TR/04/88 September 1988

Numerical conformal mapping onto a rectangle with applications to the solution of Laplacian problems

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Submitted to: J. Comput. Appl. Math.

ABSTRACT

Let F be the function which maps conformally a simple-connected domain Ω onto a rectangle R, so that four specified points on $\partial\Omega$ are mapped respectively onto the four vertices of R. In this paper we consider the problem of approximating the conformal map F, and present a survey of the available numerical methods. We also illustrate the practical significance of the conformal map, by presenting a number of applications involving the solution of Laplacian boundary value problems.

Keywords: Conformal mapping, conformal module, Laplacian problems.

1. Introduction

Let Ω be a Jordan domain in the complex z-plane $(z=x+iy)$, and consider a system consisting of Ω and four distinct points z_1, z_2, z_3, z_4 in counter-clockwise order on its boundary $\partial\Omega$. Such a system is said to be a quadrilateral Q and is denoted by

$$Q := \{\Omega;\ z_1, z_2, z_3, z_4\}.$$

The conformal module m(Q) of Q is defined as follows:

Let R_h denote a rectangle of the form

$$R_h := \{(\xi,\eta) : 0 < \xi < 1, 0 < \eta < h\},$$

in the w-plane $(w=\xi+i\eta)$. Then, m(Q) is the unique value of h for which Q is conformally equivalent to the rectangular quadrilateral

$$\{R_h : 0,\ \ 1,\ \ 1+ih,\ \ ih\}.$$

That is, for h = m(Q) and for this value only there exists a unique conformal map

$$F : \Omega \to R_h, \tag{1.1a}$$

which takes the four points Zj; j = 1,2,3,4 respectively onto the four vertices of R_h, i.e. F is such that

$$F(z_1) = 0, F(z_2) = 1, F(z_3) = 1+ih \text{ and } F(z_4) = ih. \tag{1.1b}$$

Of course, h = m(Q) is also the only value of h for which the inverse conformal map

$$F^{[-1]} : R_h \to \Omega, \tag{1.2a}$$

with

$$\begin{aligned} &F^{[-1]}(0) = z_1, \quad F^{[-1]}(1) = z_2, \\ &F^{[-1]}(1+ih) = z_3, \text{ and } F^{[-1]}(ih) = z_4, \end{aligned} \tag{1.2b}$$

exists.

This paper is concerned with the problem of determining approximations to the conformal map F (or $F^{[-1]}$) and to the corresponding conformal module m(Q). This problem has received considerable attention recently, most notably by Gaier [10] - [14] who, in particular, recognized the important role that m(Q) and other similar conformal mapping domain functionals play in many practical and theoretical investigations. The main objectives of the paper are as follows:

. To discuss briefly some areas of application of the conformal map F, and to list the main properties of the conformal module m(Q); Section 2.

- To present a survey of some of the available numerical methods for computing approximations to F and to m(Q); Section 3.
- To present two numerical examples illustrating the application of the conformal map F to the solution of Laplacian mixed boundary value problems, involving boundary singularities; Section 4.
- To report some recent results concerning a domain decomposition method for the mapping of a class of long quadrilaterals; Section 5.

2. Physical interpretation, applications and properties of m(Q).

2.1 Physical interpretation and applications.

With the notation of Section 1, let $Q := \{\Omega; z_1, z_2, z_3, z_4\}$, and assume that the boundary $\partial\Omega$ of Ω is piecewise analytic. Let now Ω represent a thin plate of homogeneous electrically conducting material of specific resistance 1, and suppose that constant voltages V_1 and V_2 are applied respectively to the boundary segments (z_1, z_2) and (z_3, z_4) , whilst the remainder of $\partial\Omega$ is insulated. Finally, let I be the current passing through the plate, and consider the problem of determining the resistance

$$r = (V_{2\_}V_1)/I. \qquad (2.1)$$

The above problem may of course be solved by determining the solution of the boundary value problem

$$\Delta_{XY} u = 0, \text{ in } \Omega \ , \qquad (2.2a)$$

$$u = V_1, \text{ on } (z_1, z_2); \ u = V_2, \text{ on } (z_3, z_4), \qquad (2.2b)$$

$$\frac{\partial u}{\partial n} = 0 \ , \text{ on } (z_{2,} z_3) \ U \ (z_4, z_5) \ , \qquad (2.2c)$$

where Δ_{XY} is the Laplace operator $\Delta_{xy} := \partial^2/\partial x2^2 + \partial^2/\partial y^2$, and $\partial/\partial n$ denotes differentiation in the direction of the outward normal. Once u is found, r may be determined from (2.1) after first computing I as a line integral of $\partial u/\partial n$ along any line running from (z_4, z_1) to (z_2, z_3); for example we may take

$$I = \int_{\gamma_1} \frac{\partial u}{\partial n}\, ds \ , \quad \gamma_1 := (z_1,\ z_2). \tag{2.3}$$

Although the boundary value problem (2.2) appears to be rather simple, its solution by standard numerical techniques may present serious difficulties due to the geometry of Ω and/or the presence of boundary singularities. For example, if $\partial\Omega$ is smooth then the solution of (2.2) has a serious singularity at each of the points Zj; j = 1,2,3,4, where the boundary conditions change from Dirichlet to Neumann. By contrast, if the conformal map $F:\Omega \rightarrow R_h$ (h = m(Q)) is available, then the solution of (2.2) can be obtained trivially from the solution of the transformed problem in R_h. More specifically, because the Laplace equation and the boundary conditions (2.2b), (2.2c) are conformally invariant, the transplanted potential $\hat{u}$ satisfies the following boundary value problem:

$$\Delta_{\xi\eta}\hat{u} = 0 \ , \ \text{ in } R_h \ ,$$

$$\hat{u} = V_1 \ , \text{ on } \eta = 0 \ , \ 0 \le \xi \le 1 \ ,$$

$$\hat{u} = V_2 \ , \text{ on } \eta = h \ , \ 0 \le \xi \le 1 \ ,$$

$$\frac{\partial \hat{u}}{\partial \xi} = 0 \ , \text{ on } \xi = 0 \text{ and } \xi = 1, \ 0 < \eta < h \ .$$

Thus, if $\hat{P} := (\xi\ ,\eta) \in R_h$ is the image under the conformal map F of a point $P \in \bar{\Omega}$, then

$$\begin{aligned} u(P) &= \hat{u}\,(\hat{P}) \\ &= h^{-1}\,(V_2 - V_1)\,\eta + V_1 \ . \end{aligned}$$

That is, the solution of (2.2) at any point $P \in \bar{\Omega}$ can be written down immediately, once the imaginary co-ordinate of the image point $\hat{P} := F(P)$ is found. Furthermore, since the integral (2.3) is conformally invariant we have that

$$I = \int_0^1 \frac{\partial \hat{u}}{\partial \eta}\, d\xi \ = \ h^{-1}\,(V_2 - V_1) \quad ,$$

and hence from (2.1) that

$$r = h = m\,(Q)\,.$$

In other words, the resistance of the conducting plate is given by the conformal module of the quadrilateral $Q := \{\Omega;\ z_1\ ,\ z_2\ ,\ z_3,\ z_4\}$. The conformal module is also closely related to the capacitance C between the boundary segments $(z_1, z_2,)$ and (z_3, z_4). This is defined as the charge on (z_1, z_2), when (z_3, z_4) is at unit potential and the remainder of $\partial\Omega$ is at zero potential; see e.g. [5]. If as before $m(Q) = h$, then it is shown in [12] that

$$C = \frac{2}{\Pi^2} \sum_{n=1}^{\infty} \left\{ (2n-1) \sinh [\, (2n-1)\, \pi h \,] \right\}^{-1}.$$

Regarding applications, we have come across a number of papers in the scientific and engineering literature which, in our conformal mapping terminology, are concerned specifically with the problem of determining conformal modules of quadrilaterals. Examples of these are references [26], [28], [43], [50], [53], and [55], in connection with applications in electromagnetic field theory, and references [3], [27], and [29], in connection with the measurement of diffusion coefficients of solid materials. We also mention a recent paper by Gaier [14], which is concerned with an area problem for quadrilaterals Q. In this, the geometry of Q is partly described by three of its four boundary segments, the conformal module m(Q) is fixed, and the fourth boundary segment is to be determined so that the area of Q is minimized. The problem is closely related to a corresponding area problem for symmetric doubly-connected domains which, according to Acker [1], has several important physical interpretations.

