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The Cauchy and Backward-Cauchy Problem for a Nonlinearly Hyperelastic/Viscoelastic Infinite Rod.

by

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In this paper we consider both simpler and more general forms of a nonlinear partial differential equation arising from a particular model of a rod vibrating longitudinally (cf. Jaunzemis [4], Love [6]). This is the Euler equation derived from a Lagrangian.

$$L = \int_{t_1}^{t_2} \int_{\Omega} [\tfrac{1}{2} u_t^2 + \tfrac{1}{2} f(u_x) u_{xt}^2 - w(u_x, f)]\, dx \quad dt\,, \qquad \Omega \subseteq IR\,. \qquad 1.$$

Setting $\delta L=0$ provides the equation of motion. Here f, $\hat{W}$ are generally nonlinear functions ($\hat{W}$ is the strain-energy density of the material and f is a function corresponding to the product of a non-linear Poisson's ratio with a variable radius of gyration about the central axis of the rod). Elsewhere ([9]) we consider $f(u_x)$ to become singular as $u_x \to -1$ for physical reasons of material inversion, but now we generalize in another direction and let $f(\phi)=\phi$.
If we define

$$\widehat{W}(u_x, u_x) = w(u_x), \ \frac{dw(u_x)}{du_x} = \sigma(u_x)\ , \qquad 2.$$

then the equation of motion obtained is

$$u_{tt} - u_{xxtt} - \sigma_x(u_x) = 0\,. \qquad 3.$$

We may consider more generally the equation

$$u_{tt} - u_{xxtt} - T_x(x,\ t, u_x, u_{xt}, u, u_t) = 0 \qquad 4.$$

(see [8]) where the nonlinear function T depends on the two dependent and independent variables as well as on the first spatial derivatives of u and u_t. However in the case of elasticity it is appropriate for T to have dependence only on x, u_x and u_{xt}, and we consider primarily homogeneous materials for which T has the form

$$T = \sigma(u_x) + \tau(u_{xt})\ . \qquad 5.$$

In places, we indicate also how one treats the more general problem 4. The paper proves local and global existence results for various spaces with $x \in IR, t \geq 0$, Some of these spaces are chosen to demonstrate possible behaviour of solutions as $x \to \pm\infty$, and are interesting in view of solitary wave solutions (see [10]) which are known to exist.

First several restrictions are imposed on σ, τ and W to permit sufficient tractability in investigating local and global existence.

Presence or absence of viscous terms does not affect the question of global existence since the equation will be shown to possess globally unique solutions even in the hyperelastic case, when $\tau \equiv 0$. Any assumptions of viscous damping such as $\phi\tau(\phi) > 0 \forall \phi \in IR$ only help to improve matters by letting solutions decay to equilibrium as $t \to +\infty$. On the other hand when $\phi\tau(\phi) < 0$ is permitted, this may produce solutions growing as $t \to +\infty$. and is equivalent to making the change of independent variable $t \to -t$ and looking at the case of $\phi\tau(\phi) > 0$, as $t \to -\infty$. In certain cases this may lead to blow-up of solutions in finite time, as in the undamped nonlinear string (Lax [5]). Here we consider three cases, generally grouped together for ease of presentation, in which blow-up does not occur i.e. for which solutions exist for all time t > 0.

Hypothesis H1) Let $\tau(.)$ be locally Lipschitz - continuous, i.e. $\forall \phi, \psi \in IR$ such that for some R>0 $|\phi| < R, |\psi| < R$, there exists a constant $\Gamma(R) > 0$, $\Gamma(R) \to +\infty$ as $R \to \infty$ with $|\tau(\phi) - \tau(\psi)| \leq \Gamma(R)|\phi - \psi|$.

H1)i) $\phi\tau(\phi) > 0, \forall \phi \in IR$,

H1)ii) $\tau(\phi) \equiv 0$,

H1)iii) $\phi\tau(\phi) = -\dfrac{\alpha\phi}{2}$, some $\alpha > 0$.

It will be seen later that the appropriate 'energy' estimate becomes less useful as we proceed from Hi) through Hii) to Hiii). Further hypotheses required for $\sigma(.)$, $W(.)$ are

H2) Let $\sigma(.)$ be locally Lipschitfz -continuous (without loss of generality we may take the same Lipschitz constant $\Gamma(R)$ as in H1), and set $\sigma(0)\equiv 0$.

H3) Assume $W(\phi)\geq 0, \forall \phi \in IR$.

We provide some notation.

$L^P(\mathbb{R})$ denotes the space of measurable real-valued functions on $\mathbb{R}$ for which

$$\| f \|_p \equiv \left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty , \; 1 \leq p < \infty ,$$

or

$$\| f \|_\infty \equiv \operatorname{ess} \sup_{x \in IR} | f(x) | < \infty , \text{ when } p = \infty .$$

The Sobolev spaces $W_o^{m,p}(IR)$, $1 \leq p < \infty$, $W^{m,\infty}(IR)$, $m \in IN$, consist of those functions in $L^P(IR)$ all of whose generalized derivatives up to and including order m belong to $L^p(IR)$.

We define norms on $W_o^{m,p}(IR)$, $W^{m,\infty}(IR)$ by

$$\| f \|_{m,p} = \left(\sum_{j=0}^{m} \| \frac{d^j f}{dx^j} \|_p^p \right)^{\frac{1}{p}}, \; 1 \leq p < \infty ,$$

or

$$\| f \|_{m,\infty} = \sum_{j=0}^{m} \| \frac{d^j f}{dx^j} \|_\infty ,$$

respectively.

