MENTAL MATHS: JUST ABOUT WHAT WE DO IN OUR HEADS?

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Introduction

Mental mathematics has, for many, negative associations with timed times table tests or questions fired out to be solved quickly but they have little memory of the explicit teaching of specific approaches to mental calculation. In this chapter, we unpick what we mean by 'mental mathematics', how it relates to mathematics as a subject and how it is conceptualised within the teaching of mathematics. We provide an overview of the English context below to exemplify the tensions and dilemmas we discuss throughout the chapter.

The English context

In the current version of the national Curriculum, developed whilst Michael Gove was Secretary of State for Education, the status of arithmetic has been raised. In a letter to the National Curriculum Expert Panel, the then UK Education Minister Michael Gove (DfE, 2012) wrote:

> In mathematics there will be additional stretch, with much more challenging content than in the current National Curriculum. We will expect children to be more proficient in arithmetic, including knowing number bonds to 20 by [age 7] and times tables up to 12 x 12 by [age 9]. The development of written methods – including long multiplication and division – will be given greater emphasis, and children will be taught more challenging content using fractions, decimals and negative numbers so that they have a more secure foundation for secondary school.

Michael Gove was determined to have a primary curriculum in place which demanded what he thought of as the highest of standards and which had gleaned best practice from the most successful schools, both in this country and abroad. However, his plans were met with strong opposition from the mathematics education community who stressed the importance of developing flexible approaches to calculation. Where to place the balance between teaching algorithms (for everything from addition to - the oft-dreaded but symbolically important - long division) and teaching such flexible approaches has been an ongoing debate within mathematics education.

Indeed, just a year before Gove's comment, the England schools inspectorate, Ofsted (2011, p.1, although now withdrawn) advocated a very different approach that identified the key aim of the teaching of calculation as developing mathematical understanding:

> It is ... of fundamental importance to ensure that children have the best possible grounding in mathematics during their primary years, number, or arithmetic, is a key component of this. Public perceptions of arithmetic often relate to the ability to calculate quickly and accurately – to add, subtract, multiply and divide, both mentally and using traditional written methods. But arithmetic taught well gives children so much more than this, understanding about number, its structures and relationships, underpins progression from counting in nursery rhymes to calculating with and reasoning about numbers of all sizes, to working with measures and establishing the foundations for algebraic thinking. These grow into the skills so valued by the world of

industry and higher education and are the best starting points for equipping children for their future lives.

Here we see a distancing from Gove's and the public's focus on calculation. And, although we might want to argue for purposes of mathematics education beyond preparing students for industry and higher education, it is clear that it is important that we give young people access to these fields. Also, research by Laurie Buxton (1981) and others on mathematics anxiety shows what a damaging long-term impact not feeling comfortable with numbers can have on people. (See Paul Ernest's chapter for further discussion about the potential harm that mathematics can do).

These two positions on calculation, represented here by Gove and Ofsted, reflect different underlying philosophies on mathematics education. Paul Ernest (1991) identified five such positions: industrial trainers, technological pragmatists, old humanists, progressive educators and public educators. Gove is typical of the 'industrial trainers' following a New Right ideology that sees mathematics as a set of absolute truths and rules to be learnt and so prioritising a drill and practice pedagogy that aims for '"Back-to-Basics" numeracy and training in social obedience' (p.139). Indeed a further development of this position is the new times table tests to be statutory from 2020. However, Ofsted, along with many mathematics education researchers, represent a 'progressive educator' position that sees mathematics as a set of absolute truths but 'with great value … attached to the role of the individual in coming to know this truth' (p.182) and so advocating a child-centred process-oriented pedagogy that aims for creativity and self-realisation. Our own sympathies lie with this position based in connection, care and empathy between humans. Thus, in this chapter, we will argue for an informal approach to mental mathematics as a way of developing 'relational understanding' of mathematics. We begin by unpacking what we mean by relational understanding.

What Do We Mean by Understanding Mathematics?

