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An Enumerative Method for the Solution of Linear Complementarity Problems

By

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Summary: In this report an enumerative method for the solution of the Linear Complementarity Problem (LCP) is presented. This algorithm completely processes the LCP, and does not require any particular property of the LCP to apply. That is the algorithm terminates after either finding all the solutions of an LCP or establishing that no solution exists. The method is extended to also process the Second Linear Complementarity Problem (SLCP), a problem which has been introduced to represent the general quadratic program involving unrestricted variables.

1. INTRODUCTION

The Linear Complementarity Problem (LCP) has become one of the important research areas in mathematical programming. This problem is stated as [2,7]

$$w = q + Mz, \quad z \geq 0, \quad w \geq 0, \quad w^T z = 0 \tag{1}$$

Three genres of algorithms for the solution of the LCP have been reported in the literature; direct, iterative and enumerative. The first two only apply to special classes of matrices and they terminate with a particular solution when it exists. Enumerative methods find all the solutions of any LCP without any assumption concerning the class of the matrix M. These latter methods are more involved and process the LCP by exploring the nodes of a tree (they are usually called tree search methods).

To date two enumerative methods have been designed for the LCP. These methods are due to Garcia and Lemke [4] and Mitra and Jahanshalou [8] and were originally designed to solve some related problems and then extended to deal with the LCP. They both consider basic feasible and infeasible solutions when the tree is explored. It is however possible to design another tree search method in which only basic feasible solutions are employed. This algorithm is described in sections 2, 3 and 4 of this paper.

One of the most important applications of the LCP is to find Kuhn-Tucker points of a quadratic program, that is, vectors which satisfy the Kuhn-Tucker conditions of this program [2]. If in a quadratic program there exist some unrestricted variables and (or) some equality constraints the Kuhn-Tucker conditions lead to a problem of the form

[6, pages 7-8]

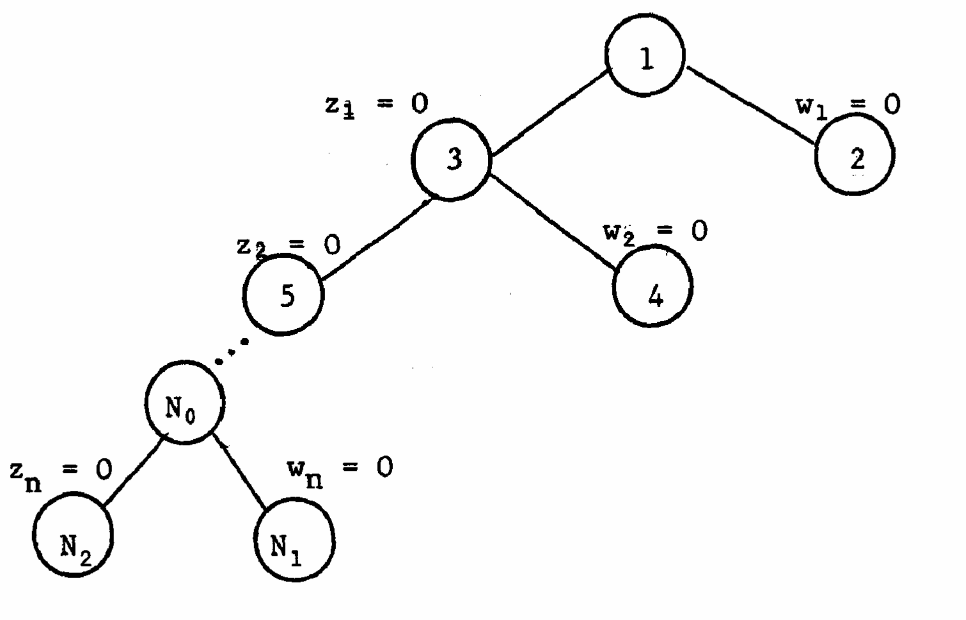
$$w = q + Mz + Nu,$$
$$0 = p + Rz + Su, \qquad (2)$$
$$z \geq 0 \quad w \geq 0, \quad z^T w = 0, \quad -\infty < u < +\infty ,$$

which is called the Second Linear Complementarity Problem (SLCP), (see [6, chapter 5] for a study of this problem). The extension of the enumerative method to this problem is presented in section 5.

In section 6 an LCP of small dimension taken from [8] is solved by the enumerative method to clarify the exposition of the algorithm. Finally, the computational experience with some test LCP's and SLCP's is presented in section 7.

2. THE NEW ENUMERATIVE METHOD FOR THE SOLUTION OF THE LCP.

The method is based on the principle that all the solutions of an LCP can be found by generating at most $\sum_{i=0}^{n} 2^i$ nodes of a tree which are defined by taking one out of the n pairs of complementary variables and then setting each of these variables to zero in turn. Therefore, by exploring the tree shown in (Dl) it is possible to either establish no solution to the LCP exists or find all the solutions of the LCP.



(D1)

If any one of the nodes N1,N2,..., at the depth n of this tree can be developed then it represents at least one solution of the LCP. This is because only feasible solutions are admitted and the n variables set to zero are taken out of n pairs of complementary variables.

If a variable is set to zero it should remain in all the subsequent nodes of the tree shown in (Dl). To indicate this fact the variable is said to be starred. So in the diagram (Dl) $z_i = 0$ ($w_i = 0$) can be replaced by z_i starred (w_i starred).