Each of the spaces $W_o^{m,p}(IR)$, $W^{m,\infty}(IR)$ is a Banach space under the given norm.

let Y be a Banach space and let A be a bounded or semibounded set in IR . Then the Banach space $C^k(A;Y), k \in N \cup \{0\}$ is the class of k-times continuously differentiable mappings $\frac{d^j u}{dt^j}(t):\ A \to Y,\ 0 \le j \le k$.

$C^k(A;Y)$ has the norm

$$| u |_{k,Y} \equiv \sum_{j=0}^{k} \sup_{t \in A} \| \frac{d^j u}{dt^j}(t) \|_Y$$

Similarly, we define the Banach space $L^1(A;Y)$ to be the class of measurable mappings $u(t): A \to Y$ such that for $u \in L^1(A;Y)$

$$\| u \|_{1,Y} \equiv \int_A \| u(t) \|_Y \, dt < \infty$$

Now we want to consider local existence of solutions to equation 4. with τ given by 5. The domain of x will be taken to be the real line over which the following Cauchy data are given -

$$u_O(x) \equiv u(x,0), u_1(x) \equiv u_t(x,0), x \in IR . \qquad 6.$$

We present the proof in two stages. The Theorem applies to the space $x \in W^{1,\infty}(IR) \cap W_O^{1,2}(IR)$ which is the broadest to which our method applies. The possibility of working in a larger space $W_O^{1,p}(IR), p < \infty$, using other techniques such as a monotonicity approach is not considered here, due partly to the restrictions thereby imposed on σ but also because there appear to be difficulties in the conservative case ($\tau=0$). The Corollary that follows the Theorem is concerned with restricting the initial data to special classes of functions which decay at various rates as $|x| \to \infty$ and showing that the solution to 4., 5., 6. behaves likewise (this is independent of the choice of subsidiary H1i), H1ii) or H1iii).

<u>Theorem 1</u> Let hypotheses H1) and H2) be satisfied, and suppose that $u_0(x),\ u_1(x) \in W^{1,\infty}(IR) \cap W_0^{1,2}(IR)$. Then there exists a solution $u(x,t)$ to 4., 5. and 6 . belonging to $C^1([0,\tau[; W^{1,\infty}(IR) \cap W_0^{1,2}(IR)$

defined on a maximal interval $[0,\tau[,\ \tau \le \infty$. If $\tau < \infty$, then $\|u(.,t)\|_{1,\infty} + \|u_t(.,t)\|_{1,\infty} \to \infty$ as $t \to \tau$,

Remark 1 The proof of the Theorem is divided into two steps, one of which we postpone until later. Here we show that equation 4. is formally equivalent to an integro-differential equation which we solve by contraction mapping. The Theorem will be proved when it is shown that the solution found in this way solves 4. in a weak sense. Henceforth we consider this to be true.

Proof of Theorem 1

We write 4., 5., as

$$(1 - \partial^2{}_x)\, u_{tt} = \partial_x (\sigma + \tau)\,. \qquad 7.$$

and we assume $u(\pm\infty, t) = \forall\ t > 0$.

Formally operating on 7. with $(1-\partial_x^2)^{-1}$ we obtain

$$u_{tt}(x,t) = \int_{-\infty}^{\infty} G(x,\xi)\partial_\xi (\sigma+\tau) d\xi$$

$$= - \int_{-\infty}^{\infty} G_\xi (x,\xi)(\sigma+\tau) d\xi \qquad 8.$$

where

$$G(x,\xi) = \frac{1}{2} e^{-|x-\xi|}, \qquad 9.$$

and, on integrating twice with respect to time, 8. becomes

$$u\,(x,t) = u_0(x) + t u_1(x) - \int_0^t \int_{-\infty}^{\infty} (t-\eta)\, G_\xi\,(x,\xi)\ (\sigma+\tau)\ d\xi\ d\eta\,. \qquad 10.$$

The derivative of 10. with respect to x is given by

$$u_x(x,t) = u'_0(x) + t u'_1(x) - \int_0^t \int_{-\infty}^{\infty} (t-\eta)\, G_{\xi x}(x,\xi)(\sigma+\tau)\, d\xi d\eta$$

$$- \int_0^t (t-\eta)(\sigma+\tau)\, d\eta \qquad 11.$$

We try to find a fixed point in $W^{1,\infty}(IR) \cap W_0^{1,2}(IR)$ for the operator equation

$$(Au)(x,t) = u_0(x) + t u_1(x) - \int_0^t \int_{-\infty}^{\infty} (t-\eta) G_\xi(x,\xi)(\sigma+\tau)\, d\xi\, d\eta \qquad 12.$$

using a standard application of the contraction mapping principle (see e.g. [1]), which then implies the existence of a unique solution for 10. To show that the fixed point exists we need to demonstrate that A maps the ball

$$B(R) \equiv \{u(x,t) \in C^1([0,T]; W^{1,\infty}(IR) \cap W_0^{1,2}(IR):$$

$$: |u|_{1,W^{1,\infty}} + |u|_{1,W^{1,2}} \le R \} \qquad 13.$$

into itself for T sufficiently small, and that A is a contraction, i.e. that for some R,T>0, 0 < θ< 1 and all u, v ∈ B(R),

$$|Au|_{1,x} \le |u|_{1,x} \quad \text{and} \quad |Au - Av|_{1,x} \le \theta\ |u-v|_{1,x}, \qquad 14.$$

where $|\ |_{1,x}$ denotes the norm inside 13.

To satisfy 14. it may in fact be shown ([8]) that because of the similarity in the steps of the calculations it is sufficient to verify that

$$|(Au)_x|_D \le |u_x|_D \quad \text{and} \quad |(Au)_x - (Av)_x|_D \le \theta |u_x - v_x|_D \qquad 15.$$

where

$$|\phi(.,.)|_D \equiv \sup_{t \in [0,T]} (\|\phi(.,t)\|_\infty + \|\phi(.,t)\|_2 + \|\phi_t(.,t)\|_\infty + \|\phi_t(.,t)\|_2) \qquad 16.$$

is the norm associated with $C^1([0,T]; L^\infty(IR) \cap L^2(IR))$.

