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by

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Abstract

Given the matrix $G(\alpha) = \begin{bmatrix} a(\alpha) & b(\alpha) \\ c(\alpha) & d(\alpha) \end{bmatrix}$ it is shown that if

$$b(\alpha) \left/ \left\{ (a(\alpha) - d(\alpha)) \pm \left[(a(\alpha) - d(\alpha))^2 + 4b(\alpha)c(\alpha) \right]^{\frac{1}{2}} \right\} \right. \quad can \ be \ exp \ ressed$$

as a ratio of polynomials then the matrix $G(\alpha)$ can be explicitly Wiener-Hopf factorized. The Wiener-Hopf factors will also have algebraic growth at infinity.

1.   Introduction

   Recently there have appeared a number of papers dealing with the explicit Wiener-Hopf factorization of special types of matrices. Lebre[1]uses a rather elaborate procedure to factorize the matrix

$$G_1(\alpha) = \begin{bmatrix} 1 & ap^2 \\ a & 1 \end{bmatrix},$$ (1.1)

where $a(\alpha)$ is an arbitrary function and $p(\alpha)$ is a rational   polynomial. Meister and Speck[2][3] factorized the matrix

(1.2)
$$G_2(\alpha) = \beta_1(\alpha)R_1(\alpha) + \beta_2(\alpha)R_2(\alpha)$$

where $\beta_1(\alpha)$ and $\beta_2(\alpha)$ are arbitrary scalar functions and $R_1(\alpha)$ and $R_2(\alpha)$ are matrices with rational elements such that

$$R_1^2 = R_1 \quad, \quad R_2 = I - R_1 \quad.$$ (1.3)

Meister and Speck applied these results to the explicit factorization of the matrix

$$G_3(\alpha) = \frac{1}{k_2^2 t_2^2} \begin{bmatrix} \alpha^2\psi - k_2^2 t_2^2 & \alpha\xi_2\psi \\ \alpha\xi_2\psi & \xi_2^2\psi - k_2^2 t_2^2 \end{bmatrix}$$ (1.4)

Where $\psi = 4\tau - 3k_2^2, \tau = \alpha^2 + \xi_2^2 - t_1 t_2,$ which occurs in elastodynamic problems Kupradze[4]. Hurd and Luneburg[5]carried out a rather complicated factorization procedure for the matrix (which occurs in a problem of diffraction by an anisotropic impedance half plane)

$$G_4(\alpha) = \begin{bmatrix} 1 + \dfrac{k^2 - \alpha^2}{k\eta\eta\Gamma} & -\dfrac{k_2\alpha}{k\eta\eta\Gamma} \\ -\dfrac{k_3\alpha}{k\eta_2\Gamma} & 1 + \dfrac{k^2}{k\eta_2\Gamma} \end{bmatrix}$$ (1.5)

where $\Gamma = \left[k^2 - \alpha^2\right]^{\frac{1}{2}}, k^2 = k^2 - k_3^2$. However, in a later paper Luneburg et al[6] they did realise that this matrix could be diagonalized by means

Of polynomial matrices. In the paper* [6] they simply state the factorization of the matrix(which occurs in a diffraction problem by a partially coated dialectric slab):

$$G5(\alpha) = \begin{bmatrix} \dfrac{k_1^2 - k_z^2}{\beta_1} + \dfrac{\mu}{\beta_2}(k_2^2 - k_z^2) & \dfrac{\alpha}{\beta_1} k_z + \dfrac{\mu}{\beta_2}\alpha k_z \\[4mm] \dfrac{\alpha k_z}{\beta_1} + \dfrac{\mu}{\beta_2}\alpha k_z & \dfrac{k_1^2 - \alpha^2}{\beta_1} + \dfrac{\mu}{\beta_2}(k_2^2 - k_z^2) \end{bmatrix} \tag{1.6}$$

where $\beta_j = (k_j^2 - \alpha^2)^{\frac{1}{2}}$, $k_j = \left(k_j^2 - k_z^2\right)^{\frac{1}{2}}$ $j = 1, 2$, in the form

$$G_5(\alpha) = T^{-1}(\alpha) \wedge (\alpha) \mathbf{T}(\alpha) \tag{1.7}$$

$$T(\alpha) = \begin{bmatrix} \alpha & k_z \\ -k_2 & \alpha \end{bmatrix}, \quad \Lambda(\alpha) = \begin{bmatrix} \dfrac{k_1^2}{\beta_1} + \dfrac{\mu k^2}{\beta_2} & 0 \\[4mm] 0 & \beta_1 + \mu\beta_2 \end{bmatrix} \tag{1.8}$$

