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A note on the diffraction by a semi-Infinite thick half plane, and a semi-infinite cylindrical rod with an absorbent end face

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by

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Abstract

The known solutions for:

(i) A plane wave incident on two rigid semi-infinite parallel plane distance 2b apart,

(ii) The radiation of an arbitrary mode from two parallel rigid semi-infinite plates 2b apart,

are used to construct the solution of the problem of the diffraction of a plane wave by a semi-infinite thick half plane (thickness 2b), whose sides are rigid and whose end face is absorbent. The method used is simple superposition of the solutions (i) and (ii). The problem solved has applications in noise barrier design. Analogously by using the known solutions for the diffraction of a plane wave by, and the radiation of an arbitrary mode from, a rigid hollow cylindrical semi-infinite duct, the solution for the diffraction of a plane wave by a thick semi-infinite rigid rod with an absorbent end face can be obtained.

1. Introduction

The diffraction of a plane wave by a thick half plane by an E or H polarized electromagnetic wave has been considered by Jones (1953). This resulted in an infinite system of equations which were approximately solved for the situation where the thickness of the half plane was small compared with the wave-length. In the rest of this paper we shall couch our language in acoustical terms. Thus the problem solved by Jones (1953) corresponded to the diffraction of a plane wave by a rigid (Neumann boundary condition on all surfaces) and a soft (Dirichlet boundary condition on all surfaces) thick half plane. Crighton and Leppington (1973), improved the approximate results obtained by Jones (1953) by an application of the powerful method of matched asymptotic expansions.

The problem we shall consider in detail here is that of the diffraction of a plane wave by a rigid thick half plane with an absorbent (Robin boundary condition) end face. This problem could no doubt be solved by Jones' approach, and possibly by the method of Crighton and Leppington. However the approach we use here is much simpler given that the solutions for the diffraction of a plane wave by, and the radiation of an arbitrary mode from, a rigid semi-infinite duct are already well worked out in the literature, see Noble (1958). The present method could be used directly on the problems considered by Jones and Crighton and Leppington. The crux of the idea of this note in a generalization of an idea of Jones (1953) who used the concept of the cancellation of the fundamental mode propagating in and out of the duct, Jones (1953) §5, see also Crighton and Leppington (1973) p325.

Jones (1954) also considered the problem of the diffraction of a plane wave by a soft and by a rigid semi-infinite thick rod. This problem could not be carried out by the method used by Crighton and Leppington unless the inner problem could be solved without the use of conformal mapping. The more general problem of the diffraction of a plane wave by a rigid (soft) cylindrical rod with an absorbent end face can be solved by the present method of superposition. Since the corresponding solutions for the diffraction of a plane wave by, and the radiation of an arbitrary mode from a rigid (soft) cylindrical duct is well known and fully worked out, Noble (1958).