Writing

$$|\psi(.)|_E \equiv \|\psi(.)\|_\infty + \|\psi(.)\|_2 \qquad 17.$$

and noticing that $(Au)_x$ is given by the right side of 11., we find that

$$|(Au)_x|_D \le |u'_0|_E + (1+T)|u'_1|_E +$$

$$+\left|\int_0^t\int_{-\infty}^{\infty}(t-\eta)G_{\xi x}(\sigma+\tau)\,d\xi\,d\eta\right|_D$$

$$+\left|\int_0^t(t-\eta)(\sigma+\tau)\,d\eta\right|_D$$

$$\leq |u_0'|_E + (1+T)\,|u_1'|_E + C\left|\int_0^t(t-\eta)(\sigma+\tau)\,d\eta\right|_D$$

where G > 0 depends only on the Greens function. Thus by 13., H1, and H2.,

$$|(Au)_x|_D \leq |u_0'|_E + (1+T)\,|u_1'|_E + C\Gamma(R)(1+T)\int_0^t |u_x|_D\,d\eta\ ,$$

that is

$$|(Au)_x|_D \leq |u_0'|_E + (1+T)\,|u_1'|_E + CR\Gamma(R)T(1+T)\ , \qquad 18.$$

where the inequalities follow from the definition of the norms and properties of the Bochner integral (Yosida [11]). Hence 18. shows that provided there exists some R > 0, T > 0 such that there holds

$$|u_0'|_E + (1+T)\,|u_1'|_E + CR\Gamma(R)T(1+T) \leqq R \qquad 19.$$

then $AB(R) \subset B(R)$

In a similar way, A can be shown to be a contraction provided $R,\ T > 0$ exist such that for some finite constant C ' > 0

$$C'\,\Gamma(R)\ T\ (1+T) \leq \theta < 1\ . \qquad 20.$$

Conditions 19. and 20. may be simultaneously satisfied by choosing large enough R and small enough T > 0, which thereby proves the existence of a unique fixed point for 12. and hence the existence of a unique solution u(x, t) to the integral equation 10., with

$u\,(C^1([0,T];\,W^{1,\infty}(IR) \cap W_0^{1,2}(IR))$.

The last part of the Theorem is proved by a routine continuation argument as found in Reed ([7]), replacing the interval of existence [0,T] by $[0,\tau[$, where τ is the supremum of the T over which existence holds.

For the Corollary to this Theorem we are interested in finding whether if the initial data approach zero at a certain rate as $x \to \pm\infty$ then the solution to the problem does so also. More clearly, by way of an example, we might ask that given $e^{\alpha|x|}(|u_0(x)|+|u_1(x)|) \to 0$ as $|x| \to \infty$, a some positive number, does $e^{\alpha|x|}(|u(x,t)|+|u_t(x,t)|) \to 0, t>0$ as $|x| \to \infty$? To treat this question more fully we define a general class of function, together with associated Banach spaces.

Let $\psi = \psi(x)$ satisfy the conditions

$$1 \le \psi(x) < e^{\alpha|x|}, 0 \le \alpha < \tfrac{1}{2}, \qquad 21.$$

$$\psi(x+y) \le \psi(x)\psi(y) \ \forall x, y \in IR, \qquad 22.$$

$\psi(x)$ continuous on R $\qquad$ 23.

and

$$-k < \psi'(0-) \le \psi'(0+) < k \text{ for some } k>0. \qquad 24.$$

Further let $J = J(x) > 0$, let $Z = C^1([0,T]; W^{1,\infty}(IR) \cap W_0^{1,2}(IR))$ and

$$Z(J) = \{u \in Z : Ju \text{ and } Ju_x \in C^1([0,T); L^\infty(IR) \cap L^2(IR))\} \qquad 25.$$

with corresponding norm (see 14.)

$$\|u\|_{z(J)} \equiv |u|_{1,x} + |Ju|_D + |Ju_x|_D \qquad 26.$$

Note that when $J(x) = 1 \ \forall x \in IR, Z(J)$ reduces to Z since the norms are then equivalent.

We now prove

<u>Corollary 1.</u> Let $\psi(x)$ satisfy 21. - 24. Provided $\psi u_0, \psi u_0', \psi u_1$ and $\psi u_1'$ belong to $L^2(IR) \cap L^\infty(IR)$, there exists a unique solution u(x, t) to

4. under the conditions of Theorem 1, with $u(x,t)\in Z(\psi)$ defined over a maximal interval of existence $[0,\tau[$. If $\tau < \infty$, then $\|u\|_{Z(\psi)} \to \infty$ as $t \to \tau^-$.

