Rational quadratic spline interpolation to monotonic data.

R. Delbourgo and J.A. Gregory

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Abstract

In an earlier paper by Gregory & Delbourgo (1982), a piecewise rational quadratic function is developed which produces a monotonic interpolant to monotonic data. This interpolant gives visually pleasing curves and is of continuity class $C^1$. In the present paper, the data is restricted to be strictly monotonic and it is shown that it is possible to obtain a monotonic rational quadratic spline interpolant which is of continuity class $C^2$. An $O(h^4)$ convergence analysis is included.
1. Introduction

A set of data points \((x_i, f_i), \ i=1,\ldots,n\), is given, with \(x_1 < x_2 < \cdots < x_n\) and such that the values \(f_i\) form a strictly monotonic sequence. In the subsequent work it will be assumed that

\[ f_1 < f_2 < \cdots < f_n, \]

since the case of a strictly decreasing sequence of function values can be treated in a similar manner.

In Gregory and Delbourgo (1982), a piecewise rational quadratic function \(s(x)\) is constructed which is monotonic on \([x_1, x_n]\) and satisfies

\[ s(x_i) = f_i, \quad s^{(1)}(x_i) = d_i, \quad i = 1, \ldots, n, \]

where the derivatives \(d_i\) are positive for strictly increasing \(f_i\).

The piecewise rational quadratic \(s(x)\) is defined as follows: Let

\[ \theta = \frac{(x - x_i)}{h_i}, \quad \Delta_i = \frac{(f_{i+1} - f_i)}{h_i} \]

Then for \(x \in [x_i, x_{i+1}]\),

\[ s(x) = \frac{f_{i+1} \theta^2 + \Delta_i^{-1}(f_{i+1} d_i + f_i d_{i+1}) \theta(1 - \theta) + f_i (1 - \theta)^2}{\theta^2 + \Delta_i^{-1} (d_i + d_{i+1}) \theta (1 - \theta) + (1 - \theta)^2}. \tag{1.1} \]

The denominator is strictly positive for all \(0 \leq \theta \leq 1\). Also, a differentiation gives the result that for \(x \in [x_i, x_{i+1}]\),

\[ s^{(1)}(x) = \frac{d_{i+1} \theta^2 + 2 \Delta_i \theta (1 - \theta) + d_i (1 - \theta)^2}{\theta^2 + \Delta_i^{-1} (d_i + d_{i+1}) \theta (1 - \theta) + (1 - \theta)^2} \tag{1.2} \]

and hence \(s^{(1)}(x) > 0\) throughout any interval \([x_i, x_{i+1}]\).
In the earlier paper by Gregory and Delbourgo (1982), the derivative values $d_i$ are determined by local approximations which involve the values $f_i$. These approximations give a $C^1[x_1, x_n]$ interpolant for which an $o(h^3)$ convergence result can be obtained. In the present paper, positive values of the derivatives $d_i$ are determined in an analogous way to cubic polynomial spline interpolation, which make $s(x) \in C^2[x_1, x_n]$. Furthermore, it is shown that an $o(h^4)$ convergence result can be obtained when accurate derivatives $d_1$ and $d_n$ are available as end conditions.

It should be noted that if the data is monotonic but not strictly monotonic, then there will be intervals $[x_i, x_{i+1}]$ where $\Delta_i = 0$. The requirement that $s(x)$ be monotonic then implies that $s(x) = f_i$, a constant, on $[x_i, x_{i+1}]$. Elsewhere, the data can be divided into strictly monotonic parts and the proposed method of this paper can be applied.

2. The Monotonic Rational Quadratic Spline

If $s(x)$ is a $C^2$ function then, necessarily, there is no jump discontinuity in the second derivatives of $s(x)$ at the interior knots $x_i, i = 2, \ldots, n - 1$. For cubic polynomial splines, such $C^2$ consistency conditions lead to a set of linear equations each relating three consecutive derivatives $d_i$. For the piecewise rational quadratic function employed here, corresponding consistency equations arise which will be non-linear. These are derived below and will then be shown to have a unique solution with all $d_i > 0$.