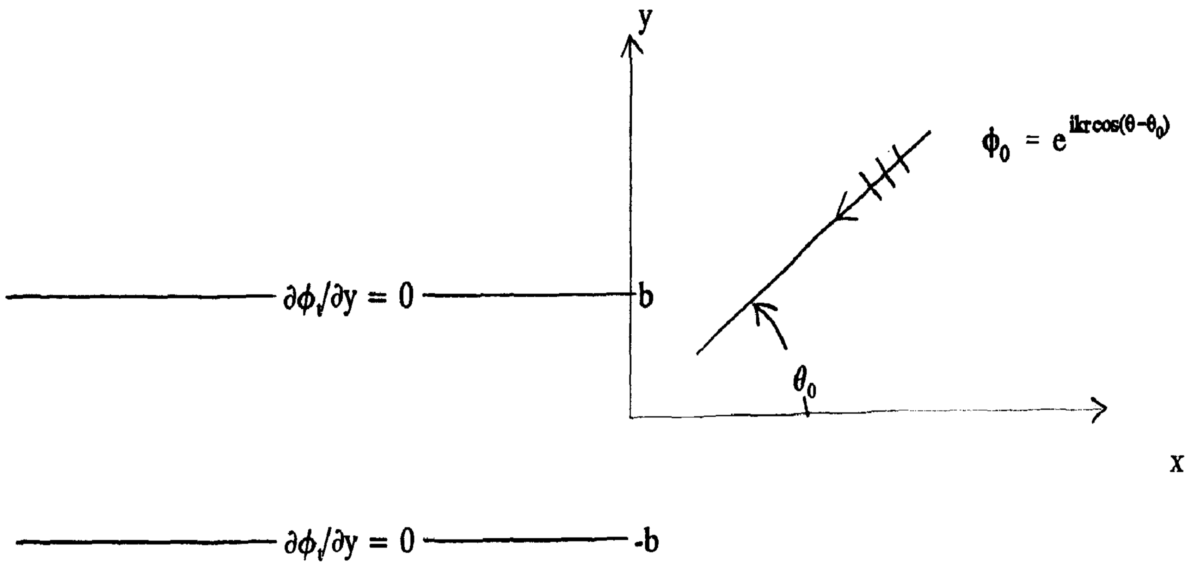
In the present work we will deal in detail with the rigid thick half plane with an absorbent end. This is a model for a noise barrier where the top of the barrier is treated with absorbent material. We shall briefly outline the solution of the corresponding cylindrical rod problem. The usefulness of the present technique obviously requires companison with numerical and experimental results. It is envisaged that a later publication will address this aspect of the problem in more detail.

In section 2 we shall set out the known solution for the diffraction of a plane wave by a semi-infinite rigid duct. In section 3 the solution to the problem of the radiation of an arbitrary mode from the same rigid duct system will be given. In section 4 a superposition of the solutions given in sections 2 and 3 will be constructed. The requirement that the boundary condition on the end face be satisfied will yield an infinite system of equations. The solution of this infinite system gives the solution of the problem of diffraction of a plane wave by rigid half plane with an absorbent end face.

The cylindrical rod problems that can be solved by the present method will be outlined briefly at the end of the paper.

2. Diffraction by a semi-infinite duct

Consider the problem of the diffraction of a plane wave ϕ_0 by a semi-infinite rigid duct The solution has been fully worked out, Noble (1958).



$$(V^2 + K^2)\phi_t = 0$$

The solution is given for an incident plane wave

$$\phi_0 = \exp[-ikr\cos(\theta - \theta_0)]e^{-i\omega i}$$

by

$$\phi_t = \phi_g + \phi \qquad (2.1)$$

where ϕ_g stands for the geometric acoustic terms at infinity and ϕ represents the diffracted field outside the duct, and the mode structure inside the duct (where $\phi_g = 0$). The time harmonic factor $e^{-i\omega t}$ will be dropped in future calculations.

Specifically

$$\phi = \sum_{m=0}^{\infty} T_m e^{-ik_m x} \cos\left[\frac{m\pi}{2b}(y - b)\right] \qquad |y| < b, \quad x < 0 \qquad (2.2)$$

$$= f(\theta, \theta_0)\frac{e^{ikr}}{\sqrt{r}} \qquad |y| > b, \quad r \to \infty. \qquad (2.3)$$

where the known quantities T_m and $f(\theta,\theta_0)$ are given by

$$T_0 = \frac{\sin(kb\sin\theta_0)}{kb\sin\theta_0 L\_(k)L_+(k\sin\theta_0)} \qquad (2.4)$$

$$T_m = \frac{kb\sin\theta_0\cos(kb\sin\theta_0)(1+i\gamma_m/k)^{1/2}(1-(-)^m)K_+(i\gamma_m)}{m\pi(1+\cos\theta_0)^{1/2}(\cos\theta_0 - i\,\lambda mk)K_+(k\cos\theta_0)\{kK(i\gamma_m)\}}$$