<u>Proof</u> The procedure of Theorem 1 can be repeated with the ball B(R) now replaced by

$$B_Z(R) \equiv \{u \in Z(\psi) : \|u\|_{Z(\psi)} \leq R\} \qquad 27.$$

It is sufficient to show that if $u \in Z(\psi)$ then $Au \in Z(\psi)$ since the fixed point argument then completes the proof as before. We therefore establish estimates corresponding to those of Theorem 1 by considering the representative term

$$\psi(x)(Au)_x(x,t) = \psi(x)u_0{}'(x) + t\psi(x)u_1{}'(x) - \int_0^t (t-\eta)\psi(x)\int_{-\infty}^{\infty} G_{\xi x}(x,\xi)(\sigma+\tau)d\xi d\eta - \int_0^t (t-\eta)\psi(x)(\sigma+\tau)d\eta. \qquad 28.$$

We note that since $\psi u_x \in B_Z(R)$ then $\psi(\sigma+\tau) \in L^1(0,T\,;\,L^2\,(IR) \cap L^\infty(IR))$ for appropriate $T > 0$. This can be seen using hypotheses H1., H2. :-

$$\int_0^t (t-\eta)\|\psi(.)(\sigma(.,\eta)+\tau(.,\eta))\|_2 d\eta$$

$$\leq \int_0^t (t-\eta)\,\|\,\psi(.)\Gamma(R)(|\,u_x(.,\eta)\,|+|\,u_{x\eta}(.,\eta)\,|)\,\|_2\; d\eta$$

for $\sup_{t\in[0,t]} \{\|\,u_x\,\|_2 + \|\,u_{xt}\,\|_2\} \leq R$.

Thus

$$\int_0^t (t-\eta)\| \psi(.)(\sigma(.,\eta) + \tau(.,\eta))\|_2 \, d\eta \le 2T^2\Gamma(R)\, R. \qquad 29.$$

A similar result may likewise be established for

$$\int_0^t (t-\eta)\| \psi(.)(\sigma(.,\eta) + \tau(.,\eta))\|_\infty \, d\eta ,$$

and so the last term of 28. is bounded in $C^1([0,T]; L^2(IR)\cap L^\infty(IR))$ for $0 \le t \le T$. We need to obtain the same type of estimates as 29. for the second last term also. These are obtained as follows. Since $\psi(\sigma+\tau) \in L^2(IR)\cap L^\infty(IR)$ for almost all t, it is only necessary to show there exists a finite constant C>0 such that

$$\| \psi(.)\int_{-\infty}^{\infty} G_{\xi x}(.,\xi) f(\xi) d\xi \|_\infty < c \| \psi(.) f(.) \|_\infty , \qquad 30.$$

and

$$\| \psi(.)\int_{-\infty}^{\infty} G_{\xi x}(.,\xi) f(\xi) d\xi \|_2 < c \| \psi(.) f(.) \|_2 , \qquad 31.$$

for all $f(x)$ with $\psi f \in L^2(IR)\cap L^\infty(IR)$.
We denote by K and M, $\| \psi(.)f(.)\|_\infty$ and $\| \psi(.)f(.)\|_2$ respectively.
Hence, for almost all $\xi, |f(\xi)| \le \dfrac{k}{\psi(\xi)}$.

Thus

$$\tfrac{1}{2}\psi(x)\int_{-\infty}^{\infty} e^{-|x-\xi|}|f(\xi)|\,d\xi \le \frac{k}{2}\int_{-\infty}^{\infty} e^{-|x-\xi|}\frac{\psi(x)}{\psi(\xi)}d\xi$$

$$\le \frac{k}{2}\int_{-\infty}^{\infty} e^{-|x-\xi|}\psi(x-\xi)d\xi \quad , \quad \text{by } 22.,$$

$$\le \frac{k}{2}\int e^{(\alpha-1)|x-\xi|}d\xi \quad , \quad \text{by } 21.,$$

$$\le C\| \psi(.)f(.)\|_\infty$$

where $C = C(\alpha) < \infty$ for $\alpha < 1$, and so certainly for $0 \le \alpha \le \frac{1}{2}$. Hence 30. follows at once on taking the essential supremum of the first and last parts of the inequality.

Next we have,

$$\| \psi(.) \int_{-\infty}^{\infty} | G_{\xi x}(.,\xi) f(\xi) | \, d\xi \|_2^2$$

$$\leq \int_{-\infty}^{\infty} \psi^2(x) \left[\frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-\xi|} | f(\psi) | \, d\xi \right]^2 dx$$

$$\leq \frac{1}{4} \int_{-\infty}^{\infty} \psi^2(x) \int_{-\infty}^{\infty} e^{-|x-\xi|} d\xi \int_{-\infty}^{\infty} e^{-|x-\xi|} f^2(\xi) \, d\xi \, dx$$

$$\leq \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^2(x) e^{-|x-\xi|} f^2(\xi) \, dx \, d\xi \ \equiv \ I,$$

where we used the Cauchy Schwartz inequality for the second last line, and in the last line interchanged the order of integration by applying Fubini's Theorem. On noting that by 22. ,

$$I \leq \frac{1}{2} \int_{-\infty}^{\infty} \psi^2(\xi) f^2(\xi) \int_{-\infty}^{\infty} \psi^2(\xi - x) e^{-|x-\xi|} dx \, d\xi$$

$$\leq \frac{1}{2} \int_{-\infty}^{\infty} \psi^2(\xi) \, f^2(\xi) \int_{-\infty}^{\infty} e^{(2\alpha-1)|x-\xi|} \, dx \, d\xi$$

$$\leq C^2 \, \| \psi(.) f(.) \|_2^2 \ ,$$

we obtain 31., where $C = C(\alpha) < \infty$ for $0 \leq \alpha < \frac{1}{2}$.

It is now straightforward to complete the proof of the Corollary using again the first part and then continuing as outlined in the proof of the Theorem. Fuller details are in [8], Yosida [11], Elcrat and Maclean [3].

We make the further remark that the above procedure may also be applied in some other equations - here for example, when $T = T(x, t, u_x, u_{xt}, u, u_t)$ - but in this case the proof becomes slightly more elaborate involving an intermediate space $Y(\psi)$ of function pairs [u, v],

$$Y(\psi) \equiv \{[u,v] \in \{([0,T];\ W^{1,\infty}(\mathrm{IR}) \cap\ W_0^{1,2}(\mathrm{IR})\}^2 \quad :$$

$$: [\psi u, \psi v],\ [\psi u_x, \psi u_x] \in \{C([0,T];\ L^\infty(\mathrm{IR}) \cap L^2(\mathrm{IR})\}^2\ \}$$

which reduces to $Z(\psi)$ once it is shown that $v = u_t$.