Consider the problem 2047 ÷ 23. Take a moment to solve this problem. How did you do this? Did you use the long division algorithm to solve this? Consider the steps that you took - how would you explain these to someone else? Do you feel that you understood what you did?

The problem in more formal classrooms would be written as in Figure 1 and the mantra heard would go something like this:

23 into 2, doesn't go

23 into 20, doesn't go

23 into 204…. hmmm how many 23s are there in 204? [and would then proceed to write down multiples of 23 discretely on the back of a piece of paper].

Figure 1. Long division algorithm for 2047 ÷ 23

				8	9
2	3	2	0	4	7
		1	8	4	↓
			2	0	7
			2	0	7
					0

An alternative approach, which has been used in schools more recently, is that of chunking. This encourages teachers to promote relational understanding (see below and Anna Llewellyn's chapter for further discussion about this) as there is no prescribed approach – pupils are encouraged to relate division to repeated subtraction where 'chunks' (multiples of the divisor) are subtracted from the dividend. The size of the chunks subtracted are not predetermined, unlike the formal long division algorithm where the largest multiple of the divisor must be subtracted. The number of chunks subtracted are totalled to find the solution. Figure 2 illustrates two different approaches to using chunking for long division. The first is a rather cautious approach, subtracting relatively easy multiples of 23. However, what is formally recorded in this illustration is what is often surreptitiously recorded on scraps of paper. The second is slightly more sophisticated in that larger chunks are subtracted. The numbers in bold are the number of chunks being subtracted from the dividend, which are totalled at the end.

Figure 2. Using chunking for long division

				8	9	
2	3	2	0	4	7	
		-	2	3	0	(**10**x23)
		1	8	1	7	
		-	2	3	0	(**10**x23)
		1	5	8	7	
		-	2	3	0	(**10**x23)
		1	3	5	7	
		-	2	3	0	(**10**x23)
		1	1	2	7	
		-	2	3	0	(**10**x23)
			8	9	7	
		-	2	3	0	(**10**x23)
			6	6	7	
		-	2	3	0	(**10**x23)
			4	3	7	
		-	2	3	0	(**10**x23)
			2	0	7	
		-	1	1	5	(**5**x23)
				9	2	
			-	6	9	(**3**x23)
				2	3	
			-	2	3	(**1**x23)
					0	

				8	9	
2	3	2	0	4	7	
		-	2	3	0	(**10**x23)
		1	8	1	7	
	-	1	1	5	0	(**50**x23)
			6	6	7	
		-	4	6	0	(**20**x23)
			2	0	7	
		-	2	0	7	(**10**x23)
					0	

Figures 1 and 2 illustrate some familiar (and maybe some unfamiliar!) strategies for 'long division'. Some people (including industrial trainers like Michael Gove) would consider this approach long-winded, ineffective with potential for errors, and resulting from 'vague generic statements of little value'. However, although we too feel that there is a significant difference in efficiency between Figures 1 and 2, we disagree. For us, the first appears to treat numbers as digits, rather than holistically and operates algorithmically whereas the second figure builds on a knowledge and understanding of mathematical relationships, for example, the relationships embedded in place value and those between multiplication and division and between division and subtraction. Perhaps speed for calculations such as this was important in the nineteenth century, for example, for clerks working in commerce, but in today's society there is no real need to emphasise efficiency. We would however argue that there remains a need for pupils to *understand* their calculations. We see this approach to calculation as being about 'mental mathematics' since there is some mental work required to select the appropriate number of chunks to subtract. Research has shown that children are often very successful in coming up with their own idiosyncratic calculation strategies which they understand (Thompson, 1994), but that they encounter problems when they try to make sense of these within the more formal written methods encouraged in school.

