USE OF QUALITY CONTROL METHODS
IN MONITORING THE PURCHASING
BEHAVIOUR OF CONSUMER PANELS

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Use of Quality Control Methods in Monitoring the Purchasing Behaviour of Consumer Panels

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1. Introduction

Consider a consumer panel of n members (which may be individuals, households, etc.), where each member provides information about his weekly purchases of a certain kind of product. We suppose that several brands of the product are available and that we wish to monitor the panel's overall preference for a particular brand B.

We assume that for each member

(i) successive purchase occasions occur as if in an independent Poisson process,

(ii) the brand purchased on a given occasion is chosen as if at random, and independently of previous brand choices, according to certain brand-choice probabilities.

For member \(i, i = 1, \ldots, n\), let \(\mu_i\) denote the mean number of purchase occasions per week, and \(p_i\) the probability of choosing brand B on a given purchase occasion. These parameters may be constant or may vary from week to week. It follows that for the panel as a whole

1) successive purchase occasions occur in a Poisson process at a rate of

\[
\mu = \sum_{i=1}^{N} \mu_i
\]

such occasions per week.

2) the brand purchased on a given occasion is independent of those purchase on previous occasions, and the probability that brand B is purchased is

\[
p = \sum_{j=1}^{n} \Pr(\text{individual } i \text{ made the purchase}) \Pr(B \text{ chosen } | i \text{ made the purchase})
\]

\[
= \sum_{i=1}^{n} \frac{\mu_i}{\mu} p_i
\]

\(p\) is thus the expected proportion of a series of purchase occasions for the panel on which brand B is purchased, provided it remains constant over these occasions.
We shall be concerned with detecting changes in the panel's overall preference for B as represented by the expected proportion $\bar{p}$, and shall present an approach for monitoring the value of $\bar{p}$ on a weekly basis using quality control methods.

2. Some Distributional Results

Let $X_t =$ overall number of purchase occasions for the panel in week $t$,

$$Z_t = \text{overall number of occasions on which brand B is purchased by the panel in week } t,$$

and $Y_t = \frac{Z_t}{X_t} =$ overall proportion of occasions on which brand B is purchased by the panel in week $t$,

where $t = 1, 2, \ldots$.

These random variables are independent between weeks.

The $Y$'s provide estimates of the value of $\bar{p}$ in their respective weeks.

The $X$'s, on the other hand, are ancillary statistics as far as $\bar{p}$ is concerned: they provide information about how accurately $\bar{p}$ can be estimated in their respective weeks but no information about its value.

We therefore consider the distributions of the random variables $Z_t$ and $Y_t$ conditionally on $X_t$.

Now, for given $X_t > 0$, $Z_t$ follows the binomial distribution $\text{Bi}(X_t, \bar{p})$ and hence

$$E(Y_t | X_t) = \bar{p}, \quad \text{var} \ (Y_t | X_t) = \frac{\bar{p}(1 - \bar{p})}{X_t}.$$

Further, if the realised $X_t$ is sufficiently large, this binomial distribution, and the conditional distribution of $Y_t$, may be approximated by normal distributions. Hence, for sufficiently large realised $X_t$, the conditional distribution of $Y_t$ given $X_t$ is approximately

$$N \left( \bar{p}, \frac{\bar{p}(1 - \bar{p})}{X_t} \right).$$

Standardising, the conditional distribution of
\[ T_t = \frac{Y_t - P}{\sqrt{\frac{P(1-P)}{X_t}}} \]

given \( X_t \) is approximation \( N(0,1) \) for sufficiently large realised \( X_t \).
Further the \( T \)'s are independent both conditionally on the \( X \)'s and unconditionally.

Approximate conditions for the validity of the normal approximation to the above binomial distribution are that we require

\[ X_t > 5 \text{ and } \frac{1}{\sqrt{X_t}} \left| \frac{\frac{P}{1-P} - \frac{1-P}{P}}{\frac{1}{2}} \right| < 0.3. \]

For example, if \( p = 0.02 \) we require a realised \( X_t > 522 \), while if \( p = 0.2 \) we require a realised \( X_t > 25 \). The realised values of the \( X \)'s will be large with high probability if the underlying overall mean weekly purchase rate \( \mu = \sum_{i=1}^{n} \mu_j \) of the panel is large in each week, which will be the case if the size \( n \) of the panel is sufficiently large.

Note that since the approximate conditional distribution of \( T_t \) given \( X_t \) does not depend on \( X_t \), it follows that this approximate distribution applies unconditionally also, i.e. for large \( \mu \), the unconditional distribution of each \( T_t \) is approximately \( N(0,1) \), independently for each \( T_t \).

Note also that the above distributional result does not depend on the panel's overall mean weekly purchase rate \( \mu \), and is therefore independent of any variation in the value of \( \mu \) from week to week such as seasonal variation.

### 3. Detecting a change in \( \bar{p} \) from an Established Value

Suppose in a given week \( t = 0 \) it is established from past data that \( \bar{p} \) has the value \( \bar{p}_0 \). Then to detect a change from this established value we may monitor the panel's weekly purchasing behaviour using Shewhart and cumulative sum (cusum) charts of the statistics

\[ T_t = \frac{Y_t - \bar{P}_0}{\sqrt{\frac{\bar{P}_0(1-\bar{P}_0)}{X_t}}} \text{, } t = 1, 2, \ldots. \]
Assuming that the value of \( p \) does not change from week to week then, conditionally on the X's or unconditionally, these statistics are independent and each is distributed approximately \( N(0,1) \). We shall assume that the overall weekly purchase rates are sufficiently large for the normal approximations to be valid.

### 3.1 The Shewhart Chart

For this chart \( T_t \) is plotted against \( t \). While \( \bar{p} \) has the value \( \bar{p}_0 \) the \( T \)'s have zero expectations, but if \( \bar{p} \) increases or decreases then they have positive or negative expectations, respectively. Large positive or negative values the \( T \)'s are therefore evidence of a change in \( \bar{p} \). To decide whether or not a value of \( T_t \) is significantly large in absolute value, the chart is provided with decision limits, which may be chosen to give an acceptable sensitivity.

We suggest that decision limits are placed at \( \pm 2.58 \); for unchanged \( \bar{p} \), each \( T_t \) has a probability of only 0.01 of falling outside these limits (a probability of 0.005 of falling beyond each limit). If, then, a value of \( T_t \) exceeds 2.58 or is less than -2.58, this is taken as evidence of an increase or decrease, respectively, in the value of \( \bar{p} \).

The sensitivity of the Shewhart chart in detecting changes in \( \bar{p} \) may be analysed as follows.

Suppose at the start of a given week the value of \( \bar{p} \) changes from \( \bar{p}_0 \) to \( \bar{p}_1 \) and remains at the new value thereafter. We shall only consider changes \( \Delta \bar{p}_0 = \bar{p}_1 - \bar{p}_0 \) which are small compared with \( \bar{p}_0 \), but which are comparable in magnitude with \( \sigma_0 = \sqrt{\frac{\bar{p}_0(1-\bar{p}_0)}{\mu}} \), which is the approximate unconditional standard deviation of \( Y \) when \( \bar{p} = \bar{p}_0 \). We shall also assume that \( \mu \) remains constant in the weeks following the change in \( \bar{p} \).