The requirement for $C^2$ continuity, namely that $s^{(2)}(x_{i+1}) - s^{(2)}(x_{i-1}) = 0$ at all the interior knots, gives

$$\frac{2}{h_i} [\Lambda_i + d_i (1 - \frac{d_i + d_{i+1}}{\Lambda_i})] + \frac{2}{h_{i-1}} [\Lambda_{i-1} + d_i (1 - \frac{d_{i-1} + d_i}{\Lambda_{i-1}})] = 0.$$
This can be written as
\[ d_i[-c_i + a_{i-1} d_{i-1} + (a_{i-1} + a_i) d_i + a_i d_{i+1}] = b_i, \]
\[ i = 2, \ldots, n-1, \]  
(2.1)

where
\[ a_i = 1 / (h_i \Delta_i), \]
\[ b_i = \Delta_{i-1} / h_{i-1} + \Delta_i / h_i, \]
\[ c_i = 1 / h_{i-1} + 1 / h_i. \]  
(2.2)

Given \( d_1 \) and \( d_n \), (2.1) gives a system of \( n-2 \) non-linear equations for the unknowns \( d_2, \ldots, d_{n-1} \). It should be noted that \( c_i > 0 \) and, for data which is strictly increasing, \( a_i > 0, b_i > 0 \) for all \( i \) in equations (2.1).

The existence and uniqueness of a solution \( d_2, \ldots, d_{n-1} \) of the non-linear equations (2.1) with all \( d_i > 0 \) will first be proved by analysing a Jacobi type of iteration. It will then be shown that a Gauss-Seidel type of iteration can be used in practice.

Each equation (2.1) is a quadratic in the variable \( d_i \). Solving for the positive root gives
\[ d_i = \frac{1}{2(a_{i-1} + a_i)} \left[ c_i - a_{i-1} d_{i-1} - a_i d_{i+1} + \left( (C_i - a_{i-1}) d_{i-1} - a_i d_{i+1} \right)^2 \right]^{1/2} + 4(a_{i-1} + a_i) b_i, \]  
(2.3)

A Jacobi iteration may be defined by the equation
\[ d_i^{(k+1)} = \frac{1}{2(a_{i-1} + a_i)} \left[ c_i - a_{i-1} d_i^{(k)} - a_i d_{i+1}^{(k)} + \left( (C_i - a_{i-1}) d_{i-1}^{(k)} - a_i d_{i+1}^{(k)} \right)^2 \right]^{1/2} + 4(a_{i-1} + a_i) b_i, \]  
(2.4)

where \( d_1^{(k=1)} = d_1^{(k)} = d_1 \) and \( d_n^{(k+1)} = d_n^{(k)} = d_n \) are given end condition.
Theorem 2.1. (Existence) For strictly increasing data and given end conditions \( d_1 \geq 0, \ d_n \geq 0 \), there exists a strictly positive solution \( d_2, \ldots, d_{n-1} \) satisfying the non-linear consistency equations.

Proof. A set of functions \( G_i, i=1, \ldots, n \), is defined initially on the domain \( \mathbb{R}^n \) by

\[
G_1(\xi) = d_1
\]

\[
G_i(\xi) = \frac{1}{2(a_{i-1} + a_i)} [c_i - a_{i-1} \xi_{i-1} - a_i \xi_{i+1} + \{(c_i - a_{i-1} \xi_{i-1} - a_i \xi_{i+1})^2
+ 4(a_{i-1} + a_i)b_i \}^{\frac{1}{2}}], \ i = 2, \ldots, n-1
\]

\[
G_n(\xi) = d_n,
\]

where \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \). Let \( G = (G_1, \ldots, G_n) \) and \( d = (d_1, \ldots, d_n) \).

Then the Jacobi iteration (2.4) assumes the form

\[
d^{(k+1)} = G(d^{(k)}).
\]

Restricting \( \xi \) to have positive components, we now show that there exist constants \( \alpha_i \) and \( \beta_i \) such that

\[
0 < \alpha_i \leq G_1(\xi) \leq \beta < \infty, \ i = 2, \ldots, n-1.
\]

Also, for \( G_1(\xi) \) and \( G_n(\xi) \), we may define \( \alpha_1 = \beta_1 = d_1 \) and \( \alpha_n = \beta_n = d_n \).

