TR/03/94 March 1994

ZERO-ONE IP PROBLEMS: POLYHEDRAL DESCRIPTIONS AND CUTTING PLANE PROCEDURES

Fatimah Abdul-Hamid
Gautam Mitra
Leslie-Ann Yarrow

Contents

0 - Abstract

A systematic way for tightening an IP formulation is by employing classes of linear inequalities that define facets of the convex hull of the feasible integer points of the respective problems. Describing as well as identifying these inequalities will help in the efficiency of the LP-based cutting plane methods. In this report, we review classes of inequalities that partially described zero-one poly topes such as the 0-1 knapsack polytope, the set packing polytope and the travelling salesman polytope. Facets or valid inequalities derived from the 0-1 knapsack and the set packing polytopes are algorithmically identified

1 – Introduction

In the first report on cutting plane methods for integer programming [Abdul-Hamid etal. (1993)], techniques for generating all valid inequalities for general integer programming (IP) and mixed integer programming (MIP) were presented. For these techniques to perform more efficiently, we concluded that it is necessary to use cuts that are strong in the sense that they define facets or even supports of the convex hull of a set of integral points. These cuts are derived by studying the facial structure of the polytope related to the problem. Applying the results of the underlying polyhedral theory to actual solving, leads to new cutting planes, known as *polyhedral cutting planes*, that are different from the classical cutting planes discussed in the previous report. While these new cuts are also valid inequalities, the facet-defining inequalities are needed for the minimal description of the polytope of the IP problem. By contrast, traditional cutting planes do not generally have this property and are not even guaranteed to intersect the convex hull of integer solutions.

Before we discuss about facet-defining inequalities of a polytope and related computational procedures to identify these inequalities we need to introduce a terminology that is frequently used in combinatorial optimization. In computational complexity theory [see, e.g., Garey and Johnson (1979)], an *instance* of a problem is a single occurance of such a problem and is specified by providing a certain *input*. The *size of an instance* is the number of characters, or binary bits required to represent the instance.

In discussing the complexity of a problem, a decision problem is sometimes more convenient than an optimization problem. A *decision problem* is the one that can be answered with a 'yes' or 'no'. For example, consider the minimum vertex cover problem. The optimization problem for this particular problem can be stated as follows. Let $G(V, E)$ be an undirected graph and let $V' \subseteq V$ be a vertex cover if any edge $e = (v_i, v_j)$ in G has the property either $v_i \in V'$ or $v_j \in V'$. Find a vertex cover with the mimimum cardinality. On the other hand, the decision problem of the minimum vertex cover is as follows. Given G and a positive integer k, decide whether G has a vertex cover whose cardinality is not larger than k. The class **P** is the set of all decision problems which can be solved polynomially.

That is, for each problem $P \in \mathbf{P}$, there must exist an algorithm and a polynomial $p(\ell)$ such that an instance of P whose encoding is of length t can be solved by the algorithm in at most $p(\ell)$ elementary steps.

The most important class of problems is in the class **NP**. These are problems for which a 'yes' answer can be verified in a polynomial amount of time, provided that some extra information called *certificate* is given. For each instance the length of this certificate must be polynomially bounded in the length of the corresponding input. Consider the question of determining whether a graph G is Hamiltonian. The input is some encoding of G. To the best of our knowledge it is known that no algorithm will solve this problem in a polynomial number elementary steps. But, the problem is in **NP** and a certificate consists of a list of the edges belonging to a Hamiltonian cycle. Given this information, it can be verified that a graph is Hamiltonian.

A problem is **NP**-*complete* if it is in **NP**, and showing that it is in **P** would imply **P** = **NP**. More specifically, a problem is **NP**-*complete* if a polynomial bounded algorithm for solving it could be used once as a subroutine to obtain a polynomially bounded algorithm for every problem in **NP**. A problem is **NP**-*hard* if a polynomially bounded algorithm for it would result in a polynomially bounded algorithm for every problem in **NP**. Examples of **NP**-*hard* optimization problems include the travelling salesman problem and knapsack problem.

The idea of using facial structure to determine strong valid inequalities was first introduced by Dantzig, Fulkerson and Johnson (1959) . Since then, strong valid inequalities have been obtained for a variety of specially structured problems, such as the node-packing polytope, the zero-one (0-1) knapsack polytope and the symmetric travelling salesman polytope. Most of the known results are for problems having zero-one variables only. However, the use of structure to obtain a polyhedral representation of the constraint set is limited by the inherent complexity of the problem. For **NP**-*hard* IP problems, complete descriptions of convex hulls of feasible solutions by way of linear inequalities are often not known. However, several experimental studies based on polyhedral theory indicate that partial description of the convex hull of integer solution can be of considerable practical help for the solution of an IP problem [see, e.g., Padberg (1979)].

This report is divided into three main sections. Section one is the introduction followed by the definitions of classes of strong valid inequalities. Section two involves the studies of the polyhedral structure of problems such as the zero-one knapsack, the set packing and the symmetric travelling salesman. This includes, derivation of valid inequalities (and facets) and discussion on how they can define high dimensional facets of the polyhedron. Section three consists of procedures to solve the constraint identification problem, that is, algorithmically identifying violated facet-defining inequalities. Given a fractional solution to the LP-relaxation of an integer program, algorithms for identifying facet-defining inequalities *violated* by the solution will be considered. The summary or the conclusion can be found at the end of this report.