The present paper shows that all the above matrices $G_j(\alpha)$, $j = 1, 2, 3, 4, 5$ fall into a more general class which can be Wiener-Hopf factorized by elementary means. We also give a simple criterion on the coefficients of the general matrix $G(\alpha)$ given by

$$G(\alpha) = \begin{bmatrix} a(\alpha) & b(\alpha) \\ c(\alpha) & d(\alpha) \end{bmatrix} \tag{1.9}$$

which will ensure that the matrix can be polynomially diagonalized. That is that G(a) can be represented in the form

$$G(\alpha) = P(\alpha) D(\alpha) P^I(\alpha) \tag{1.10}$$

where $P(\alpha)$ and $P^I(\alpha)$ have only polynomial elements. $P^I(\alpha) = \det(P) P^-$

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* They represent these factors as $G_-(\alpha) = T^{-1}_{(\alpha)} \Lambda_-(\alpha), G_+(\alpha) = \Lambda_+(\alpha) T(\alpha)$

Their factorization would seem to be not strictly correct because $T^{-1}(\alpha)$ will have poles at a = ±ik. However, in the application they eventually only require G_+ α) and $G_-^{-1}(\alpha) = \wedge(\alpha)\mathbf{T}(\alpha)$ to be analytic and regular in their regions of reguality, and hence their procedure gives the correct results.

is the "formal inverse", and $D(\alpha) = \text{diag}\{d,(\alpha),d_2(\alpha)\}$ where $d,(\alpha)$ and

$d_2(\alpha)$ are arbitrary functions of α. Obviously once such a decomposition

as (1.10) has been achieved then the Wiener-Hopf factorization can be

carried out quite simply as

$$G(\alpha) = G_+(\alpha) G_-(\alpha) \tag{1-11}$$

where

$$G_+(\alpha) = P(\alpha) D_+(\alpha) \qquad G_-(\alpha) = D_-(\alpha) P^I(\alpha) \tag{1.12}$$

$$D_\pm = \text{diag}\{d_{1\pm}(\alpha), d_{2\pm}(\alpha)\}, \qquad d_1 = d_{1+}(\alpha) d_{1-}(\alpha), \qquad d_2(\alpha) = d_{2+}(\alpha) d_{2-}(\alpha)$$

We also observe that the factorization processes ensures that the

factors will have algebraic growth at infinity, and thus enable complete

solution of the relevant Wiener-Hopf problem to be carried out.

2. **Polynomial diagonalization**

Consider the matrix

$$G(\alpha) = \begin{bmatrix} a(\alpha) & b(\alpha) \\ c(\alpha) & d(\alpha) \end{bmatrix} \tag{2.1}$$

The characteristic equation of this matrix is given by $\det(G-\lambda I) = 0$,

i.e. $\lambda^2-\lambda(a+d)+ad-bc=0$. Thus the characteristic values of G are given

by

$$\lambda_1(\alpha) = \frac{(a+d)-\{(a-d)^2+4bc\}^{\frac{1}{2}}}{2}, \quad \lambda_2(\alpha) = \frac{(a+d)-\{(a-d)^2+4bc\}^{\frac{1}{2}}}{2}. \tag{2.2}$$

The corresponding eigenvetors are given by the solutions of the equations

$$(a-\lambda_1)v_{11}+bv_{21} = 0, \quad cv_{11}+(d-\lambda_1)v_{21} = 0 \qquad \text{for } \lambda_1 \tag{2.3}$$

$$(a-\lambda_2)v_{12}+bv_{22} = 0, \quad cv_{12}+(d-\lambda_2)v_{22} = 0 \qquad \text{for } \lambda_2. \tag{2.4}$$