Now, for \( i=2, \ldots, n-1 \), examination of \( G_i(\xi) \) in the two cases

\[
0 \leq a_{i-1} \xi_{i-1} + a_i \xi_{i+1} \leq c_i \text{ and } a_{i-1} \xi_{i-1} + a_i \xi_{i+1} > c_i \]

gives

\[
\beta_i = \frac{1}{2(a_{i-1} + a_i)} [c_i + \{c_i^2 + 4(a_{i-1} + a_i)b_i \}^{\frac{1}{2}}].
\]

Finding a strictly positive value for \( \alpha_i \) is slightly more complicated but it can be shown that

\[
\alpha_i = \min \left\{ \frac{-c_i + \{c_i^2 + 4(a_{i-1} + a_i)b_i \}^{\frac{1}{2}}}{2(a_{i-1} + a_i)}, \ \frac{2b_i}{N_i + \{N_i^2 + 4(a_{i-1} + a_i)b_i \}^{\frac{1}{2}}}, \right\},
\]

where \( N_i = \max \{0, -c_i + (a_{i-1} + a_i) \ max_{2 \leq i \leq n-1} \beta_i \} \). Thus if \( I_i = [\alpha_i, \beta_i] \),
i=1, ..., n, then the map $G$ can be restricted to the $n$-dimensional interval $I = I_1 \times \ldots \times I_n$, where $G : I \to I$ and hence maps positive vectors into positive vectors.

Next, $G$ is shown to be a contraction mapping on $I$: Let $\xi, \eta \in I$ and let

$$X_i = c_i - a_i - 1 \xi_{i-1} - a_i \xi_i + 1, \quad Y_i = c_i - a_i - 1 n_{i-1} - a_i n_{i+1}.$$ 

Then, for $i = 2, \ldots, n-1$,

$$G_i(\xi) - G_i(\eta) = \frac{1}{2(a_{i-1} + a_i)} (X_i - Y_i + \{X_i^2 + 4(a_{i-1} + a_i) b_i\}^{\frac{1}{2}} - \{Y_i^2 + 4(a_{i-1} + a_i) b_i\}^{\frac{1}{2}})$$

and $G_1(\xi) - G_1(\eta) = 0, G_n(\xi) - G_n(\eta) = 0$. Now

$$\frac{|X_i - Y_i|}{(a_{i-1} + a_i)} \leq \|\xi - \eta\|_{\infty}, \quad \text{and} \quad \frac{|X_i + Y_i|}{\{X_i^2 + 4(a_{i-1} + a_i) b_i\}^{\frac{1}{2}} + \{Y_i^2 + 4(a_{i-1} + a_i) b_i\}^{\frac{1}{2}}} \leq \frac{|X_i| + |Y_i|}{\{\{X_i\} + |Y_i|\}^2 + 8(a_{i-1} + a_i) b_i}$$

$$\leq \frac{1}{\{1 + L\}^2}$$

where, since each of $|X_i|$ and $|Y_i|$ has an upper bound $c_i + a_{i-1} \beta_{i-1} + a_i \beta_{i+1}$,

$$L = 2 \min_{2 \leq i \leq n-1} (a_{i-1} + a_i) b_i / \max_{2 \leq i \leq n-1} (c_i + a_{i-1} \beta_{i-1} + a_i \beta_{i+1})^2 > 0.$$ 

Hence

$$\|G(\xi) - G(\eta)\|_{\infty} \leq \frac{1}{2} \left[ 1 + 1 / (1 + L)^2 \right] \|\xi - \eta\|_{\infty},$$

from which it follows that $G$ is a contraction mapping on $I$. Thus the
Jacobi iteration converges to a unique fixed point \( \overline{d} \in I \), i.e. \( \overline{d} = G(\overline{d}) \), and it follows that \( \overline{d} \) is a solution of (2.1), which thus completes the proof.

Equations (2.3) are derived from (2.1) by solving for the positive root. The alternative choice of negative root must lead to a \( d_i < 0 \), if such a solution exists. Thus uniqueness of a positive solution of (2.1) follows directly from the uniqueness of the solution of \( \overline{d} = G(\overline{d}) \), where \( G \) is a contraction map. Alternatively, uniqueness of a positive solution of (2.1) may be proved directly as follows:

**Theorem 2.2.** (Uniqueness) The solution of the non-linear consistency equations which satisfies the monotonicity conditions \( d_i > 0 \) is unique.