1.1 - Classes of valid inequalities

Consider the convex polytope in $\mathbf{R}^n$ defined by

$$P = \{x \in \mathbf{R}^n \mid Ax \leq b), \tag{1}$$

where A and b are mxn and $mx1$ matrices respectively, with arbitrary rational coefficients. Let

$$P_I = conv\{x \in P : x_j \in Z_+^n\}, \tag{2}$$

(where Z_+^n is the set of nonnegative integers) denote the convex hull of the integer points of P (that is, the smallest convex space that contains all feasible integer solutions to **IP**).

An inequality

$$\pi x \leq \pi_0 \tag{3}$$

[or (π, π_0)] is called a *valid inequality* for P_I if it is satisfied by all points in P_I. If (π, π_0) is a valid inequality for P_I and

$$F = \{x \in P_I \mid \pi X = \pi_0\}, \tag{4}$$

F is called 2. *face* of P_I, A face F is said to be *proper* if $F \neq$ and $F \neq P_I$. F is nonempty if and only if $\max\{\pi x \mid x \in P_I\} = \pi_0$. When F is nonempty we say that the inequality $\pi x \leq \pi_0$ *supports* $P_{I;}$ for an example, an inequality is supporting if it is valid and satisfied as an inequality by at least one $x \in P_I$. A face F of P_I is a. *facet* of P_I if $\dim(F) = \dim(P_I) - 1$. Specifically, if $\dim(P_I) = d$, there exist exactly d affinely independent vertices x^i of P_I satisfying $\pi x^i = \pi_0$, i=1,... ,d. Faces of dimension zero are called vertices of the polyhedron and faces of highest dimension are termed facets.

Example 1.1:

Let P={(0,0),(0,1),(1,0)}.

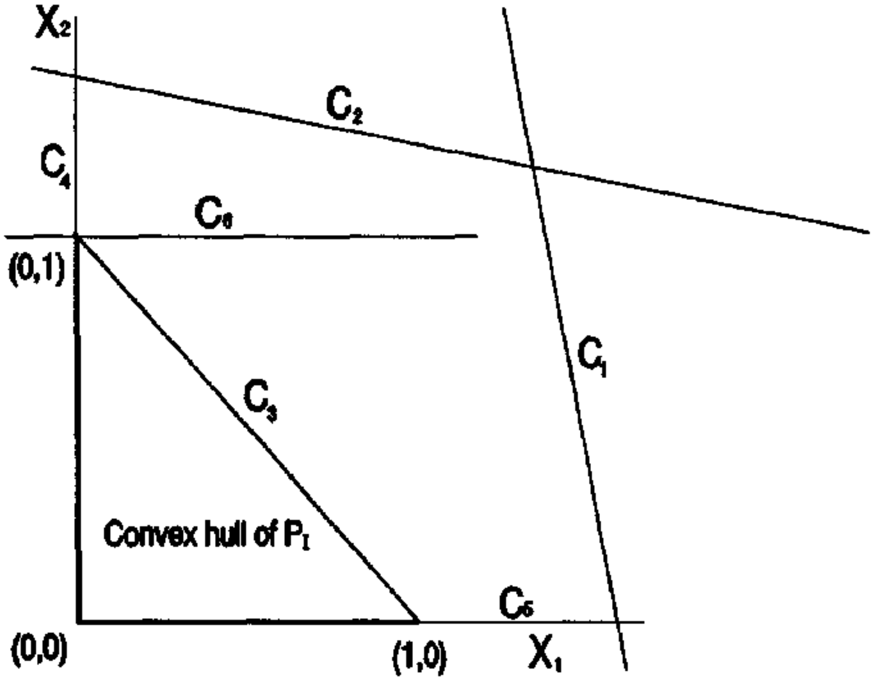


Figure 1

In this example, P_I is fully dimensional (dim (P_I) = dim (R^2) = 2); C_1 and C_2 are valid inequalities; C_3, C_4 and C_5 are valid inequalities, define supports and they define facets; C_6 is a valid inequality that supports but is not a facet-defining for P. ■

Example 1.2:

Consider $\mathbf{R}^3$ polytope.

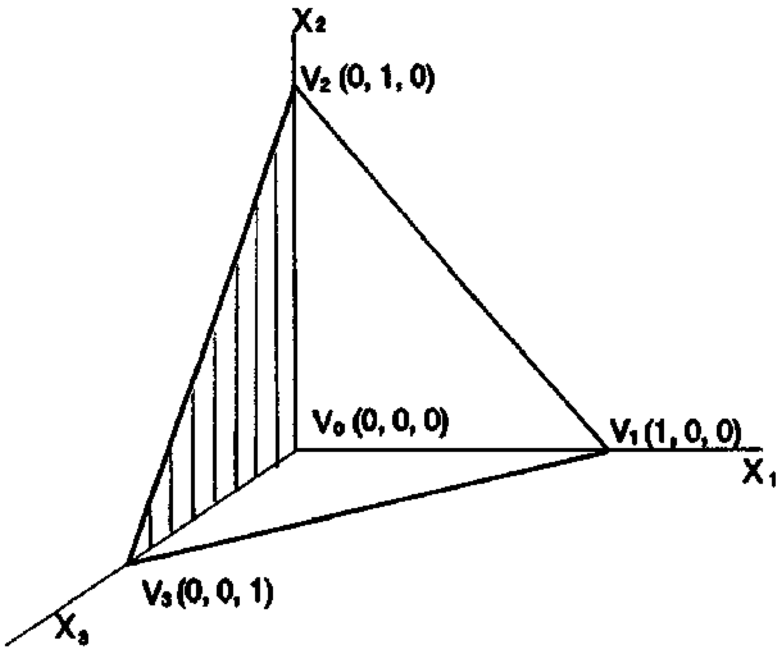


Figure 2

In this example, faces of dimension zero are the vertices V_0, V_1, V_2 and V_3, and the plane $\{V_1, V_2, V_3\}$ is an example of a facet (that is, a face of highest dimension). Any line joining two vertices is a face. ■