A more general application of the full conformal map $F: \Omega \to R_h$ (or $F^{[-1]}$) concerns the computer generation of orthogonal curvilinear co-ordinate systems for the finite-difference solution of partial differential equations. Examples of this can be found in [2], [30] and [56], and in the review paper by Thompson et al [48] which includes fourteen pages of discussion on the use of conformal transformations for numerical grid generation; see also [49].

2.2 Properties of m(Q).

The properties of conformal modules are studied in detail in Section 6.11 of the recent book by Henrici [19]. Here, we merely state without proofs six basic results, which are important in computational work for estimating m(Q) and for comparing the modules of different quadri-aterals.

P 2.1. If $Q := \{\Omega;\ z_1,\ z_2,\ z_3,\ z_4\}$ and $Q^1 := \{\Omega;\ z_2,\ z_3,\ z_4,\ z_1\}$, then

$$m(Q') = 1/m(Q)\ ;$$

see [10] and [19: p.432]. (The quadrilateral Q^1 is said to be the conjugate (or reciprocal) quadrilateral to Q.) □

P 2.2. Let $Q := \{\Omega;\ z_1,\ z_2,\ z_3,\ z_4\}$ and assume the following: (a) The domain Ω is symmetric with respect to the straight line ℓ joining the points z_1 and z_3. (b) The point z_4 is the mirror image in ℓ of the point z_2. Then,

$$m(Q) = 1;$$

see [21] and [19: p433]. (A quadrilateral of the form described by (a) and (b) is said to be a symmetric quadrilateral.) □

P 2.3. *Variational property of m(Q).*

Let $Q := \{\Omega;\ z_1,\ z_2,\ z_3,\ z_4\}$, let $h := m(Q)$ and let K be the class of real valued functions u which are continuous in $\bar{\Omega}$, attain the boundary values $u = 0$ on the segment (z_1, z_2) and $u = 1$ on (z_3, z_4), and are in the Sobolev space $W,(\Omega,)$i.e.

$$K := \{u\ :\ u\varepsilon C(\Omega)\cap W_1(\Omega)\ ,\ u=0\ \text{ on }\ (z_1\ ,\ z_2)\ ,u=1\ \text{on}\ (z_3\ ,\ z_4)\}. \tag{2.4}$$

Also, let $D_\Omega[u]$ denote the Dirichlet integral of $u \in K$ with respect to Ω, i. e

$$D_\Omega[u] := \iint_\Omega (u_X^2 + u_Y^2)dxdy\ . \tag{2.5}$$

Then

$$\begin{aligned} h^{-1} &= \min\{D_\Omega[u]\ :\ u \in K\} \\ &= D_\Omega[u]\ , \end{aligned}$$

where u_0 is the solution of the Laplacian problem (2.2) corresponding to the boundary values $V_1 = 0$ and $V_2 = 1$; see [10:pl80], [13:p66], [19: p434] and [58:p73]. □

P 2.4 Let $Q := (\Omega;\ z_1,\ z_2,\ z_3,\ z_4\}$, let $\hat{z}$, be any point other than z_1, z_4 on the boundary segment $(z_4,\ z_1,)$, and let $\hat{Q} := \left\{\Omega; \hat{z}_1, z_2, z_3, z_4\right\}$.

Then,

$$m(Q) > m((\hat{Q});$$

see [19: p436o]. □

P 2.5. Let $Q := \{\Omega;\ z_1,\ z_2\ ,\ z_3\ ,\ z_4\}$ and $\hat{Q} := \left\{\hat{\Omega}:\ z_1\ \ z_2,\ z_3,\ z_4\right\}$ be two quadrilaterals which have the two boundary segments (z_2, z_3) and (z_4, z_1) in common, and which are such that Q c H, the inclusion being proper. Then,

$$m(Q) < m(\hat{Q});$$

(see [19: p436]). □

P 2.6. Let z_j ; $j = 1,2,...,6$, be six points in counter-clockwise order on $\partial\Omega$ and, by means of a cross-cut Γ from z_3 to z_s, decompose Ω into two disjoint Jordan domains Ω_1, and Ω_2 so that

$$\partial\Omega_1 = (z_1,z_2) \cup (z_2,z_3) \cup \Gamma \cup (z_6,z_1)\ \ ,$$

and

$$\partial\Omega_2 = (z_3,z_4) \cup (z_4,z_6) \cup (z_{5,}z_6) \cup \Gamma.$$

If $Q := \{\Omega:\ z_1\ ,\ z_2\ ,\ z_4\ ,\ z_5\}$, $Q_1 := \{\Omega_1\ :\ z_1,\ z_2\ ,\ z_3\ ,\ z_6\}$ and $Q_2 := \{\Omega_2\ ,\ z_6\ , z_3\ ,\ z_4\ ,\ z_5\}$, then

$$m(Q) \geq m(Q_1) + m(Q_2),$$

and equality occurs when the cross-cut Γ is an equipotential of the solution of problem (2.2), corresponding to the boundary values $V_1 = 0$ and $V_2 = 1$; see [19: p437]. □

Several other interesting results on conformal modules are derived in the two papers by Hersch [20,21] where, in particular, the modules of various non-trivial quadrilaterals are determined by elementary methods.

3. Numerical methods.

3.1 Methods based on approximating the conformal map of Ω onto the unit disc.

Let $Q := \{\Omega; z_1, z_2, z_3, z_4\}$,, let $h := m(Q)$, and let f be the function which maps conformally Ω onto the unit disc $D := \{\zeta : |\zeta| < 1\}$ so that $f(z^*) = 0$ and $f'(z^*) > 0$, where z^* is some fixed point in Ω. Then, the conformal map $F: \Omega \to R_h$ can be expressed as

$$F = S o f, \tag{3.1}$$

where $S: D \to R_h$ is a simple Schwarz-Christoffel transformation. In fact, once the images $\zeta_j = f(z_j)$; $j = 1,2,3,4$, of the four boundary points z_j are found, S can be written down in terms of an inverse Jacobian elliptic sine, and the conformal module h can be determined by computing the ratio of two complete elliptic integrals of the first kind; see e.g. [4], [32] and [35]. For this reason, in theory at least, the problem of approximating F may be regarded as solved once a suitable approximation to $f : \Omega \to D$ is found. In particular, if f is known exactly then, in theory, (3.1) gives the exact conformal map F. In practice however, the application of (3.1) is restricted by a well-known numerical difficulty which is caused by a certain crowding phenomenon. This can be described as follows.

The points $\xi_j = f(z_j)$; $j = 1,2,3,4$, divide the unit circle into the four arcs $\gamma_j := (\zeta_j, \zeta_{j+1})$; $j = 1,2,3$ and $\gamma_4 := (\zeta_4, \zeta_1)$. Let φ_1 be the length of the smaller of the two arcs γ_1 and γ_3 , and let φ_2, be the length of the smaller of γ_3 and γ_4. Then, the numerical difficulty mentioned above is due to the fact that φ_1, becomes very small even when the conformal module h is only moderately large, and φ_2 becomes very small even when h is only moderately small. More precisely, it can be shown that if h is "large" then

$$\varphi_1 \sim \exp(-\pi h/2), \tag{3.2}$$

and if h is "small" then

$$\varphi_2 \sim \exp(-\pi/2h) . \tag{3.3}$$

Therefore, if h is either large or small then some of the images of the points zj; j = 1,2,3,4, on the unit circle will be very close to each other. This crowding of points may be regarded as a form of ill-conditioning, in the sense that a numerical procedure based on the use of (3.1) may fail to provide a meaniningful approximation to $F:\Omega \to R_h$, even if an accurate approximation to $f:\Omega \to D$, is used. In particular, the process will break down completely if, due to the crowding, the computer fails to recognize the points ζ_j ; j = 1,2,3,4, in the correct order. For example, if h = 12 then it can be shown that $\varphi_1, < 5.3x10^{-8}$. Thus, in this case, the procedure will fail on a computer with precision 10^{-7}, even if the conformal map f is performed exactly. (A more detailed discussion of the above can be found in [35: §2]. See also [10: p179], [19: p428], the remarks of Trefethen in his preface of [54: p4], and the paper by Zemach [60] which concerns a similar but more general conformal mapping difficulty.)