Suppose that a variable is required to be starred. This obviously can be done if the variable is nonbasic. Note that if a nonbasic variable is starred its corresponding column can never be pivot column of any pivot transformation and the term "starred" is used to indicate this fact. If the variable is basic, however, it is not always possible to star it. Let i be the row of this variable; by applying a simplex type algorithm with the rth row as the objective row either this variable is made nonbasic or it is shown that it is not possible to do so. When the latter occurs let the corresponding tableau be set out in the Tableau form Tl.

$$\begin{array}{cc|c|c|} & & & -x_j \\ \cline{3-4} y_i = & & \bar{b}_i & \bar{a}_{ij} \\ \cline{3-4} \end{array}$$

T1

Then one of the two following properties must hold:

(P1) $\bar{b}_i = 0$ and $\bar{a}_{ij} = 0$ for all the nonstarred j ,

(P2) $\bar{b}_i > 0$ and $\bar{a}_{ij} \leq 0$ for all the nonstarred j ,

In the second case the minimum value of the variable is positive and the variable cannot be starred. In the first case the variable has zero value and this value does not change in any subsequent tableau obtained from any pivot transformation, whence the variable can be starred. This simplex type procedure is called "minvar" and is fully described later. So a variable "can be starred" if

(i) it is a nonbasic variable,

(ii) it is a basic variable, the minvar procedure is applied and at the end of this procedure either the variable is nonbasic or satisfies (P1).

A variable "cannot be starred" if it is a basic variable which satisfies (P2) at the end of the minvar procedure.

Therefore some of the nodes of the tree shown in (Dl) cannot be generated, which implies that the complete enumeration given by the tree is usually avoided.

The method uses two types of tableaus as stated in the following definition.

Definition 1. A tableau is Complementary if and only if either (a) or (b) holds:

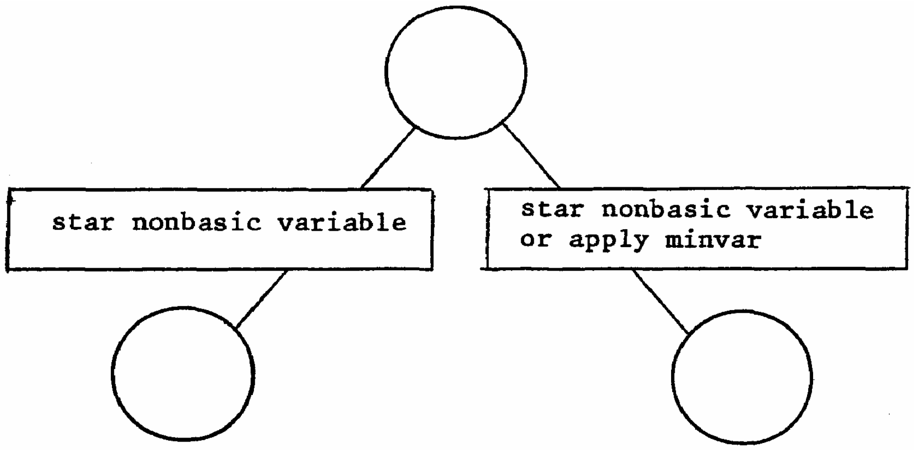
(a) - there is no i such that z_i and w_i are both basic

(b) - if z_i and w_i are both basic then one of the variables is starred and satisfies (P1).

A tableau is Noncomplementary if it is not complementary.

It follows from this definition that if there are n starred variables then the tableau must be complementary. In this case the tableau usually provides one complementary solution. However, if one or more starred variables satisfy (P1) then it might lead to an infinite number of solutions. In fact in this latter case there is at least one nonbasic nonstarred variable and this variable can take any non negative value which does not force any basic variable to become negative.

If a tableau is complementary then any pair of complementary variables such that neither of them are starred has at least one nonbasic variable. Therefore the branches for a complementary tableau are of the form set out in D2.



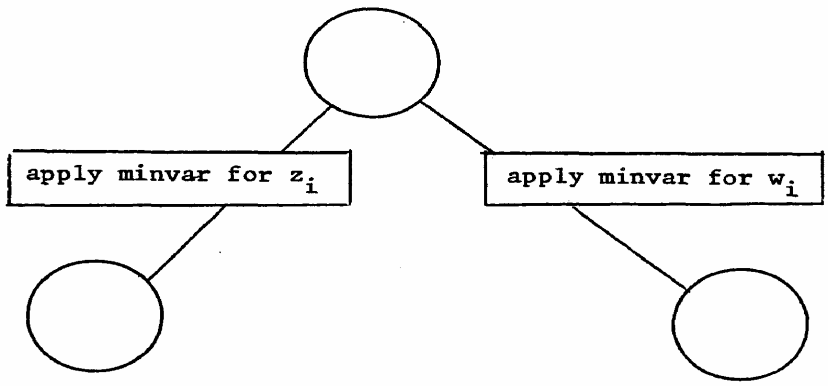
(D2)

Branching in Complementary Tableau.

Note that in the right hand branch minvar is usually applied.

When one or more starred variables satisfy (P1) both the variables of a complementary pair may be nonbasic. In this case minvar is not applied.

If the tableau is noncomplementary there is at least one pair of complementary nonstarred basic variables (z_i , w_i) and in order to achieve complementarity each variable is starred in one of the two branches. Hence minvar is applied to generate both the nodes and the branches are of the form set out in D3.



(D3)

Branching in Noncomplementary Tableau.

At any node of the tree if a basic variable, say z_i, satisfies (P2), that is, its minimum value is positive, its complementary, w_i, must be starred in order that $z_i w_i = 0$. If w_i cannot be starred, that is, the minimum value of w_i is also positive, there are no solutions of the LCP beyond this node and the search does not proceed further down the tree.

A number of preliminary concepts related to the enumerative method have been discussed and a complete description of the algorithm now follows. To make the description concise the symbol y_i is introduced to represent any basic z_i or w_i variable as is convenient and x_i represents any non basic variables, see Tableau T1.

STEP 0 - Initialization - Set NODE = 1 (the node under current investigation), NNODE = 1 (total number of nodes to be investigated), NSOL = 0 (Number of solutions of the LCP). Apply the Phase 1 of the simplex method. If no feasible solution exists go to EXIT.

STEP 1 - Check tableau state - See if the tableau is complementary. If it is set COMPL = TRUE. Otherwise set COMPL = FALSE.