The inequalities $x_j \geq 0$ are *trivial* facets of P_I provided that $0 \leq a_{ij} \leq b_i$ for all $j \in N = \{1,..., n\}$ and for all i. For any nontrivial facet of P_I, where again $P = \{x \in R, Ax \leq b\}$ and A is *mxn* matrix, we have $x_j \geq 0$, $j = 1,\ldots,n$ and $x_0 > 0$.

Given two valid inequalities $\pi x \leq \pi_0$ and $\gamma x \leq \gamma_0$ that are not scalar multiples of each other, we say that $\pi x \leq \pi_0$ is *stronger or dominates* $\gamma x \leq \gamma_0$ if $\pi \geq \gamma$, $\pi_0 \geq \gamma_0$ and at least one of the inequalities is strict.

2 - Polyhedral descriptions of the zero-one polytopes

A systematic way to obtain a tighter formulation of an IP problem is to study classes of linear inequalities that define facets of the convex hull of the feasible integer points of the respective problem. According to a result on convex polyhedra due to Weyl (1935), there of feasible integer points P_I which can be described by a system of linear inequalities, that is,

$$P_I = \{x \in P \mid 1x \leq 1_0 \ \forall \ (1, 1_0) \in \mathcal{L} \} \qquad (5)$$

where $\mathcal{L}$ is a finite family of linear inequalities. Moreover, $\mathcal{L}$ (a minimal system of inequalities that describes P_I completely) can be chosen such that each inequality of $\mathcal{L}$ induces a different facet of P_I.

In this section, facet-defining inequalities for special structured zero-one polytopes such as the zero-one knapsack, the set packing and the symmetric travelling salesman problem are discussed.

2.1 - Facets for the zero-one knapsack polytope

The facial structure of the knapsack polytope has been studied simutaneously by Balas (1975), Padberg (1975), Hammer, Johnson and Peled (1975) and Wolsey (1975). However, a complete list of the linear inequalties that define the knapsack polytope still remains unknown.

Zero-one problems and knapsack problems

Consider the zero-one programming problem:

$$Maximize \ \sum_{j-1}^{n} c_j x_j$$

$$subject\ to \ \sum_{j-1}^{n} a_{ij} x_j \leq b_r \qquad i = 1, 2,\ldots, m, \qquad (6)$$

$$x_j = 0 \text{ or } 1 \qquad j = 1, 2,..., n.$$

In matrix form, this can be written as

$$\max\{\ \mathrm{c}x : Ax \leq b, x \in \{0, 1\}^n\} \qquad (7)$$

where A is *mxn* matrix with arbitrary rational entries, and b and c are vectors of length *m* and *n* respectively, with rational entries. The zero-one problem with a single linear constraint (where m = 1) is called the *knapsack problem.* Let (a^i, b_i) denote the ith constraint of zero-one problem (6) and let

$$P_I^i = conv\{x \in \mathbf{R}^n \mid a^i x \leq b_{i,}\ x_j = 0 \ or\ 1$$

$$for\, j = 1, 2,..., n) \qquad (8)$$

denote the convex hull of the zero-one solutions to the single inequality $dx \leq b_i$ where $i \in$ {1, 2,..., m}. That is, P_I^i is the knapsack poly tope associated with constraint i of problem (6). Likewise, again let

$$P_I = conv\{x \in \mathbf{R}^n \mid Ax \le b, x_j = 0 \ or \ 1 \ for \ j = 1,\ldots, n) \tag{9}$$

denote the convex hull of zero-one solutions to the entire constraint set of problem (6). If P_I is the zero-one polytope associated with problem (6), then we have

$$P_I \subseteq \bigcap_{i=1}^{m} P_I^i \tag{10}$$

In other words, P_I is equal to or contained in the intersection of all the knapsack polytopes P_I^i, i = 1,..., m. Thus, all inequalities that are valid with respect to P_I^i are also valid for P_I.

If the problem is a large-scale zero-one programming problem with a sparse matrix *A* and with no apparent special structure, it is reasonable to expect that intersection of the *m* knapsack polytopes provides a fairly good approximation to the zero-one polytopes. This is the working hypothesis used by Crowder et al. (1983) and was strongly supported by their computational study to be a reasonable assumption. With this assumption, we can concentrate on the individual rows of the constraint set of problem (6) when deriving valid inequalities for the polytope P_I.

Valid inequalities for the 0-1 knapsack polytope

By complementing variables, an individual constraint of a zero-one problem can be expressed as a zero-one knapsack problem. Specifically, the ith constraint can be restated as

$$\sum_{j \in N} | a_{ij} | x_j^i \le b_i - \sum_{j \in N} a_{ij}, x^i \in \{0,1\} \tag{11}$$

where $x_j^i = x_j$ if $j \in N_+$ and $x_j^i = 1 - x_j$ for $j \in N$. (N_+ denotes the index set of coefficients

a_{ij} with positive value and $N\_$ denotes the index set of coefficients a_{ij} with negative value). This transformation enables one to use valid inequalities or facet-defining inequalities for the zero-one knapsack problem as valid inequalities or facet-defining inequalities for the general zero-one IP problems.