It is useful to draw on the distinction that Richard Skemp made between instrumental understanding of mathematics and relational understanding. He wrote about pupils developing an 'instrumental' understanding of mathematics who would follow 'rules without reasons' (Skemp, 1978). The alternative approach is to develop 'relational' understanding which encourages pupils to know *both* what to do with a calculation *and* why. A useful analogy to illustrate the difference between these two types of understanding is a comparison between someone who only uses the underground to travel around London and a London taxi driver. Let us say that the underground traveller is attempting to navigate between Buckingham Palace and the Science Museum. They would be able to use the underground (tube) map to get themselves between the two but they would struggle if the tube trains were not running. The London taxi driver however, would be aware of numerous possible routes between the two landmarks and further, would be able to cope with finding alternative routes due to traffic restrictions or jams. The underground user is like the pupil with instrumental understanding; able to follow rules to go between different concepts within mathematics – for example, able to follow an algorithm to solve long division. The London taxi driver on the other hand, is the pupil with relational understanding – able to navigate in a multitude of ways between different mathematical concepts. Using the long division example – a pupil with relational understanding would be able to solve the problem in numerous ways, make a reasonable estimate of the solution and know how to check their solution.

Do we always have to teach so that pupils have relational understanding? Consider some examples, such as multiplication of negative numbers, division of fractions and the angle sums of polygons. What does relational understanding mean to you in each case?

Deciding what constitutes relational understanding of any aspect of mathematics is a complicated (perhaps impossible) business. However, we are not trying to claim that there is a clear-cut distinction between relational and instrumental understanding. For example, in our example above, the taxi driver has probably acquired their knowledge of London through 'rote learning' and the underground user's understanding of the underground network is functional and is likely to be based on a deeper understanding of urban transportation. However, just as the taxi driver and tourist have different relationships to space and different feelings about navigating London, we would suggest that how a pupil understands mathematics affects their relationship with the subject and the distinction between relational and instrumental understanding is helpful in discussing this.

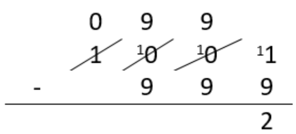
We conducted a study to compare the approaches to long division of primary and secondary student teachers (Babbar and Ineson, 2013). They were asked to solve the problem (207 ÷ 23) themselves, then outline how they would support a pupil encountering difficulty in solving it. Secondary student teachers were found to be more secure in the approaches that they used themselves, but struggled to think of alternative ways to support pupils in developing approaches for long division. Their favoured approach was that of an algorithm, even when the numbers involved were near multiples of 10. Although most secondary student teachers recognised that 23 x 10 = 230, which is near 207, nevertheless, their initial response was to write down the algorithm. Primary student teachers, on the other hand, were found to be less likely to be able to find the accurate solution but could suggest a range of alternative approaches for supporting learners. This raises questions about the kind of understanding the secondary student teachers had and how they would teach division. These are questions that we explore through focusing on mental mathematics in the rest of this chapter.

What is the Place of Mental Mathematics in the Mathematics Curriculum?

> *Think back to your own experiences of mental mathematics: What memories do they conjure up. Do you remember mental mathematics tests? What impact did these have on you? Do you remember any other aspect of mental mathematics? Whilst considering these questions, how would you define mental mathematics and what does this mean to you?*

Before we consider current explanations of mental mathematics, it is useful to first provide some background. For over a century there has been concern over the 'standards' achieved by school leavers in the UK and less than favourable international comparisons. In addition to this, educationalists have been alarmed by the apparent over-reliance on formal algorithms for relatively simple problems. For example, when faced with problems such as 1001 – 999, pupils tended to write these numbers out vertically then laboriously use formal written methods involving decomposition to find the solution (illustrated in Figure 3). Because of this over-reliance on formal strategies (sometimes described as 'comfort blankets') pupils failed to consider the numbers involved and whether, therefore, there was another more effective strategy.

Figure 3. Using decomposition to solve 1001-999



A Task Group was set up in 1997 to explore the possibilities of raising mathematical achievement through the review of research and theory and although concerns were expressed in professional corners about the range and scope of the Task Group's review, it led to the establishment of the National Numeracy Strategy (NNS) in 1998. This strategy emphasised the development of mental calculation in primary schools and the approach advocated in this document was radical. Prior to the implementation of the NNS, the basis of teaching about calculation in primary schools was the formal written approaches illustrated in Figure 3. This in itself gives rise to another question: why the need to change and focus on mental calculation? The new emphasis placed on mental computation was therefore of considerable significance. For example, the NNS stated that pupils should not be taught a standard method of written calculation until they could reliably use addition and subtraction mentally for any pair of two digit numbers. This flexibility in using mental calculation strategies is what is lost in Gove's proposed new primary mathematics curriculum.