STEP 2 - Analyse the current tableau to check for variables with positive minimum value - Let

$$R = \{y_i : y_i \text{ satisfies (P2) and its complementary variable is not starred}\} \qquad (4)$$

(a) If $R \neq \emptyset$, go to (b). If COMPL = TRUE go to Step 3. If COMPL = FALSE go to Step 4.

(b) If for any $y_i \in R$ its complementary variable cannot be starred go to Step 6. Otherwise star all the variables complementary to the variables in the set R. Now:

(i) if the number of starred variables is equal to n go to Step 5 (at least one solution is generated);

(ii) if no pivot transformation is performed go to Step 2(a);

(iii) if at least one pivot transformation is carried out go to Step1 (a new tableau is obtained and it is necessary to analyse its state in Step 1).

STEP 3 - Branching for complementary tableaus - Branch by starring a nonbasic variable in any column of the tableau in node (NNODE + 2) and designating its complementary to be starred in node (NNODE + 1). Store the tableau, set NNODE = NNODE + 2 and NODE = NNODE. If the number of starred variables is equal to n go to Step 5. Otherwise go to Step 2.

STEP 4 - Branching for noncomplementary tableaus – Let y_t and y_r be two basic variables which constitute a pair of complementary variables. Designate y_t and y_r to be starred in nodes (NNODE + 1) and (NNODE + 2), store the tableau, set NNODE= NNODE + 2 and NODE = NNODE and go to Step 7.

STEP 5 - Generation of a solution - Set NSOL = NSOL + 1. If all the nonbasic variables of the tableau are starred this tableau yields exactly one solution of the LCP given by ($y_i = \bar{b}_i$, $x_i = 0$). Otherwise a family of solutions to the LCP can be obtained from this tableau. Add the solution to the list and go to Step 6.

STEP 6 - Backtrack - If NODE = 1 go to EXIT. Otherwise set NODE = NODE –1. If node NODE has already been processed go to Step 6. Otherwise extract the last tableau stored and go to Step 7.

STEP 7 - Generation of the node - If the chosen variable to be starred cannot be starred go to Step 6. Otherwise star this variable. If the number of starred variables is equal to n go to Step 5. Otherwise go to Step 1.

EXIT - If NSOL = 0 the LCP has no solution. Otherwise all the solutions of the LCP have been enumerated and there are NSOL or an infinite number of solutions.

Note that if the basis matrices are used to implement the algorithm only the basic and nonbasic variables are stored in the branching steps.

If in Step 5 there is exactly one starred basic variable it is possible to write explicitly the family of solutions. Let y_r be this variable and let s by the column of its complementary nonbasic and nonstarred variable. Then $\bar{a}_{rs} = 0$ and the family of solutions is given by

$$x_s \in [0, \alpha_0] \ , \ y_r = 0, \ y_i \in [b_i, \ b_i + \beta_0] \ , \ i \neq r \tag{5}$$

where

$$\begin{cases} +\infty \text{ if } a_{is} \leq 0 \\ \\ \min\left\{ \frac{b_i}{\bar{a}_{is}} : \bar{a}_{is} > 0 \right\} \end{cases} \qquad \beta_0 = \begin{cases} 0 \text{ if } a_{is} = 0 \\ -\infty \text{ if } \alpha_0 = +\infty \text{ and } \bar{a}_{is} > 0 \\ +\infty \text{ if } \alpha_0 = +\infty \text{ and } \bar{a}_{is} < 0 \\ \bar{a}_{is}\,\alpha_0 \quad \text{if } \alpha_0 \text{ is a real number} \end{cases} \tag{6}$$

Obviously if $\alpha_0 = 0$ $(y = \bar{b}, x = 0)$ is the unique solution given by the tableau.

<u>Minvar Procedure</u>

STEP 0 – Set $k = 1$, $A^k = A$, $b^k = b$, where A and b are the entries of the given tableau. Let t be the row of the basic variable chosen for the minvar procedure.

STEP 1 – If $b_t^k = 0$, go to Step 3. If $b_t^k > 0$ and $a_{tj}^k \leq 0$ for all the nonstarred columns j , set TERM = 3 and go to EXIT. Otherwise let

$$s = \min\{\text{nonstarred } j \ : \ a_{tj}^k > 0\} \tag{7}$$

and go to Step 2

STEP 2 - determine the row r by

$$r = \min \left\{ i : \frac{b_i^k}{a_{is}^k} = \min \left\{ \frac{b_j^k}{a_{js}^k} : a_{js}^k > 0 \right\} \right\} \tag{8}$$

Perform a single pivot transformation with a_{rs}^k as the pivot. If r = t, set TERM = 1 and go to exit. Otherwise set k = k + 1 and go to Step 1.

STEP 3 - If $a_{tj}^k = 0$ for all the nonstarred columns j , set TERM = 2 and go to EXIT. If there is an $a_{tj}^k \neq 0$ perform a single pivot transformation with a_{tj}^k as the pivot, set TERM =1 and go to EXIT.

EXIT - If TERM = 1 the variable is nonbasic. If TERM = 2 the variable is basic and satisfies (P1). If TERM = 3 the variable is basic and satisfies (P2), that is, the minimum value of the variable is positive.

Note that Bland's double least index rule (7) and (8) [1] has been implemented in the algorithm to avoid the possibility of cycling.

3. AN EXTENSION OF THE BRANCHING STRATEGY FOR COMPLEMENTARY TABLEAUS

Consider a complementary tableau and suppose that there are n_1 ($n_1 < n$) starred variables. Then there are at least $\ell = n - n_1$ nonstarred nonbasic variables. As outlined in the algorithm no pivot transformation is performed until the number of starred variables is equal to n. These variables are starred either in the branch procedure of Step 3 or in Step 2 when $R \neq \phi$. The algorithm looks at the same tableau at most ℓ times and generates at most ℓ branches to achieve the node which yields the solution. It is however possible to reduce the number of branches and the number of times the tableau is inspected for variables belonging to the set R by an improved strategy for the choice of the variables for branching.

