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A note on methods with $0(H^4)$ and $0(H^6)$ phase lags for periodic initial value problems.

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ABSTRACT

Two families of computational methods are discussed for the solution of second order periodic initial value problems.

The first is a family with $0(H^4)$ phase-lag which contains the recently published "Numerov made explicit" method of Chawla [2]. The second is a family with $0(H^6)$ phase-lag and periodicity interval given by $H^2 \in (0,12)$.

1. INTRODUCTION

Consider the second order nonlinear periodic initial value problem

(1) $$y''(t) = f(t,y(t)),\ y(t_0) = y_0,\ y'(t_0) = z_0$$

which arises in the theory of orbital mechanics and in the study of wave equations (see Lambert and Watson [5] and Twizell [6]).

Usually, linear multistep methods are used for the numerical solution of (1). Lambert and Watson [5] have shown that such methods cannot have order of accuracy greater than two if they are to be P-stable. Furthermore, those linear multistep methods which are P-stable are implicit and a nonlinear algebraic system must be solved in order to find the solution of (1).

The need to solve a nonlinear algebraic system is obviated by recourse to the approach followed by Chawla [2] in his "Numerov made explicit" formulation. The method developed in [2] is a member of two families of methods for the solution of (1) The first family is the family of multiderivative methods of Twizell and Khaliq [7], the method of Chawla [2] having the same local truncation error and periodicity polynomial as the method based on the (0,4) Padé approximant in [7]. The second family is developed in §2. A sixth order family is developed in §3.

2. A FAMILY OF METHODS WITH $0(H^4)$ PHASE-LAG

Suppose that the independent variable t is discretized into steps of size k such that $t_n = t_0 + nk (n = 0,1,2,...)$ and that y_n is the solution of some numerical method when $t = t_n$.

Consider now the explicit linear multistep method

(2) $$y_{n+1}^{[p]} = 2y_n - y_{n-1} + \alpha k^2 f_n,$$

where $f_n = f(t_n, y_n)$ is determined from (1) in which $\alpha > 0$ is a parameter. Equation (2) with $\alpha = 1$ is the classical second order explicit method which was used by Chawla [2].

Differentiating (2) gives

(3) $$f_{n+1}^{[p]} = 2y''_n - y''_{n-1} + \alpha k^2 f''_n$$

and this value of $f_{n+1}^{[p]}$ is used in

(4) $$y_{n+1}^{[c]} = 2y_n - y_{n-1} + k^2(af_{n+1}^{[p]} + bf_n + af_{n-1})$$

in which $a > 0$ and $b > 0$ are parameters, to give an improved value of Y_{n+1}. Equation (4) with $a = \frac{1}{12}$ and $b = \frac{5}{6}$ is Numerov's method which was used in [2] by Chawla.

It is easy to show that (4) is a fourth order method provided

(5) $$2a + b = 1 \text{ and } a\alpha = \frac{1}{12},$$

the error constant (see Lambert [4; p.253]) being $c_5 = \frac{1}{360}$; the error constant of the Numerov method is $c_6 = -\frac{1}{240}$

The usual choice of model periodic initial value problem is the test equation

(6) $$y'' = -\lambda^2 y, \ \lambda > 0 \text{ real}$$

with the initial conditions as for (1). The periodicity polynomial associated with (4) and (6) is

(7) $$\Omega(s,H^2) = A(H)s^2 - 2B(H)s + A(H),$$

where $H = \lambda k$, $A(H) = 1$ and $B(H) = 1 - (a + \frac{1}{2}b)H^2 + \frac{1}{2}a\alpha H^4 = 1 - \frac{1}{2}H^2 + \frac{1}{24}H^4$

(from (5)), and the interval of periodicity is determined by computing the range of values of H^2 for which s_1 and s_2, the zeros of the periodicity equation

(8) $$\Omega(s,H^2) = 0$$

satisfy

(9) $$s_1 = e^{i\theta(H)}, \quad s_2 = e^{-i\theta(H)},$$

where $\theta(H) \in \mathbb{R}$ is an approximation for H . It is easy to show that the interval of periodicity for the family of methods given by (4) is given by $H^2 \in (0,12)$.