As might be expected, all our remarks concerning crowding also apply to the use of the composition

$$F^{[-1]} = f^{[-1]} \circ S^{[-1]}, \tag{3.4}$$

for computing approximations to the inverse map $F[-1] : R_h \to \Omega$. Thus, the crowding phenomenon is a serious numerical drawback of procedures based on the use of both (3.1) and (3.4). However, such procedures deserve strong consideration for the following reasons:

(i) Methods based on the use of (3.1) and (3.4) benefit from a very important advantage. This is connected with the fact that the problems of determining the conformal maps $f:\Omega \to D$ and $f^{[-1]} : D \to \Omega$ are very well-studied. As a result, there are many efficient numerical methods for computing approximations to f and to f [-1]. For details of such methods we refer the reader to the following: (a) The classic monograph by Gaier [9] which, although written in 1964, is still very relevant. (b) Volume III of Henrici's *Applied and Computational Complex Analysis* [19]. (c) The collection of recent papers on numerical conformal mapping, which was edited in 1986 by Trefethen [54]. (In particular, this collection contains a survey [18] of almost all the known methods for approximating f [-1], and a report [41] of recent developments for dealing with corner and pole-type singularities in numerical methods for f.)

(ii) Crowding difficulties can be anticipated by using the quantity

$$C = \begin{cases} 8\exp\{-\pi/(2h)\}\ , & \text{if } h \text{ is "small"}, \\ 8\exp\{-\pi h/2\}\ , & \text{if } h \text{ is "large"}, \end{cases} \tag{3.5}$$

as a measure; see (3.2) - (3.3) and the discussion in [35: §2]. Although the conformal module h is not known a priori, a reliable indication of the extent of crowding can be provided by using a crude estimate of h in (3.5). It is often possible to determine such crude estimates, by using the properties of m(Q) listed in Section 2.2.

Unless C is small by comparison to the precision of the computer, or to the accuracy of the available approximation to f (or f [-1]), the use of (3.1) or (3.4) will not present any crowding difficulties. For example, if $h \in [0.4, 2.5]$ then (3.5) gives that $C > 0.157$. There fore, for such values of h there will be no difficulties due to crowding, unless the approximation to f (or f [-1]) is very inaccurate.

(iii) For an important class of quadrilaterals the numerical difficulties assocated with crowding can be overcome by using a domain decomposition method; see [38,39] and the discussion in Section 5 below.

(iv) Procedures based on the use of (3.1) and (3.4) have been applied successfully to many problems, for determining the conformal maps of non-trivial quadrilaterals. Examples of such applications can be found in the following: (a) References [22] and [42], where numerical methods based on the integral equation formulation of Symm [46] are used for the approximation of the conformal map $f : \Omega \to D$. (b) References [33] and [37], where an orthonormalization method, based on the properties of the Bergman Kernel function of Ω, is used for the approximation of f. (c) The recent paper by Trefethen [53], who considers polygonal quadrilaterals, and uses his Schwarz-Christoffel package [51,52] for approximating the conformal map f [-1] : D $\to \Omega$.

3.2 *Methods based on approximating the conformal map of an associated doubly-connected domain onto a circular annulus.*

Methods of this type can be used only in cases where the quadrilateral Q := $\{\Omega;\ z_1,\ z_2,\ z_3,\ z_4\}$ has one of the two special forms illustrated in Fig 1.

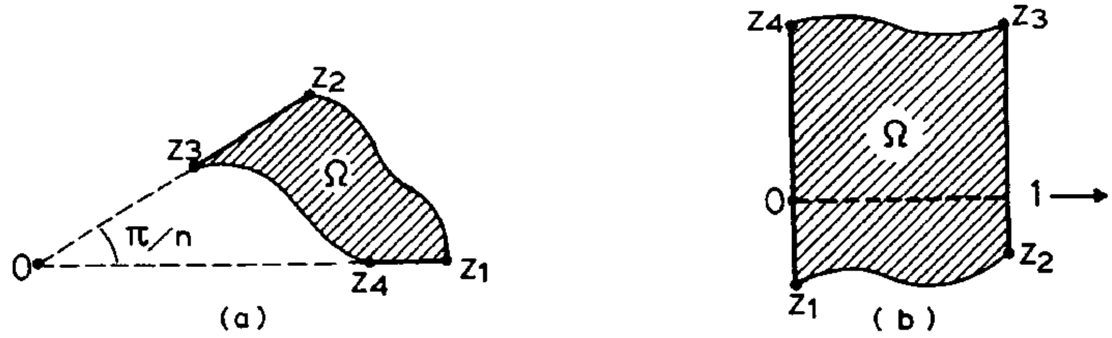

Figure 1

Following [35: §3] we consider first the case where Q is of the form illustrated in Fig l(a). That is, Ω, is bounded by a segment L_1 : = (z_4, z_1) of the real axis, a straight line L_2 := $(z_2, z_3.,)$ inclined at an angle π/n to L_1, with $n \geq 1$ an integer, and two Jordan arcs Γ_1 and Γ_2. Proceeding exactly as in [35], we assume that the arcs Γ_1 and Γ_2 are given in polar co-ordinates by

$$r_j := \{z : z = p_j(\theta)e^{i\theta}, 0 \leq \theta \leq \pi/n\}; \ j = 1,2,$$

with $0 < p_2(\theta) < p_1(\theta)$, $\theta \in [0, \pi/n]$, and denote by Ω_d the 2n-fold symmetric doubly-connected domain obtained by first reflecting Ω about the straight line L_2. That is,

$$\Omega_d := \operatorname{Int}(\hat{\Gamma}_1) \cap \operatorname{Ext}(\hat{\Gamma}_2), \tag{3.6a}$$

where

$$\hat{\Gamma}_j := \{z : z = \hat{p}_j(\theta)e^{i\theta}, \ 0 \leq \theta \leq 2\pi\}; \ j = 1,2, \tag{3.6b}$$

with

and
$$\left.\begin{array}{ll} \hat{p}_j(\theta) = p_j(\theta), & \theta \in [0, \pi/n], \\ \hat{p}_j(k\pi/n + \theta) =, \hat{p}_j(k\pi/n + \theta), & \theta \in [0, \pi/n], \ k = 1(1)2n-1. \end{array}\right\} \tag{3.6c}$$

Then, for a certain value of q, $0 < q < 1$, Ω_d is conformally equivalent to the annulus

$$A_q := \{\zeta : q < |\zeta| < 1\},$$

and the reciprocal of the inner radius, i.e. the value $M := 1/q$, is called the conformal modulus of Ω_d .

Let g be the function which maps Ω_d conformally onto Aq so that the curve $\hat{\Gamma}_1$ is mapped onto the unit circle $|\zeta|=1$. Also, let

$$T(\zeta(:=\{n\log\zeta\}/i\pi ,$$

and let S_q denote the sector

$$S_q := \{\zeta : \zeta = re^{i\varphi}, q < r < i,\ 0 < \varphi < \pi/n\} .$$

Then, the function T maps Sq conformally onto the rectangle

where

$$\left.\begin{aligned} &R_h := \{w = \xi + i\eta : 0 < \xi < 1, 0 < \eta < h\}, \\ &h = -(n\ \log q)/\pi o \end{aligned}\right\}$$

so that the four corners of Sq are mapped respectively onto those of R_h. It follows from the above that

$$\begin{aligned} m(Q) &= h \\ &= -(n\ \log q\ )/\pi , \end{aligned}$$

and that the conformal map $F:\Omega \to R_h$ can be expressed as

$$F = T\,o\,g .$$

In other words, the problem of determining F is equivalent to that of determining the conformal map $g:\Omega_d \to A_q$.

Consider now the case where Q is of the form illustrated in Fig l(b), and let the arcs (z_1, z_2) and (z_3, z_4) have cartesian equation $y=\tau_j\{x\}$: j = 1,2 . That is, let

$$Q := \{\Omega\ ;\ z_1,\ z_2,\ z_3,\ z_4\} , \tag{3.7a}$$

where

$$\Omega := \{(x,y) : 0 < x < 1, \tau_1(x) < y < \tau_2(x)\} , \tag{3.7b}$$

and

$$\left.\begin{aligned} &z_1 = i\tau_1(0) , \quad z_2 = 1 - i\tau_1(1) , \\ &z_1 = 1 + i\tau_2(1),\ z_3 = i\tau_2(0) . \end{aligned}\right. \tag{3.7c}$$

Then, the function

$$v(z) := \exp(i\pi z),$$

maps Ω conformally onto the upper half of a symmetric doubly-connected domain Ω_d which has the form (3.6) with $n = 1$ and

$$p_j(\theta\theta = \exp\{-\pi\tau_j(\theta\theta/\pi); \; j = 1,2 \; .$$

Therefore,

$$F = T \circ g \circ V,$$

and the equivalence of the conformal maps $F : \Omega \to R_h$ and $g : \Omega_d \to A_q$. persists.