For such a change in \( \bar{p} \), and for large \( \mu \), the distribution of \( T_t \) is approximately \( N\left( \frac{\Delta \bar{p}_0}{\sigma_0}, 1 \right) \) independently for each \( T_t \), both unconditionally and conditionally on the realised \( X_t \) (see Appendix 1).
Suppose \( \bar{p} \) increases and let

\[
\theta = \Pr \left( T_T > 2.58 \frac{\Delta}{\sigma_0} \right) = 1 - \Phi \left( 2.58 - \frac{\Delta}{\sigma_0} \right).
\]

Let \( R \) denote the number of weeks that elapse following the increase before an increase is detected, i.e. before the upper decision limit is reached; \( R \) has the geometric distribution

\[
\Pr(R = r) = (1 - \theta)^{r-1} \theta, \quad r = 1, 2, \ldots.
\]

Hence the probability that the increase is detected by the \( r \)th week is

\[
\Pr(R \leq r) = 1 - (1 - \theta)^r, \quad r = 1, 2, \ldots,
\]

and the expected number of weeks that elapse before the increase is detected is

\[
E(R) = \frac{1}{\theta},
\]

which is called the average run length (ARL) in quality control.

Corresponding results apply for the detection of a decrease in \( \bar{p} \).

The sensitivity of the chart in detecting a change in \( \bar{p} \) depends on the value of \( |\Delta| \frac{\bar{p}}{\sigma_0} \), i.e. on the change in \( \bar{p} \) expressed as a multiple of \( \sigma_0 \).

Table 1 gives approximate values for the probability of the detection of an increase or a decrease, as appropriate, by the \( r \)th week, and of the ARL, for \( |\Delta| \frac{\bar{p}}{\sigma_0} = 0, 0.5, 1, 2, 3 \). Table 2 gives the values of \( |\Delta| \frac{\bar{p}}{\sigma_0} \) corresponding to these multiples of \( \sigma_0 \) for two pairs of values of \( \bar{p} \) and \( \mu \) typical of those found in practice.

In the case of no change in the value of \( \bar{p} \) (\( \Delta \bar{p}/\sigma_0 = 0 \)), the starting point for calculating the detection probabilities is arbitrary. In such a case, of course, the detection of an increase or a decrease constitutes an error. The ARL both between consecutive detections of an increase and consecutive detections of a decrease is 200 weeks; hence if \( \bar{p} \) remains unchanged the ARL between false alarms is 100 weeks. (These ARL's do not depend on the assumption of constant \( u \) between weeks.) The choice of decision limits was in fact based on this choice of the ARL between false alarms.
### TABLE 1

Detection Probabilities and ARL's for the Shewhart Chart

<table>
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<tr>
<th>$\Delta p_0$</th>
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<th>6</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>ARL</th>
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<td></td>
<td></td>
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<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.5</td>
</tr>
</tbody>
</table>

### TABLE 2

Values of $|\Delta p_0|$ corresponding to $|\Delta p_0|/\sigma_0$

| $\Delta p_0$ | $\Delta p_0$ | $\mu$ | $|\Delta p_0|$ |
|--------------|--------------|-------|----------------|
| $\sigma_0$   | $p_0$        |       |                |
| 0.5          | 0.02         | 1000  | 0.0022         |
|              | 0.20         | 1000  | 0.0063         |
| 1.0          | 0.02         | 1000  | 0.0044         |
|              | 0.20         | 1000  | 0.013          |
| 2.0          | 0.02         | 1000  | 0.0089         |
|              | 0.20         | 1000  | 0.025          |
| 3.0          | 0.02         | 1000  | 0.013          |
|              | 0.20         | 1000  | 0.038          |
3.2 The Cusum Chart

For this chart the cumulative sum of the T's,

$$C_t = \sum_{j=1}^{t} T_j,$$

is plotted against $t$.

While $\bar{p}$ has the value $\bar{p}_0$, the T's have zero expectations, and the expected value of $C_t$ does not change with $t$; hence the path of the cusum tends to be roughly horizontal. However, if $\bar{p}$ increases, the T's have positive expectations, and the expected value of $C_t$ begins to increase with $t$; hence the path of the cusum then tends to slope upwards. Similarly if $\bar{p}$ decreases, the path of the cusum then tends to slope downwards.

Thus for the cusum chart, changes in the mean of $T_t$ (and hence in $p$) are indicated by changes in the slope of the cusum path, the magnitude of the slope indicating the value of the mean of the T's at the corresponding location.

Changes in the slope of the cusum path may occur by chance even if $\bar{p}$ does not change, but a marked upward or downward slope in the path which persists for a sufficiently long period is evidence that the T's corresponding to that period have non-zero means, and hence that $\bar{p}$ has changed from the value $\bar{p}_0$. To assess the significance of the slope of the cusum path the following decision rule is used.

To detect an increase in the mean of the T's, and hence an increase in $\bar{p}$, a so-called 'reference value' $k > 0$ and a 'decision interval' $h > 0$ are chosen and a modified 'cusum' $U_t$ of the T's is formed as follows: $U_t$ is defined to be zero until a value of $T_t$ occurs, say in week $t_1$, which exceeds $k$; from this week onwards $U_t$ is then defined to be the cusum

$$U_t = \sum_{j=t_1}^{t} (T_j - k), \quad t = t_1, t_1 + 1, ..., t,$$

provided the cusum is positive; if the cusum returns to zero or becomes negative, then $U_t$ is again defined to be zero until a further value of $T_t$ occurs which exceeds $k$; the above cycle is then repeated. Thus for any value of $t \geq 1$, $U_t$ is given by
\[ U_t = \max(U_{t-1} + T_t - k, 0) \ (U_0 = 0) \]

An increase in the mean of the \( T \)'s, and hence in \( \bar{p} \), is then signalled if the modified cusum \( U_t \) exceeds the limit \( h \). We shall call \( U_t \) the upper 'decision interval' (d.i.) cusum.

The values of \( k \) and \( h \) are chosen to give an acceptable sensitivity in detecting an increase in the mean of the \( T \)'s. This sensitivity is expressed in terms of ARL's. If the mean of the \( T \)'s remains at zero, the detection of an increase is an error, and so we require a large ARL between such false detections. On the other hand, if a significant increase in the mean of the \( T \)'s occurs, we would want to detect this quickly, and so require the ARL to detection to be small. There is a trade-off between these two requirements.

For the changes in \( \bar{p} \) that we consider the (unconditional or conditional)

distribution of \( T_t \) becomes approximately \( N(\frac{\Delta \rho_0}{\sigma_0}, 1) \) The required
sensitivity in detecting the increase \( \frac{\Delta \rho_0}{\sigma_0} \) in the mean of the \( T \)'s may be
expressed by specifying a suitably small ARL for a particular 'critical' increase for which quick detection is desired, together with a suitably large ARL for the case of no change in the mean. Values of \( k \) and \( h \) may then be found, using the nomograms in BS5703 Part 3 (1982), which meet the above ARL specifications as closely as possible.

An alternative approach is to set \( k \) equal to half the 'critical' increase in the mean of \( T \)'s and then to determine \( h \), using the nomogram, to give the required ARL in the case of no change in the mean. This usually also gives a satisfactory ARL in the case of the 'critical' increase, but if this ARL is found to be unsatisfactory, \( h \) can be varied until an acceptable compromise between the two ARL's is achieved.

We suggest that we regard an increase of \( \frac{\Delta \rho_0}{\sigma_0} = 1 \) in the mean of the \( T \)'s as a 'critical' increase and take the values of \( k \) and \( h \) to be

\[ k = 0.5 \quad , \quad h = 3.5 \]

These values give an ARL of 200 in the case of no change in the mean and 7.4 in the case of a 'critical' increase of 1.
A corresponding lower d.i. cusum $L_t$ of the $T$'s is used to detect a decrease in their mean, and hence a decrease in $\bar{p}$. Thus, if $k$ and $h$ denote positive quantities, we calculate

$$L_t = \min(L_{t-1} + T_t + k, 0), \ t \geq 1 \ (L_0 = 0).$$

If $L_t$ goes below $-h$, this is taken as evidence that the mean has become negative and hence that $p$ has decreased. The values of $k$ and $h$ are chosen in the same way as before and the above suggested values for $k$ and $h$ apply also for the detection of a decrease in $\bar{p}$.