In the previous primary framework, teachers were asked to focus on encouraging these skills in their pupils:

- remembering number facts and recalling them without hesitation;
- understanding and using the relationship between the 'four rules' to work out answers and check results: for example, 24 ÷ 4 = 6, since 6 × 4 = 24;
- drawing on a repertoire of mental strategies to work out calculations like 81 -26, 23 × 4 or 5% of £3000, with some thinking time;
- solving problems like the following mentally: 'Can I buy three bags of crisps at 35p with my £1 coin?' or 'Roughly how long will it take me to go 50 miles at 30 m.p.h.?' (DfEE, 1998a, p.6)

Following the establishment of the NNS in 1998, the Secondary National Strategy was developed to support with progression from primary to secondary. In the secondary framework, the following examples are given as ways of building opportunities to develop mental mathematics skills:

- remember number facts and recall them without hesitation;
- draw on a repertoire of mental strategies to work out calculations such as 326 – 81, 223 × 4 or 2.5% of £3000, with some thinking time;
- understand and use the relationships between operations to work out answers and check results: for example, 900 ÷ 15 = 60, since 6 × 150 = 900;
- approximate calculations to judge whether or not an answer is about the right size: for example, recognise that 1⁄4 of 57.9 is just under 1⁄4 of 60, or 15;
- solve problems such as: 'How many CDs at £3.99 each can I buy with £25?' or: 'Roughly how long will it take me to go 50 miles at 30 mph (DfEE, 2001, p.10)

Consider the progression in mental mathematics in the primary phase to the secondary phase – does this seem appropriate?

Liping Ma (1999) carried out research to explore the way in which effective mathematics teachers understand mathematics. She came up with the term Profound Understanding of Fundamental Mathematics (PUFM) to describe the type of understanding that effective teachers had. This included seeing the connections between different mathematical concepts (for example, the relationship between decimals and fractions, or the number operations), the ability to be flexible when calculating and recognising the coherence of the mathematics curriculum. So, in considering the place of mental mathematics in the mathematics curriculum, we suggest that almost all of mathematics could be described as 'mental' in the sense that engaging in a mathematical task involves thinking. Mental mathematics is about more than the recall of facts; it is about having the confidence and competence to deal with numbers. Ma would emphasise the teacher's ability to be flexible when calculating in order to encourage flexibility in their pupils. Many of the examples of skills listed above for both primary and secondary pupils are about developing flexible strategies for calculating. This reflects the views of those Paul Ernest called 'Progressive Educators', that mathematics education should allow pupils to gain confidence and not rely solely on formal procedures.

As research shows, promoting mental mathematics is not without dangers (Beishuizen 1999; Denvir and Askew, 2001). On the one hand, the NNS suggests that the development of mental capability is more than drill and practice, procedural understanding and memorising number facts; teachers are also encouraged to promote pupils' understanding. On the other hand, research studies also consistently show that there is a tendency, when carrying out daily mental work (as encouraged in the NNS) for teachers to emphasise the procedural at the expense of understanding (Gray and Tall, 1994; Denvir and Askew, 2001). Denvir and Askew (2001) explored the behaviour of pupils during numeracy lessons and found that teachers tended to emphasise the need for speed and accuracy, rather than mathematical thinking. They suggested that these pupils were 'participating' rather than 'engaging' in the mathematical activities. In these and similar contexts, an emphasis on rote memory to recall number facts often has a negative effect, compounding the problems faced, in particular, by those who we construct as lower achieving pupils, who initially have difficulty in remembering and so develop a reliance on counting strategies, rather than developing a flexible approach to calculation (Gray and Tall, 1994).