Since the determinant of $G-\lambda I$ vanishes, the 2 equations in (2.3) are

linearly dependent. Hence taking any one, say (2.3), we have

$$\frac{v_{11}(\alpha)}{v_{21}(\alpha)} = \frac{-2b}{(a-d)-\{(a-d)^2+4bc\}^{\frac{1}{2}}} \tag{2.5}$$

Now, if we can diagonalise, the first column of the transformation matrix will consist of $v_{11}(\alpha)$ and $v_{12}(\alpha)$, and we require them to be polynomials. Thus if

$$\frac{-2b}{(a-d)-\{(a-d)^2+4bc\}^{\frac{1}{2}}} = \frac{P_{11}(\alpha)}{P_{21}(\alpha)}. \tag{2.6}$$

where $P_{11}(\alpha)$ and $P_{21}(\alpha)$ are polynomials, then we can choose

$$v_{11}(\alpha) = P_{11}(\alpha), \qquad v_{21}(\alpha) = P_{21}(\alpha).$$

Similarly, solving either of the equations (2.4) we get for the components of the eigenvector corresponding to λ_2 that

$$\frac{v_{12}}{v_{22}} = \frac{-2b}{(a-d)+\{(a-d)^2+4bc\}^{\frac{1}{2}}}; \tag{2.7}$$

and if

$$\frac{-2b}{(a-d)+\{(a-d)^2+4bc\}^{\frac{1}{2}}} = \frac{P_{12}(\alpha)}{P_{22}(4\alpha)}, \tag{2.8}$$

where $P_{12}(\alpha)$ and $P_{22}(\alpha)$ are polynomials then we can choose

$$v_{12}(\alpha) = P_{12}(\alpha), \qquad v_{22}(\alpha) = P_{22}(\alpha). \tag{2.9}$$

Thus if (2.6) and (2.8) (where these expressions must have distinct values in order to be able to diagonalize) are satisfied then $G(\alpha)$ can be represented by

$$G(\alpha) = P(\alpha)D(\alpha)P^{I}(\alpha) \tag{2.10}$$

where

$$P^{I}(\alpha) = \begin{bmatrix} P_{22}(\alpha) & -P_{12}(\alpha) \\ -P_{21}(\alpha) & P_{11}(\alpha) \end{bmatrix}, \quad P(\alpha) = \begin{bmatrix} P_{11}(\alpha) & P_{12}(\alpha) \\ P_{21}(\alpha) & P_{22}(\alpha) \end{bmatrix}, \quad D(\alpha) = \begin{bmatrix} \dfrac{\lambda_1(\alpha)}{\det(P)} & 0 \\ 0 & \dfrac{\lambda_2(\alpha)}{\det(P)} \end{bmatrix},$$

and the Wiener-Hopf factorization can be carried out to give (1.11) and (1.12).

3. Special cases of the general matrix

(a) If we let $a(\alpha)=1$, $d(a)=1$, $b(\alpha)=c(\alpha)p^2(\alpha)$ where $p(\alpha)$ is a rational

polynomial then we get the matrix considered by Lebre[l]. The
expressions (2.5) and (2.7) now become

$$\frac{v_{11}(\alpha)}{v_{21}(\alpha)} = \rho, \qquad \text{and} \qquad \frac{v_{12}(\alpha)}{v_{22}(\alpha)} = -\rho, \tag{3.1}$$

so clearly we can polynomially diagonalize.

(b) If we let $\hfill$ (3.2)

$$a(\alpha) = \beta_1(\alpha)r_{11}(\alpha) + \beta_2(\alpha)\tilde{r}_{11}(\alpha), \qquad b(\alpha) = \beta_1(\alpha)r_{12}(\alpha) + \beta_2(\alpha)\tilde{r}_{12}(\alpha)$$

$$c(\alpha) = \beta_1(\alpha)r_{21}(\alpha) + \beta_2(\alpha)\tilde{r}_{21}(\alpha) \qquad d(\alpha) = \beta_1(\alpha)r_{22}(\alpha) + \beta_2(\alpha)\tilde{r}_{22}(\alpha)$$

where $r_{ij}(\alpha)$ and $\tilde{r}_{ij}(\alpha)$ are the rational elements of the matrices R_1 and