Consider a zero-one knapsack problem,

$$\sum_{j \in N} a_j x_j \le a_0 \tag{12}$$

where $0 < a_j \le a_0$ are positive integers and $x_j = 0$ or $1, j \in N = \{1, 2, ..., n\}$. Let the coefficient be ordered monotonically so that $a_1 \ge a_2 \ge ... \ge a_n$. It is known that the knapsack polytope P_I^i is fully dimensional polytope since $a_j \le a_0$.

Two classes of inequalities for knapsack polytopes that can be used to characterize facets of P_I^i are being considered. One of these classes is known as the *minimal* cover *inequalities* and was introduced in 1975 [see e.g., Balas (1975)].

Let $S \subseteq N$ such that

$$\sum_{j \in S} a_j > a_0 \quad and \quad \sum_{j \in S} a_j - a_k \le a_0 \text{ for all } k \in S \tag{13}$$

hold. Then the set S is called a *minimal cover* with respect to (12); it has been shown [see e.g., Balas and Jeroslow (1972)] that every zero-one solution to (12) satisfies the inequality

$$\sum_{j \in S} x_j \le |S| - 1 \tag{14}$$

where $|S|$ denote the cardinality of the set S. The inequalities (14) define facets of the associated knapsack polytope whenever $S = N$ holds. For any subset $H \subseteq N$, let P_H be the convex hull of zero-one solutions with respect to

$$\sum_{j \in H} a_i x_j \leq a_0 \tag{15}$$

It is known [see, Balas (1975), Padberg (1975) and Wolsey (1975)] that if S is a minimal cover for (12), then the inequality (14) defines a facet of the polytope Ps. When S = N, then P_S is exactly the original knapsack polytope. If, however, $S \subset N$, then a procedure for lifting inequalities in Section 2.4 is needed to generate a facet for the original polytope.

The next class of inequalities for the knapsack polytope are due to Padberg (1979, 1980). This class of inequalities was shown to define facets for the knapsack polytope with zero-one vertices only. In addition to the minimal cover inequalities, Crowder et al. (1983) used inequalities in this class to optimality solve a number of large-scale pure zero-one problems. Suppose that any set $S^* \subseteq N$ and any index $t \in (N \setminus S^*)$ satisfying

$$\begin{gathered} \sum_{j \in S^*} a_j \leq a_0 \ , \\ \textit{and } Q \cup \{\mathrm{t}\} \textit{ is a minimal cover} \\ \textit{for every } Q \subset S^* \textit{ with } |Q| = k, \end{gathered} \tag{16}$$

where k is any integer satisfying $2 \leq \mathrm{k} \leq |\mathrm{S}^+|$. Due to the one-element role of the index t and since k is some integer number, the set $S^* \cup \{t\}$ is called a *(1, k)~configuration* with respect to (12). It was proven that every zero-one solution to (12) satisfies the inequalities that are associated with a (1, k)-configuration given by:

$$(r-k+1)x_i + \sum_{j \in T(r)} x_j \leq r_i \ , \tag{17}$$

where $T(r) \subseteq S^*$ is any subset of cardinality r of S^* and r is any integer satisfying $k < r <$ $| S^* |$. Minimal cover inequalities are obtained when $r = k$. If $k = | S^* |$ holds in (16) then a (1, k)-configuration is a minimal cover. In general, the class of inequalities associated with (1, k)-configurations properly contains the class of inequalities associated with minimal covers. The inequalities (17) define facets of the associated knapsack polytope whenever $N = S^* \cup \{t\}$.

So far, to the best of our knowledge, the minimal cover inequalities and (1, k)-configuration inequalities are the only configurations that describe the knapsack polytope and procedures for finding these two classes of inequalities are discussed in Section 3.2.

2.2 - Facets for the set packing polytope

The set packing problem is a specially structured zero-one IP, The facets of this polytope are related to certain subgraphs of an associated graph. Several classes of facet-defining inequalities for the set packing polytope have been identified (see, Padberg (1973), Nemhauser and Trotter (1974) and Trotter (1974)). Derivation of some of these inequalities will now be discussed.

Set partitioning, set packing and set covering problems

Consider the (weighted) set-partitioning problem (SPP):

$$\begin{aligned} &\textit{minimise} && \sum_{j=1}^{n} c_j x_j && (18)\\ &\textit{subject to} && \mathrm{A}x = e_m \\ & && x_j \in \{0, 1\} \end{aligned}$$

where $A \in \mathrm{Z}^{mxn}$ of zeros and ones, and e_m is the vector having m unit entries.

The SPP problem has two close relatives, the set packing problem (SP),

$$\begin{aligned} &\textit{maximise} && \sum_{j=1}^{n} c_j x_j \\ &\textit{subject to} && Ax \le e_m && (19)\\ & && x_j \in \{0, 1\} \end{aligned}$$

and the set covering problem (SC),

$$\begin{aligned} &\textit{minimise} && \sum_{j=1}^{n} c_j x_j \\ &\textit{subject to} && Ax \ge e_m && (20)\\ & && x_j \in \{0, 1\} \end{aligned}$$

where A and e_m are defined as in SPP.

The SP problem, like the SPP problem is a "tightly constrained" problem (that is, each constraint requires at most one, or exactly one, of many variables to be one), whereas the SC problem is a "loosely constrained" problem (that is, each constraint requires at least one of the many variables to be one). Any SPP can be reduced to a SP problem by a suitable change in the objective function [see e.g., Darby-Dowman and Mitra (1985)].