**Proof.** Assume that \( d_1, \ldots, d_n \) and \( e_1, \ldots, e_n \) are two sets of values each satisfying the consistency equations, where \( d_1 = e_1 \geq 0 \) and \( d_n = e_n \geq 0 \) are given and \( d_i > 0, e_i > 0, i = 2, \ldots, n - 1 \). Then

\[
\begin{align*}
b_i / d_i + c_i - a_{i-1}d_{i-1} - (a_{i-1} + a_i)d_i - a_id_{i+1} &= 0, \\
b_i / e_i + c_i - a_{i-1}e_{i-1} - (a_{i-1} + a_i)e_i - a_ie_{i+1} &= 0, i = 2, \ldots, n - 1.
\end{align*}
\]

Substraction gives

\[
(e_i - d_i) [b_i / (d_ie_i) + a_{i-1} + a_i] = a_{i-1}(d_{i-1} - e_{i-1}) + a_i(d_{i+1} - e_{i+1})
\]

Consider the \( j \)th equation, where \( j \) is chose so that

\[
|e_j - d_j| = \max_{2 \leq i \leq n-1} |e_i - d_i|.
\]

Then taking moduli gives

\[
|e_j - d_j| |b_j / (d_je_j) + a_{j-1} + a_j| \leq (a_{j-1} + a_j)|e_j - d_j|,
\]

and thus

\[
|e_j - d_j| |b_j / (d_je_j)| \leq 0.
\]
Hence \( d_j = e_j \) and so \( d_i = e_i, \ i = 2, \ldots, n = 1 \).

In practice a Gauss-Seidel type of iteration can be used to solve (2.3). This iteration is defined by

\[
d_{1}^{(k+1)} = \frac{1}{2(a_1a_{-1} + a_i)} (c_i - a_i - d_i^{(k+1)} - a_i d_{i+1}^{(k)}) + \frac{1}{4}(a_i - a_i + a_1 b_i^2), \quad i = 2, \ldots, n - 1,
\]

where \( d_{1}^{(k+1)} = d_{1}^{(k)} = d_1 \) and \( d_{n+1}^{(k+1)} = d_{n}^{(k)} = d_n \) are given end conditions.

A convenient starting vector \( d_1^{(0)} \) for this iteration is given by

\[
d_1^{(0)} = \frac{1}{2} (b_i / (a_1a_{-1} + a_i)) d_1, \quad i = 2, \ldots, n - 1.
\]

**Theorem 2.3.** The Gauss-Seidel iteration (2.6) converges to the unique positive solution of the non-linear consistency equations.

**Proof.** By Theorems 2.1 and 2.2 there exist unique \( d_i > 0 \) satisfying

\[
b_i / d_i + c_i - a_i - d_i^{(k)} - (a_i - a_1 d_i^{(k+1)} - a_i d_{i+1}^{(k)}) = 0, \ i = 2, \ldots, n - 1.
\]

Also, the Gauss-Seidel iterates satisfy

\[
b_i / d_1^{(k+1)} + c_i - a_i - d_i^{(k+1)} - (a_i - a_1 d_i^{(k+1)} - a_i d_{i+1}^{(k)}) = 0, \ i = 2, \ldots, n - 1.
\]

Subtract and write \( d_1^{(k+1)} = d_i + \epsilon_1^{(k)} \) Then

\[
[b_i / (d_1^{(k+1)}) + a_i - a_i + \epsilon_1^{(k+1)}] \epsilon_1^{(k+1)} = -a_i - \epsilon_1^{(k)} - a_i \epsilon_1^{(k)}
\]

Since \( d_i + \epsilon_1^{(k+1)} = d_1^{(k+1)} > 0 \), on taking moduli we obtain

\[
[b_i / (d_1^{(k+1)}) + a_i - a_i + \epsilon_1^{(k+1)}] \epsilon_1^{(k+1)} \leq a_i - [\epsilon_1^{(k+1)} + a_i \epsilon_1^{(k)}].
\]

Consider the \( j \)th inequality, where \( j \) is chosen so that

\[
[\epsilon_1^{(k+1)}] = \max_{2 \leq i \leq n - 1} \epsilon_1^{(k+1)} = \|\epsilon_1^{(k+1)}\|_\infty.
\]

Then
which reduces to
\begin{align*}
\|g^{(k+1)}\|_{\infty} \leq \frac{a_j \|g^{(k)}\|_{\infty}}{a_j + b_j / \{d_j(d_j + \|g^{(k)}\|_{\infty})\}}
\end{align*}

It follows that
\begin{align*}
\|g^{(k+1)}\|_{\infty} \leq \beta \|g^{(k)}\|_{\infty},
\end{align*}

where
\begin{align*}
\beta = \frac{a_j}{a_j + b_j / \{d_j(d_j + \|g^{(0)}\|_{\infty})\}}
\end{align*}

And $0 < \beta < 1$. Thus $\|g^{(k)}\|_{\infty} \to 0$ as $k \to \infty$ and hence $d_i^{(k+1)} \to d_i$, $i=2, \ldots, n-1$.