Estimates of the probability of the detection of an increase or decrease in $p$, as appropriate, by the $r$th week following a change, and of the corresponding ARL, are given in Table 3 for values of $|\Delta p_0|/\sigma_0 = 0, 0.5, 1, 2, 3$. The probability estimates were obtained by simulation using 1000 runs and a 'worst possible' starting value of 0 for the d.i. cusum in each case; the ARL's are given in BS5703 Part 3 (1982). (See Table 2 for changes $|\Delta p_0|$ in $\bar{p}$ corresponding the above values of $|\Delta p_0|/\sigma_0$).

If $\bar{p}$ remains unchanged, the ARL both between consecutive detections of an increase and consecutive detections of a decrease is 200 weeks, and hence the ARL between false alarms is 100 weeks; this is the same as for the Shewhart chart discussed earlier.

**TABLE 3**

Detection Probabilities and ARL's for the Cusum Chart

| $|\Delta p_0|/\sigma_0$ | $r = 2$ | 4 | 6 | 8 | 10 | 15 | 20 |
|------------------------|--------|---|---|---|----|----|----|
| 0                      | 0.0010 | 0.013 | 0.023 | 0.044 | 0.056 | 0.13 | 0.17 | 200 |
| 0.5                    | 0.0050 | 0.062 | 0.14 | 0.22 | 0.32 | 0.51 | 0.64 | 22  |
| 1                      | 0.044  | 0.26 | 0.51 | 0.72 | 0.83 | 0.94 | 0.98 | 7.4 |
| 2                      | 0.36   | 0.90 | 0.99 | 1.0 | 1.0 | 1.0 | 1.0 | 3.0 |
| 3                      | 0.86   | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 2.0 |

Except for relatively very large changes, the cusum chart has greater sensitivity in detecting a change in $\bar{p}$ than the Shewhart chart, as can be seen by comparing Tables 1 and 3.
When a change is detected, an estimate of the week in which the change occurred can be obtained from the ordinary cusum chart of the T's which makes the cusum especially useful. This is provided by the point at which the slope of the cusum path changes in the weeks prior to the point of detection.

3.3 An Illustration

The data shown in Table 4 concern purchases of packets of tea bags by a consumer panel over a period of 52 weeks. The weekly total number of purchases of such packets is given together with the number, and proportion, of purchases of a particular brand B. The panel's overall preference for this brand is to be monitored.

For illustration purposes, we shall take the proportion of purchases of B for the first 10 weeks as the established value \( \bar{p} \) of \( \bar{p} \). Over this period there is no trend in the weekly estimates \( Y \) of \( \bar{p} \), and so the value of \( \bar{p} \) appears to be stable.

Thus we take \( \bar{p}_0 = 0.1933 \), and monitor the data from week 11 onwards for a change from this values.

Figure 1 shows a time plot of the weekly proportions of B purchased by the panel. There is no clear indication of a change in the level of \( \bar{p} \); any such changes are obscured by the variation in the data.

Figure 2 shows the Shewhart chart - a time plot of statistics T - from week 11 onwards. Neither decision limit is reached, and so no change in the value of \( \bar{p} \) is detected by this chart.

3. \( \bar{p} \) is clearly signalled at week 37.

Referring to the ordinary cusum chart (figure 3) we see that, in the weeks immediately preceding week 37, the slope changes from roughly horizontal to increasing in week 31. This, then, is our estimate of the week in which the change in \( \bar{p} \) first occurred.

Further, the slope of the cusum path remains approximately constant from
<table>
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<tr>
<th>Week</th>
<th>Total Number of purchases (X)</th>
<th>Number of purchases of brand B (Z)</th>
<th>Proportion of purchases of B (Y)</th>
</tr>
</thead>
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<td>1</td>
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FIGURE 1.
PROPORTION OF PURCHASES OF BRAND B
FIGURE 2.
SHEWHART CHART

DECISION LIMIT
FIGURE 3. ORDINARY CUSUM CHART

CUSUM C

WEEK
FIGURE 4.
DECISION INTERVAL CUSUM CHART
wek 31 to week 39, after which it begins to decrease. Hence, the increase-
ed level of $\overline{P}$ appears to be held over this period, after which it begins to fall off. The proportion of B purchased over this period is 0.2059, which provides an estimate of the increased level of $\overline{P}$.

4. Adapting to a Gradual Change in the Panel's Preference for Brand B

(Here we denote the values of $\mu$ and $\overline{P}$ in week $t$ by $\mu_t$ and $p_t$, respectively.)

Apart from rapid changes in the value of $\overline{P}$ that may occur during promotion-al activity, the value of $\overline{P}$ may change slowly over a comparatively long period of time due to many small background influences on the panel's overall preference for B, or because the membership of the panel gradually changes.

The above procedures for detecting a change in the value of $\overline{P}$ from an est-
ablished value $p$ will eventually detect a gradual drift away from $\overline{P}_0$ when it becomes sufficiently large. If we wish to detect only rapid, local local changes in $p$, then we shall need to compare the estimate of $p$ in a
given week with an estimate of the current value of $\overline{P}$ just prior to the
given week.

An estimate of the current value of $\overline{p}$ at a given time, which adapts to
gradually changing $\overline{p}$, is provided by an exponential smoothing of the week-
ly estimates. At week $t$, the exponentially smoothed value $\tilde{Y}_t$ of the avail-
able weekly estimate $Y_1, \ldots, Y_t$ is given by

$$\tilde{Y}_t = (1-\alpha)Y_t + \tilde{Y}_{t-1}$$

where $0 < \alpha < 1$ is a chosen smoothing constant; the value of $\alpha$ controls
the rate at which $\tilde{Y}$ adapts to changing $p$, although the more rapidly it is
made to adapt the greater its variance becomes. To start up the smoothing process, the initial smoothed value $\tilde{Y}_0$ is taken to be the 'established'
value $\overline{p}_0$ of $\overline{p}$ at that time, as given, say, by the mean of the $Y$'s over
preceeding weeks.

The statistics $T$ may then be modified to measure only local changes in the
weekly estimates $Y$ of $\overline{p}$: we replace $\overline{p}_0$ in the statistic $T_t$ corresponding
to week $t$ by the estimate $\tilde{Y}_{t-1}$ of the current value of $\overline{p}$ at week $t - 1$, 
We now consider weekly changes in \( \bar{p} \) which are small compared with the value of \( \bar{p} \), but comparable in magnitude to \( \sqrt{\frac{p(1-p)}{\mu}} \), the approximate unconditional standard deviation of \( Y \). We shall express the changes as multiples of the approximate standard deviation of \( Y \) in week 0, and write

\[
\bar{p}_j - \bar{p}_{j-1} = \lambda_j \sqrt{\frac{p_0(1-p_0)}{\mu_0}}, \quad j = 1, 2, \ldots
\]