To study the facial structure of the SP polytope, one associates with the zero-one matrix A, the finite undirected *intersection graph* $G = (V, E)$ defined as follows:

> *G has a node for every column of A, and an edge for every pair of nonorthogonal columns of A, that is, (i, j) $\in$ E if and only if dd ≥ 1 (where a^1 is the ith column of A).*

Let A_G be the edge-node incidence matrix of G, and let the (weighted) node-packing problem (NPP) whose weight C_j are the same for each node of SP be

$$\begin{aligned} &\textit{maximise} \quad \sum_{j=1}^{n} c_j x_j \\ &\textit{subject to} \quad A_G x \leq e_q \\ &\qquad\qquad\quad x_j \in \{0,1\} \end{aligned} \tag{21}$$

where e_q is the vector q ones corresponding to the edges of G. It can be verified that each feasible solution to (21) (i.e., every *independent* (*stable*) node set in G) is a feasible solution to (19) and vice versa. Moreover, for every optimal solution to (21), there exists a corresponding feasible integer solution that is optimal for (19). Thus, any SP problem is equivalent to an NPP on a finite undirected graph. Therefore, one way of solving SP problems is to solve the associated NPP.

<u>Example 2.1</u>:

Let A be the A-matrix of a SP problem given by

$$\boldsymbol{A} = \begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{vmatrix}$$

The intersection graph G constructed from A is as follows:

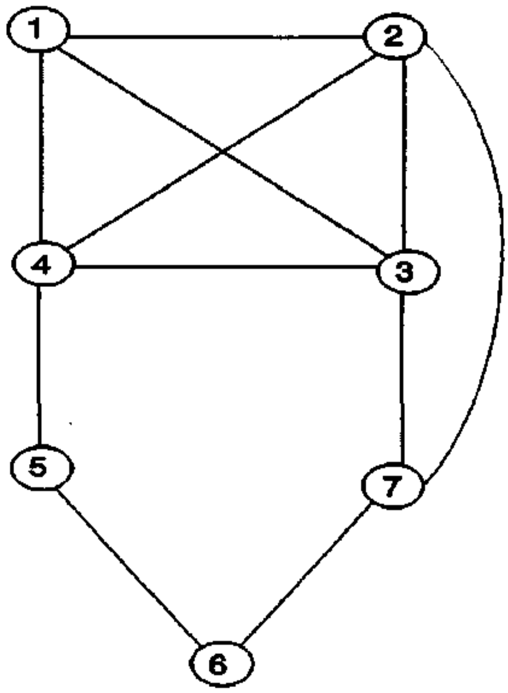


Figure 3

and the corresponding A_G is

$$\boldsymbol{A_G} = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

where A_G has exactly two ones in each row.

Valid inequalities for the set-packing polytope

Assume that A has no zero rows or zero columns. Denote by P, the polyhedron given by the feasible solutions of the LP-relaxation associated with SP, that is,

$$P = \{x \in \mathbf{R}^n \mid Ax \leq e, 0 \leq x \leq 1\} \tag{22}$$

where A is the coefficient matrix of SP, and let P_I be the associated convex hull of zero-one solutions satisfying the constraints of SP:

$$P_I = conv\{x \in P \mid x \in \{0, 1\}^n\}. \tag{23}$$

We note that $\dim(P) = \dim(P_\mathrm{I}) = n$ (both P and P_I are fully dimensional) and that $P_\mathrm{I} \subseteq P$. Recalling that the NPP polytope is defined by the feasible zero-one solutions to (21), we further define P_I^G to be the convex hull of zero-one solutions to this polytope. Since $P_I = P_I^G$ every facet of P_I is a facet of P_I^G and vice versa. In order to identify facets of P_I, one may then restrict one's attention to facet identification for NPP. Certain subgraphs of G give rise to classes of facet-defining inequalities that have been proven to be useful in solving NPP.

The first class of graphs or subgraphs that give rise to facet defining inequalities of NPP (and hence, SP polytope) are *cliques*. A set $K \subseteq V$ is called a clique if each pair of nodes in K is joined by an edge. That is, a clique is a maximal complete subgraph of the intersection graph G. The following result is due to Padberg (1973):

An inequality

$$\sum_{j \in K} x_j \leq 1 \tag{24}$$

where $K \subseteq V$, is a facet-defining for P, if and only if K is the node set of a clique in G where G is the associated intersection graph.

Example 2.2:

Consider the intersection graph G in Figure 3. A maximal clique in G is $K = \{1, 2, 3, 4\}$ with the corresponding clique constraint that is

$$x_1 + x_2 + x_3 + x_4 \leq 1,$$

This is facet-defining for the associated NPP polytope.

Other types of graph structures that generate facet-inequalities of NPP (hence SP) polytopes are *odd holes*, and the *odd anti-holes* in the intersection graph G. Subset $V^l \subseteq V$ induces a subgraph $G^l = (V', E')$, where $(i, j) \in E'$ if and only if $i \in V'$, $j \in V'$, $(i, j) \in$ E. The *complement* $\overline{G}$ of a graph $G = (V, E)$ is the graph $G = (V, \widetilde{E})$, where $(i, j) \in \widetilde{E}$ if and only if $(i, j) \notin E$. A *chordless cycle* C in G is a cycle each of whose nodes is adjacent to exactly two other nodes of C. A cycle is called *odd* or even according to whether it is of odd or even length. A cycle of length three is obviously *chordless* and is a clique. A chordless cycle of length greater than three is called a *hole*, its complement an *anti-hole*.

