TR/07/85 April 1985

The failure of Bernstein's theorem for polynomials on C(K) spaces

by

Andrew Tonge

The failure of Bernstein's theorem for polynomials on C(K) spaces

Summary. Bernstein's theorem asserts that if $p : C \to C$ is a polynomial of degree m then its derivative p' satisfies the inequality

$$\| P' \|_\infty \leq M \|P\|_\infty ,$$

where the symbol $\| \ \|_\infty$ denotes the supremum norm taken over the unit disc. Harris [1] proved an analogous inequality for the Fréchet derivative of polynomials on Hilbert space. In his commentary to problem 73 in the Scottish Book [2, pp. 144-145] he asked whether there is a similar result for polynomials on $C(K)$ spaces. The purpose of this note is to give a negative answer, even for polynomials of degree 2.

The example. We recall the definition of a polynomial on a Banach space. If $L : E\times\dots\times E \to c$ is a continuous symmetric m-linear form on the Banach space E, we define the map $\hat{L} : E \to C$ by $\hat{L}(x) = L(x,\dots,x)$ A map $p : E \to C$ is

(a) a homogeneous polynomial of degree 0 if p is constant,

(b) a homogeneous polynomial of degree $m \geq I$ if $p = \hat{L}$ for some continuous symmetric m- linear form L on E, and

(c) a polynomial of degree m if $p = p_o + \dots + p_m$, where p_i is a homogeneous polynomial of degree i $(0 \leq i \leq m)$ and $p_m \neq 0$.

If $p : E \to C$ is a polynomial with Fréchet derivative Dp, we define

$$\| p\|_\infty - \sup\{|p(x)| : \| x \| \leq 1\} \quad \text{and}$$

$$\| Dp \|_\infty = \sup\{ \| D_P(x) \| : \| x \| \leq 1 \} = \sup \{| Dp(x)(y) | : \| x \| \leq l , \| y\|\leq 1\}$$

We use the standard notation ℓ_∞^n for the space C^n equipped with the norm $\|x\|_\infty = \| (x_1,\dots,x_n) \| - \max\{ |x_1| , \dots , |x_n| \}$

In [1], Harris proved that if $p : E \to C$ is a polynomial of degree m then $\|Dp\|_\infty \leq m \|p\|_\infty$, provided that E is a Hilbert space or that

$E = \ell_\infty^2$. This prompted him to ask whether the same inequality holds whenever $E = C(K)$, the Banach space of continuous functions on the compact Hausdorff space K , under the usual uniform norm. We shall give an example of a homogeneous polynomial of degree 2 on the space $E = \ell_\infty^3$ for which the proposed inequality fails.

Define a symmetric bilinear form $L : \ell_\infty^3 \times \ell_\infty^3 \to c$ by

$$L(x,y) = x_1 y_1 + x_2 + y_2 + x_3 y_3 - x_1 y_2 - x_2 y_1 - x_2 y_3 - x_3 y_1 - x_1 y_3 ,$$

where $x = (x_1,x_2,x_3)$ and $y = (y_1,y_2,y_3)$. The corresponding polynomial, which has already been useful in the investigation of von Neumann's inequality [4], is given by

$$\hat{L}(x) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_1$$

In [4] Kaijser and Varopoulos used elementary calculus to show that $\| \hat{L} \|_\infty = 5$. (The norm is attained when $x = (1,1,-1)$.) Consequently we need to show that $\| D\hat{L} \|_\infty > 10$,

However, if we set $x = (1,w,w^2)$ and $y = (1,w^2,w)$, where $w = \exp(2\pi i/3)$, we obtain $DL(x)(y) = 2L(x,y) = 12$. It follows that $||| D\hat{L} \|_\infty \geq 12$.

<u>Comments and open problems.</u>

It has been shown that if p is a homogeneous polynomial of degree 2 on a $C(K)$ space then $\|Dp\|_\infty \leq \sqrt{(27/4)}\, \| p\|_\infty$ .A proof may be found (implicitly) in [1] or [3]. Computer experiments strongly suggest that this constant is best possible, but we know of no proof.

It would be interesting to classify the Banach spaces for which Bernstein's theorem does hold.

<u>References</u>.

[1] L.A. Harris, Bounds on the derivatives of holomorphic functions of vectors. Colloque d'analyse, Rio de Janeiro, 1972. (Ed. L. Nachbin) Herman Paris, Act. Sc. et Ind., 1367 (1975), pp. 145-163.

[2] R.D. Mauldin (Ed.), The Scottish Book (Mathematics from the Scottish Cafe). Birkhauser, Boston, 1981.

[3] A.M. Tonge, Polarization and the two dimensional Grothendieck inequality. Math, Proc. Cambridge Philos. Soc. 95 (1984), pp.313-318.

[4] N.Th. Varopoulos, On an inequality of von Neumann and an application of the metric theory of tensor products to operator theory. J. Functional Analysis, 16 (1974), pp. 83-100.