$$+\frac{\sin\theta_0\sin(kb\sin\theta_0)(1+(-)^m)L_+(i\gamma_m)}{2kb(1+\cos\theta_0)(1-i\gamma_m/k)(\cos\theta_0 - i\gamma_m/k)L_+(k\cos\theta_0)\{KL^{'}(i\gamma_m)\}} \qquad (2.5)$$

$$m\geq 1$$

$$f(\theta,\theta_0) = \left\{\frac{e^{-i\pi/4}}{\sqrt{(2\pi k)}}\cdot\frac{2i\sin(\theta_0/2)\sin(\theta/2)}{(\cos\theta_0+\cos\theta)}\right\} *$$

$$* \left[\operatorname{sgn}\theta\frac{\cos(kb\sin\theta_0)\cos(kb\sin\theta)}{K_+(k\cos\theta_0)K_+(k\cos\theta)} - \right.$$

$$\left. - \frac{i\sin(kb\sin\theta_0)\sin(kb\sin\theta)}{2kb\cos(\theta_0/2)\cos(\theta 2))L_+(k\cos\theta_0)L_+(k\cos\theta k}\right] ,$$

$$\cos\theta_0 + \cos\theta \neq 0, -\pi < \theta < \pi ; \qquad (2.6)$$

Where

$$K_0=k,\; k_m = i\gamma_m \equiv i\left(\left(\frac{m\pi}{2b}\right)^2 - k^2\right)^{1/2},$$

$$K'(\alpha)\frac{d}{d\alpha}K(\alpha), \qquad L'(\alpha)\frac{d}{d\alpha}L(\alpha).$$

$$K(a) = K_+(\alpha)K\_(\alpha) = e^{ikb}\cos xb = e^{-\gamma b}\cosh\gamma b$$

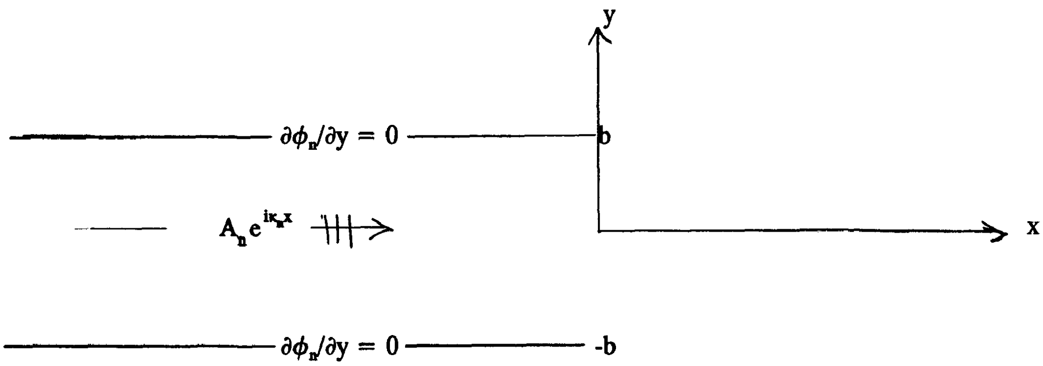
$$L(\alpha) = L_+(\alpha)L\_(\alpha) = e^{ikb}\frac{\sin kb}{kb} = e^{-\gamma b}\frac{\sinh\gamma b}{\gamma b} \qquad (2.7)$$

$$k = (k^2-\alpha^2)^{1/2} = i\gamma = i(\alpha^2-k^2)^{1/2}.$$

The properties of the split functions (2.7) have been fully worked out, Noble (1958).