We end this section by making the following remarks in connection with the use of the compositions

$$F = T \circ g, \quad F = T \circ g \circ V, \tag{3.8}$$

And

$$F^{[-1]} = g^{[-1]} \circ T[^{[-1]}, F^{[-1]} = V^{[-1]} \circ g^{[-1]} \circ T^{[-1]}, \tag{3.9}$$

for computing approximations to F and to the inverse conformal map $F^{[-1]}$: $R_h \to \Omega$.

(i) As was previously remarked, the application of (3.8) and (3.9) is restricted to quadrilaterals having one of the two special forms illustrated in Fig 1. We note however that the mapping of such quadrilaterals has received considerable attention recently; see e.g. [6], [15], [30], [45], and [56].

(ii) Procdures based on the use of (3.8) and (3.9) are not affected by crowding of the form described in Section 3.1.

(iii) The use of (3.8) or (3.9) for approximating F or $F^{[-1]}$ depends on the availability of a suitable approximation to $g : \Omega_d \to A_q$.or $g^{[-1]}$: $A_q \to \Omega_d$. Although the conformal mapping of doubly-connected domains has received much less attention than that of simply-connected domains, there are several good numerical methods for approximating g and $g^{[-1]}$; see e.g. [7], [8], [9: Kap.V], [15], [19: §17.2-17.5], [23], [34], [47] and

[57]. Furthermore, some of these methods involve more or less the same computational effort as the corresponding methods for approximating $f:\Omega \to D$ or $f^{[-1]}$: $D \to \Omega$. This can be seen by comparing, for example, the following: (a) The numerical methods for f and g that arise from the integral equation formulations of Symm [46,47]; see in particular [23: §2.3]. (b) The Bergman kernel method for f and the corresponding orthonormalization method for g; see [33,34] and [40]. (c) The Theodersen method for $f^{[-1]}$ and the Garrick method for g[-1] ; see e.g. [9: pp 61-107, 194-207], [19: pp 404-408, 474-478] and [15].

3.3 A finite element method

Let h := m(Q), where as before Q := { Ω.; z_1, z_2, z_3, z_4}, and recall the variational property P2.3, i.e.

$$
\begin{aligned}
h^{-1} &= \min\{D_\Omega[u] : u \in k\} \\
&= D_\Omega [u], \qquad (3.10)
\end{aligned}
$$

where K, $D_\Omega[u]$ and u_0 are respectively the class of functions (2.4), the Dirichlet integral (2.5) and the solution of problem (2.2) with $V_1 = 0$ and $V_2 = 1$. Also, let

$$
k' := \{u : u \; c(\overline{\Omega}) \cap w_1(\Omega\Omega) u = 0 \text{ on } (z_2,z_3),\ u = 1 \text{ on } \{z_4,z_1\}\ ,
$$

and observe that

$$
\begin{aligned}
h &= \min\{D_\Omega[u] : u \in k'\} \\
&= D_\Omega[u_0']. \qquad (3.11)
\end{aligned}
$$

where u_0' is the solution of the Laplacian problem

$$
\Delta_{xy} u_0' = 0, \quad \text{in } \Omega \qquad (3.12a)
$$

$$
u_0' = 0, \text{ on } (z_2,z_3);\ u_o' = 1, \text{ on } (z_4,z_1) \qquad (3.12b)
$$

$$
\frac{\partial u_0'}{\partial n} = 0, \text{ on } (z_1,z_2) \cup (z_2,z_4)\ . \qquad (3.12c)
$$

(This follows by applying P2.3 to the conjugate quadrilateral

$Q' := \{\Omega; z_2, z_3, z_4, z_1\}$, and recalling that $m(Q') = h^{-1}$; see property P2.1). Finally, let

$$\alpha = \min\{D_\Omega[u] : u \in \hat{K}\} \ , \tag{3.13a}$$

and

$$\beta = \min\{D_\Omega[u] : u \in \hat{K}'\} \ , \tag{3.13b}$$

where $\hat{K}$ and $\hat{K}'$ are respectively subclasses of K and K'. Then, it follows at once from (3.10) and (3.11) that

$$\alpha^{-1} \le h \le \beta \ . \tag{3.13c}$$

The above observations form the basis of a finite element method due to Gaier [10], for computing approximations to $h := m(Q)$ in cases where Ω is a polygonal domain. More precisely, the method of [10] determines upper and lower bounds to h of the form (3.13), by computing finite element solutions to the two Laplacian problems associated with the quadrilaterals Q and Q'; i.e. to problem (2.2) with $V_1 = 0$ and $V_2 = 1$, and to problem (3.12). The finite element discretization used in [10] involves the following: (a) Partitioning the polygonal region $\overline{\Omega}$ into regular triangular (or rectangular) elements, so that each of the points Z_j; $j = 1,2,3,4$, coincides with a node of the subdivision, (b) Taking $\hat{K}$ and $\hat{K}'$ to be finite- dimensional spaces of linear (or bilinear) functions.

The method of [10] has been analyzed fully, particularly by Weisel [58], and estimates of the order of convergence are given in both [10] and [58]; see also [13: pp.69-70]. This is of course an important theoretical advantage of the method. Unfortunately however, the speed of convergence is in general slow, because in any non-trivial application the Laplacian problems associated with Q and Q' contain boundary singularities, i.e. "corner" singularities of the type that arise frequently in the study of elliptic mixed boundary value problems. In particular, serious singularities occur when the polygon $\partial\Omega$ contains re-entrant corners, or when one or more of the points Z_j; $j = 1,2,3,4$, do not coincide with corners of $\partial\Omega$. The damaging effect of such singularities is predicted by the analysis of [58], and also by the more general finite element theory of Laplacian boundary value problems which has been developed in recent years; see e.g. [17]. Furthermore, it is now well-known from this general theory that the situation can be improved by the use of singular elements. This approach has been applied successfully by Weisel [58] in connection with the use of a different, but similar, method for computing approximations to the conformal modules of doubly-connected domains.

3.4 A Fourier series method

Let Q be of the form illustrated in Fig. l(b) and assume that (z_1,z_2) is a segment of the real axis. That is, let

$$Q := \{ \Omega.; z_1, z_2, z_3, z_4\}, \tag{3.14a}$$

where

$$\Omega := \{(x,y): 0 < x < 1,\ 0 < y < \tau(x)\}, \tag{3.14b}$$

And

$$z_1 = 0,\quad z_2 = 1,\quad z_3 = 1 + i\tau\tau(1\quad, z_4 = i\tau\tau(0) \tag{3.14c}$$

Also, let h := m(Q) and, as before, let $F^{[-1]}$ denote the conformal map $F^{[-1]}: R_h \to \Omega$. Then, if follows from the Schwarz reflection principle that the conformal map $F^{[-1]}$ can be extended to map the infinite strip $\{(\xi(\eta): -\infty < \xi < \infty, 0 < \eta < h\}$ onto the infinite domain bounded by the x-axis and the curve $y = \hat{\tau}(x)$, where $\hat{\tau}$ is the periodic function defined by $\hat{\tau}(\pm x) = \tau(X)$, $0 \le x \le 1$, and $\hat{\tau}(2+x) = \hat{\tau}(x)$. This implies that the function $F^{[-1]}(W)$ - w is periodic with period 2, and that $F^{[-1]}$ has a series representation of the form

$$F^{[-1]}(w) = w + \sum_{k=1}^{\infty} c_k \ \sin k\pi i \quad , \tag{3.15}$$

where the coefficients c_k are all real.

Let

$$x(\xi(:= \operatorname{Re} F^{[-1]}(\xi + ih) \quad ,$$

so that

$$F^{[-1]}(\xi + ih) = x(\xi) + i\tau\tau(x(\xi))$$

Then, it is easy to see that the Fourier series expansions of the functions

$x(\xi)$ and $\tau(x(\xi))$ are of the form

$$x(\xi) \sim \xi + \sum_{k=1}^{\infty} a_k . \; \coth k\pi h \; \sin k\pi i \quad , \tag{3.16}$$

and

$$\tau(x(\xi)) \sim h + \sum_{k=1}^{\infty} a_k . \; \cos k\pi o \; , \tag{3,17}$$

where the coefficients a_k are related to those of (3.15) by

$$a_k = c_k \sinh k\pi h \; , \tag{3.18}$$

Also, from (3.17) we have that

$$h = \int_0^1 \tau(\chi(\xi)) d\xi \quad . \tag{3.19}$$

The above results lead to a numerical conformal mapping method for quadrilaterals of the form (3.14), which was proposed, but not analyzed, by Challis and Burley [6]. This method of [6] is based on solving a set of nonlinear equations that arise from the discrete Fourier series analogues of (3.16), (3.17) and (3.19), corresponding to the nodes $\xi_r := r/N$; $r = 0,1,\ldots,N$. More precisely, the method involves solving iteratively, by a Jacobi type algorithm, the equations

$$\tilde{x}(\xi_r) = \xi_r + \sum_{K=1}^{N-1} \tilde{a}_k \; \coth k\pi \tilde{h} \; \sin k\pi \xi_r ; \quad r = 0,1,\ldots., N \; ,$$

$$\tilde{a}_k = \tfrac{2}{N} \sum_{r=0}^{N}{}' \tau(\tilde{x}(\xi_r)) \cos k\pi \xi_r ; \quad k = 1,2,\ldots., N \; ,$$

$$\tilde{h} = \tfrac{1}{N} \sum_{r=0}^{N} \tau(\tilde{x}(\xi_r)) \quad .$$

for the unknowns $\tilde{a}_k$; $k = 1,2,\ldots,N$, and $\tilde{h}$. Once these unknowns are found, the conformal map $F^{[-1]}$ is approximated by

$$\tilde{F}^{[-1]}(w) = w + \sum_{k=1}^{N} \tilde{a}_k \text{ cosech} k\pi \tilde{h} \text{ sin} k\pi i \quad ;$$

See Eqs (3.15) and (3.18).