If G_H is a subgraph of G, with nodeset $H \subseteq V$, we see that there is a matrix A_H that corresponds with the nodes of H and which is made up of a subset of the columns of A_G. Let P_H denote the polytope associated with the feasible solutions to the problem defined as:

$$\begin{aligned} A_H x &\leq e_t \\ x_j &\in \{0, 1\} \end{aligned} \tag{25}$$

where e_t is the vector t ones corresponding to the edges of G_H. The following two results are due to Padberg (1973) and Nemhauser and Trotter (1974), respectively.

(i) Let G_H be an odd hole of G. Denote by $H \subseteq V$ the node set of G_H and let $h=|H|$. Then

$$\sum_{j \in H} x_j \leq \frac{(h-1)}{2} \tag{26}$$

is a facet of P_H.

(ii) If instead G_H is an odd anti-hole and $H \subseteq V$, then

$$\sum_{j \in H} x_j \leq 2 \tag{27}$$

is a facet of P_H.

<u>Example 2.3</u>:

Consider the intersection graph G in Figure 3. The only odd hole is $H = \{3, 4, 5, 6, 7\}$ (see Figure 4) and the corresponding facet defining inequality is

$$x_3 + x_4 + x_5 + x_6 + x_7 \leq 2.$$

This inequality is facet-defining for the polytope P_H.

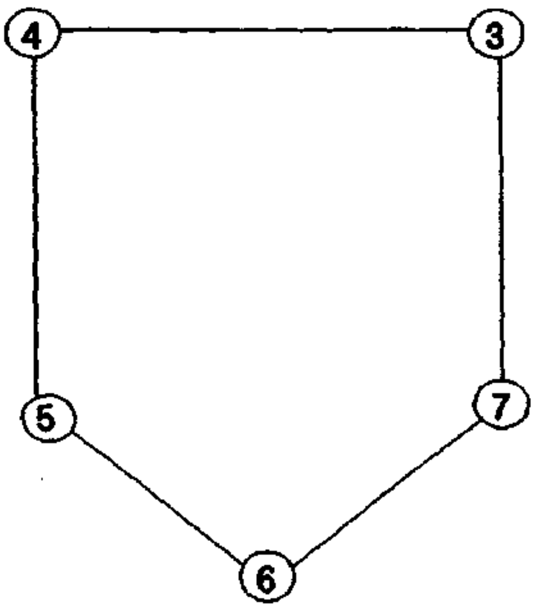


Figure 4

The odd anti-hole (Figure 5) is $H = \{3, 4, 5, 6, 7\}$ gives the corresponding odd anti-hole inequality

$$x_3 + x_4 + x_5 + x_6 + x_7 \leq 2.$$

which is a facet-defining for the associated polytope P_H.

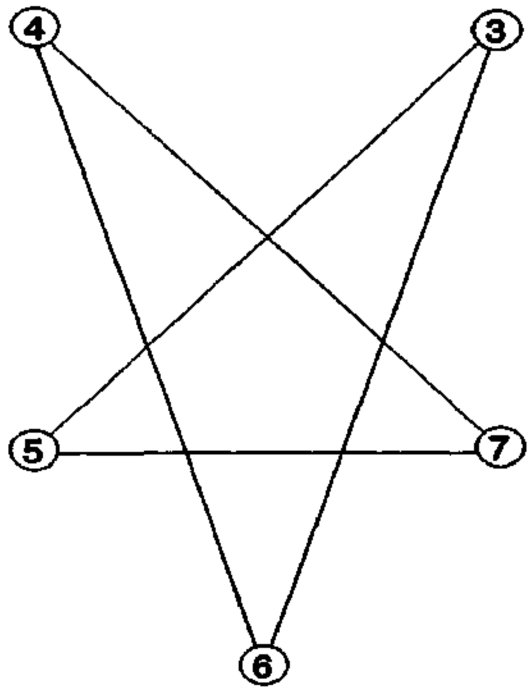


Figure 5

In addition to cliques, odd hole and odd anti-hole Trotter (1974) specifies two further classes of facet defining inequalities derived from a *web* of a graph. This is a generalization of cliques, odd holes and odd anti-holes. However, as asserted by Padberg (1979), the implementation of this idea appears to be hopelessly difficult and Nemhauser and Sigismondi (1992) claimed that they still had not found efficient procedures to find violated members of this family including the odd anti-hole family. Thus, facet-defining inequalities derived from such graph structures will not be discussed in this report.

2.3 - Facets for the symmetric travelling salesman polytope

Another specially structured zero-one IP that uses a graph to derive facet-defining inequalities is the symmetric travelling salesman problem (STSP). This is the problem of finding the shortest *hamiltonian cycle or tour* in a weighted undirected finite graph without loops and multiple edges. In other words, tours are the feasible solutions to the STSP. In the most common interpretation of this problem, the nodes of the graph represent cities, the edges represent the routes between the cities and the weights the distances between pairs of cities.