We show in Appendix 2 that, for large values of the \( \mu \)'s, both unconditionally and conditionally on the realised values of the \( X \)'s, the statistic

\[
\tilde{T}_t = \frac{\bar{Y}_t - \bar{Y}_{t-1}}{\sqrt{\frac{1}{\mu_t + C_t^2}}}
\]

is distributed approximately \( N \left( \frac{\delta_t}{\sqrt{1 + \gamma_t^2}}, 1 \right) \), where

\[
C_t^2 = \frac{1 - \alpha}{1 + \alpha} X_t, \quad S_t \left( \frac{1}{X}; \alpha^2 \right),
\]

\[
Y_t^2 = \frac{1 - \alpha}{1 + \alpha} \mu_t, \quad S_t \left( \frac{1}{\mu}; \alpha^2 \right),
\]

\[
\delta_t = \left( \frac{\mu}{\mu_0} \right)^{\frac{1}{2}} \sum_{j=0}^{t-1} \alpha^j \lambda_{t-j},
\]

and where the \( S \)'s, which denote the exponentially smoothed values of the quantities shown in brackets together with the smoothing constants, are given by

\[
S_t \left( \frac{1}{X}; \alpha^2 \right) = \left( 1 - \alpha^2 \right) \frac{1}{X^2} + \alpha^2 S_{t-1} \left( \frac{1}{X}; \alpha^2 \right), \quad S_0 \left( \frac{1}{X}; \alpha^2 \right) = 0,
\]

\[
S_t \left( \frac{1}{\mu}; \alpha^2 \right) = \left( 1 - \alpha^2 \right) \frac{1}{\mu^2} + \alpha^2 S_{t-1} \left( \frac{1}{\mu}; \alpha^2 \right), \quad S_0 \left( \frac{1}{\mu}; \alpha^2 \right) = 0.
\]

### 4.1 Case where \( a \) is close to 1

If \( a \) is close to 1, then \( c_t^2 \) and \( y_t^2 \) are both close to zero, and hence the statistic \( \bar{T}_t \) is distributed approximately \( N \left( \delta_t, 1 \right) \) for large \( \mu \)'s, both unconditionally and conditionally on the realised values of the \( X \)'s.
Further, is $\bar{p}$ is changing so slowly that $\lambda_j \approx 0$, $j = 1, 2, \ldots$, and $\delta_t$ is of negligible size, then, under the above conditions, the distribution of $\tilde{T}_t$ is approximately $N(0,1)$.

Suppose now that, in week $s$, $\bar{p}$ changes by a 'large' amount, and thereafter continues to change very slowly, so that $\lambda_j = 0$ except for $j = s$. Then for $t \geq s$, $T_t$ is distributed approximately $N\left(\frac{\mu_t}{\mu_t} \frac{1}{\lambda_s}, 1\right)$, under the above Conditions. Thus the mean of $\tilde{T}_t$ suddenly changes to $\frac{\mu_t}{\mu_t} \frac{1}{\lambda_s}$ in the week of the change, and then gradually decays back to zero as the estimator $\tilde{Y}$ gradually adapts to the new 'level' of $\bar{p}$.

Thus to detect only rapid, local changes in the value of $\bar{p}$, we may monitor the panel's weekly purchasing behaviour using Shewhart and cusum charts of the modified statistics $\tilde{T}$.

Unlike the $T$'s the $\tilde{T}$'s are autocorrelated. We show in Appendix 2 that if $a$ is close to 1, then for large $\mu$'s, both unconditionally and conditionally on the realised $X$'s, the correlation between $\tilde{T}_{t+1}$ and $\tilde{T}_t$ is approximately

$$\rho = \left(\frac{\mu_t}{\mu_t} \frac{1}{\lambda_s} \right)^2 \left(1-\alpha\right)^{t-1} + \left(\frac{\mu_t}{\mu_t} \frac{1}{\lambda_s} \right)^2 S_{t-1} \left(\frac{1}{\mu_t} \frac{1}{\mu_t} \frac{1}{\lambda_s} \frac{1}{\lambda_s} \right) \left(1-\alpha\right)^{t-1} \left(1+\alpha\right)^{t-1}$$

$$= -\left(\frac{1-\alpha}{1+\alpha}\right) \left(1+\alpha^{2t-1}\right), \text{if the }\mu\text{'s are approximately equal.}$$

These autocorrelations are in fact quite small. For example, if $a = 0.9$, and if the $\mu$'s are approximately equal, the autocorrelation at lag 1 (the largest autocorrelation) ranges from about -0.1 for $t = 1$ to about -0.05 for large $t$. Hence, for large $a$, the presence of autocorrelations amongst the $T$'s should have only a slight effect on the behaviour of the Shewhart and cusum charts of these statistics.

4.2 Choice of $\alpha$

The choice of $a$ depends on what magnitude of gradual change in the value of $\bar{p}$ we do not wish to detect.

Suppose that we do not wish to detect a change in $\bar{p}$ if the weekly amounts
by which it changes are such that $|\lambda_j| \leq \lambda_m, j = 1,2,\ldots$. For such changes, the mean of $T_t$ is bounded (for large $\mu$'s) as follows

$$|E(\widetilde{T}_t)| = \left| \frac{\delta_t}{\sqrt{1+\gamma^2_t}} \right| \leq |\delta_t|$$

$$= \left( \frac{\mu_t}{\mu_0} \right)^{1/2} \sum_{j=0}^{t-1} \alpha_j \lambda_{t-j} \leq \lambda_m \left( \sqrt{\frac{\max \mu_j}{\mu_0}} \right)^{1/2}.$$

Now, if $p$ does not change, the means of the $\widetilde{T}$'s are zero. Hence, if $p$ is changing as above, to keep the risk of detecting a change with a cusum chart close to what it would be for unchanging $p$, we need to choose a so that the means of the $\widetilde{T}$'s are close to zero, say less than 0.1 in absolute value. Thus we should choose $a$ so that

$$\lambda_m \left( \sqrt{\frac{\max \mu_j}{\mu_0}} \right)^{1/2} < 0.1,$$

$$\Rightarrow a < 1 - 10 \lambda_m \left( \sqrt{\frac{\max \mu_j}{\mu_0}} \right)^{1/2}.$$

If $|\nabla p|$ denotes the maximum weekly change in $p$ that we do not wish to detect, then

$$\lambda_m = \left| \nabla p \right| / \sqrt{p_0(1-p_0) / \mu_0},$$

and hence we require

$$a < 1 - \frac{10 \left| \nabla p \right|}{\sqrt{p_0(1-p_0) / \mu_0}},$$

where $\mu_m = \max \mu$.

As an illustration, suppose $\nabla p = 0.2$ and $\mu = 1000$. Then, if we do not wish to detect a change if $p$ gradually changes by 0.005 over 50 weeks, we should take $a < 0.92$; whereas, if we do not wish to detect a change if $p$ gradually changes by 0.015 over 50 weeks, we should take $a < 0.76$.

If the required value of $a$ is not sufficiently close to 1 to justify taking
$C_t^2 = 0$, then we should have to use the statistics $\tilde{T}_t = \frac{T_t}{\sqrt{1 + C_t^2}}$ instead of $\tilde{T}_t$.

### 4.3 Modified Decision Rules for Cusum Charts of the $\tilde{T}$'s

The sensitivity of the cusum of the $\tilde{T}$'s in detecting a change in their means is slightly different to that for the $T$'s.

Thus, if $\tilde{p}$ is constant or is only slowly changing, and if $\mu$ is approximately constant, the small negative autocorrelations amongst the $\tilde{T}$'s (for large $a$) slightly reduce the variances of their upper and lower d.i. cusums compared with the independent $T$'s. Hence each of these cusums reaches its decision limit slightly less frequently than those for independent $T$'s, and hence the ARL between false alarms is slightly greater for the $\tilde{T}$'s than for the $T$'s.

On the other hand, if $\tilde{p}$ suddenly changes to a new level, the appropriate d.i. cusum moves towards the decision limit slightly less rapidly for the $\tilde{T}$'s than for the $T$'s. This is because the statistics $\tilde{Y}$ gradually adapt to the new level of $\tilde{p}$ and so gradually reduce the magnitude of the $\tilde{T}$'s, whereas the $T$'s are not so affected. Hence the sensitivity in detecting a sudden change in $\tilde{p}$ is slightly less for the cusum of the $\tilde{T}$'s than for the $T$'s.

We can increase the sensitivity of the cusum of the $\tilde{T}$'s in detecting a sudden change in $p$ by reducing the value of the decision interval $h$. A preliminary simulation study indicates that for $a = 0.9$, and taking $k = 0.5$, $h$ should be reduced to about 3.2 to maintain an ARL of about 100 weeks between false alarms when $\tilde{p}$ remains unchanged and $y$ is constant.