3. **Convergence Analysis of Rational Quadratic Spline**

We begin by quoting a theorem which was given with proof in the earlier paper Gregory and Delbourgo (1982) and which will be required in the subsequent work.

**Theorem 3.1** Let $f(x) \in C^4[x_1, x_n]$ and $f^{(1)}(x) > 0$ on $[x_1, x_n]$. Let $s(x)$ be the piecewise rational quadratic interpolant such that $s(x_i) = f(x_i)$ and $s^{(1)}(x_i) = d_i \geq 0$. Then for $x \in [x_i, x_{i+1}]$, $i = 1, \ldots, n-1$

\begin{align*}
|f(x) - s(x)| & \leq \frac{h_i}{4c} \|f^{(1)}\| \max \left\{f^{(1)}_i - d_i, f^{(1)}_{i+1} - d_{i+1}\right\} \\
& \quad + \frac{h_i^4}{384c} \{2 \|f^{(1)}\| + \|f^{(1)}_i\| + \frac{2}{3} h_i \|f^{(3)}\|^2 + 2 \|f^{(2)}\| \|f^{(3)}\|\}, \quad (3.1)
\end{align*}

Where $h_i = x_{i+1} - x_i$, $c$ is a constant independent of $i$ whose value is at least
\[
\frac{1}{2} \min_{x_1, x_n} f^{(1)}(x) \text{ and } \|\cdot\| \text{ denotes the uniform norm on } [x_1, x_n].
\]

The next theorem establishes an upper bound for \( \max_{2 \leq i \leq n-1} \left| f^{(1)}_i - d_i \right| \) when the \( d_i \) are the solutions of the non-linear consistency conditions (2.1).

**Theorem 3.2** Let \( d_i = f^{(1)}_n \) and \( d_n = f^{(1)}_n \) in the rational quadratic spline interpolant. Then, with the assumptions of Theorem 3.1 and for \( h \) sufficiently small,

\[
\max_{2 \leq i \leq n-1} \left| f^{(1)}_i - d_i \right| \leq \frac{h^3 K(h)}{2m^3} \frac{\|f^{(1)}\|}{\|f^{(1)}\| - h^3 K(h)}, \quad (3.2)
\]

where

\[
k(h) = \frac{1}{12} \left\{ 7 \left\| f^{(1)} \right\| + \left\| f^{(4)} \right\| + \left\| f^{(2)} \right\| + \left\| f^{(3)} \right\| \right\} + o(h), \quad (3.3)
\]

and \( h = \max h_i, \quad m = \min_{[x_1, x_n]} f^{(1)}(x) > 0 \) \( (3.4) \)

Thus \( \max_{2 \leq i \leq n-1} \left| f^{(1)}_i - d_i \right| = o(h^3) \).

**Proof.** Consider the consistency equations

\[
b_i / d_i + c_i - a_{i-1} d_{i-1} - (a_i - 1 + a_1) d_i - a_i d_{i+1} = 0
\]

and let

\[
b_i / f^{(1)}_i + c_i - a_{i-1} f^{(1)}_{i-1} - (a_i - 1 + a_1) f^{(1)}_i - a_i f^{(1)}_{i+1} = E_i,
\]

\[
i = 2, \ldots, n-1. \quad (3.5)
\]

where, from (3.4), \( 0 < 1 / f^{(1)}_i < 1 / m. \) Subtracting and writing

\[
d_i - f^{(1)}_i = \lambda_i
\]

gives

\[
b_i \lambda_i / \left( f^{(1)}_i (f^{(1)}_i + \lambda_1) \right) + a_{i-1} \lambda_{i-1} + (a_i - 1 + a_1) \lambda_i + a_i \lambda_{i+1} = E_i,
\]

\[
i = 2, \ldots, n-1, \quad (3.7)
\]
where we require a bound on \( \max_{2 \leq i \leq n-1} |\Delta_i| \). Now, from (3.5) and the definitions (2.2), it follows that

\[
E_i \Delta_{i-1} \Delta_i = (h_i \Delta_{i-1}^2 \Delta_i + h_{i-1} \Delta_{i-1} \Delta_i^2) / f_i^{(1)} + (h_i + h_{i-1}) \Delta_{i-1} \Delta_i
\]

\[
- h_i \Delta_i (f_{i-1}^{(1)} + f_i^{(1)}) - h_{i-1} \Delta_{i-1} (f_i^{(1)} + f_{i+1}^{(1)}).
\]