Given a graph $G = (V, E)$ with $n = |V|$ labelled nodes and $m = |E|$ labelled edges, a tour is a subset of E given by a Hamiltonian cycle of G. Let R^m be the space of real vectors whose components are indexed by the elements of E. With every tour t of G, we associate an *incidence vector* $x^\tau \in \mathrm{R}^m$ with components

$$x_{\mathrm{e}}^{\tau} = \begin{cases} 1 & if \;\; e \in \tau, \\ 0 & if \;\; e \notin \tau. \end{cases} \tag{28}$$

Much work has been done on the study of the STSP polytope by Grotschel and Padberg (1979, 1985). The STSP polytope is the convex hull of the incidence vector of all tours of the complete graph K_n having $m = ½\, n(n\text{-l})$ edges. Let

$$Q^n = conv\{x^{\tau} \in \mathrm{R}^m \mid \tau \; is \; a \; tour \; in \; K_n\} \tag{29}$$

be the STSP polytope. Since every node is met by exactly two edges of a tour, Q^n is contained in the polytope

$$Q^n{}_A = conv\{x \in \mathrm{R}^m \mid Ax = 2, \;\; 0 \leq x \leq 1\}, \tag{30}$$

where A is the node-edge incidence matrix of K_n,0, 1, and 2 are suitably dimensioned vectors having all components equal to 0, 1 and 2, respectively. The equalities $Ax = 2$ are called the *degree equalities.*

Let $c \in R^m$ be a vector that associates with each edge of the complete graph K_n a real number c_e, the length of edge e. The optimal solution of STSP is given by a minimum length tour. Since there are only a subfamily $\mathcal{L}'$ of the facet inducing inequalities $\mathcal{L}$ for the STSP polytope is known, the following relaxed problem:

$$\begin{aligned} &minimise \quad cx \\ &subject \quad to \quad Ax = 2, \\ &\qquad lx \le l_0 \quad (l_2\ l_0) \in \mathcal{L}^* \in \mathcal{L}^l, \\ &\qquad 0 \le x \le 1 \end{aligned} \tag{31}$$

(where $\mathcal{L}^* \subset \mathcal{L}' \subset \mathcal{L}$) is generally solved. We now discuss known classes of inequalities in $\mathcal{L}$'.

Valid inequalities for the STSP polytope

There are four families of inequalities valid for the STSP polytope. The best known linear inequalities that are satisfied by all tours are due to Dantzig et al. (1954). For $W \subseteq V$, let $E(W) = \{e \in E$: both ends of e are in $W\}$. If $E' \subseteq E$ and $|E' \cap E(W)| \ge |W|$, the subgraph $G' = (V, E')$ contains at least one subtour. This yields the *subtour elimination inequalities*:

$$\sum_{c \in E(W)} x_e \le |W| - 1 \quad \forall\, W \subseteq V,\ 2 \le |W| \le n - 1. \tag{32}$$

These inequalities were shown to define facets of the convex hull of tours Q^n by Grötschel and Padberg (1979).

These subtour elimination inequalities are not enough to describe the symmetric travelling salesman polytope. Moreover, there exists fractional solutions satisfying the degree equalities and the subtour elimination inequalities which leads researchers to study different classes of facet defining inequalities for the symmetric travelling salesman polytope.

The second class of valid inequalities for Q^n is called the 2-matching inequalities and is due to Edmonds (1965). A 2-matching in a graph is a set of edges such that every node is an endpoint of exactly two edges. Clearly every tour (subset of a tour) is a 2-matching. Thus the 2-matching inequality given by

$$\begin{gathered}\sum_{e\in E(H)} x_e + \sum_{e\in E^1} x_e \leq |H| + \tfrac{1}{2}(|E^1| - 1) \\ \textit{for all } H \subset V \textit{ and all } E^l \subset E \textit{ satisfying} \\ \text{(i) } |e \cap H| = 1 \ \forall \mathrm{e} \in E^l \\ \text{(ii) } e_i \cap e_j = \varnothing,\ e_i \neq e_j \in E^l \\ \text{(iii) } |E^l| \geq 3 \textit{ and odd,}\end{gathered} \tag{33}$$

where the set H is called *handle* and the edges of set E' are called the *teeth.* The graphical configuration of a 2-matching inequality is shown in Figure 6.

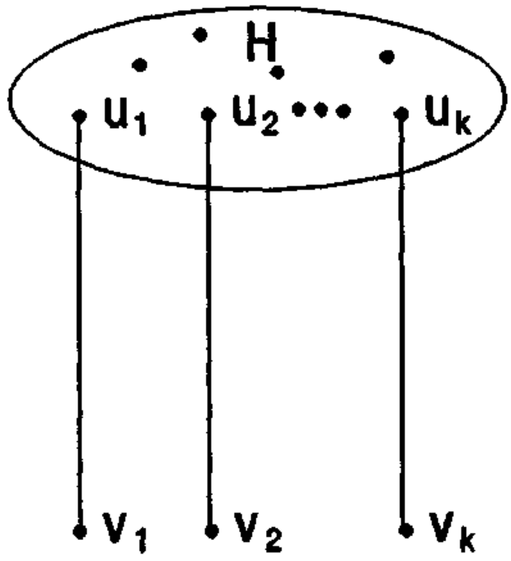


Figure 6

__Example 2.4__:

Suppose we have the subgraph given in Figure 7a. The numbers given are values for a current LP-relaxation solution of the problem (that is, the variable corresponding to edge *AB* currently has value 3/4 etc.) Since the subtour elimination inequalites:

$$x_{AB} + x_{BC} + x_{CD} + x_{BD} + x_{AC} = 2\tfrac{3}{4} \leq |4| - 1 = 3$$

is strictly satisfied by subgraph (7a), we must consider the 2-matching inequality given in (33) corresponding to (7b) to cut off the fractional solution, that is,