It turns out that the above method of Challis and Burley is equivalent to a known method of the type discussed in Section 3.2, although this is not immediately apparent from the work of [6]. More specifically, the method of [6] is equivalent to expressing $F^{[-1]}$ in the composite form given by the second equation in (3.9), i.e. as

$$F^{[-1]} = V^{[-1]} \circ g^{[-1]} \circ T^{[-1]}, \tag{3.20}$$

and using the well-known method of Garrick ([9], [16], [19], [25]) for approximating the conformal map $g^{[-1]} \; : \; A_q \to \Omega_d$. This equivalence is established in [15], where also the theory of the method of Garrick is used to provide some theoretical justification for the numerical method proposed in [6].

We end this section by making the following remarks:

(i) The doubly-connected domain Ω_d. corresponding to (3.14) is of the form (3.6) with $n = 1$, $\rho_1(\theta)=1$ and $\rho_2(\theta)=\exp\{-\pi\tau(\theta/\pi)\}$. That is, for quadrilaterals of the type considered in [6], the outer boundary component of Ω_d. is the unit circle and, because of this, the Garrick algorithm simplifies considerably. This simplified algorithm is particularly efficient if use is made of the fast Fourier transform (FFT). The algorithm is then fast, in the sense that it requires the application of only two FFT's in each iterative step; see [15: 4.2,5] and [18: p.74].

(ii) The method of Garrick can, of course, also be used in conjunction with (3.20) for the mapping of quadrilaterals having the more general form (3.7). In this case however, the associated doubly-connected domain Ω_d does not display any special simplifying features, apart from symmetry. Thus, the general Garrick algorithm must be used, and this involves the application of four FFT's in each iterative step; see [15].

(iii) The results of several numerical experiments illustrating the above comments and also the convergence properties of the method of Garrick, are given in [15: Sect.6]; see also [35: Sect.4].

(iv) A Fourier series method has also been used by Wanstrath et al [56], for the mapping of quadrilaterals of the form (3.7). It is of interest to note that their series representation of the mapping function $F^{[-1]}$ can also be derived from the theory of the Garrick method, via the use of (3.20).

3.5 A finite difference method

Let $Q := \{ \Omega.; z_1, z_2, z_3, z_4\}$, where the domain Ω. is bounded by four Jordan arcs, and the points z_1, z_2, z_3, z_4 are the corners where these arcs intersect. That is, the quadrilaterals under consideration are of the form illustrated in Fig. 1, except that now all four sides of Q are allowed to be curved.

The conformal mapping of quadrilaterals of the above special form has been considered recently by Seidl and Klose [45]. Their numerical method involves the use of an iterative algorithm which solves by finite differences a coupled pair of Laplacian mixed boundary value problems in R_h, for the unknown functions

$$x(\xi(\eta) = ReF^{[-1]}(\xi,\eta) \text{ and } y(\xi(\eta) = ImF^{[-1]}(\xi,\eta) \ .$$

(Of course, the height of the rectangle R_h, i.e. the conformal module $h := m(Q)$, is also an unknown of the two Laplacian problems.)

The method of [45] is based to a large extent on experimental observations. Also, in the numerical examples the authors are mainly interested in comparing the efficiencies of methods for solving the discretized Laplacian problems; e.g. the SLOR and the multigrid.

3.6 A modified Schuarz-Christoffel transformation method

This is a recent method due to Howell and Trefethen [24], for computing approximations to $h := m(Q)$ and to $F^{[-1]} : R_h \to \Omega$ in cases where Ω is a polygonal domain. The method is designed to overcome the crowding difficulties which, when h is large, affect the use of procedures based on (3.4); see Section 3.1. This is done by using an infinite strip, instead of the unit disc, as the intermediate canonical domain. More precisely the method of [24] is based on expressing $F^{[-1]}$ in the form

$$F^{[-1]} = \hat{f}^{[-1]} \text{ o } \hat{s}^{[-1]}, \tag{3.21}$$

where $\hat{s}^{[-1]}$ and $\hat{f}^{[-1]}$ are as follows:

(a) $\hat{S}^{[-1]}$ denotes the conformal map of R_h onto the infinite strip $\sigma := \{(s,t) : -\infty < s < \infty,\ 0 < t < 1\}$, (This map is known exactly in terms of h and the logarithm of a Jacobian elliptic sine.)

(b) $\hat{f}^{[-1]} : \sigma \rightarrow \Omega$ is a modified Schwarz-Christoffel transformation that maps the strip σ onto the polygon Ω. (An algorithm for constructing $\hat{f}^{[-1]}$ is described with full computational details in [24].)

The efficiency of methods based on the use of (3.21) is illustrated in [24] by several impressive examples, involving the conformal map of highly elongated quadrilaterals.

4. Numerical examples

In this section we present three numerical examples, chosen to illustrate the following:

. The application of the conformal map $F : \Omega \rightarrow R_h$ to the solution of "singular" Laplacian boundary value problems of the form (2.2); Examples 1 and 2.

. The remarks (i), (ii) and (iv) made in Section 3.1, in connection with the use of procedures based on expressing the conformal maps F or F[¯'] in the composite forms (3.1) or (3.4); Examples 1 and 2.

. The crowding difficuties associated with the use of procedures of the type described in Section 3.1, in cases where the conformal module h is "large" or "small"; Example 3.

. The efficiency of procedures of the type described in Section 3.2, for the mapping of quadrilaterals having one of the two special forms illustrated in Fig. 1; Example 3.

Example 1. Consider the solution of the Laplacian problem (2.2) in the case where:

(a) The quadrilateral $Q := \{\Omega; z_1 , z_2 , z_3 , z_4\}$ consists of the rectangle

$$\Omega : \{(x,y) : |x| < 1 ,\ 0 < y < 1\} ,$$

and the points

$$z_1 = -1,\ z_2 = 0,\ z_3 = 1,\ z_4 = 1 + i .$$

(b) The Dirichlet boundary values are

$V_1 = 500$ and $V_2 = 1{,}000$.

This problem has a serious boundary singularity at the point $z_2 = 0$, in the sense that the first derivatives of its solution become unbounded at z_2. More specifically, the singularity occurs because in the neighbourhood of z_2 the solution u has a series expansion of the form

$$u(r\cos\theta, r\sin\theta) = 500 + \sum_{n=0}^{\infty} a_n r^{n+\frac{1}{2}} \cos(n+1/2)\theta , \tag{4.1}$$

where (r, θ) are polar coordinates. (In fact, it can be shown that the expansion (4.1) is valid everywhere in Ω; see [44].)

The above problem is often referred to as the "Motz problem", because it was first considered by Motz [31] in 1946, as an example for solving singular Laplacian problems by a modified finite difference technique. Since then it has become a standard test problem, for comparing the performances of numerical methods in the presence of boundary singularities. Here, we consider the solution of the Motz problem by conformal transformations and, in particular, by expressing the conformal map $F : \Omega \to R_h$ in the composite form (3.1), i.e. as

$$F = S \circ f . \tag{4.2}$$

The use of (4.2) recommends itself because in this case the conformal map $f : \Omega \to D$ is known exactly in terms of a Jacobian elliptic sine; see e.g. [4: p44], [32: p280] and [59]. In addition, the conformal module h is known exactly in terms of two complete elliptic integrals of the first kind; see [44: Eq. (6.23)] and [59]. In fact, to nine decimal places,

$$h = 1.469\ 218\ 032 .$$

Hence the measure of crowding (3.5) is

$$C = 0.795\ 805\ 280 .$$

Therefore, we can conclude that, in this case, the implementation of (4.2) will not be affected by crowding of the form described in Section 3.1.