The phase-lag $\Phi(H)$ of (4) is the leading term in the expansion of

(10) $$|\{\theta(H) - H\}/H|,$$

see Brusa and Nigro [1]. It is easy to deduce from (7), (8), (9) that

(11) $$\tan\theta(H) = [\{A(H)\}^2 - \{B(H)\}^2]^{\frac{1}{2}}/B(H),$$

and then, by substituting for A(H} and B(H), that

(12) $$\tan\theta(H) = H(1 + \frac{1}{3}H^2 + \frac{19}{144}H^4 + \frac{181}{3456}H^6 \ldots).$$

Using the expansion

(13) $$\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{2}{35}x^7 + \ldots$$

it may then be shown that the phase-lag of each of the family of fourth order methods {(4),(5)} is

$$\Phi(H) = \frac{1}{720}H^4.$$

It must be noted that a recent method of Chawla and Rao [3] cannot be generalized to the same extent as Chawla's "Numerov made explicit" method.

Writing

(14) $$y_n^{[p]} = y_n - \alpha k^2(f_{n+1} - 2f_n + f_{n-1})$$

and then

(15) $$y_{n+1}^{[C]} = 2y_n - y_{n-1} + k^2 (af_{n+1} + bf_n + bf_n^{[p]} + af_{n-1}),$$

where

(16) $$f_n^{[p]} = y_n'' - \alpha k^2 (2f_n'' + f_{n-1}''),$$

examination of the local truncation error of (15) shows that a and b must take the unique values $\frac{1}{12}$ and $\frac{5}{6}$ respectively, to ensure consistency.

Equation (15),with (16),is then fourth order accurate with $C_6 = \frac{5\alpha}{6} - \frac{1}{240}$; $\alpha = 0$ gives the standard fourth order Numerov method while $\alpha = \frac{1}{200}$ gives the sixth order method M_4 $\frac{1}{200}$ of Chawla and Rao [3].

3. A FAMILY OF METHODS WITH $0(H^6)$ PHASE-LAG

If, instead of (14) , the implicit formula

(17) $$y_{n+1}^{[p]} = 2y_n - y_{n-1} + k^2(\alpha f_{n+1} + \beta f_n + \alpha f_{n-1})$$

is used, where $\alpha > 0$ and $\beta > 0$ are parameters, equation (16) becomes

(18) $$f_{n+1}^{[p]} = 2y_n'' - y_{n-1}'' + k^2(\alpha \alpha_{n+1}'' + \beta f_n'' + \alpha f_{n-1}).$$

Equation (4), with $f_{n+1}^{[p]}$ given by (18) , is then seen to represent a family of sixth order methods each with phase-lag

$$\varphi(H) = \frac{1}{120960} H^6$$

provided

(19) $$2a + b = 1,\ a\alpha = \frac{1}{360} \text{ and } a\beta = \frac{7}{90}.$$

It may also be shown that the periodicity polynomial associated with (18) and (6) takes the form of (7) with

$$A(H) = 1 - a\alpha H^4 = 1 - {}^1/_{360}\, H^4 \text{ (from 19)}$$

and

$$B(H) = 1 - (a + \frac{1}{2} b)H^2 + \frac{1}{2} abH^4 = 1 - \frac{1}{2} H^2 + \frac{7}{180} H^4 \text{(from (19))}.$$

The periodicity interval for the family of sixth order methods is then easily shown to be given by $H^2 \in (0,12)$.

The family of methods developed in this section of the present paper clearly has two advantages over the family of methods derived by Chawla and Rao [3]: each method has $0(H\ )$ phase-lag and each has stability interval given by $H^2 \in (0,12)$ compared with $H^2 \in (0,7.35)$, approximately, for the sixth order method of Chawla and Rao [3] .

4. SUMMARY

Two families of numerical methods have been discussed for the solution of the general second order nonlinear periodic initial value problem.

The methods of the first, explicit, family each have $0(H^4)$ phase-lag; this family was seen to contain the "Numerov made explicit" method of Chawla [2] . The methods of the second, implicit, family each have phase—lag of $0(H^6)$ and stability interval given by $H^2 \in (0, 12)$ which is superior to a recent sixth order method of Chawla and Rao [3].

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