Two cases (A) and (B) are considered below.

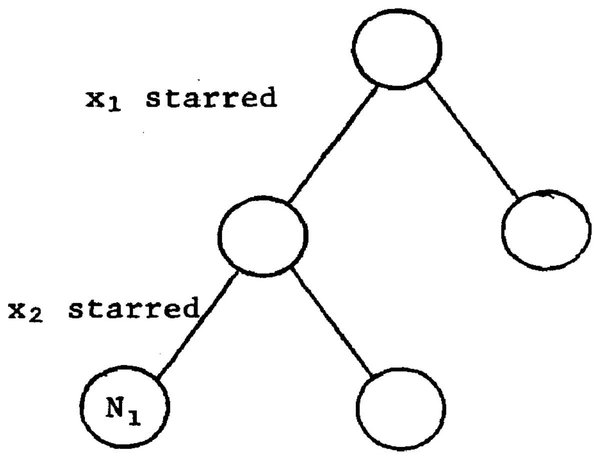
(A) There are exactly ℓ nonbasic nonstarred variables.

In this case all the variables already starred are nonbasic. Suppose that the tableau is of the form

	1	$-x_1$	$-x_2$	$-x_3$	...	$-x_j$	...	$-x_\ell$	$-x_{\ell+1}$	...	$-x_n$
$y_i =$	+	+	+	−	...	−	...	−			

(9)

where $x_{\ell+1},\ldots,x_n$ are the starred variables, x_j and y_i constitute a pair of complementary variables and + and – represent respectively a positive and a nonpositive entry. If the usual branching step is used the pair (x_j,y_i) can be chosen and $y_i \in R$ in the subsequent nodes with the same tableau, since x_j is a starred variable. However, if the first two branches are of the form



then in node N_1, $y_i \in R$ and x_j is starred. Note that the entry $\bar{a}_{ij}$ of the tableau must be nonpositive. Otherwise x_j has to be starred for y_i to satisfy (P2), whereby $y_i \notin R$.

Consider now the sets of variables

$S = \{y_i$: y_i has positive value and its complementary variable is not starred$\}$

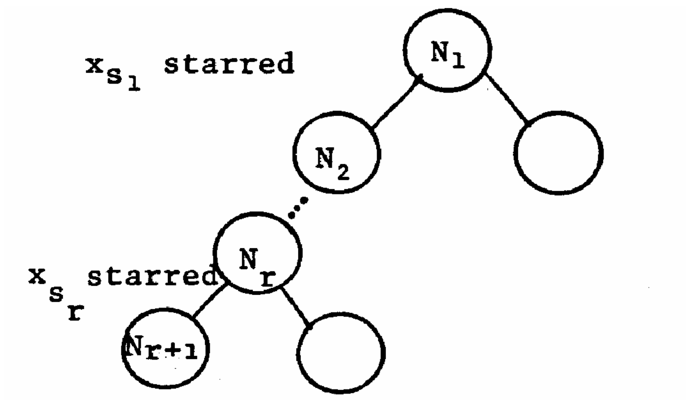
$T = \{ y_i \in S : \bar{a}_{ij} \leq 0$ where j is the column of the complementary variable of $y_i\}$

and suppose that $T \neq \emptyset$. For any $y_i \in T$ consider the set of variables indices

$$U_i = \{ j : j \text{ is a nonstarred column of the tableau and } \bar{a}_{ij} > 0\} \qquad (11)$$

If $U = \{s_r,\ldots,s_r\}$ is the set with the least elements among the sets U_i then it follows from the definitions of the sets U_i that

(i) r branches of the form



provides a variable $y_i \in R$ in node N_{r+1} (N_1 is the present node),

(ii) there are no variables $y_i \in R$ in the nodes $N_1,\ldots,N_r$, whence Step 2 does not have to be used before reaching node N_{r+1}.

On the other hand if T = Ø then no matter what the variables $x_1, \ldots, x_\ell$ are chosen for starring the set R is empty in all the nodes generated by starring these variables. Therefore in this case it is not necessary to use Step 2 until all these variables are starred, which means that Step 2 is used ℓ less times than in the original algorithm.

(B) The number of nonstarred nonbasic variables is greater than ℓ .

In this case there is at least one starred basic variable at zero level and there exists at least a nonbasic pair of complementary variables. Let r and t be the columns of such a pair, $y_i \in T$, $a_{ir} > 0$ and suppose that x_r is not starred. Hence $r \in U_i$. But if x_t is already starred or $\bar{a}_{it} > 0$ then either x_r and x_t are both starred or the definition of the set T should be modified. But two variables of the same pair cannot be starred in the same node since the method is based on the tree shown in (Dl). So the definition of the set T is modified to rule out this case as follows:

$$T = \{ y_i \in S : \text{property (P3) holds}\} \tag{12}$$

where

(P3) (i) $a_{ij} \leq 0$, where j is the column of the complementary variable of y_i,

(ii) if x_s is a nonbasic nonstarred variable such that $a_{is} > 0$ then its complementary is not starred and either it is basic or nonbasic with a nonnegative entry in the ith row.

Therefore Step 2 and Step 3 are modified in order to improve the efficiency of the enumerative method. The modifications in Step 2 are presented in the next section.

Before presenting the modified Step 3 a further extention is discussed. Suppose that the following property holds:

(P4) Number of starred variables is equal to (n-1) and there is exactly one nonbasic nonstarred variable.

In this case the tableau is of the form

$$
\begin{array}{ccc}
 & & \begin{array}{cccc} 1 & \text{star} & -x_s & \text{star} \end{array} \\
\begin{array}{c} \vdots \\ y_t \\ \vdots \end{array} & = & \begin{array}{|c|c|c|c|} \hline \bar{b}_t & \quad & \bar{a}_{ts} & \quad \\ \hline \end{array}
\end{array}
\tag{13}
$$

where y_t and x_s are complementary variables.