On the right the following Taylor expansions are made:

\[
\Delta_{i-1} = f_i^{(1)} - \frac{1}{2} h_i f_i^{(2)} + \frac{1}{6} h_i^2 f_i^{(3)} - \frac{1}{24} h_i^3 f_i^{(4)},
\]

\[
\Delta_i = f_i^{(1)} + \frac{1}{2} h_i f_i^{(2)} + \frac{1}{6} h_i^2 f_i^{(3)} + \frac{1}{24} h_i^3 f_i^{(4)},
\]

\[
f_{i-1}^{(1)} = f_i^{(1)} - h_{i-1} f_i^{(2)} + \frac{1}{2} h_{i-1} f_i^{(3)} - \frac{1}{6} h_{i-1}^2 f_i^{(4)},
\]

\[
f_{i+1}^{(1)} = f_i^{(1)} - h_i f_i^{(2)} + \frac{1}{2} h_i f_i^{(3)} + \frac{1}{6} h_i^2 f_i^{(4)},
\]

where \( f_i^{(1)} \) means \( f_i^{(1)} (\lambda_i - \alpha h_{i-1}), 0 < \alpha < 1 \), etc. After some algebra, the result of these substitutions gives

\[
E_i \Delta_{i-1} \Delta_i = f_i^{(1)} \left( \frac{1}{8} h_i^2 f_i^{(4)}_{i+\beta} - h_{i-1}^2 f_i^{(4)}_{i-\alpha} \right) - \frac{1}{6} \left( h_i^2 f_i^{(4)}_{i+\delta} - h_{i-1}^2 f_i^{(4)}_{i-\gamma} \right)
\]

\[
+ \frac{1}{12} (h_i^2 - h_{i-1}^2) f_i^{(2)} f_i^{(3)} + o(h^3).
\]

Now \( \Delta_{i-1} \Delta_i \geq m^2 \), where \( m \) is defined by (3.4). Thus it follows that

\[
m^2 |E_i| \leq \frac{1}{12} h^2 \left( \|f_i^{(1)}\| \|f_i^{(4)}\| + \|f_i^{(2)}\| \|f_i^{(3)}\| \right) + o(h^3).
\]

Hence

\[
|E_i| \leq m^{-2} h^2 K(h), \quad (3.8)
\]

where \( K(h) \) is defined by (3.3). We now consider equation (3.7) with index \( i = j \) taken so that \( |\lambda_j| = \max_{2 \leq i \leq n-1} |\lambda_i| \). Then

\[
|b_j / (f_j^{(1)} (f_j^{(1)} + \lambda_j) + a_{j-1} + a_j) | = \lambda_j = E_j - a_{j-1} \lambda_{j-1} - a_j \lambda_{j+1},
\]

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where $|\lambda_j| = ||\lambda||_\infty$, since $\lambda_1 = 0 = \lambda_n$. Taking moduli and noting that

$0 < f_j^{(1)} + \lambda_j \leq f_j^{(1)} + ||\lambda||_\infty$ gives

$$[b_j / \{f_j^{(1)} (f_j^{(1)} + ||\lambda||_\infty)\} + a_{j-1} + a_j] ||\lambda||_\infty \leq |E_j| + (a_{j-1} + a_j) ||\lambda||_\infty.$$  

This inequality reduces to

$$||\lambda||_\infty \leq f_j^{(1)} \left( \frac{b_j}{|E_j| / \{f_j^{(1)} - |E_j|\}} \right),$$  

under the assumption that the denominator is positive. Now

$$b_j / f_j^{(1)} = (\Delta_{j-1} / h_{j-1} + \Delta_j / h_j) / f_j^{(1)},$$  

$$= (\theta^{(1)} / h_{j-1} + \phi^{(1)} / h_j) / f_j^{(1)}$$  

for some $0 < \theta, \phi < 1$.

Thus, from (3.8)

$$b_j / f_j^{(1)} - |E_j| \geq 2m / \{h \left\| f^{(1)} \right\| \} - m^{-2} h^2 \kappa(h)$$  

which is positive for $h$ sufficiently small. Finally, substituting (3.10) and (3.8) in (3.9) gives the desired result.