3. Radiation from a semi-infinite duct

Consider the problem of the radiation of an arbitrary plane wave mode out of a semi-infinite duct The solution has been fully worked out, Noble (1958).



$$(\nabla^2+k^2)\,\phi_n = 0$$

The solution for an arbitrary incident mode

$$\phi_{0n} = A_n e^{ik_n x}, \quad A_n \text{ arbitrary} , \tag{3.1}$$

is given by ϕ_n where

$$\phi_n = A_n\left\{\cos\left[\frac{n\pi}{2b}(y-b)\right]e^{ik_n x} + \sum_{m=0}^{\infty} R_m^n \cos\left[\frac{m\pi}{2b}(y-b)\right]e^{ik_m x}\right\}$$

$$|y| < b, \quad x < 0 \tag{3.2}$$

$$= A_n g(\theta(\theta_0)\frac{e^{ikr}}{\sqrt{r}} \qquad |y| > b, \quad r \to \infty \tag{3.3}$$

where the known quantities R_m^n and $g(\theta,\theta_0)$ are given by

$$R_m^n = \in_m \left\{ \frac{(1+(-)^m)}{2} \cdot \frac{(1+i\gamma_m/k)(1+i\gamma_n)/k)L_+(i\gamma_m)L_+(i\gamma_m)}{((\gamma_n+\gamma_m)/k)(\gamma_m/k)} \right.$$

$$\left. + \frac{i(1-(-)^m)}{2} \cdot \frac{(1+i\gamma_m/k)^{1/2}(1+i\gamma_n/k)^{1/2} K_+(i\gamma_n)K_+(i\gamma_m)}{b((\gamma_n+\gamma_m)/k)(\gamma_m/k)} \right\}, \qquad (3.4)$$

$$\in_m = \tfrac{1}{2} \quad \text{for} \quad m = 0 \;, \quad \in_m = 2 \quad \text{for} \quad m \geq 1$$

$$g(\theta,\theta_0) = \left\{ \frac{e^{-i\pi/4}}{\sqrt{(2\pi k)}} \cdot \frac{e^{-ikb|\sin\theta|}}{(\cos\theta - i\gamma_n/k)} \right\} \quad *$$

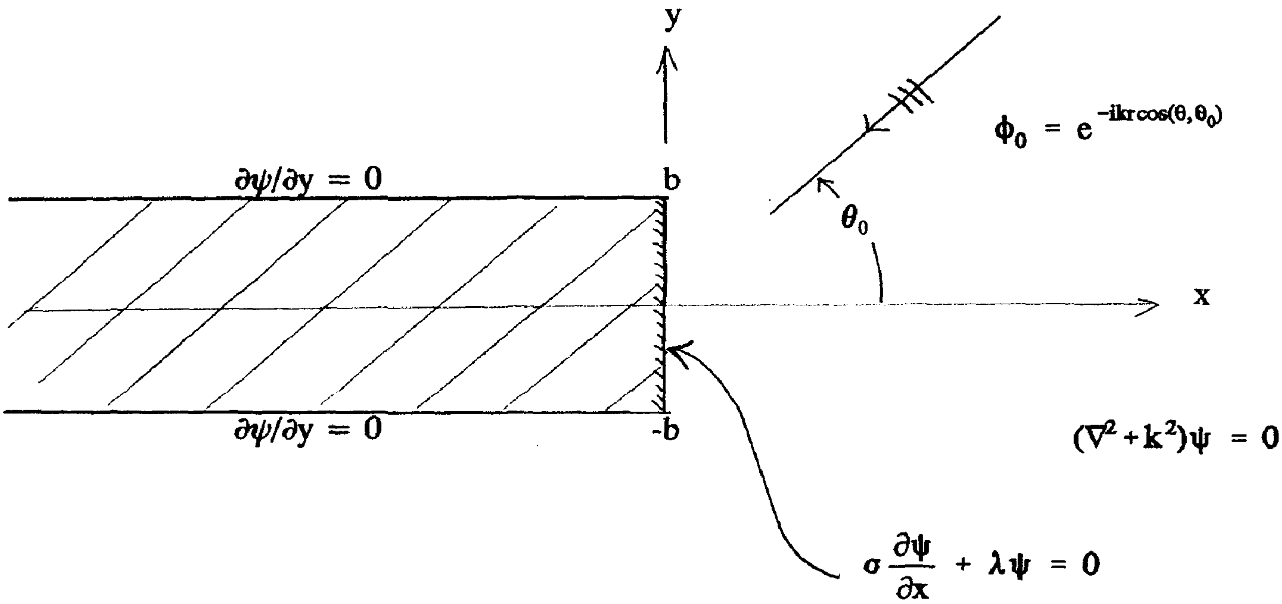
$$* \left[kb(1+i\gamma_n/k)(\cos\theta-1)L_+(i\gamma_n)L_-(k\cos\theta) \right. \qquad (3.5)$$