We revisit now, Gove's quote at the beginning of the chapter, and his suggestion that pupils should be 'more proficient in arithmetic…. including knowing the times tables up to 12 x 12 by the end of year 4'. Despite the research evidence about the negative impact on the emphasis on rote memory, Gove has had his way and a new times table test, which becomes mandatory in 2020, will be taken by all 8/9 year-olds. This will be an on-screen test and will check whether pupils have quick instant recall of the multiplication tables, up to 12 x 12. Whilst few would argue with the need for pupils to learn their multiplication tables, many have voiced their concern about the emphasis, through this assessment strategy, on instrumental understanding.

How do you feel about mental mathematics being tested in this way? What kind of understanding do you think this promotes?

We feel that this promotes the industrial trainer approach which sees mathematics as a fixed body of knowledge which is best tested in this way. It also reflects the status of mathematics and its power to act as a gatekeeper. However, we feel that there is a distinction to make when working mentally and in this situation the emphasis is on working in the head, rather than with the head. So, for example, focusing on quick recall or memorising of number facts focuses on *in* the head thinking, whereas some of the strategies that were discussed earlier such as using facts such as 20 times a number to solve 19 times a number encourage *with* the head thinking. This would not be a 'known fact' but we could use our heads to find the solution, without reliance on an algorithm. Whilst there are clearly benefits of quick recall of number facts it is also important to focus on the importance of developing strategies - *with* the head (Beishuizen, 1997). This has parallels with Skemp's concern over instrumental understanding which has a tendency to focus on *in* the head work. Research has shown that pupils who have a large bank of known facts (i.e. those that they can instantly recall) are able to make use of these for more derived facts (Gray, 1991). Conversely, the same researchers found that pupils who rely heavily on counting strategies (usually because they do not have a bank of quickly recalled known facts) are likely to be slower and less accurate in their solutions (Gray 1991).

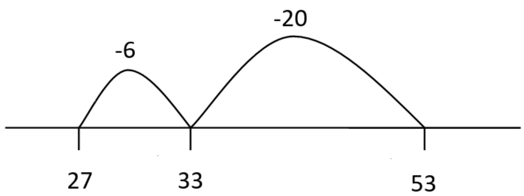
We end this chapter by using the example of the empty number line to illustrate this tension between the *in* the head and *with* the head ways of thinking.

The Empty Number Line: Encouraging *with* the head thinking

As mentioned previously, mental mathematics is not just about what is done *in* the head. In both primary and secondary settings informal jottings are very much encouraged and these usually draw on visual images that pupils have built up from an early age. The Empty Number Line (ENL) was introduced as a model in schools in the Netherlands, which helped pupils visualise the quantity value of numbers. Ian Thompson (1999) makes the useful distinction between the *quantity* value of numbers and the *column* value of digits. He suggests that it is important for pupils to understand the relative size of numbers, rather than focusing on which column the digits are in. The ENL replaced the practice of partitioning using base ten that encouraged pupils to concentrate on the column value of numbers.

The ENL is exactly what its name describes; a line which is empty. The emptiness is significant because it discourages pupils from counting in ones (which we identified above as unhelpful) and provides the scope for flexible approaches. It is also a useful image on which to record calculation steps. Figure 4 uses the ENL to solve 53 – 26.

Figure 4. One way of using the ENL to solve 53-26



In this example the ‘take away’ approach to subtraction is illustrated (rather than the approach which would focus on the ‘difference’ between the two numbers). The pupil has started with 53 and counted back first 20, then a further 6. Beishuizen (1999) explains that the benefits of this approach in terms of general mathematics competence are that this supports pupils’ understanding of number and operations: ‘dealing with whole numbers supports pupils’ understanding and insight into number and number operations much more than the early introduction of vertical algorithms dealing with isolated digits’ (p.159).