A more extensive simulation study needs to be carried out to obtain estimates, as in Table 3, of the ARL's to detection, and detection probabilities, following a sudden change in the value of $\tilde{p}$ of various magnitudes.

### 4.4 An Illustration

We use the data of the previous illustration (section 3.3) to demonstrate use of Shewhart and cusum charts of the modified statistics $\tilde{T}$. 

##
FIGURE 5.
ADAPTIVE SHEWHART CHART
FIGURE 6.
ADAPTIVE ORDINARY CUSUM CHART

CUSUM

WEEK

0  5  10  15  20  25  30  35  40  45  50

0  1  2  3  4  5  6  7  8  9  10
FIGURE 7.
ADAPTIVE DECISION INTERVAL CUSUM CHART
We take as the value of the smoothing constant $\alpha = 0.9$, which will allow for a gradual change in $\bar{p}$ of about 0.5% over a period of a year.

We take as the established value of $\bar{p}$ at the start of the smoothing process $\bar{p}_0 = 0.1933$ (the proportion of purchases of B for the first 10 weeks), and monitor the data for a rapid change in $\bar{p}$ from week 11 onwards.

The results are very similar to those for the T's. No change in $p$ is detected by the Shewhart chart (figure 5), but the d.i. cusum chart (figure 7) clearly detects an increase at week 39. Also from the ordinary cusum chart (figure 6), as before, we estimate that the sudden increase in $\bar{p}$ first occurred in week 31, and that the increased level was approximately held until week 39.

5. Monitoring by Weight of Product Purchased

Suppose that the product is available in various packet sizes. We may regard the different packet sizes of a given brand as brands in their own right; suppose that there are $g$ such brands and that brands 1 to $b$ correspond to different packet sizes of the particular brand B of interest. Let $w_r$ be the weight of a packet of the $r^{th}$ brand, $r = 1, \ldots, g$.

Let $\mu_i$, $i = 1, \ldots, n$, and $\mu$ be as previously defined, but here let $p_{rij}$, $i = 1, \ldots, n$, $r = 1, \ldots, g$, denote the probability of member $i$ choosing the $r^{th}$ brand on a given purchase occasion, $\sum_{r=1}^{g} p_{rij} = 1$. The probability of the panel choosing the $r^{th}$ brand on a given occasion is then

$$\bar{p}_r = \sum_{i=1}^{n} \frac{\mu_i}{\mu} p_{rij}, \quad r = 1, \ldots, g, \quad \sum_{r=1}^{g} \bar{p}_r = 1.$$  

Let $X_t$, $t = 1, 2, \ldots$, be as previously defined but here let

$$Z_{rt} = \text{overall number of occasions on which the } r^{th} \text{ brand is purchase by the panel in week } t.$$
\[
\gamma_{rt} = \frac{Z_{rt}}{X_t}
\]
overall proportion of occasions on which the \(r^{th}\) brand is purchased by the panel in week \(t\).

and \(W_{rt} = \frac{W_r Z_{rt}}{\sum_{s=1}^{g} W_s Z_{st}} = \frac{W_r Y_{rt}}{\sum_{s=1}^{g} W_s Y_{st}}\) overall proportion by weight of the \(rt^{th}\) brand purchased in week \(t\).

The overall proportion by weight of brand \(B\) purchased in week \(t\) is then \(W^B_t = \sum_{r=1}^{b} W_{rt}\).

We show in Appendix 3 that for sufficiently large realised \(X\), the conditional distribution of \(W^B_t\) given \(X_t\) is approximately normal with

\[
E(W^B_t | X_t) = \sum_{r=1}^{b} P^w_r = \sum_{r=1}^{b} \frac{W_r \bar{P}_r}{\sum_{s=1}^{g} W_s \bar{P}_s} = P^B, \text{ say}
\]
and

\[
\text{var}(W^B_t | X_t) = x \left( \sum_{r=1}^{b} W_r \bar{P}_r^w - P^B \left( \sum_{r=1}^{b} W_r \bar{P}_r^w \right) \right) = (\sigma^B)^2, \text{ say}
\]

where

\[
p^w_r = \frac{W_r \bar{P}_r}{\sum_{s=1}^{g} W_s \bar{P}_s}.
\]

\(P^B\) is thus the approximate expected proportion by weight of brand \(B\) purchased by the panel in a given week \(t\), and does not depend on \(X_t\).

The value of \(P^B\) may be monitored from week to week in the same way as \(P\). Thus to detect a change from an established value \(P^B_0\) (corresponding to established values \(P^r_0\) of \(P^r\), \(r = 1, \ldots, g\)), the monitoring would be based on the statistics

\[
T^w_t = \frac{W^B_t - P^B}{\sigma^B_0},
\]
where \( u_p \) is the value of \( o_R \) corresponding to the established values of the \( p_r \).

If the situation remains stable, then for large \( \mu 's \), both unconditionally and conditionally on the realised \( X 's \), the \( T^W 's \) are independent and each is distributed approximately \( N(0,1) \); and if there is a change in \( p^B \) the mean of the \( T^W 's \) changes accordingly.

To adapt to a gradually changing \( p^B \) in order to detect only rapid, local changes in its value, the monitoring would be based on the statistics

\[
\tilde{T}_t^W = \frac{\tilde{W}_t^B - \tilde{p}_{t-1}^B}{\tilde{\sigma}_{t-1}^B},
\]

where \( \tilde{p}_{t-1}^B \) and \( \tilde{\sigma}_{t-1}^B \) are estimates of the current values of \( p^B \) and \( \sigma^B \) at week \( t-1 \). The estimates of \( p^B \) and \( \sigma^B \) at week \( t \) are obtained by replacing the \( \tilde{p}_r, r=1,...,g \), in the expressions for these quantities by the corresponding exponentially smoothed estimates \( \tilde{Y}_{r,t} \), given by

\[
\tilde{Y}_{r,t} = (1-\alpha) Y_{r,t} + \alpha \tilde{Y}_{r,t-1}, r=1,...,g.
\]

If \( p^B \) is constant or slowly changing, and if \( a \) is close to 1, then for large \( \mu 's \), both unconditionally and conditionally on the realised \( X 's \), the \( \tilde{T}^W 's \) are distributed approximately \( N(0,1) \); and if \( p^B \) changes rapidly the means of the \( \tilde{T}^W 's \) change accordingly.

Note that, apart from the b 'pseudo-brands' which make up the particular brand \( B \) of interest, the other 'pseudo-brands' may be regrouped in terms of packet size, which may be computationally more convenient.

The behaviour of these procedures for monitoring by weight (which can also be applied to expenditure) have not yet been fully investigated.

Reference

APPENDIX 1

**Distribution of T, when \( \bar{p} = \bar{p}_1 \)**

Now \( T \) can be written

\[
T = \frac{Y_t - \bar{p}_0}{\sqrt{\frac{\bar{p}_0(1 - \bar{p}_0)}{X_t}}} = \sqrt{\frac{\bar{p}_1(1 - \bar{p}_1)}{\bar{p}_0(1 - \bar{p}_0)}} \left[ \frac{Y_t - \bar{p}_1}{\sqrt{\frac{\bar{p}_1(1 - \bar{p}_1)}{X_t}}} \right] + \frac{X_t}{\mu} \left[ \frac{\bar{p}_1 - \bar{p}_0}{\sqrt{\frac{\bar{p}_0(1 - \bar{p}_0)}{\mu}}} \right].
\]

We consider a change in \( \bar{p} \) of magnitude

\[
\Delta \bar{p}_0 = \bar{p}_1 - \bar{p}_0 = \lambda \sqrt{\frac{\bar{p}_0(1 - \bar{p}_0)}{\mu}}, \text{ where } \lambda \text{ is fixed.}
\]

We have the following limit results

(i) If \( \bar{p} = \bar{p}_1 \), then as \( \mu \to \infty \)

\[
\frac{Y_t - \bar{p}_1}{\sqrt{\frac{\bar{p}_1(1 - \bar{p}_1)}{X_t}}} \to Z \sim \mathcal{N}(0, 1) \text{ unconditionally, or conditionally on the realised } X_t
\]

(ii) Now \( X_t \sim \text{Po}(\mu) \).