$$\left. - i\,\mathrm{sgn}(\sin\theta s(1+i\gamma_n/k)^{1/2}(1-\cos\theta)^{1/2}(1-\cos\theta)^{1/2} K_+(i\gamma_n)K_-(k\cos\theta k \right]$$

$$-\pi < \theta < \pi$$

4. Diffraction by a thick half plane with an absorbent end

Consider the diffraction of a plane wave ϕ_0 by a semi-infinite thick half plane which has rigid sides, but whose end face has an absorbent material lining.



The solution to the above problem can be constructed by simple superposition of the solutions obtained in sections *2* and 3. We only require that the addition of the solutions is such that the boundary condition

$$\sigma \frac{\partial \psi}{\partial x} + \lambda \psi = 0 \quad \text{on} \quad x = 0 \ , \quad |y| < b \qquad (4.1)$$

should be satisfied. To achieve this we choose for ψ the form

$$\psi = \phi + \sum_{n=0}^{\infty} \phi_u \qquad (4.2)$$

such that

$$\sigma \frac{\partial \psi}{\partial x} + \lambda \psi = 0 \quad \text{on} \quad x = 0 \ , \quad |y| < b$$

Note each ϕ_n has an arbitrary magnitude A_n which will no longer be arbitrary on the imposition of the boundary condition (1). Thus the requirement (4.1) gives

$$\sigma \frac{\partial}{\partial x}\left(\phi + \sum_{n=0}^{\infty} \phi_n \right) + \lambda \left(\phi + \sum_{n=0}^{\infty} \phi_n \right) = 0 \qquad \text{on} \quad x=0$$

and on substituting the expression for ϕ and ϕ_n given by (1.2) and (2.2) respectively gives:

$$\sigma\left(\sum_{m=0}^{\infty} T_m(-ik_m)\cos\left[\frac{m\pi}{2b}(y-b)\right] + \sum_{n=0}^{\infty} A_n\left\{(ik_n)\cos\left[\frac{n\pi}{2b}(y-b)\right] + \sum_{m=0}^{\infty}(-ik_m)R_m^n\cos\left[\frac{m\pi}{2b}(y-b)\right]\right\}\right)$$
$$+\lambda\left(\sum_{m=0}^{\infty} T_m\cos\left[\frac{m\pi}{2b}(y-b)\right] + \sum_{n-0}^{\infty} A_n\left\{\cos\left[\frac{n\pi}{2b}(y-b)\right] + \sum_{m=0}^{\infty} R_m^n\cos\left[\frac{m\pi}{2b}(y-b)\right]\right\}\right) = 0, \qquad |y| < b.$$

$$\sum_{m=0}^{\infty}\left[(\lambda - i\sigma k_m)T_m + A_n(\lambda + i\sigma k_m) + \sum_{n=0}^{\infty}(\lambda - i\sigma k_m)A_n R_m^n\right]\cos\left[\frac{m\pi}{2b}(y-b)\right] = 0, \ |y| < b,$$

where we have assumed we can interchange orders of summation. This last result implies that

$$(\lambda - i\sigma k_m)T_m + A_m(\lambda + i\sigma k_m) + \sum_{n=0}^{\infty}(\lambda - i\sigma k_n)A_n R_n^m = 0, \quad m = 0,1,2,.... (4.3)$$