In another study by Gwen (Ineson 2019) primary student teachers were encouraged to use the Empty Number Line to solve simple numerical problems in a variety of ways in preparation for teaching in primary schools. They were encouraged to use the ENL as a model to visualise the relationship between the numbers involved in specific calculation problems and as a tool to note steps in their calculations – so that they were working *with* their heads. Initially many student teachers were sceptical about the benefits of such an approach as they felt comfortable and confident using the more traditional approaches to calculation that they had grown up with. However, after spending time in school, many reported that they found that pupils were using the ENL well to support their approach to calculation and were beginning to change their opinion about it. Some even claimed to be using the ENL to support their own calculations. However, during one activity focusing on using the ENL for different approaches of subtraction one student teacher was heard to say, ‘I’m confused, which numbers do I write on the line?’ This suggests that the student teacher was beginning to use the ENL approach as a procedure, or algorithm, rather than as a conceptual model and was focusing on working *in* the head. This may have been because he was so unfamiliar and uncomfortable with an alternative approach to calculation that his only way to embrace it was to ‘learn’ it as an algorithm because that is how he had become used to operating within mathematics. Given the level of anxiety that mathematics creates for people, it is not surprising that they will seek the apparent security of algorithms.

> *a) Have you seen the empty number line being used in the primary classroom? Can you think of alternative ways to use the ENL to solve 53 - 26? Can you think of examples where it could be used to support calculations in secondary classrooms? What alternative jottings have you seen secondary pupils use to support their calculations?*
>
> *b) How would you solve 2001 - 999? It was pupils’ perceived over reliance on formal written methods for problems such as this that was one of the prompts to a focus on mental calculation. Thinking about the arguments in this chapters, how important is it that teachers emphasise alternative approaches to formal written methods?*

Summary

In this chapter, we have investigated some of the debates surrounding mental mathematics, including what it is, its place in the curriculum and how pupils might be encouraged to engage with it. We offer the viewpoint that it is important for secondary teachers of mathematics to understand and build on the approaches taken to calculating in primary settings. Furthermore, it is important for them to be fluent in the strategies that pupils are likely to make use of so that they can encourage them to adopt a more flexible approach, ultimately enabling them to develop relational understanding, without an over-reliance on formal written algorithms. Being taught, and subsequently being able to use, specific strategies for mental calculation equips pupils with the ability to make choices and have a ‘toolkit’ at their fingertips. So even if they have learnt facts and can recall them (*in* the head), they feel sufficiently confident that they are able to apply what they have learnt to unknown situations (*with* the head).

To conclude, we would like to bring the debate back to where we started, with the current National Curriculum. The emphasis on rote learning times tables, as well as specific *approved* formal written algorithms, suggests a move towards industrial training. Primary and secondary mathematics teachers are tasked with the job of ensuring that children and young people leave school with relational understanding of at least some concepts through an emphasis on what they are doing *with* their heads.

Further Reading

Ma, L (1999) *Knowing and Teaching Elementary Mathematics*. Mahwah, NJ: Lawrence Erlbaum: This book has quickly become a classic in mathematics education. Through a comparative study of Chinese and North American primary school teachers, Liping Ma develops her theory that they need a Profound Understanding of Fundamental Mathematics.

Plunkett. S, (1979) 'Decomposition and all that rot', *Mathematics in School*, 8, 3, 2-5: Ever thought about how/why we teach decomposition? A classic article querying the teaching of algorithms.

Skemp, R. R. (1978) 'Relational understanding and instrumental understanding' *Mathematic Teaching,* 77, 20–26: This provides further distinction between these two different ways of understanding mathematics from the originator of the terms.

Thompson, I. (2010) 'Subtraction in Key Stage 3: Which algorithm?' *Mathematics in School*, 39, 1, 29-31: Ian Thompson has written prolifically about mental mathematics but this article continues the debate started in this chapter about algorithms and instrumental understanding.

Murphy, C. (2011). Comparing the use of the empty number line in England and the Netherlands. British Educational Research Journal, Vol 37, 1, 147-161. This article explores the reason for introducing the ENL as a model to support mathematics education in the Netherlands as part of the Realistic Mathematics Education programme. Carol Murphy critiques the way in which this has been adopted in England and suggests that this has led to an algorithmic approach to teaching mental calculation.

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