To finally complete the proof of Theorem 1 it is only necessary to verify that the solution to the integral equation 10. solves the differential equation 4. (5., 6.). As is evident, we have so far considered solutions which can generally only be interpreted in a weak sense for 4. Thus we have the following definitions and a Lemma to Theorem 1.

Definition Let $\varphi = \phi(x,t),\ \psi = \psi(x,t) \in L^2(]0,T[\times \mathrm{IR}),\ 0 \le t_1 < t_2 \le T$

and $$< \phi(.,.),\ \psi(.,.) > \equiv \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \phi(x,t)\,\psi(x,t)\,dx\,dt \qquad 32.$$

denote the inner product on $L^2(]0,T[\times \mathrm{IR})$.

Let f =(x), g = g(x) $\in L^2(\mathrm{IR})$

and $$(f(.), g(.) \equiv \int_{-\infty}^{\infty} f(x)\,g(x)\,dx \qquad 33.$$

denote the inner product on $L^2(R)$

Lemma 1 The unique solution u(x, t) to equation 10. satisfies

$$u(x,t) \in C^2([0,\tau[; W^{1,\infty}(\mathrm{IR}) \cap W_0^{1,2}(\mathrm{IR}))$$

and for every $\phi(x,t) \in C^1([0,T]; W_0^{1,1}(\mathrm{IR}))$ there holds

$$< u_t, \varphi_t > + < u_{xt}, \phi_{xt} > - < \sigma + \tau, \phi_x >$$

$$= (u_t, \phi)\big|_{t_1}^{t_2} + (u_{xt}, \phi_x)\big|_{t_1}^{t_2}, \forall T < \tau. \qquad 34.$$

Proof The first part may be seen by differentiating the right side of 10. twice with respect to t and noticing that all the terms are continuous in t by H1., H2. and Theorem 1.

The second part follows on substituting u_t and u_{xt} from 10. into the first two terms of 34. and integrating by parts (see [8]) for $\phi(x,t) \in C_0^\infty(]0,T[\times \mathrm{IR})$, then using density to include all $\phi(x,t)$.

<u>Remark 2</u> Results on regularity may now be found when hypotheses H1. and H2. are strengthened and initial data are made smoother but since these are straightforward reapplications of the contraction mapping principle or simple 'bootstrap' arguments for time dependence, we avoid stating them and again refer to [8] for details.

<u>Remark 3</u> It is possible to prove by a continuation argument that the interval of existence $[0,\tau[$ of the regular solutions in Remark 2 is the same as that in Theorem 1 under the same hypotheses on the initial data and $\sigma(.)$ and $\tau(.)$, and the same may hold for solutions corresponding to data as given in Corollary 1. Therefore it is only necessary to obtain global existence (i.e. $\tau = \infty$) of solutions in $W^{1,\infty}(\mathbb{R}) \cap W_0^{1,2}(\mathbb{R})$ to infer global existence in any of the other spaces alluded to above under the appropriate conditions.

Lemma 1 leads immediately to a result concerning continuous dependence of solutions on their initial data :-

<u>Lemma 2</u> Suppose u(x, t) and $u_{mn}(x, t)$ are solutions of 5., 6, corresponding to initial data 7., and $u_{0m}(x)$, $u_{1n}(x)$ respectively, where $\{u_{0m}\}$, $\{u_{1n}\}$ are bounded sequences in $W^{1,\infty}(\mathrm{IR})$ such that

$$u_{0m}(.) \to u_0(.) \text{ in } W_0^{1,2}(\mathrm{IR}), \qquad 35.$$

$$u_{1n}(.) \to u_1(.) \text{ in } W_0^{1,2}(\mathrm{IR}), \qquad 36.$$

as $m,n \to \infty$.

Then for some $\tau > 0$,

$$u(x,t), u_{mn}(x,t) \in C^2([0,\tau[; W_0^{1,2}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})).$$

Further, for all $t \in [0,T], T < \tau$, as $m,n \to \infty$

$$u_{mn}(.,t) \to u(.,t) \text{ in } W_0^{1,2}(\mathbb{R}), \qquad 37.$$

$$u_{mnt}(.,t) \to u_t(.,t) \text{ in } W_0^{1,2}(\mathbb{R}), \qquad 38.$$

$$u_{mn}(.,t) \xrightarrow{*} u(.,t) \text{ weak-star in } W^{1,\infty}(\mathbb{R}) , \qquad 39.$$

$$u_{mnt}(.,t) \xrightarrow{*} u_t(.,t) \text{ weak-star in } W^{1,\infty}(\mathbb{R}) , \qquad 40.$$

<u>Proof</u> Every bounded sequence in $W^{1,\infty}(\mathbb{R})$ contains a subsequence which converges weak-* to a member of $W^{1,\infty}(\mathbb{R})$ (e.g. [11]). If a sequence converges to a member in the norm topology of $W_0^{1,2}(\mathbb{R})$ and is bounded in $W^{1,\infty}(\mathbb{R})$ then we may show that the entire sequence converges weak-* to that element in $W^{1,\infty}(\mathbb{R})$ ([2], P.76).