It follows from the above that (4.2) gives the exact solution of the Motz problem, in terms of elliptic functions and integrals. Surprisingly, this does not appear to be generally recognized, althoug the exact solution has been available in [59], since 1972.

The conformal map (4.2) has also been used by Rosser and Papamichael [44], for developing a procedure that computes accurate approximations to the coefficients a_n of the expansion (4.1). This procedure is essentially based on expressing the coefficients a_n, in terms of the coefficients in the series expansions of the various elliptic functions and integrals involved in the conformal maps f and S. The computed approximations to the values $a_n/500$; $n = 0,1,\ldots,19$, corresponding to the first twenty coefficients of (4.1), are listed in pages 34 and 35 of [44]. From these we can conclude, for example, that the exact values of the first four coefficients are given, to fourteen significant figures, by:

$$a_0 = 401.16\,24537\,4523\,,$$

$$a_1 = 87.655\,92019\,5088\,,$$

$$a_2 - 17.237\,41507\,9446\,,$$

and

$$a_3 = -8.0712\,15259\,6981\,.$$

Example 2. We consider again the solution of a Laplacian problem of the form (2.2), where now:

(a) The quadrilateral $Q := \{\Omega;\ z_1,\ z_2,\ z_3,\ z_4\}$ consists of the L-shaped domain

$$\Omega := \{(x,y): -1 < x < 3,\ \ |y| < 1\} \cup \{(x,y):\ |x|\ < 1, -1 < y < 3\},$$

and the points

$$z_1 = 3 - i,\ \ z_2 = 3 + i,\ \ z_3 = -1 + 3i,\ z_4 = -1 - i\,.$$

(b) The Dirichlet boundary values are

$$V_1 = 0\ \text{ and } V_2 - 1\,.$$

The above problem also has a boundary singularity, which this time occurs because the first derivatives of the solution u become unbounded at the re-entrant corner

$$z_c = 1 + i .$$

In this case the exact conformal map $f : \Omega \to D$ is not known. However, by using elliptic integrals or symmetry arguments, it can be shown that

$$h := m(Q) = 1/\sqrt{3} ;$$

see [10], and [20,21], That is, the conformal module of Q is given to six decimal places by

$$h = 0.577\ 350 .$$

Also, although the exact solution of the Laplacian problem is not known, it is easy to show by using symmetry arguments that the value of u at the re-entrant corner z_c is

$$u_c = 2/3 .$$

For the numerical solution of the problem we use again the composite transformation (3.1). However, since in this case the conformal map $f : \Omega \to D$ is not known exactly, we perform the transformation $F : \Omega \to R_h$ approximately by means of

$$\tilde{F} = S o \tilde{f} , \tag{4.3}$$

where $\tilde{f}$ is an approximation to f. More specifically, the approximation $\tilde{f}$ is obtained by using the Bergman kernel method (BKM), i.e. an orthonormalization method based on the properties of the Bergman kernel function of Ω. Full details of the BKM procedure used can be found, for example, in [33]. Here, we only note that the BKM leads to approximations of the form

$$f_n(z) = \sum_{n=1}^{\infty} \alpha_n v_n(z) ,$$

that in our case $\tilde{f} := f_{26}$, and that the coefficients α_n and basis functions v_Ω; n = 1,2,...,26, involved in the formula for $\tilde{f}$, are listed in [36: Ex. 2.3]. We also note that the estimate of the maximum error in $|\tilde{f}|$ is

$$E = 2.2 \times 10^{-5} \tag{4.4}$$

Regarding crowding, in this case (3.5) gives

$$C = 8 \exp\{-\pi\sqrt{3/2}\} = 0.526\ 629\ ,$$

i.e. there are no crowding difficulties. We can therefore conclude that the estimate (4.4) also gives a good indication of the accuracy of the approximation (4.3) and of our numerical solution to the Laplacian problem. This is confirmed by the computed approximations to the conformal module h and to the value u_C, which are as follows:

$$\tilde{h} = 0.570\ 340 \text{ and } \tilde{u}_c = 0.666\ 674\ ,$$

i.e., $h - \tilde{h} = 1.0\times10^{-5}$ and $\tilde{u}_c - u_c = 7.0\times10^{-6}$. (The computed approximations to u at several other points in $\overline{\Omega}$ are listed in [37: p82].) By comparison, the finite element method outlined in Section 3.3 gives the following lower and upper bounds to h:

$$0.577\ 25 < h < 0.577\ 45\ .$$

These bounds were obtained in [10: p191], by solving a set of 12,351 linear equations corresponding to a discretization based on the use of rectangular elements and bilinear test functions.

Further examples, illustrating the application of the composite transformation (3.1) to the solution of Laplacian and other more general elliptic boundary value problems, can be found in [37] and [42].

Example 3. Let Q_ℓ, denote the quadrilateral illustrated in Fig. 2. That is, $Q_\ell := \{\Omega_\ell\ ; z_1, z_2, z_3, z_4\}$ where:

(a) Ω_ℓ is the trapezium bounded by the real and imaginary axes and the lines $x = \ell$ and $y = x + \ell - 1$, where $\ell > 1$.

(b) The points z_1, z_2, z_3, z_4 are the corners of Ω_ℓ, i.e.

$$Z_1 = 0,\quad z_2 = 1,\quad z_3 = 1 + i\ell,\quad z_4 = i(\ell - 1)\ .$$

Also, let $h_\ell := m(Q_\ell)$ and observe that the exact values of h_ℓ are known in terms of elliptic integrals; see [4: p104]. For example, the exact values of $h_{2.5}$, $h_{5.0}$ and $h_{10.0}$ are given, to nine decimal places, by:

$$h_{2.5} = 1.779\ 359\ 959,\quad h_{5.0} = 4.279\ 364\ 400\ ,$$

and

$$h_{10,0}=9.279\ 364\ 400\ .$$

In this example we compute approximations to h_ℓ; $\ell = 2.5, 5.0, 10.0$, by using each of the following two methods:

Method 1: This method is based on using the composite form (3.1), i.e. on approximating the conformal map $f_\ell : \Omega_\ell \to D$; see Section 3.1. As in Example 2, the approximation to f_ℓ is obtained by using the BKM.

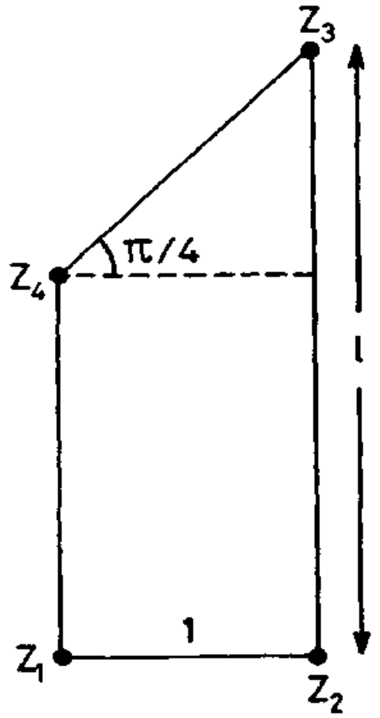

Figure 2

Method 2: This method is based on using the composite form (3.8), i.e. on approximating the conformal map $g_\ell : \Omega_{(\ell)d} \to A_q$, where $\Omega_{(\ell)d}$ is the doubly-connected domain associated with the quadrilateral Q_ℓ ; see Section 3.2. The approximation to g_ℓ is obtained by an orthonormalization method (ONM), which may be regarded as the generalizaiton of the BKM to the mapping of doubly-connected domains. (The details of the ONM procedure used can be found in [34].)

The approximations $\bar{h}_\ell$ to h_ℓ; $\ell = 2.5$, 5.0 , 10.0 , computed by each of the above methods are listed in Tables 1 and 2 respectively. In the tables we also list the values E£ and e^, whose meanings are as follows:

. E_ℓ denotes the estimate of the maximum error in modulus in the BKM approximation to $f_\ell : \Omega_\ell \to D$ (Table 1), or the ONM approximation to $g_\ell : \Omega_{(\ell)d} \to A_q$ (table 2).

. ε_ℓ denotes the absolute error in the approximation $\tilde{h}_\ell$, i.e. $\varepsilon_\ell :=$ $|\tilde{h}_\ell - h_\ell|$.

In addition, in Table 1 we list the values

$$c_\ell := 8\exp\{-\pi h_\ell/2\} \; ; \; \ell, = 2.5, 5.0, 10.0 \, ,$$

which measure the "crowding" associated with the use of Method 1; see Eq. (3.5).