Hence

\[
E \left[ \frac{X_t}{\mu} \right] = 1 \text{ for all } \mu \text{ and }
\]

\[
\text{Var} \left[ \frac{X_t}{\mu} \right] = \frac{1}{\mu} \to 0 \text{ as } \mu \to \infty
\]

\[
\Rightarrow \frac{X_t}{\mu} \to 1 \text{ as } \mu \to \infty.
\]

(iii) As \( \mu \to \infty \), \( \Delta p_0 \to 0 \) and hence

\[
\frac{\bar{p}_1(1 - \bar{p}_1)}{\bar{p}_0(1 - \bar{p}_0)} \to 1.
\]

Hence as \( \mu \to \infty \)
Hence for large $\mu$, both unconditionally and conditionally on the realised value of $X_t$, $T_t$ is distributed approximately $N(\lambda, 1)$, where $\lambda = \frac{\Delta \tilde{p}_0}{\tilde{p}_0 (1 - \tilde{p}_0) / \mu}$.

For the typical values $\mu = 1000$, $\lambda = 1$, $\tilde{p}_0 = 0.02$ and 0.2, we note that:

1) if $\tilde{p} = \tilde{p}_1$, the distribution of $Y_t - \tilde{p}_1$ is well approximated by the $N(0,1)$ distribution;

2) since $X_t$ is approximately normally distributed, $X_t / \mu$ will almost certainly fall in the range $1 \pm 3/\sqrt{\mu} = 1 \pm 0.095$, and so $\sqrt{X_t / 2\mu}$ will almost certainly fall in the range $1 + 0.046$ and so be close to the above limiting value;

3) $\sqrt{\frac{\tilde{p}_1 (1 - \tilde{p}_1)}{\tilde{p}_0 (1 - \tilde{p}_0)}} = \begin{cases} 1.10 & \text{if } \tilde{p}_0 = 0.02 \\ 1.02 & \text{if } \tilde{p}_0 = 0.2 \end{cases}$; these values are reasonably close to the above limiting value for this quantity.

Thus the above approximation to the distribution of $T_t$ should be adequate for the values of $\mu$, $\tilde{p}_0$ and $\lambda$ which we consider.
APPENDIX 2

2.1 Asymptotic Distribution of $\tilde{T}_t$

Here we denote the values of $\mu$ and $p$ in week $t$ by $\mu_t$ and $p_t$, respectively.

Now $\tilde{Y}_t$ is given by

$$Y_t = (1-\alpha)(\tilde{Y}_t + \alpha Y_{t-1} + \alpha^2 Y_{t-2} + \ldots + \alpha^{t-1} Y_1 + \alpha^t - Y_0).$$

We shall take $\tilde{Y}_0$ to be a constant 'starting' value $p_0$, representing the 'established' value of $p$ in week 0. Hence

$$E(Y_t | X_1, \ldots, X_t) = (1-a) (\tilde{p}_t + \alpha \tilde{p}_{t-1} + \alpha^2 \tilde{p}_{t-2} + \ldots + \alpha^{t-1} \tilde{p}_1) + \alpha^t - \tilde{p}_0 = S_t(\tilde{p}; \alpha),$$

the exponentially smoothed value of $p$ at week $t$ with smoothing constant $\alpha$, where $S_0(\tilde{p}; \alpha) = \tilde{p}_0$.

And $\text{var} (Y_t | X_1, \ldots, X_t) = (1-\alpha)^2 \left\{ \frac{\tilde{p}_t (1-\tilde{p}_t)}{X_t} + \alpha^2 \frac{\tilde{p}_{t-1} (1-\tilde{p}_{t-1})}{X_{t-1}} + \alpha^4 \frac{\tilde{p}_{t-2} (1-\tilde{p}_{t-2})}{X_{t-2}} + \ldots + \alpha^{2t-2} \frac{\tilde{p}_1 (1-\tilde{p}_1)}{X_1} \right\} = \frac{1-\alpha}{1+\alpha} S_t \left( \frac{\tilde{p}(1-\tilde{p})}{X}; \alpha^2 \right)$,

where

$$S_t \left( \frac{\tilde{p}(1-\tilde{p})}{X}; \alpha^2 \right) = (1-\alpha^2) \frac{\tilde{p}_t (1-\tilde{p}_t)}{X_t} + \alpha^2 S_{t-1} \left( \frac{\tilde{p}(1-\tilde{p})}{X}; \alpha^2 \right)$$

is the exponentially smoothed value of $\frac{\tilde{p}(1-\tilde{p})}{X}$ at week $t$ with smoothing constant $\alpha^2$, where $S_0 \left( \frac{\tilde{p}(1-\tilde{p})}{X}; \alpha^2 \right)$

Now $\tilde{T}_t$ can be written

$$\tilde{T}_t = \frac{Y_t - \tilde{Y}_{t-1}}{\sqrt{Y_{t-1}(1-\tilde{Y}_{t-1})}} = b_1 (U_t' - C_t' V_t' + \delta_t')$$
where

\[
U_t' = \frac{Y_t - \overline{P}_t}{\left\{ \overline{P}_t(1 - \overline{P}_t) \right\}^{1/2}},
\]

\[
V_t' = \frac{\tilde{Y}_{t-1} - S_{t-1}(\overline{P}; \alpha)}{\left\{ \frac{1- \alpha}{1+ \alpha} S_{t-1} \left( \frac{\overline{P}(1-\overline{P})}{X}; \alpha^2 \right) \right\}^{1/2}},
\]

\[
b_t = \left\{ \frac{\overline{P}_t(1 - \overline{P}_t)}{Y_{t-1}(1-Y_{t-1})} \right\}^{1/2},
\]

\[
C_t' = \left\{ \frac{(1- \alpha)S_{t-1} \left( \frac{\overline{P}(1-\overline{P})}{X}; \alpha^2 \right) + 1+ \alpha \frac{\overline{P}_t(1-\overline{P}_t)}{X}}{X_t} \right\},
\]

\[
r_t = \left\{ \frac{X_t}{\mu_t} \right\}^{1/2},
\]

\[
\delta_t = \frac{\overline{P}_t - S_{t-1}(\overline{P}; \alpha)}{\left\{ \frac{\overline{P}_t(1-\overline{P}_t)}{\mu_t} \right\}^{1/2}}.
\]

Note that \(U_t'\) and \(V_t'\) are independent both unconditionally and conditionally on \(X_1, \ldots, X_t\).

We have the following limit results.

(i) \(\text{As } \mu_t \to \infty, U'_t \xrightarrow{d} \mathcal{N}(0, 1), \) both unconditionally and conditionally on the realised value of \(X_t\).

(ii) \(\text{As } X_1, \ldots, X_{t-1} \to \infty, V'_t \xrightarrow{d} \mathcal{N}(0, 1) \) conditionally on \(X_1, \ldots, X_{t-1}\).

Also, since \(X_1, \ldots, X_{t-1} \xrightarrow{p} \infty \) as \(\mu_1, \ldots, \mu_{t-1} \to \infty\), it follows that \(V'_t \xrightarrow{d} \mathcal{N}(0, 1)\) unconditionally as \(\mu_1, \ldots, u_{t-1} \to \infty\).