The solution of this infinite set of equations (4.3) determines the A_n, and hence the ϕ_n, uniquely, in the solution ψ (4.2). The exact solutions for the duct problems are for arbitrary kb so, unless the solution of the infinite system is in some way restrictive, the solution for the thick half plane problem should be for any kb. Approximate solutions of the infinite system (4.3) can be achieved by the method of truncation. That is, if A_m is the solution of the finite system

$$(\lambda - i\sigma k_m)T_m + A_m^N\ (\lambda - i\sigma k_m) + \sum_{n=0}^{N}(\lambda - i\sigma k_n)A_n^N R_n^m = 0$$

$$m = 0,1,\ldots,N$$

then

$$A_m = \lim_{N\to\infty} A_m^N \ .$$

In the application of this procedure it is hoped that A_m^N tends rapidly to A_m for sufficiently low values of N. It will simplify the actual numerical calculations considerably by using asymptotic estimates for T_m and R_n^m for kb >> 1 or for kb << 1.

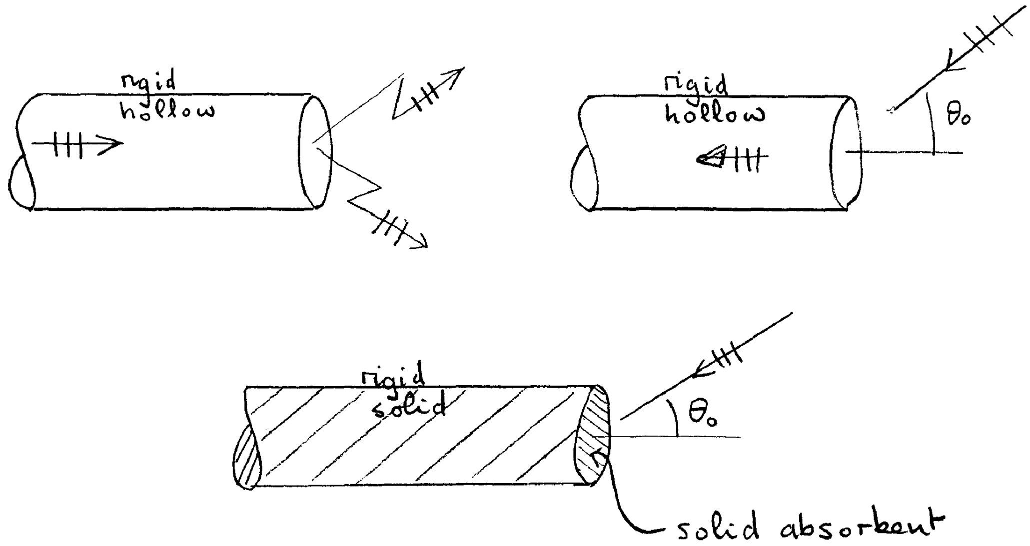
We note that special cases of the problem we have considered are

(a) for a rigid thick half plane $\sigma \neq 0,\ \lambda = 0$. This problem was considered by Jones (1953) and Crighton and Leppington (1973)

(b) for a thick half plane with rigid sides and a soft end $\sigma = 0,\ \lambda \neq 0$. This problem has been considered by McIver and Rawlins (1993).

(c) For a rigid half-plane with an acoustically absorbent end $\sigma = 1, \lambda = ik\beta,\ \operatorname{Re}\beta > 0$.

We note that the present method can also be used for a cylindrical rod with an absorbent end. The solutions for a semi-infinite rigid cylindrical duct with an incident plane wave, or an arbitrary mode excitation are known, Noble (1958).



This cylindrical problem cannot be solved by the method of Crighton and Leppington because there is no conformal mapping technique to obtain the inner problem. Jones (1953) has considered the situation where the end face is rigid. Finally problems in electromagnetism can also be solved where E polarized waves are diffracted by a thick half plane and a thick semi-infinite cylindrical rod with an absorbent end face. These problems would use the known solutions for the acoustically soft duct and an acoustically soft cylindrical duct respectively.

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