R_2 respectively, then we obtain the matrix (1.2).The condition (1.3)

requires that the elements of R_1 and R_2 satisfy:

$$r_{11} + \tilde{r}_{11} = 1, \; r_{22} + \tilde{r}_{22} = 1, r_{12} + \tilde{r}_{12} = 0, \; r_{21} + \tilde{r}_{21} = 0, \; r_{11}^2 + r_{12}r_{21} = r_{11},$$

$$r_{12}r_{21} + r_{22}^2 = r_{22}, \tag{3.3}$$

$$r_{12}(r_{11} + r_{22} - 1) = 0, \; r_{21}(r_{11} + r_{22} - 1) = 0 \tag{3.4}$$

Substituting (3.2) into the expressions (2.5) and (2.7) and using the

fact that $r_{12} = -\tilde{r}_{12}, r_{21} = -\tilde{r}_{21}, \tilde{r}_{11} = 1 - r_{11}, \tilde{r}_{22} = 1 - r_{22}$ we get

$$\left. \begin{array}{c} v_{11}(\alpha)/v_{21}(\alpha) \\ v_{12}(\alpha)/v_{22}(\alpha) \end{array} \right\} = \frac{-2r_{12}(\beta_1 - \beta_2)}{(\beta_1 - \beta_2)(r_{11} - r_{22}) \mp \{(\beta_1 - \beta_2)^2 (r_{11} - r_{22})^2 + 4(\beta_1 - \beta_2)^2 r_{21}r_{12}\}^{\frac{1}{2}}}$$

$$= -2r_{12} / \{(r_{11} - r_{22}) \mp [(r_{11} - r_{22})^2 + 4r_{12}r_{21}]^{\frac{1}{2}}\}$$

Now the conditions (3.4) require either (i) $r_{12} = r_{21} = 0$ or (ii) $r_{11} + r_{22} = 1$.

If (i) holds then $r_{12} = r_{21} = \tilde{r}_{12} = \tilde{r}_{21} = 0$ and the original matrix is
already in diagonal form (this corresponds to $v_{11}/v_{21} = v_{12}/v_{22} = 0$).

If (ii) holds then from (3.3) we have $r_{21}r_{12} = -r_{22}^2 + r_{22} = -(1 - r_{11})^2 + (1 - r_{11})$

giving $r_{11}^2 - r_{11} + r_{12}r_{21} = 0$ so that $r_{11} = \{1 + \sqrt{1 - 4r_{12}r_{21}}\}/2$. Also from (3.3)

$r_{11} - r_{22} = 2r_{11} - 1$. Thus (2.15) becomes

$$\left.\begin{array}{c} v_{11}(\alpha)/v_{21}(\alpha) \\ v_{12}(\alpha)/v_{22}(\alpha) \end{array}\right\} = \frac{-2r_{12}}{(r_{11}+r_{22})\mp\left[\left(\pm\sqrt{1-4r_{12}r_{21}}\right)^2+4r_{12}r_{21}\right]^{\frac{1}{2}}} = \frac{-2r_{12}}{r_{11}-r_{22}\mp 1}$$

$$(3.6)$$

Clearly (3.6) satisfies the conditions (2.6) and (2.8) and therefore polynomial diagonalization can be carried out. The remaining matrices G_i, i=3,4,5 can also be shown to satisfy the conditions (2.6) and (2.8) and thus be Wiener-Hopf factorized by polynomial diagonalization.

In conclusion we suggest that given any 2x2 matrix that is required to be Wiener-Hopf factorized, the first test one should apply is that the elements satisfy (2.6) and (2.8). If these conditions are not satisfied, then consider the more sophisticated methods of factorization, see for example Heins[7], Daniele[8], Jones[9], Krapkov[10), Hurd[ll], Rawlins and Williams[12], Rawlins[13], Williams[15]. A similar approach can be carried out for 3x3 matrices, however, although the cubic characteristic equation can be solved explicitly, the resulting formulae are complicated. It is probably better to apply the general principle of section 2 of this paper to 3x3 or indeed nxn matrices. This principle being that when one is solving the eigenvector equations one should choose the elements $v_{ij}(\alpha)$ of the transformation matrix such that they solve the eigenvalue equations and they are polynomials.

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