Thus 35. and 36. imply

$$u_{0m}(.) \xrightarrow{*} u_0(.) \text{ weak-* in } W^{1,\infty}(\mathbb{R})$$

and

$$u_{1n}(.) \xrightarrow{*} u_1(.) \text{ weak-* in } W^{1,\infty}(\mathbb{R}) .$$

Similarly 39. and 40. follow when 37. and 38. are proved. These latter are obtained by setting

$$W_{mn}(x,t) = u_{mn}(x,t) - u(x,t). \qquad 41.$$

Then by Lemma 1 with <,> defined in $L^2(]0,T[\times R)$,

$$< W_{mn_t}, \phi_t > + < W_{mn_{xt}}, \phi_{xt} > - < \sigma_{mn} + \tau_{mn} - \sigma - \tau, \phi_x >$$
$$= (Wmn_t, \phi)\big|_0^T + (Wmn_{xt}, \phi x)\big|_0^T \qquad 42.$$

where the meaning of σ_{mn}, τ_{mn} is evident. We may substitute

$\phi = W_{mn_t}$ in 42. to obtain

$$\| W_{mn_t}(.,t) \|_{1,2}^2 \Big|_0^T + 2 < \sigma_{mn} + \tau_{mn} - \sigma - \tau, W_{mn_{xt}} > = 0 . \qquad 43.$$

Letting

$$\| u_{0m} - u_0 \|_{1,2}^2 = \delta_{0m} , \qquad 44.$$

$$\| u_{1n} - u_1 \|_{1,2}^2 = \delta_{1n} \qquad 45.$$

and using H1., H2., 43. shows

$$\| W_{mn_t}(.,T)\|^2_{1,2} \le \delta_{1n} + c\int_0^T [\| W_{mn}(.,\eta)\|^2_{1,2} + \| W_{mn\eta}(.,\eta)\|^2_{1,2}]\,d\eta \qquad 47.$$

where $C = C(\Gamma(R))$ (see the proof of Theorem 1).

But $W^2_{mn}(x,t) = W^2_{mn}(x,0) + 2\int_0^t W_{mn}(x,\eta)W_{mn\eta}(x,\eta)\,d\eta$

and so

$$\| W_{mn}(.,T)\|^2_{1,2} \le \delta_{0m} + \int_0^T [\| W_{mn}(.,\eta)\|^2_{1,2} + \| W_{mn\eta}(.,\eta)\|^2_{1,2}]\,d\eta \qquad 48.$$

Adding 47. and 48. finally gives an inequality to which Gronwall's Lemma may be applied, and so

$$\| W_{mn}(.,T)\|^2_{1,2} + \| W_{mn_t}(.,T)\|^2_{1,2} \le (\delta_{0m} + \delta_{1n})\exp((C+1)T), \qquad 49.$$

giving the desired results, since δ_{0m}, $\delta_{1n} \to 0$ as $m,n \to \infty$.

Before turning to the final question of global existence, we investigate a simple property concerning the propagation of initial discontinuities in the first derivatives of the given data.

We define the 'jump' $[\phi(.,.)](x)$ in $\phi(x.,)$ at a point x by the relation

$$[\phi(.,.)](x) = \phi(x^+,.) - \phi(x^-,.) \quad . \qquad 50.$$

<u>Lemma 3</u> Let u(x, t) be the solution of 4. -6. and suppose $u_0(x) \in C^1(IR \setminus \chi_{0m}), u_1(x) \in C^1(IR \setminus \chi_{1n})$ where $\chi_{0m}, \chi_{1n} \subset IR$ are arbitrary sets of $m,n \in N \cup \{0\}$ points at which $u_0'(x)$, respectively $u_1'(x)$ has a discontinuity. Then the only points at which discontinuities in $u_x(x, t)$, $u_{xt}(x, t)$ may occur for t>0 are contained in $\chi_{0m} \cup \chi_{1n}$.

<u>Proof</u> By Theorem 1, the solution u(x, t) for 4. -6. satisfies

$$u_x(x,t) = u_0'(x) + tu_1'(x) - \int_0^t \int_{-\infty}^{\infty} (t-\eta)G_{\xi x}(x,\xi)(\sigma(u_\xi) + \tau(u_{\xi\eta}))\,d\xi\,d\eta$$

$$- \int_0^t (t-\eta)(\sigma(u_x) + \tau(u_{x\eta}))\,d\eta \qquad 51.$$

for almost every x. It is easy to show that the map

$\phi(x) \to \int^{\infty} G_{\xi X}(x,\xi)\phi(\xi)d\xi$ takes $L^{P}(\mathbb{R})$ into $C_B(\mathbb{R})$ $1 \le p \le \infty$,

and so the third term in 51.is continuous. Thus

$$[u_x(.,t)](x) = [u_0'(.)](x) + t[u_1'(.)](x)$$

$$\int_0^t (t-n)([\sigma(u_x(.,n))](x) + [\tau(u_{x\eta}(.,n))](x))dn . \qquad 52.$$

Similarly,

$$[u_{xt}(.,t)](x) = [u_1'(.)](x) - \int_0^t ([\sigma(u_x(.,n))](x) + [\tau(u_{xn}(.,n))](x))dn . \qquad 53.$$

By hypotheses HI., H2., adding 52., 53. implies that for $t \in [0,T] \subset [0,\tau[$,

$$|[u_x(.,t)](x)| + |[u_{xt}(.,t)](x)|$$

$$\le |[u_0'(\cdot)](x)| + (1+T)|[u'(\cdot)](x)|$$

$$+ (1+T)\Gamma(R)\int_0^t (|[u_x(.,n)(x)| + |[u_{xn}(.,n)](x)|)$$

$$\le \left(|[u_0'(\cdot)](x)| + (1+T)[u_1'(\cdot)](x)|\right)\exp(T(1+T)\Gamma(R)) \qquad 54.$$

where the last inequality is a consequence of Grŏnwall's lemma.
Hence for $x \in \chi_{om} U \chi_{1n}$ the right side is zero, and the result follows immediately.
Finally we show that under mild additional hypotheses there appear solutions to 4. - 6. which exist globally in time.