TABLE 1 (Method 1)

ℓ	E_ℓ	C_ℓ	$\tilde{h}_\ell$	$\in_\ell$
2.5	$3{,}5\times10^{-6}$	4.9×10^{-1}	1.779 360 754	8.0×10^{-7}
5.0	1.4×10^{-6}	9.6×10^{-3}	4.279 424 389	6.0×10^{-5}
10.0	1.1×10^{-4}	3.7×10^{-6}	*Method fails*	-

TABLE 2 (Method 2)

ℓ	E_ℓ	$\tilde{h}_\ell$	$\in_\ell$
2.5	1.8×10^{-6}	1.799 359 961	2.0×10^{-9}
5.0	6.3×10^{-7}	4.279 364 401	1.0×10^{-9}
10.0	1.3×10^{-5}	9.279 364 291	1.1×10^{-7}

We make the following remarks in connection with the results of Method I, listed in Table 1:

. The value$C_{2.5}$ indicates that no crowding occurs in the case $\ell = 2.5$. For this reason, Method 1 gives an accurate approximation to $h_{2.5}$, with $\varepsilon_{2.5} < E_{2.5}$

. Although the value $C_{5.0}$ indicates a noticeable amount of crowding, the method leads to a perfectly adequate approximation $\tilde{h}_{5.0}$. This occurs because $C_{5.0}$ is substantially larger than $E_{5.0}$ However, the computed approximation is contaminated somewhat by the effects of crowding and, as a result, $\varepsilon_{5.0} > E_{5.0}$

When $\ell = 10.0$, the crowding on $|\zeta| = 1$ is severe. In this case $C_{10.0} < E_{10.0}$ and , not surprisingly, the BKM fails to compute the images of the points Zj in the correct order. For this reason, Method 1 fails completely in this case.

The results to Table 2 illustrate the high accuracy that can be achieved by methods of the type discussed in Section 3.2. We note in particular that, for each of the three values of ℓ, the error ε_ℓ is substantially smaller than the corresponding error estimate E_ℓ. This is due to the fact that, in general, numerical methods for the conformal map $g : \Omega_d \to A_q$(or $g^{[-1]} : A_q \to \Omega_d$)approximate q (and hence h) more accurately than g (or $g^{[-1]}$).

Further numerical examples, similar to the one considered here, can be found in [35: Sect.4].

5. A domain decomposition method for long quadrilaterals

In this section we discuss briefly some recent work, concerning a domain decomposition method for the mapping of quadrilaterals of the form illustrated in Fig. l(b).

In order to motivate our discussion, we consider again the quadrilateral Q_ℓ illustastrated in Fig. 2 and, with the notation of Example 3, we list below the exact values of $h_\ell := m(Q_\ell)$; ℓ = 2.0,2.5,4.0,5.0,10.0, which were computed to twelve decimal places from the formulae of Bowman [4]:

$$h_{2.0} = 1.279\ 261\ 571\ 171 \ ,$$

$$h_{2.5} = 1.779\ 359\ 959\ 478 \ ,$$

$$h_{4.0} = 3.279\ 364\ 399\ 489 \ ,$$

$$h_{5.0} = 4.279\ 364\ 399\ 847 \ ,$$

$$h_{10.0} = 9.279\ 364\ 399\ 847 \ ,$$

The above values indicate that $h_{2.5}$, $h_{4.0}$, $h_{5.0}$, and $h_{10.0}$ are given respectively, to three, five, eight and twelve decimal places by $h_{2.0} + 0.5$, $h_{2.5} + 1.5$, $h_{4.0} + 1.0$ and $h_{s.0} + 5.0$. In fact, a closer inspection suggests that, for "large" h_ℓ,

$$h_{\ell+c} - (h_\ell + c) - \exp\{-2\pi\pi_\ell\} \ , \quad c > 0 \ .$$

This is an example of a much more general phenomenon, which can be described as follows:

Let $Q := \{\Omega;\ z_1, z_2, z_3, z_4\}$ be a quadrilateral of the form illustrated in Fig. l(b), where:

$$\left.\begin{array}{l} \Omega := \{(x, y : 0 < x < 1,\ \tau,(x) < y < \tau_2\ (x)\} \\ \tau_j(x) > 0\ , x \in [0,1]\ ;\quad j = 1, 2 \end{array}\right\}$$

and

$$\left.\begin{array}{l} z_1 = -i\tau_1(0)\ ,\quad z_2 = 1 - i\tau_1(1)\ , \\ z_3 = 1 + i\tau_2(1)\ ,\quad z_4 = i\tau_2(0)\ . \end{array}\right\}$$

Also let

$$\Omega_1 := \{(x, y : 0 < x < 1,\ -\tau,(x) < y < 0\}\ ,$$

and

$$\Omega_2 := \{(x, y : 0 < x < 1,\ \tau, 0 < y < \tau_2\ (x)\}\ ,$$

So that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, and let

$$Q_1 := \{\Omega_1;\ z_1, z_2, 1, 0\}\ \text{and}\ .\ Q_2 := \{\Omega_2;\ 1, 0, z_3, z_4\}\ .$$

Finally, let $h := m(Q)$, $h_j := m(Q_j)$; $j = 1,2$, and $h^* = \min(h,, h_2)$. Then for large h^*,

$$h - (h_1 + h,) \sim \exp\{-2\pi h^*\} \quad . \tag{5.1}$$

That is, if Q is a "long" quadrilateral, then $h := m(Q)$ can be approximated closely by the sum $h_1 + h_2$ of the conformal modules of the two smaller quadrilaterals Q_1 and Q_2 . In fact, by imposing certain smoothness conditions on the functions τ_J; $j = 1,2$, it is possible to obtain precis estimates of $h - (h_1 + h_2)$. Such estimates are derived in [39], where also the analysis of a decomposition method, for determining the full conformal map $F : \Omega \to R_h$ in terms of $F_1 : \Omega_1 \to R_{h_1}$ and $F_2 : \Omega_2 \to R_{h2}$, is given; see also [38].

To illustrate the practical significance of (5.1), we recall the numerical results which were obtained in Example 3, by the use of Method 1. In particular, we recall that the crowding on the unit circle had affected the accuracy of the approximation to $h_{5\ 0}$, and had caused the procedure to fail completely in the case $\ell = 10.0$. These crowding difficulties can be overcome, quite simply, by domain decomposition. For example, $h_{5.0}$ and $h_{10.0}$ can be approximated respectively by

$$\widehat{h}_{5.0} := \widetilde{h}_{2.5} + 2.5 = 4.279\ 360\ 754\ ,$$

and

$$\widehat{h}_{10.0} := \widetilde{h}_{2.5} + 7.5 = 9.279\ 360\ 754\ ,$$

where $\widetilde{h}_{2.5}$ is the accurate Method 1 approximation to $h_{\ 2.5}$ listed in Table 1 In this way we find that the error in both $\widehat{h}_{5.0}$ and $\widehat{h}_{10.0}$ is $\hat{\varepsilon} = 3.6 \times 10^{-6}$, i.e. $\hat{\varepsilon} \simeq E_{2.5}$.

References

[1] A. Acker, Heat flow inequalities with applications to heat flow optimization problems, SIAM J. Math. Anal. 8 (1977) 604-618.

[2] W.D. Barfield, Numerical method for generating orthogonal curvilinear meshes, J. Comput. Phys. 5 (1970) 23-33.

[3] G.E. Bell and J. Crank, Influence of imbedded particles on steady-state diffusion, J.C.S. Faraday Trans II 70 (1974) 1259-1273.

[4] F. Bowman, *Introduction to Elliptic* Functions (English University Press, London 1953).

[5] J.B. Campbell, Finite difference techniques for ring capacitors, J. Engng Math. 9 (1975) 21-28.

[6] N.V. Challis and D.M. Burley, A numerical method for conformal mapping, IMA J. Numer. Anal. 2 (1982) 169-181.

[7] H.D. Däppen, Die Schwarz-Christoffel – Abbildung für zweifach usammenhängend Gebiete mit Anwendungen, Diss. No.8495, Swiss Federal Institute of Technology, zürich 1988.

[8] B. Fornberg, A numerical method for conformal mapping of doubly connected regions, SIAM J. Sci. Stat. Comput. 5 (1984) 771-783.

[9] D. Gaier, *Konstruktive Methoden der konformen Abbildung* (Springer, Berlin, 1964).

[10] D. Gaier, Ermittlung des konformen Moduls von Vierecken mit Differenzenmethoden, Numer. Math 19 (1972) 179-194.