There are two cases as stated below.

(i) $\bar{b}_t = 0$ – the node (NNODE + 1) is generated by starring the variable y_t. Then either y_t satisfies (PI) or must become nonbasic. In the first case $\bar{a}_{ts} = 0$ and the tableau (13) represents a family of solutions of the form (5),(6). Further any possible solution obtained in node (NNODE + 2) is included in the family of solutions of node (NNODE + 1). Hence it is not necessary to branch but y_t must be starred. On the other hand if $\overline{a_{ts}} = 0$, y_t can be made nonbasic after a single pivot transformation with $\bar{a}_{ts}$ as the pivot. So the nodes (NNODE + 1) and (NNODE + 2) give the same solution, whereby it is not necessary to branch but instead x_s is starred.

(ii) $\bar{b}_t > 0$ - in this case $\bar{a}_{ts} > 0$, since otherwise y_t satisfies (P2) and x_s is starred. Let ℓ be defined by

$$\ell = \min \left\{ i : \frac{\bar{b}_i}{\bar{a}_{is}} = \min \left\{ \frac{\bar{b}_j}{\bar{a}_{js}} : \bar{a}_{js} > 0 \right\} \right\} \tag{14}$$

If $\ell = t$ or $\ell \neq t$ and $\gamma = \bar{b}_t - \dfrac{\bar{a}_{ts}\bar{b}_\ell}{\bar{a}_{\ell s}} = 0$ then two different solutions can be obtained in nodes (NNODE + 1) and (NNODE + 2) if the usual branching procedure is followed. However there are no more solutions in these nodes, and it is not necessary to branch but one simply adds these two solutions to the list of solutions already enumerated. Therefore in this case set NSOL = NSOL + 2 and add the solutions ($y = \bar{b}, x = 0$) and the one obtained from this solution by a pivot transformation with $\bar{a}_{\ell s}$ as the pivot. On the other hand if $\ell \neq t$ and $\gamma > O$ then y_t satisfies (P2) and x_s must be starred.

The modified Step 3 is now described where NU denotes the number of elements of the set U .

STEP 3 (a) If NU > O go to (c). If (P4) holds go to (b). Otherwise branch by starring a nonbasic variable in any column of the tableau in node (NNODE + 2) and designating its complementary to be starred in node (NNODE + 1). Store the tableau and set NNODE = NNODE + 2 and NODE = NNODE. If the number of starred variables is equal to n go to Step 5. Otherwise go to Step 3.

(b) Let s be the index of the nonstarred column and t be the index of the row of the basic variable complementary to $x_{s.}$

(i) If $b_t > O$ go to (ii). If $\bar{a}_{ts} = O$ star y_t. If $a_{ts} \neq O$ star x_S. Go to Step 5.

(ii) Determine the row ℓ by (14). If $\ell \neq t$ and $\bar{b}_t - \dfrac{\bar{a}_{ts}b_\ell}{\bar{a}_{\ell s}} > O$, star x_s and go to Step 5.

Otherwise set NSOL = NSOL + 2 and add the solutions ($y = \bar{b}$, $x = O$) and the basic solution obtained from the previous solution by a pivot transformation with $a_{\ell S}$ as the pivot to the list of solutions. Go to step 6.

(c) NU > O - Branch by starring the nonbasic variable in any column of the set U in node (NNODE + 2) and design-nating its complementary to be starred in node (NNODE + 1). Store the tableau and set NNODE = NNODE + 2, NNODE = NNODE and NU = NU - 1. If NU = 0 go to Step 2. Otherwise go to Step 3 (c).

4. AN EXTENSION OF THE BRANCHING STRATEGY FOR NONCOMPLEMENTARY TABLEAUS

If a tableau is concomplemtary there is at least a basic pair of complementary variables y_r and y_t taking positive values. The branching procedure specified in the algorithm designates y_r and y_t as the variables to star in the nodes (NNODE + 1) and (NNODE + 2) and then apply minvar to generate these nodes. Obviously the minvar procedure could be performed prior to the branching. This is not done since the value of y_t might increase when minvar is applied to star y_r increasing the number of pivot transformations necessary to generate (NNODE +1). In certain cases, however, it is possible to make nonbasic one of the variables and decrease or at least do not increase the value of the other. This occurs if

(i) one of the variables has zero value and the pivot row is the row of this variable,

(ii) both the variables y_r and y_t have positive values and the pivot column is a column s such that either $\bar{a}_{rs} > 0$ and $\bar{a}_{ts} > 0$ or $a_{rs} \geq 0$, $a_{ts} \geq 0$ and one of these entries is positive.

A modification of the minvar procedure, which can be called "joint minvar", can be designed following the rules (i) and (ii). At the end of this procedure one of the variables either satisfies (P1) or is nonbasic. However, in the two cases below the joint minvar procedure terminates unsuccessfully:

(iii) at least one of the variables, say y_r, satisfies (P2),

(iv) if $\bar{a}_{rs} > 0$ ($\bar{a}_{ts} > 0$) then $\bar{a}_{ts} < 0$ ($\bar{a}_{rs} < 0$) For any column s.

In case (iii) the complementary variable of y_r has to be starred for which minvar is applied. In case (iv) the Step 4 of the algorithm should be applied.

Suppose now that at the end of the joint minvar procedure one of the variables, say z_i, either satisfies (P1) or is nonbasic. Since the value of its complementary variable has been reduced it might satisfy (P2) in which case z_i is starred. If w_i does not satisfy (P2) it is advisable to see whether the current tableau is complementary before branching. If this tableau is complementary then the branching is different and Step 2 is applied without starring z_i. If the tableau is noncomplementary two branches are developed such that z_i is starred in node (NNODE + 2) and w_i is designated to star in node (NNODE + 1). After the branching Step 2 is applied since it is known that the current tableau is noncomplementary.