Theorem 2 In addition to the conditions of Theorem 1 let hypotheses H3. and H1 i), ii) or iii) hold. Then the solution u(x,t) of 4, - 6. belongs to the class $C^2([0,\tau[; W^{1,\infty}(R) \cap W_0^{1,2}(\mathbb{R}))$ for every finite $\tau > 0$.

Proof We may use Lemma 1 to derive an 'energy' estimate for u(x,t) on replacing $\phi(x,t)$ by $U_t(x,t)$ in the expression 34. This delivers

$$\tfrac{1}{2}\int_{-\infty}^{\infty}(u_t^2 + U_{xt}^2)\,dx + \int_{-\infty}^{\infty} W(u_x)\,dx + \int_0^t\int_{-\infty}^{\infty} u_{xt}\tau(u_{xt})\,dx\,dt$$

$$= \tfrac{1}{2}\int_{-\infty}^{\infty}(u_1^2 + u_1'2)\,dx + \int_{-\infty}^{\infty} W(u_0')\,dx \equiv E_0 \ , \qquad 55.$$

where we have let $t_1 \to 0$ and $t_2 = t < \tau$ in 34.

In cases Hl. i), ii) we therefore have the a priori bounds

$$\tfrac{1}{2}\| u_t \|_{1,2}^2 (t) + \int_{\infty}^{\infty} W(u_x)dx(t) < E_0 \qquad 55\ i)$$

and

$$\tfrac{1}{2}\| u_t \|_{1,2}^2 (t) \to \int_{-\infty}^{\infty} w(u_x)dx(t) = E_0 \qquad 55\ ii)$$

In cases Hl. iii) we rewrite 55. and use Grönwall's lemma.

$$\tfrac{1}{2}\| u_t \|_{1,2}^2 (t) + \int_{-\infty}^{\infty} W(u_x)dx(t) = E_0 + \tfrac{\alpha}{2}\int_0^t\int_{-\infty}^{\infty} u_{xt}^2\,dx\,dt .$$

implies

$$\| u_{xt} \|_2^2 (t) \le E_0 e^{\alpha t}$$

and, in turn, therefore

$$\tfrac{1}{2}\| u_t \|_{1,2}^2 (t) + \int_{\infty}^{\infty} W(u_x)dx\ (t) \le E_0 (1 + \tfrac{1}{2}e^{\alpha t}) . \qquad 55\ iii)$$

The bounds 55. are in themselves insufficient except in a special case when σ is uniformly Lipschitz continuous. Here we must supplement them by pointwise bounds on u_x, u_{xt} valid almost everywhere. To do this most conveniently, we add two further hypotheses which may however be relaxed, or disposed of entirely when the rod is finite.

H4. Let $0 \le K < \infty$ be such that o, W are assumed to satisfy,
for $|\phi| \ge K$,

$$|\phi| \le |\sigma(\phi)| \le W(\phi) .$$

H5. (case Hl. i)) Let $P \in N$, $0 \le K < \infty$ be such that for $|\Phi| \ge K$,

$$\tau(\phi) = \alpha\phi^{2p-1}, \quad \text{some} \quad \alpha > 0 .$$

Now we differentiate 10. twice with respect to t and once with respect to x, giving

$$u_{ttx} + \sigma(u_x) + \tau(u_{xt}) = -\int_{-\infty}^{\infty} G_{\xi x}(\sigma(u_\xi) + \tau(u_{\xi t}))d\xi . \qquad 56.$$

Thus in case H1. i), using 55.,

$$|u_{ttx} + \sigma + \tau| \le \left\{\int_{-\infty}^{\infty} G_{\xi x}(|\sigma|\tau|d\xi\right\}|u_\xi|,|u_{\xi t}| < K$$

$$+ \left\{\int_{-\infty}^{\infty} G_{\xi x}(|0| + |\tau|d\xi\right\}|u_\xi|,|u_{\xi t}| \ge K$$

$$\le 2\Gamma(K)\int_{-\infty}^{\infty} G_{\xi x}(x,\xi)d\xi$$

$$+ \int_{-\infty}^{\infty} W(u_\xi)d\xi + \alpha\int_{-\infty}^{\infty}|u_{\xi t}|^{2p-1} d\xi$$

$$\le 2\Gamma(K) + E_0 + \sigma\int_{-\infty}^{\infty}|U_{\xi t}|^{2p-1} d\xi \qquad 57.$$

Next we multiply both sides of 57. by $|u_{xt}|$ and integrate with respect to time to obtain

$$\tfrac{1}{2}U_{xt}^2(x,t) + W(U_x(x,t)) + \int_0^t u_{xn}\,\tau\, dn(x)$$

$$\le \tfrac{1}{2}u_1'^2(x) + W(u_0'(x)) + \int_0^t |u_{x\eta}(2\Gamma(K) + E_0 + \sigma\int_{-\infty}^{\infty}|u_{\xi\eta}|^{2p-1} d\xi)dn(x) . \quad 58.$$