(iii) \(\text{As } X_1, \ldots, X_{t-1} \to \infty, \text{var}(Y_{t-1} | X_1, \ldots, X_{t-1}) \to 0, \) and hence \(Y_{t-1} \xrightarrow{p} S_{t-1}(\overline{p}; a) \) conditionally on \(X_1, \ldots, X_{t-1}\). And since
\(X_1, \ldots X_{t-1} \to \infty\) as \(\mu_1, \ldots, \mu_{t-1} \to \infty\), it follows that
\(\tilde{Y}_{t-1} \to S_{t-1}(p^*; a)\) unconditionally on \(\mu_1, \ldots, \mu_{t-1} \to \infty\). (The convergence is uniform in the values of \(p_1, \ldots, p_{t-1}\).)

Hence as \(\mu_1, \ldots, \mu_{t-1} \to \infty\),
\[
\beta_t \to \left( \frac{\bar{p}_1(1-\bar{p}_1)}{S_{t-1}(p; a)(1-S_{t-1}(p; a))} \right)^{\frac{1}{2}} = \beta_t, \text{ say},
\]
both unconditionally and conditionally on the realised values of \(X_1, \ldots, X_{t-1}\) (and uniformly in the \(p^*\)’s).

(iv) As \(\mu \to \infty\), \(r_t \to 1\) (see Appendix 1).

(v) Let \(\mu_0, \ldots, \mu_t \to \infty\) with fixed ratios \(\theta_j = \mu_j / \mu_0, 0 < \theta_j < \infty, j = 1, \ldots, t\).

Then
\[
\gamma_t = \left(1-\alpha\right) \frac{\mu_t S_{t-1} \left(1-\bar{p}(1-\bar{p}); p^*\alpha^2 \right)}{\mu_0 X_{t} (1-\bar{p}(1-\bar{p}); p^*\alpha^2 \right)} \frac{1}{2} = \gamma_t, \text{ say}
\]
(The convergence is uniform in the \(p^*\)’s.)

(vi) Consider now weekly changes in \(p\) of magnitude
\[
V \bar{p} j = \bar{p}_j - \bar{p}_{j-1} = \lambda_j \sqrt{\frac{p_0(1-\bar{p}_0)}{\mu_0}},
\]
j = 1, 2, ..., where the \(\lambda_j\)’s are fixed.
Then as \(\mu_0, \mu_1, \ldots, \to \infty\), \(V \bar{p}_j \to 0\) and hence \(\bar{p}_j \to \bar{p}_0 j = 1, 2, \ldots\); and as \(\bar{p}_j \to \bar{p}_0, j = 1, \ldots, t, \beta_t \to 1, \) since \(S_{t-1}(\bar{p}; a) \to \bar{p}_0\), and
\[
\gamma_t \to \left(1-\alpha\right) \theta_t S_{t-1} \left(1-\bar{p}(1-\bar{p}); p^*\alpha^2 \right) \frac{1}{2} = \gamma_t, \text{ say}.
\]
Since, for fixed \( p \)'s, the convergence of \( b_t \) to \( \beta_t \), and of \( c_t \) to \( \gamma' \), is uniform in the \( p \)'s, it follows that if \( \mu_0, \mu_1, \ldots, \mu_t \to \infty \) with fixed ratios \( \theta_j = \mu_j / \mu_0 \) and with \( p_j - p_{j-1} = \lambda_j \cdot \frac{p_0(1-p_0)}{\mu_0} \), where \( \lambda_j \) is fixed, \( j = 1, \ldots, t \), then

\[
\begin{align*}
\beta_t \to 1 & \quad \text{and} \quad \gamma_t \to \gamma_t,
\end{align*}
\]

both unconditionally and conditionally on the realised values of \( X_1, \ldots, X_t \).

(vii) For weekly changes in \( p \) as in (vi),

\[
\begin{align*}
p_t - S_{t-1}(p; \alpha) &= p_t - (1-\alpha)p_{t-1} - aS_{t-2}(p; \alpha) \\
&= V p_t + \alpha(p_t - S_{t-2}(p; \alpha)) \\
&= V p_t + \alpha V p_{t-1} + \alpha^2 V p_{t-2} + \ldots + \alpha^{t-1} V p_1 \\
&= \sum_{j=0}^{t-1} \alpha^j \lambda_{t-j} \frac{p_0(1-p_0)}{\mu_0} \left( \frac{1}{2} \right)
\end{align*}
\]

Therefore as \( \mu_0, \mu_1, \ldots, \mu_t \to \infty \) with \( \theta_j \) fixed, \( \lambda_j = 1, \ldots, t \),

\[
\delta_t \to \theta_t^2 \sum_{j=0}^{t-1} \alpha^j \lambda_{t-j} = \delta_t, \quad \text{say}.
\]

The following asymptotic distribution for \( T_t \) follows from the above results.

**Theorem**

Let \( \mu_0, \mu_1, \ldots, \mu_t \to \infty \) with fixed ratios \( \theta_j = \mu_j / \mu_0 \), \( 0 < \theta_j < \infty \), \( j = 1, \ldots, t \), and let the weekly changes in \( p \) be given by

\[
p_j - p_{j-1} = \lambda_j \sqrt{\frac{p_0(1-p_0)}{\mu_0}} \quad \text{where} \quad \lambda_j \quad \text{is fixed},
\]

\( j = 1, \ldots, t \).

Then
\[ T_t \sim N \left( \delta_t, 1 + \gamma_t^2 \right), \]
both unconditionally and conditionally on the realised values of \( X_1, \ldots, X_t \).

**Proof** Under the limiting conditions of the theorem, we have from the above results that, both unconditionally and conditionally on the realised values of the \( X \)'s,

\[
\begin{align*}
U_t' & \overset{b}{\longrightarrow} U_t ', \quad \text{where } U_t \sim N(0,1), \\
V_t' & \overset{b}{\longrightarrow} V_t ', \quad \text{where } V_t \sim N(0,1), \\
b_t & \overset{p}{\longrightarrow} 1, \quad C_t' \overset{p}{\longrightarrow} \gamma_t \quad \text{and} \quad r_t \overset{p}{\longrightarrow} 1.
\end{align*}
\]

Also \( \delta_t' \to \delta_t \), and \( U_t \) and \( V_t \) are independent.

Hence

\[
T_t = b_t (U_t' - c_t' V_t' + r_t \delta_t') \overset{d}{\longrightarrow} U_t - \gamma_t V_t + \delta_t \sim N(\delta_t, 1 + \gamma_t^2).
\]

**Corollary**

Under the above conditions

\[
\begin{align*}
\tilde{T}_t \overset{b}{\longrightarrow} N \left( \frac{\delta_t}{(1 + \gamma_t^2)^{1/2}}, 1 \right), \\
\quad C_t = \left( \frac{(1-\alpha) X_t' S_{t-1} \left( \frac{1}{X_t}; \alpha^2 \right)}{1 + \alpha} \right)^{1/2} \overset{p}{\longrightarrow} \gamma_t.
\end{align*}
\]

Hence

\[
\tilde{T}_t \overset{b}{\longrightarrow} N \left( \frac{\delta_t}{(1 + \gamma_t^2)^{1/2}}, 1 \right),
\]
both unconditionally and conditionally on the realised values of \( X_1, \ldots, X_t \).

It follows from this corollary that for large values of the \( \mu \)'s,

\[
\frac{\tilde{T}_t}{\gamma_t} \overset{b}{\longrightarrow} N \left( \frac{\delta_t}{(1 + \gamma_t^2)^{1/2}}, 1 \right).
\]
is distributed approximately $N\left(\frac{\delta_t}{(1+\gamma_t)^2}, 1\right)$ both unconditionally and conditionally on the realised values of $X_t, ..., X_T$.