Before presenting the modified Step 4 it must be noted that Step 2 is applied as part of Step 4 for a pair of complementary variables. Furthermore Step 2 is quite time consuming since it is necessary to look at almost all the rows of the tableau in order to find variables which belong to the set R. Finally if n is large and the number of starred variables is small then usually $R = \emptyset$. Hence it is not advisable to apply Step 2 for noncomplementary tableaus. This implies that if the branching is performed then Step 4 follows instead of going back to Step 2. On the other hand

Step 2 should be used for complementary tableaus not only to look for variables with positive minimum value but also for the branching procedure of Step 3. The modified Steps 2 and 4 are set out below.

STEP 2 (only used with complementary tableaus).

(a) Let R be defined by (4). If $R = \emptyset$ go to (b). Otherwise star all the (nonbasic) variables complementary to the variables in the set R. If the number of starred variables is equal to n go to Step 5. Otherwise go to Step 2.

(b) Let NU = 0 (the number of elements of the set U) and S and T be given by (10) and (12) respectively. If $T = \emptyset$ go to Step 3. Otherwise consider for any $y_i \in T$ the set U_i given by (11) and let U be the set with least elements among the sets U_i. and NU be the number of its elements. Go to Step 3.

STEP 4 Branch for noncomplementary tableaus - Consider a basic pair of complementary variables $y_r = z_i$ and $y_t = w_i$.

(a) If both the variables y_r and y_t have positive value go to (b). Otherwise one of these variables either satisfies (PI) or a pivot transformation with any nonzero entry of its row as the pivot makes it nonbasic. Go to (d).

(b) Let

RCOL = the first nonstarred column whose entry in the r^{th} row is positive,

TOOL = the first nonstarred column whose entry in the t^{th} row is positive,

NNCOL = the first nonstarred column such that both entries in the r^{th} and the t^{th} rows are nonnegative and at least one of them is positive,

PSCOL = the first nonstarred column such that both the entries in the r^{th} and t^{th} columns are positive.

If PSCOL exists go to (i). Otherwise go to (ii).

(i) Let s = PSCOL and calculate the row ℓ which satisfies (14). If $\bar{b}_\ell > 0$ go to (c). Otherwise (degenerate case)

$$\left.\begin{array}{l} \text{if PSCOL} = \min \{\text{RCOL, TCOL}\} \text{ go to (c)} \\ \text{if PSCOL} > \text{NCCOL go to (iii)} \\ \text{if PSCOL} = \text{NNCOL go to (iv)} \end{array}\right\} \qquad (15)$$

(ii) If both RCOL and TCOL exist go to (iii). If RCOL and TCOL do not exist go to Step 6. If RCOL (TCOL) does not exist go to Step 7 with y_t (y_r) as the variable to be starred.

(iii) If NNCOL exists, set PSCOL = NNCOL and go to (i). Otherwise go to (iv).

(iv) Branch by designating the variables y_r and y_t to be starred in nodes (NNODE + 2) and (NNODE + 1) and store the current tableau. Set NNODE = NNODE + 2 and NODE + 2 and NODE = NNODE. Go to Step 7.

(c) Perform a single pivot transformation with $\bar{a}_{\ell s}$ as the pivot. If $\ell \neq r$ and $\ell \neq t$ go to (a). Otherwise one of the variables is nonbasic after the pivot transformation and go to (d).

(d) One of the variables of the pair $(z_i, w_i.)$, say $z_i.$, is nonbasic or satisfies (PI) — if its complementary variable w_i does not satisfy (P2) go to (e). Otherwise star $z_{i..}$ If the number of starred variables is equal to n go to Step 5. Otherwise go to Step 1.

(e) See if the tableau is complementary. If it is set COMPL = TRUE and go to Step 2. Otherwise set COMPL = FALSE and go to (f).

(f) Branch by starring z_i in node (NNODE + 2) and designating w_i to star in node (NNODE + 1). Store the tableau and set NNODE = NNODE +2, NODE = NNODE. Go to Step 4.

Note that the joint minvar procedure and its unsuccessful terminations are incorporated in the parts (a), (b) and (c). On the other hand (15) is used to follow Bland's double least index rule [1]. In fact if PSCOL = min {RCOL,TCOL} then PSCOL is the first positive column of the function sum of the two variables which decreases in the nondegenerate case in any iteration of joint minvar. Otherwise cycling might occur and either NNCOL < PSCOL and NNCOL might be the pivot column or it is better to terminate the joint minvar procedure.

5. EXTENSION OF THE ENUMERATIVE METHOD TO THE SLCP

The enumerative method for the LCP can be extended to the SLCP with minor modifications. For the SLCP the initial step looks for a feasible solution to the system $y = b + A(-x)$, where initially the y-variables are artificial or nonnegative and the x-variables are unrestricted or nonnegative. This can be done by the general Phase 1 of the simplex method (see [6, pages 109-112] for instance). In this procedure whenever an unrestricted (artificial) variable becomes basic (nonbasic) it is not allowed to become nonbasic (basic) again. The artificial variable and the respective column are then starred. Further the unrestricted variable and the respective row are usually called Flagged. The feasible solution sought is a basic solution which satisfies the following properties:

(i) any variable required to be nonnegative assumes nonnegative value.

(ii) any unrestricted (artificial) variable is either basic (nonbasic) or nonbasic in a column (basic in a row) with zero entries in all the nonflagged rows (nonstarred columns).

If no feasible solution can be found SLCP has no solutions. Otherwise let NNBV be the number of nonstarred nonnegative nonbasic variables at the end of Phase I. Then the three possible relations NNBV=n,, NNBV < n, NNBV > n are satisfied if and only if at the end of Phase I the number of basic artificial variables is respectively equal to, less than or greater than the number of nonbasic unrestricted variables. Since NNBV < n can occur there might exist a tableau in which all the nonbasic variables are starred but this number is smaller than n . In this case there exists at least a basic pair of complementary variables (z_i,w_i). If for any pair of basic complementary variables at least one of the variables of this pair has zero value then the tableau represents a solution of the SLCP. So the enumerative method has to be modified in order to incurporate this possibility. This modification can be done by generalizing the definition of complementary tableau and then modifying Step 1 of the enumerative method.