In particular therefore, for $t \in [0,T]$,

$$\int_0^t u_{x\eta}^{2p} \ dn(x) \le E_1(x) + (2\Gamma(K) + E_0)\int_0^t |u_{x\eta}|\, dn(x)$$

$$+ \sigma\int_0^t |u_{x\eta}| \int_{-\infty}^{\infty} |u_{\xi\eta}|^{2p-1} d\xi\, dn(x)$$

where $E_1(x) \equiv \frac{1}{2} U_1'^2(x) + W(u_0'(x))$

So by Hölder's inequality and 55.,

$$\int_0^t u_{x\eta}^{2p} \ dn(x) \le E_1(x) + (2\Gamma(K) + E_0)T\left\{\int_0^t |u_{x\eta}|^{2p} dn(x)\right\}^{\frac{1}{2p}}$$

$$+\sigma\left\{\int_0^t |u_{x\eta}|^{2p} dn(x)\right\}^{\frac{1}{2p}}\left\{\int_0^t \int_{-\infty}^{\infty} |u_{\xi\eta}|^{2p} d\xi dn\right\}^{1-\frac{1}{2p}}$$

$$\le E_1(x) + (2\Gamma(K) + E_0)T\left\{\int_0^t |u_{x\eta}|^{2p} dn(x)\right\}^{\frac{1}{2p}}$$

$$+\sigma\left\{\int_0^t |u_{x\eta}|^{2p} dn(x)\right\}^{\frac{1}{2p}} E_0^{\ 1-\frac{1}{2p}} \qquad 59.$$

It follows that the term on the left side of 59. is uniformly bounded for almost every x and each $t \in [0,T]$. By 58. and H4., t is immediate that also $u_x(x,t)$ and $u_{xt}(x,t)$ remain uniformly bounded, i.e. we have that for some J, $0 < j < \infty$,

$$\| u_x(.,t)\|_\infty,\ \| u_{xt}(.,t)\|_\infty,\ \| \int_0^t u_{xn}\, \tau\, dn \|_\infty \le J,\ \forall t \in [0,T]. \qquad 60.$$

Using these estimates in 10. shows

$$\| u(.,t)\|_{1,\infty}, \| u_t(.,t)\|_{1,\infty} \le f(t) < \infty \qquad 61.$$

where f(t) is uniformly bounded on every finite interval [0,T], Similarly, in case H1. ii) when $\tau = 0$, 56. leads to

$$|u_{ttx} + \sigma| \le \Gamma(K) + \int_{-\infty}^{\infty} W(u_\xi) d\xi$$

$$\le \Gamma(K) + E_0 \qquad 62.$$

and 58. becomes

$$\tfrac{1}{2} u_{xt}^{2}(x,t) + W(u_x(x,t))$$

$$\leq E_1(x) + (\Gamma(K) + E_0)(\tfrac{1}{2}\int_0^t (1+u_{x\eta}^2)\ dn(x)) \qquad 63.$$

Therefore, by Grönwall's inequality, for $t \in [0,T]$,

$$u_{xt}^{2}(x,t) \leq \{2E_1(x) + [\Gamma(K) + E_0]T\}\exp[\Gamma(K) + E_0)t] \qquad 64.$$

and we obtain, as before,

$$\|u(.,t)\|_{1,\infty}, \|u_t(.,)\|_{1,\infty} \leq g(t) < \infty \qquad 65.$$

where g(t) grows at most exponentially in time.

In case H1. iii) when $\tau = -\frac{\alpha\phi}{2}$, $\alpha > 0$ we have

$$|u_{ttx} + \sigma| \leq |\int_{-\infty}^{\infty} G_{\xi x}(\sigma - \frac{\sigma u}{2}\xi t)\ d\xi| + \frac{\sigma}{2}|u_{xt}|$$

$$\leq \Gamma(K) + \int_{-\infty}^{\infty} W(u_\xi)d\xi + \frac{\sigma}{2}\|G_{\xi x}(x.,)\|_2\ \|u_{\xi t}(.,t)\|_2$$

$$+ \frac{\sigma}{2}|u_{xt}|$$

$$\leq \Gamma(K) + E_0 + \frac{\alpha}{2}E_0^{\frac{1}{2}}e^{\frac{\alpha}{2}t} + \frac{\alpha}{2}|u_{xt}|. \qquad 66$$

where we used the inequality preceding 55.iii). Multiplying by $|u_{xt}|$ and integrating,

$$\tfrac{1}{2}u_{xt}^{2}(x,t) + W(U_x) \leq E_1(x) + \int_0^t \Gamma\left((K) + E_0 + \frac{\sigma}{2}E_0^{\frac{1}{2}}e^{\frac{\sigma}{2}t}\right)|u_{x\eta}(x,n)|\,dn$$

$$+\frac{\sigma}{2}\int_0^t u_{xn}^{2}(x,n)\,dn$$

$$\leq E_1(x) + \tfrac{1}{2}\left(\Gamma(K)T + E_0T + E_0^{\frac{1}{2}}\ e^{\frac{\sigma}{2}T}\right)$$

$$+\tfrac{1}{2}\left((\Gamma(K) + E_{-0} + \alpha + \frac{\sigma}{2}E_{,0}^{\frac{1}{2}}\ e^{\frac{\sigma}{2}T}\right)\int_0^t u_{xn}^{2}(x,n)dn \qquad 67.$$

from which, again by Grtönwall's lemma

$$u_{xt}^{2}(x,t) \le \left(2E_1(x) + \Gamma(K)T + E_0T + E_0^{\frac{1}{2}} e^{\frac{\sigma}{2}T} \right) \exp\left\{ \left(\Gamma(K) + E_0 + \sigma + \frac{\sigma}{2} E_0^{\frac{1}{2}} \ e^{\frac{\sigma}{2}T} \right) t \right\} \quad 68.$$

and we have

$$\| u(.,t) \|_{1,\infty} , \| u_t(.,t) \|_{1,\infty} \le h(t) < \infty \quad 69.$$

where h(t) is exponentially bounded on every finite interval [0,T]. We have therefore found in each case that an a priori bound for $| u |_{1}, {}_{W}1,\infty + | u |_{1}, {}_{W}1,2$ (see Theorem 1) exists on every finite interval of the form [0,T] . A standard continuation argument therefore proves that these solutions exist for all time $t \in [0,\infty]$ (see, for example, Reed [7], or [8]).

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