Remark. When the results of Theorems 3.1 and 3.2 are taken together, it can be seen that $f(x) - s(x) = 0(h^4)$ on the assumption that $d_1 = f_1^{(1)}$ and $d_n = f_n^{(1)}$ are given end conditions.

4. Numerical Results and Discussion

Our first set of results is concerned with the order of convergence of the interpolation scheme. Tables 1 and 2 show the interpolation errors arising from the application of the rational quadratic spline scheme to
f(x) = \exp(x) over [0,1] when the exact choice of end conditions
\[ c_1 = f^{(1)}(0) = 1 \quad \text{and} \quad c_1 = f^{(1)}(1) = 1 \quad a = \exp(1) \] is made. The knots are taken to be equally spaced with four choices of interval lengths, namely h = 0.2, 0.1, 0.05, 0.025. In one experiment, the errors \( e_1, e_2, e_3, e_4 \) corresponding to these four choices of h are evaluated at \( \theta = 1/3 \), where, for each h, the interval of interpolation is that containing the point x = 0.86. In a second experiment the four intervals containing the point x = 0.86 are selected with \( \theta = 2/3 \).

<table>
<thead>
<tr>
<th>error ( e_i ) (h = 0.2)</th>
<th>error ( e_1 ) (h = 0.1)</th>
<th>error ( e_3 ) (h = 0.05)</th>
<th>error ( e_4 ) (h = 0.25)</th>
<th>( e_1/e_2 )</th>
<th>( e_2/e_3 )</th>
<th>( e_3/e_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-.45217 \times 10^{-5}</td>
<td>-.26477 \times 10^{-6}</td>
<td>-.16973 \times 10^{-7}</td>
<td>-.1046 \times 10^{-8}</td>
<td>17.08</td>
<td>15.60</td>
<td>16.22</td>
</tr>
</tbody>
</table>

Table 1. Rational quadratic spline interpolation errors at \( \theta = 1/3 \) in interval containing x = 0.26, f(x) = \exp(x).

<table>
<thead>
<tr>
<th>error ( e_i ) (h = 0.2)</th>
<th>error ( e_1 ) (h = 0.1)</th>
<th>error ( e_3 ) (h = 0.05)</th>
<th>error ( e_4 ) (h = 0.25)</th>
<th>( e_1/e_2 )</th>
<th>( e_2/e_3 )</th>
<th>( e_3/e_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-.84774 \times 10^{-5}</td>
<td>-.47378 \times 10^{-6}</td>
<td>-.30788 \times 10^{-7}</td>
<td>-.1902 \times 10^{-8}</td>
<td>17.89</td>
<td>15.39</td>
<td>16.19</td>
</tr>
</tbody>
</table>

Table 2. Rational quadratic spline interpolation errors at \( \theta = 2/3 \) in interval containing x = 0.86, f(x) = \exp(x).

The theory of Section 3 shows that a convergence rate of \( O(h^4) \) is expected and this is confirmed by both tests which clearly show the tendency of the ratios \( e_k / e_{k+1} \) to approach the value \( 2^4 \).

Our second set of results is concerned with the application of the rational spline scheme to the monotonic data sets of Tables 3, 4, and 5.

<table>
<thead>
<tr>
<th>x</th>
<th>7.99</th>
<th>8.09</th>
<th>8.19</th>
<th>8.7</th>
<th>9.2</th>
<th>10</th>
<th>12</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>0</td>
<td>2.76429 \times 10^{-5}</td>
<td>4.37498 \times 10^{-5}</td>
<td>0.169183</td>
<td>0.469428</td>
<td>0.943740</td>
<td>0.998636</td>
<td>0.999919</td>
<td>0.999994</td>
</tr>
</tbody>
</table>

Table 3. Monotonic Data Set 1 [Fritsch & Carlson (1980)]
Table 4. Monotonic Data Set 2 [Pruess (1979)]

<table>
<thead>
<tr>
<th>x</th>
<th>22</th>
<th>22.5</th>
<th>22.6</th>
<th>22.7</th>
<th>22.8</th>
<th>22.9</th>
<th>23</th>
<th>23.1</th>
<th>23.2</th>
<th>23.3</th>
<th>23.4</th>
<th>23.5</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>523</td>
<td>543</td>
<td>550</td>
<td>557</td>
<td>565</td>
<td>575</td>
<td>590</td>
<td>620</td>
<td>860</td>
<td>915</td>
<td>944</td>
<td>958</td>
<td>986</td>
</tr>
</tbody>
</table>

Table 5. Monotonic Data Set 3 [Akima (1970); Fritsch & Carlson (1980)]

Both the Fritsch-Carlson radio-chemical data of Table 3 and the Akima data of Table 5 are used in Gregory & Delbourgo (1982) in connection with the piecewise rational quadratic $C^1$ scheme proposed there. These data sets are also used by Fritsch & Carlson (1980), where the need for good monotonic interpolants is clearly illustrated by the poor behaviour of other interpolation methods.