Definition 2. A tableau is complementary if (i) or (ii) holds:

(i) it is complementary in the sense of definition 1,

(ii) if all the nonbasic variables are starred and the number of starred variables is less than n , then any pair of basic complementary variables has at least one variable with zero value.

STEP 1 See if the tableau is complementary. Set COMPL = TRUE if the tableau is complementary and COMPL = FALSE if the tableau is noncomplementary. If all the nonbasic variables are starred go to Step 5 if COMPL = TRUE or to Step 6 if COMPL - FALSE. Otherwise go to Step 2 if COMPL - TRUE or to Step 4 if COMPL = FALSE.

Furthermore if an unrestricted variable is nonbasic at the and of Phase 1, then all the nonflagged entries in the column of this variable are zero for any subsequent tableau. Hence this unrestricted variable can assume any real value in any solution of the SLCP which is enumerated in Step 5.

6. A WORKED EXAMPLE ILLUSTRATING THE ENUMERATIVE METHOD

In this section an example taken from [8] is solved by the enumerative method with the modified steps. Consider the LCP given by the following tableau

	1	$-z_1$	$-z_2$	$-z_3$	$-z_4$
$w_1 =$	2	-2	1	3	-4
$w_2 =$	-4	-10	-1	1	-1
$w_3 =$	3	-1	2	-1	2
$w_4 =$	-6	-20	-3	1	3

The Phase I of the simplex method requires one pivot transformation to get an initial feasible solution represented by the tableau:

	1	$-z_1$	$-z_2$	$-z_3$	$-z_4$
$w_1 =$	2.8	-0.2	1.2	2.8	-3.8
$w_2 =$	0.4	-0.1	0.1	-0.1	0.1
$w_3 =$	2.6	0.1	1.9	-0.9	1.9
$w_4 =$	2.0	-2.0	1.0	-1.0	5.0

This tableau is noncomplementary (r = 1, t = 2). So step 4 is applied and RCOL = TCOL = PSCOL = 2 and s = 2. Therefore a joint minvar step is used and the values of W_1 and z_1 decrease, since $\ell = 3$ and $\bar{b}_3 \neq 0$. The pivot transformation with $\bar{a}_{32}$ as the pivot leads to the following tableau

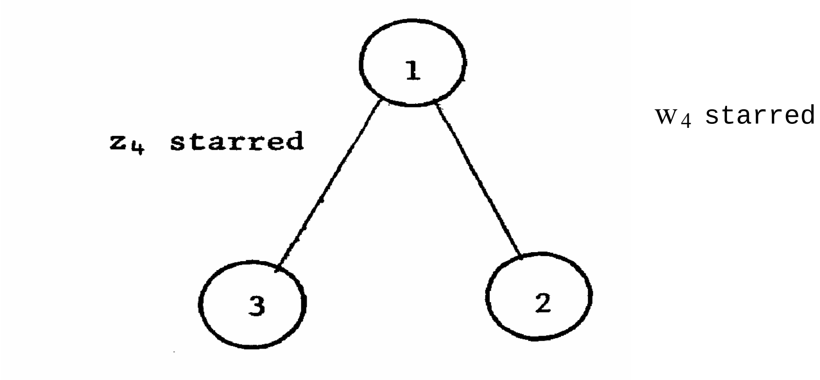
	1	$-w_2$	$-w_3$	$-z_3$	$-z_4$
$w_1 =$	1.15	-0.26	-0.63	3.36	-5
$-z_1 =$	0.26	-0.1	-0.05	-0.05	0
$z_2 =$	1.37	0.05	0.52	-0.47	1
$w_4 =$	3.36	-1.0	0.52	1.47	6

Now RCOL = 3 and TCOL = 0. So w_1 should be starred (Step 4(ii)) using the minvar procedure. This procedure chooses $\bar{a}_{13}$ as the pivot and makes w_1 nonbasic in one iteration yielding the tableau

	1	$-w_2$	$-w_3$	$-w_1$	$-z_4$
$-z_3 =$	0.39	-0.07	-0.18	-0.29	-1.48
$-z_1 =$	0.28	-0.1	-0.06	-0.01	-0.07
$z_2 =$	1.53	0.01	0.43	0.14	0.29
$w_4 =$	3.87	-2.06	0.25	0.43	3.81

(16)

This is a complementary tableau. On the other hand z_3 satisfies (P2) whereby the nonbasic variable w_3 is starred. Step 2 is applied again and $S = \{z_2, w_4\}$, $R = \emptyset$ and $T = \emptyset$. Hence NU = 0 and Step 3 creates the two following branches



Further more tableau (16) is stored for generating node 2. After this NODE takes the value 3. But in node 3 the number of starred variables is equal to 3 = n - 1 and there is one nonbasic nonstarred variable. So property (P4) holds and s = 1, t = 3, $\bar{b}_t = 1.53 > 0$. Furthermore by (14) $\ell = t = 3$. Hence node 3 provides two solutions which are

Solution 1 – $z_1 = 0.28$, $z_2 = 1.53$, $z_3 = 0.39$, $w_4 = 3.87$,
$w_1 = w_2 = w_3 = z_4 = 0$.

Solution 2 – (obtained from solution 1 by a pivot transformation with a_{31} as the pivot) – $z_1 = 10.99$, $w_2 = 97.99$,

$Z_3 = 7.99$, $w_4 = 205.99$, $w_1 = z_2 = w_3 = z_4 = 0$.