[11] D. Gaier, Determination of conformal modules of ring domains and quadrilaterals, Lecture Notes in Mathematics 399 (Springer, New York, 1974) pp.180-188.

[12] D. Gair, Capacitance and the conformal module of quadrilaterals, J. Math. Anal. Appl. 70 (1979) 236-239.

[13] D. Gaier, Numerical methods in conformal mapping, in: H. Werner et al, Eds, Computational Aspects of Complex Analysis (D. Reidel, Dordecht, 1983) pp.51-78.

[14] D. Gaier, On an area problem in conformal mapping. Results in Mathematics, 10 (1986) 66-81.

[15] D. Gaier and N. Papamichael, On the comparison of two numerical methods for conformal mapping, IMA J, Numer. Anal. 7 (1987) 261-282.

[16] I.E. Garrick, Potential flow about arbitrary biplane wing sections, Report 542, NACA, 1936.

[17] P. Grisvard, Elliptic problems in nonsmooth domains (Pitman, Boston, 1985).

[18] M.H. Gutknecht, Numerical conformal mapping based on function conjugation, in: L.N. Trefethen, Ed., Numerical Conformal Mapping, (North-Holland, Amsterdam 1986) 31-77.

[19] P. Henrici, Applied and Computational Complex Analysis, Vol.III (Wiley, New York, 1986).

[20] J. Hersch, Représentation conforme et symmétries: une détermination élémentaire du mudule d'un quadrilatere en forme de L, El. Math. 37 (1982) 1-5.

[21] J. Hersch, On harmonic measures, conformal moduli and some elementary symmetry methods, J. Analyse Math. 42 (1982/83) 211-228.

[22] D.M. Hough and N. Papamichael, The use of splines and singular functions in an integral equation method for conformal mapping, Numer. Math. 37 (1981) 133-147.

[23] D.M. Hough and N. Papamichael, An integral equation method for the conformal mapping of interior, exterior and doubly-connected domains, Numer. Math. 41 (1983) 287-307.

[24] L.H. Howell and L.N. Trefethen, A modified Schwarz-Christoffel transformation for elongated regions, Numerical Analysis Report 88-5, Dept. of Maths, Massachusetts Institute of Technology, Cambridge, Mass., 1988.

[25] D.C, Ives, A modern look at conformal mapping including multiply connected regions, AIAA J. 14 (1976) 1006-1011.

[26] H.B. Keller and D. Sachs, Calculations of the conductivity of a medium containing cylindrical inclusions, J. Appl. Phys. 35 (1964) 537-538.

[27] J. Larsen, On the measuring of diffusion coefficients of solid materials, ZAMP 32 (1981) 229-231.

[28] P.A.A. Laura and L.E. Luisoni, An application of conformal mapping to the determination of the characteristic impedance of a class of coaxial systems, IEEE Trans MTT 25 (1977) 162-164.

[29] J.H. Lienhard, Heat conduction through "Yin-Yang" bodies, J. Heat Transfer 103 (1981) 600-601.

[30] C.D. Mobley and R.J. Stewart, On the numerical generation of boundary-fitted orthogonal curvilinear coordinate systems, J. Comput. Phys. 34 (1980) 124-135.

[31] H. Motz, The treatment of singularities of partial differential equations by relaxation methods, Q. Appl. Math. 4 (1946) 371-377.

[32] Z. Nehari, *Conformal Mapping* (McGraw-Hill, New York, 1952).

[33] N. Papamichael and C.A. Kokkinos, Two numerical methods for the conformal mapping of simply-connected domains, Comput. Meths Appl. Mech. Engng 28 (1981) 285-307.

[34] N. Papamichael and C.A. Kokkinos, The use of singular functions for the approximate conformal mapping of doubly-connected domains, SIAM J. Sci. Stat. Comput. 5 (1984) 684-700.

[35] N. Papamichael, C.A. Kokkinos and M.K. Warby, Numerical techniques for conformal mapping onto a rectangle, J. Comput. Appl. Math. 20 (1987) 349-358.

[36] N. Papamichael and A. Sideridis, Formulae for the approximate conformal mapping of some simply-connected domains. Tech. Report TR/72, Dept of Mathematics, Brunel University 1977.

[37] N. Papamichael and A. Sideridis, The use of conformal transformations for the numerical solution of elliptic boundary value problems with boundary singularities, J. Inst. Maths Applics 23 (1979) 73-88.

[38] N. Papamichael and N.S. Stylianopoulos, On a domain decomposition method for the computation of conformal modules, Appl. Math. Letters. (To appear)

[39] N. Papamichael and N.S. Stylianopoulos, A domain decomposition method for conformal mapping onto a rectangle, Tech. Report, Dept of Mathematics and Statistics, Brunel University. (To appear)

[40] N. Papamichael and M.K. Warby, Stability and convergence properties of Bergman kernel methods for numerical conformal mapping, Numer. Math. 48 (1986) 639-669.

[41] N. Papamichael, M.K. Warby and D.M. Hough, The treatment of corner and pole-type singularities in numerical conformal mapping techniques in:L.N. Trefethen, *Ed., Numerical Conformal Mapping,* (North Holland, Amsterdam 1986) 163-191.

[42] N. Papamichael and J.R. Whiteman, A conformal transformation method for harmonic mixed boundary value problems in polygonal domains, ZAMP 24 (1973) 304-316.

[43] G.J. Rogers and G.K. Cambell, The piece-by-piece solution of elliptic boundary value problems, J. Phys. D: Appl. Phys. 8 (1975) 1615-1623.

[44] J.B. Rosser and N. Papamichael, A power series solution for a harmonic mixed boundary value problem, MRC Tech. Summary Report #1405, Univ. of Wisconsin-Madison 1974.

[45] A. Seidl and H. Klose, Numerical conformal mapping of a towell-shaped region onto a rectangle, SIAM J. Sci. Stat. Comput.6 (1985) 833-842.

[46] G.T. Symm, An integral equation method in conformal mapping, Numer. Math. 9 (1966) 250-258.

[47] G.T. Symm, Conformal mapping of doubly-connected domains, Numer. Math. 13 (1969) 448-457.

[48] J.F. Thompson, Z.U.A. Warsi and C.W. Mastin, Boundary fitted coordinate systems for numerical solution of partial differential equations - A review, J. Comp. Phys. 47 (1982) 1-108.

[49] J.F. Thompson, Z.U.A. Warsi and C.W. Mastin, *Numerical grid generation* (North-Holland, Amsterdam, 1985).

[50] J.C. Tippet, Radiation characteristics of electriclaly small devices in a FEM transmission cell, IEEE Trans EMC 18 (1976) 134-140.

[51] L.N. Trefethen, Numerical computation of the Schwarz-Christoffel transformation, SIAM J. Stat. Comput. 1 (1980) 82-102.

[52] L.N. Trefethen, SCPACK Version 2 User's Guide, Internal Rep.24, Inst. Computer Applies in Sci. and Engrg, NASA Langley Res. Ctr. Hampton, Va. 1983.

[53] L.N. Trefethen, Analysis and design of polygonal resistors by conformal mapping, ZAMP 35 (1984) 692-704.

[54] L.N. Trefethen, Ed., *Numerical Conformal Mapping* (North-Holland, Amsterdam, 1986); reprinted form: J. Comput. Appl. Math. 14 (1986).

[55] W. Versnel, Analysis of the Greek cross, a Van Der Paw structure, Solid-State Electronics 22 (1979) 911-914.

[56] J.J. Wanstrath, R.E. Whitaker, R.O. Reid and A.C. Vastano, Storm surge simulation in transformed co-ordinates, Tech. Report 76-3, U.S. Coastal Engineering Research Center, Fort Belvoir, Va. 1976.

[57] R. Wegmann, An iterative method for the conformal mapping of doubly connected regions, in: L.N. Trefethen, Ed., *Numerical Conformal Mapping,* (North-Holland, Amsterdam 1986) 79-98.

[58] J. Weisel, Lösung singulärer Variationsprobleme durch die Verfahren von Ritz und Galerkin mit finiten Elemervten - Anwendugen in der konformen Abbildung, Mitt, Math. Sem. GieBen 138 (1979) 1-150.

[59] J.R. Whiteman and N. Papamichael, Treatment of harmonic mixed boundary value problems by conformal transformation methods, ZAMP 23 (1972) 655-664.

[60] C. Zemach, A conformal map formula for difficult cases, in: L.N. Trefethen, Ed., *Numerical Conformal Mapping* (North-Holland, Amsterdam, 1986) 207-215.