In general, to apply the $C^2$ rational spline scheme of this paper, it is necessary to set the end derivatives $d_1$ and $d_n$ to suitable non-negative values. Two possible methods are explored below. It should be noted that for the Akima data, $s(x)$ is constant over the interval [0, 8] and the rational spline scheme is applied only over [8, 15]. The condition $d_1 = 0$ is then imposed at the left hand end point $x = 8$ of this interval, where $s(x)$ will be $C^1$.

Method 1. This is based on the three point difference approximations

$$d_1 = \Delta_1 + (\Delta_1 - \Delta_2) \frac{h_1}{(h_1 + h_2)},$$

if the expression on the right is positive, otherwise $d_1$ is set to zero;

$$d_n = \Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2}) \frac{h_{n-1}}{(h_{n-2} + h_{n-1})},$$

if the expression on the right is positive, otherwise $d_n$ is set to zero.

Here each of $f_1^{(1)} - d_1$ and $f_n^{(1)} - d_n$ is $O(h^2)$. 
Method 2  Non-linear approximations for $d_1$ and $d_n$ are given by

$$d_1 = \Delta_1 \left( \Delta_1 \left( \left( f_3 - f_1 \right) \left( x_3 - x_1 \right) \right) \right)^{h_1} / h_2,$$

$$d_n = \Delta_{n-1} \left( \Delta_{n-1} \left( \left( f_n - f_{n-2} \right) \left( x_n - x_{n-2} \right) \right) \right)^{h} \left( n - 1 / h \right),$$

Here, as in Method 1, each of $f_1^{(1)} - d_1$ and $f_n^{(1)} - d_n$ each $O(h^2)$, as can be shown by a Taylor expansion argument. These approximations are an improvement on the non-linear end conditions quoted in Gregory & Delbourgo (1982) and are identical with these conditions in the case of equal intervals.

Figures 1, 2, and 3 show the results of applying the rational spline scheme to the three given data sets. The scheme is implemented with the end conditions described by Method 2. (End conditions- based on Method 1 gives graphs little different from those shown.) For the purposes of comparison, the $C^1$ piecewise cubic interpolant using the $L_2$ monotonicity region recommended by Fritsch & Carlson is shown. Also; the $C^1$ piecewise rational quadratic interpolant based on the second method of derivative approximation recommended by Gregory & Delbourgo (1982) is shown. For the Data Set 1, the extra degree of continuity of the rational spline scheme is apparent at the knot $x = 10$ when compared with the $C^1$ schemes. The Data Set 2 illustrates a behaviour which is to be expected of any spline Scheme. Here, due to the nature of the data, the $C^2$ constraint has lead to more variation in the curve than that given by the rational quadratic $C^1$ scheme. However, in general it can be seen that the rational spline scheme produces good curves.

6. Conclusion

A method of constructing a $C^2$ monotonic interpolant to given monotonic data has been described. This method is based on a rational quadratic spline
representation and involves the solution of a non-linear system of consistency
equations. The iterative solution of this system means that the method
involves more work than existing \( C^3 \) methods. However, the method seems
to produce visually pleasing curves which have the advantage of being twice
continuously differentiable and \( O(h^4) \) convergent.

References

Akima, A. 1970 A new method of interpolation and smooth curve fitting based


Gregory, J.A. & Delbourgo, R. 1982 Piecewise rational quadratic interpolation

Comp. 33, 1273-1281.
Fig. 1. Results for monotonic data set 1. (i) Fritsch-Carlson; (ii) $C^1$ piecewise rational quadratic; (iii) $C^2$ rational quadratic spline.
Fig. 2. Results for monotonic data set 2. (i) Fritsch-Carlson; (ii) C piecewise rational quadratic; (iii) $C^2$ rational quadratic spline.
Fig. 3. Results for monotonic data set 3. (i) Fritsch-Carlson; (ii) $C^1$ piecewise rational quadratic; (iii) $C^2$ rational quadratic spline.