### 2-2 Correlation Structure of the $\tilde{T}$'s

First note that $V_t'$ can be written

$$V_t' = \left(1-\alpha\right)\sum_{j=0}^{t-2} \alpha^j (Y_{t-j} \bar{P}_{t-j})$$

$$= \sum_{j=0}^{t-2} f_{t,j} U_{t-1-j} ,$$

where

$$f_{t,j} = \left(\frac{1-\alpha}{1+\alpha} \right) S_{t-1} \left(\frac{\bar{P}(1-\bar{P})}{X}; \alpha^2\right)^{1/2}.$$

And under the limiting conditions of the above theorem

$$f_{t,i} \xrightarrow{p} \left(\frac{1-\alpha}{\theta} \right) S_{t-1} \left(\frac{1}{\theta};\alpha^2\right)^{1/2}$$

$$\alpha^j = \phi_{t,j}, \text{ say.}$$

and hence

$$V_t' \xrightarrow{b} \sum_{j=0}^{t-2} \phi_{t,j} U_{t-1-j} ,$$

both unconditionally and conditionally on the realised values of the $X$'s.

Hence, for the asymptotic distribution, the covariance between $\tilde{T}_{t+T}$ and $\tilde{T}_t$, $T \geq 0$, is

$$a.cov(\tilde{T}_{t+T}, \tilde{T}_t) = a.cov(\bar{U}_t + \bar{Y}_{t+T} V_{t+T} + \delta_{t+T} U_t - Y_t V_t + \delta_t)$$

$$= a.cov\left(\sum_{j=0}^{t+T-2} \phi_{t+T,j} U_{t+T-1-j}, \sum_{j=0}^{t-2} \phi_{t,j} U_{t-1-j}\right).$$
\[
\begin{align*}
= -\gamma_{t+\tau} \varphi_{t+\tau, t+\tau-1} + \gamma_{t+\tau} \gamma_t \sum_{j=0}^{t-2} \varphi_{t+\tau, t+j} \varphi_{t, j} \\
= -\left(\frac{\theta_{t+\tau}}{\theta_t}\right)^2 (1-\alpha) \tau^{-1} + \left(\theta_t \theta_{t+\tau}\right)^\tau (1-\alpha)^2 \alpha^\tau \sum_{j=0}^{t-2} \frac{\alpha^{2j}}{\theta_{t-1-j}} \\
= -\left(\frac{\theta_{t+\tau}}{\theta_t}\right)^2 (1-\alpha) \tau^{-1} + \left(\theta_t \theta_{t+\tau}\right)^\tau \left[\frac{1-\alpha}{1+\alpha}\right] \alpha^\tau \sum_{j=0}^{t-1} \left[\frac{1}{\theta_{t-1-j}} \alpha^2\right].
\end{align*}
\]

The asymptotic correlation coefficient between \(\tilde{T}_{t+\tau}\) and \(\tilde{T}_t\) is then

\[
\text{a. corr} \left[\tilde{T}_{t+\tau}, \tilde{T}_t\right] = \frac{\text{a. cov} \left[\tilde{T}_{t+\tau}, \tilde{T}_t\right]}{\sqrt{\text{a. var} \left[\tilde{T}_{t+\tau}\right] \text{a. var} \left[\tilde{T}_t\right]}} = \frac{\text{a. cov} \left[\tilde{T}_{t+\tau}, \tilde{T}_t\right]}{\sqrt{(1+\gamma^2_{t+\tau})(1+\gamma^2_t)}}.
\]

For large \(\mu\)'s, the correlation between \(\tilde{T}_{t+\tau}\) and \(\tilde{T}_t\) may be approximated by their asymptotic correlation, with \(\theta_j\) replaced by \(\frac{\mu j}{\mu 0}\).
APPENDIX 3

Conditional Distribution of $W^B_t$ given $X_t$

For given $X_t > 0$, $Z_{1,t}, \ldots, Z_{g,t}$ have a multinomial distribution with index $X_t$ and probability parameters $p_1, \ldots, p_g$.

Hence, for $Y_{rt} = Z_{rt}/X_t$,

$$E(Y_{rt} \mid X_t) = \frac{P_r}{X_t} , \text{ var } (Y_{rt} \mid X_t) = \frac{P_r (1 - P_r)}{X_t} , \text{ cov } (Y_{rt}, Y_{st} \mid X_t) = -\frac{P_r P_s}{X_t} , \ r \neq s.$$  

Now

$$w^B_t = \frac{\sum_{r=1}^b w_{rt} Y_{rt}}{\sum_{s=1}^g w_{st} Y_{st}} = \frac{U}{V} , \text{ say.}$$  

Where

$$E(U \mid X_t) = \sum_{r=1}^b W_r \frac{P_r}{X_t} ,$$

$$\text{ var } (U \mid X_t) = \sum_{r=1}^b \frac{W_r^2 P_r - \left( \sum_{r=1}^b W_r P_r \right)^2}{X_t} ,$$

with corresponding expressions for $E(V \mid X_t)$ and $\text{ var } (V \mid X_t)$, and where

$$\text{ cov } (U, V \mid X_t) = -\frac{\left( \sum_{r=1}^b W_r P_r \right) - \left( W_r - \sum_{s=1}^g W_s P_s \right)}{X_t}.$$  

For sufficiently large realised $X_t$, the conditional variances of $U$ and $V$ are small, and hence we may approximate $W^B_t$ by the linear terms of the Taylor expansion of $U/V$ about the conditional expectations of these variables. Thus

$$w^B_t = \frac{E(U \mid X_t)}{E(V \mid X_t)} + \frac{1}{E(V \mid X_t)} \left( v - E(V \mid X_t) \right) \left( v - E(V \mid X_t) \frac{E(U \mid X_t)}{E^2(V \mid X_t)} \right).$$

Hence, for large realised $X_t$, ...
\[ E\left( W_t^B \mid X_t \right) = \frac{E(U \mid X_t)}{E(v \mid X_t)} = \frac{\sum_{r=1}^{b} W_r \bar{p}_r}{\sum_{s=1}^{g} W_s \bar{p}_s}, \text{and} \]

\[ \text{Var}\left( W_t^B \mid X_t \right) = \frac{1}{E^2(v \mid X_t)} \var(U \mid X_t) + \frac{E^2(U \mid X_t)}{E^2(v \mid X_t)} \text{Var}(v \mid X_t) \]

\[ = \frac{2E(U \mid X_t)}{E^3(v \mid X_t)} \text{COV}(U \mid X_t) \]

\[ = \frac{1}{X_t} \sum_{s=1}^{g} W_s \bar{p}_s - p^B \left[ 2 \sum_{r=1}^{b} W_r \bar{p}_r - p^B \sum_{s=1}^{g} W_s p_s^w \right] \]

(after a little algebra), where \( p_r^w = \frac{W_r \bar{p}_r}{\sum_{s=1}^{g} W_s} \) and \( p^B = \sum_{r=1}^{b} p_r^w \).

Further, for sufficiently large realised \( X_t \), the multinomial conditional distribution of \( Z_{1t}, \ldots, Z_{gt} \), and hence the conditional distribution of \( Y_{1t}, \ldots, Y_{gt} \), may be approximated by multivariate normal distributions. And

since, for large \( X_t \), \( W_t^B \) is well approximated by a linear combination of \( Y_{1t}, \ldots, Y_{gt} \) it follows that the conditional distribution of \( W_t^B \) given \( X_t \) is approximately normal.

Note also that, since \( X_t \xrightarrow{p} \infty \) as \( \mu \to \infty \), and since the approximate conditional distribution of \( W_t^B \) given large realised \( X_t \) does not depend on \( X_t \), it follows that, for large \( \mu \), this approximate distribution applies unconditionally also.