear mapping that commutes with Γ, and let $W \subset V$ be a Γ-irreducible subspace. Then $A(W)$ is Γ-invariant and either $A(W) = \{0\}$ or the representations of Γ on W and $A(W)$ are isomorphic.*

Proof. Let $z \in A(W)$, so that $z = A(w)$ for $w \in W$. Since A commutes with Γ, we have

$$\gamma z = \gamma A(w) = A(\gamma w)$$

so $\gamma z \in A(W)$. Also, $\ker A$ is Γ-invariant since $A(v) = 0$ implies that $A(\gamma v) = \gamma A(v) = \gamma 0 = 0$. Then, $\ker A \cap W$ is a Γ-invariant subspace of W, and irreducibility implies that, either $W \subset \ker A \cap W = \{0\}$. In the first case, $A(W) = \{0\}$. In the second, $A(W)$ is isomorphic to W as a vector space, the isomorphism being A. Since Γ commutes with A, we see that A is a Γ-isomorphism between A and $A(W)$. □

D.6 Orbits and Isotropy Subgroups

Let a group Γ acts on a vector space V. Then the set of points γx generated by all the actions of the group Γ on the point $x \in V$ is called the group orbit of x (or Γ-orbit). We denote the orbit of x by $\mathcal{O}(x)$. Certain properties evaluated along its orbit are the same.

Lemma D.9. *Let $f : V \to W$ be a Γ-equivariant map. When f vanishes at x, it vanishes on $\mathcal{O}(x)$.*

Proof. If $f(x) = 0$, then $f(\gamma_V x) = \gamma_W f(x) = \gamma_W 0 = 0$. □

A group action decomposes a set into orbits, and the group acts on each orbit. Orbits are disjoint with no sub-orbits. The decomposition into orbits may be seen as a factorisation of the set into irreducible parts for the group action. When there is only one orbit, the action is termed transitive.

D.6.1 Isotropy Subgroup

The isotropy subgroup of x is the maximal set of group actions which maps a point to itself. Let $\Gamma \times V : (\gamma, x) \mapsto \gamma x$ be an action. The isotropy subgroup of $x \in V$ is

$$\Sigma_x = \{ \ \gamma \in \Gamma : \ \gamma x = x \ \}.$$

That is, $\Sigma_x \subset \Gamma$ contains the symmetries that fix the point $x \in V$. Note that either $\Gamma \cdot x = \Gamma \cdot y$ or $\Gamma \cdot x \cap \Gamma \cdot y = \emptyset$.

Lemma D.10. *The isotropy subgroup Σ_x is a subgroup for which $f_x := \{ \ \gamma\Sigma_x \mapsto \gamma x \ \} : \Gamma/\Sigma_x \to V$ is a bijection.*

Proof. Let $\alpha, \beta \in \Sigma_x$. Then

$$\begin{aligned} e \cdot x &= x, \\ \alpha\beta \cdot x = \alpha \cdot (\beta \cdot x) &= \alpha \cdot x = x, \\ \alpha^{-1} \cdot x = \alpha^{-1}(\alpha \cdot x) &= \alpha^{-1}\alpha \cdot x = e \cdot x = x. \end{aligned}$$

So $\Sigma_x \subset \Gamma$. Let $\gamma, \delta \in \Gamma$, since

$$\begin{aligned} (\gamma\alpha) \cdot x &= \gamma \cdot (\alpha \cdot x) = \gamma \cdot x, \\ f : \gamma\Sigma_x &\to \gamma x \in \Gamma/\Sigma_x \end{aligned}$$

is a well defined mapping. If $\gamma \cdot x = \delta \cdot x$, then

$$(\gamma^{-1}\delta)x = \gamma^{-1} \cdot (\delta \cdot x) = \gamma^{-1} \cdot (\gamma \cdot x) = e \cdot x = x,$$

i.e. $\gamma^{-1}\delta \in \Sigma_x$, hence f is injective. Take $y \in V$, then by transitivity, there exists $\gamma \in \Gamma : \gamma x = y$. Thereby, $f(\gamma\Sigma_x) = \gamma \cdot x = x$, i.e. f is surjective. $\square$

Isotropy relates to point-wise invariance whilst the stabilizer relates to set-wise invariance.

D.7 Fixed-Point Subspaces

Nonlinear Γ-equivariant mappings have invariant subspaces which correspond to certain subgroups of Γ. Let $\Sigma \subset \Gamma$ be a subgroup. The fixed-point subspace of Σ is defined by

$$\text{Fix}(\Sigma) = \{ \ x \in V : \ \sigma x = x \quad \text{for all} \quad \sigma \in \Sigma \ \}.$$

$\mathrm{Fix}(\Sigma)$ is a subspace since

$$\mathrm{Fix}(\Sigma) = \bigcap_{\sigma \in \Sigma} \ker(\sigma - Id),$$

and each kernel is a subspace. The simplest fixed-point subspaces are $\mathrm{Fix}(1) = V$ and $\mathrm{Fix}(\Gamma) = \{0\}$, by hypothesis.

Lemma D.11. *Let $f : V \to V$ be Γ-equivariant. Let $\Sigma \subset \Gamma$ be a subgroup. Then $f(\mathrm{Fix}(\Sigma)) \subset \mathrm{Fix}(\Sigma)$.*

Proof. Let $\sigma \in \Sigma$, $x \in \mathrm{Fix}(\Sigma)$. Then,

$$f(x) = f(\sigma x) = \sigma f(x)$$

with the first equality following from the definition of $\mathrm{Fix}(\Sigma)$ and the second from equivariance. From (D.7) we see that σ fixed $f(x)$. Hence $f(x) \in \mathrm{Fix}(\Sigma)$. $\square$

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