Step 6 (Backtrack) follows and node 2 is generated by starring the variable w_4. To do this, tableau (16) is considered and minvar is applied. One iteration of minvar is sufficient to make w_4 nonbasic and the following complementary tableau is obtained

	1	$-w_2$	$-w_3$	$-w_1$	$-z_4$
$-z_3 =$	1.85	-0.88			0.38
$-z_1 =$	0.36	-0.15			0.02
$z_2 =$	1.22	0.17			-0.07
$z_4 =$	1.01	-0.54			0.26

Note that it is not necessary to update the second and the third columns since the variables w_3 and w_1 are starred. At this stage the number of starred variables is equal to 3 and there is exactly one nonstarred nonbasic variable. Hence (P4) holds and s = l, t = 3, $\bar{b}_t$, = 1.22 > 0, ℓ = t = 3. So node 2 provides two more solutions which are

Solution 3 – $z_1 = 0.36$, $z_2 = 1.22$, $z_3 = 1.85$, $z_4 = 1.01$,
$w_1 = w_2 = w_3 = w_4 = 0$.

Solution 4 – $z_1 = 1.41$, $w_2 = 6.97$, $z_3 = 7.99$, $z_4 = 4.79$,
$w_1 = z_2 = w_3 = w_4 = 0$.

After this NODE = 1 and the algorithm stops. Therefore the LCP has four solutions. Note that only 3 nodes (the complete enumeration consists of 31 nodes) and 6 pivot transformations (2 of them only update the right-hand side coefficients) have been performed.

7 COMPUTATIONAL EXPERIENCE AND DISCUSSION OF RESULTS

In this section a set of test LCPs and SLCPs either constructed or taken from a known source are presented in Table 1 together with the results of the application of the enumerative method to these problems.

In Table 1 the problem number is presented in the first column. The source of the respective problem is given in the second column by a parameter which is explained at the end of the table. The first fourteen SLCPs and LCPs are the Kuhn-Tucker conditions of different nonconvex quadratic programming problems. The dimensions of these different quadratic programs are set out in the third column under the headings which are explained below.

V - number of nonnegative variables,
U - number of unrestricted variables,
I - number of inequality constraints,
E - number of equality constraints.

The order of the matrix of the SLCP or LCP is presented in the fourth column. Finally the results (number of solutions, iterations and nodes) of the application of the enumerative method to the test problems are presented in the last three columns.

Problem	Source	Dimensions V	U	I	E	Matrix Order	Solutions	Itera-tions	Nodes
P1	S1	4	0	3	0	7	3	23	11
P2	S2	4	0	3	0	7	3	17	15
P3	S3	4	0	3	0	7	11	43	25
P4	S4	9	2	6	0	17	15	663	239
P5	S4	6	0	12	2	20	1	204	79
P6	S4	10	2	9	0	21	4	610	163
P7	S4	9	0	15	0	24	3	2090	479
P8	S4	10	0	15	0	25	3	3534	785
P9	S4	16	0	10	1	27	3	4548	971
P10	S5	8	2	7	1	18	7	49	15
P11	S5	8	1	12	2	23	3	314	117
P12	S5	9	0	12	3	24	37	720	335
P13	S5	11	2	9	3	25	376	6509	2603
P14	S5	12	0	8	0	20	589	6761	2835
P15	S6					20	7	293	69
P16	S6					25	62	482	169
P17	S6					30	29	513	117
P18	S6					33	44	429	101
P19	S6					35	66	1568	503
P20	S6					37	40	1407	407
P21	S6					40	117	2282	711
P22	S6					42	29	1904	465
P23	S6					45	50	2479	735
P24	S6					48	33	1686	605
P25	S6					50	27	1640	521

TABLE 1 - Problems and results

List of Sources for the test problems:

SI, S2. S3 - Hansen and Mathiesen problems 1.1, 1.4 and 1.5 [5]

S4 - Hansen and Matheiesen random problems [5]

S5 - Let the quadratic program be of the form

$$\text{Minimize} \quad p^T S + \frac{1}{2} z^T Dz,$$

$$\text{subject to } Az \begin{pmatrix} \geq \\ = \end{pmatrix} b, \; z \in R^{\ell}, \; b \in R^{m}$$

where the variables may be nonnegative or unrestricted. Then these test problems are obtained by using coefficient values

$$p_i = -1 \quad , \qquad d_{ij} = \begin{cases} -2 \text{ if } i = i \\ \;\; 1 \text{ if } |i - j| = 1 \;, \; i, j = 1,\ldots, \;\; \ell \\ \;\; O \text{ otherwise} \end{cases}$$

$$b_i = -1 \quad , \qquad a_{ij} = -\alpha_0 \quad , \; i = 1,\ldots,m, \quad j = 1,\ldots \ell$$

where α_0 is a random number drawn from a uniform distribution in the interval [0,1]

S6 - These are LCPs whose matrix $M \in R^{nxn}$ and vector $q \in R^n$ are given by

$$q_i = 5 \quad , \; m_{ij} = \begin{cases} i + j - 2 \;, & \text{if } i + j - 2 < n_{i,j=1,\ldots n} \\ i + j - 2 - n \;, & \text{otherwise} \end{cases}$$

where α_0 is a random number drawn from a uniform distribution in the interval [0,1].

Brief discussion of the results

As it was referred in section 1 two other enumerative methods to the LCP have been reported in the literature. However, no computational experience with these methods is available. On the other hand the results indicate that this enumerative method is clearly superior to Hansen and Mathiesen approach [5] to find Kuhn-Tucker points. It seems that the enumerative method performs well for the quadratic programs with small number of Kuhn-Tucker points.

If the matrix of an LCP is nonnegative then for any i such That $q_i > 0$ the variable z_i. can be starred before starting the algorithm. Because of this property only nonpositive vectors q are considered in the LCPs of source 6. Note that the matrices of these LCPs are not L-matrices [3], whence Lemke's. method [7] cannot be applied to these problems. The results presented in Table 1 for these LCPs are quite encouraging and also show that the enumerative method does not perform poorly with an increase of the dimension of the LCP.

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