L.H.S.

$$\sum_{e \in E(H)} x_e = \tfrac{3}{4} + \tfrac{1}{4} + 1 + \tfrac{1}{2} + \tfrac{1}{4} = 2\tfrac{3}{4}$$

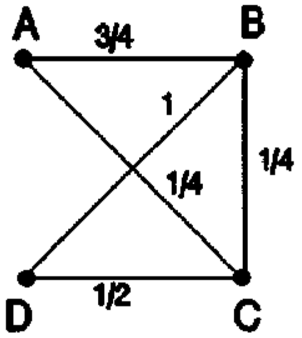
$$\sum_{x \in E^1} x_e = x_{AG} + x_{DF} + x_{CE} = 1 + 1 + \tfrac{1}{2} = 2\tfrac{1}{2}$$

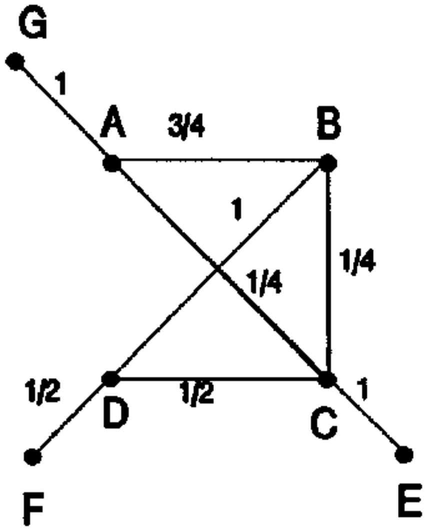
$$i.e. \quad \sum_{e \in E(H)} x_e + \sum_{x \in E^1} x_e = 5\tfrac{1}{4}$$

R.H.S.

$$|H| + \tfrac{1}{2}(|E^I| - 1) = |4| + 1 = 5$$

hence we have a violated 2-matching constraint.





7(a) 7(b)

Figure 7

Grotschel and Padberg (1979) generalised the 2-matching inequalities and called *comb inequalities*:

$$\sum_{e\in E(H)} x_e + \sum_{i=1}^{k} \sum_{e\in E(T_1)} x_e \leq |H| + \sum_{i=1} (|T_l| - \tfrac{1}{2}(k+1)$$

for all $H, T_{1,\ldots,} T_k \subset V$ *satisfying*

(i) $|T_i \cap H| \geq 1, \quad i = 1,\ldots.k,$ (34)

(ii) $|T_t \setminus H| \geq 1, \quad i = 1,\ldots.k,$

(iii) $\mathrm{T_t} \cap \mathrm{T_i} = \text{ø}, \mathrm{i} \neq \mathrm{j}, \ 1 \leq \mathrm{i} \leq \mathrm{k},$

(iv) $k \geq 3$ *and odd.*

The set H is called handle and sets T_i are called teeth. Chvatal (1973) considered the simple comb; that is, where (i) is satisfied with equality for all T_i, i = 1, 2,...k. The graphical configuration of a comb is shown in Figure 8. Here the teeth T_i for i = 1,..., k, can contain more than two nodes and can have more than one node in common with the handle. Specifically a *comb C* is a subgraph generated by a node set {H, T_i..., T_k} satisfied by the four properties given in (34).

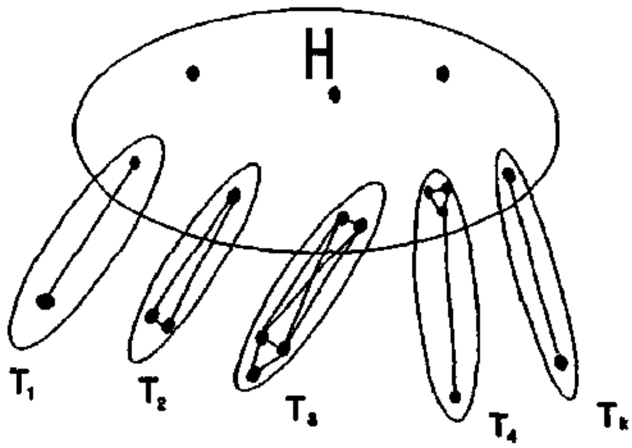


Figure 8

A comb C with $k = 1$ and $| H | = 1$ is a subtour elimination inequality, while a comb inequality is a 2-matching inequality if the inequalities of both (i) and (ii) in (34) hold as strict equalities. This class of comb inequalities was shown to define facets by Grötschel and Padberg (1979).

Grotschel and Pulleyblank (1986) generalised comb inequalities to give the following facet-defining *clique tree inequalities*:

$$\sum_{i=1}^{r} \sum_{e \in E(H_1)} x_e + \sum_{i=1}^{k} \sum_{e \in E(T_i)} x_e \leq \sum_{i=1}^{r} | H_i | + \sum_{i=1}^{k} (| T_i | - t_i) - \tfrac{1}{2}(k+1), \tag{35}$$

where t_i is the number of handles met by T_i V $H_{i,\dots,}$ $H_r \subseteq V$ and $T_{1,\dots,}$ $T_k \subseteq V$ which are the handles and teeth, respectively, of a clique tree. A clique tree is a connected subgraph of K_n whose cliques satisfying the following properties:

(i) the cliques are partitioned into two sets, the set of handles and the set of teeth,

(ii) no two teeth intersect,

(iii) no two handles intersect,

(iv) each tooth contains at least two and at most n-2 nodes and at least one node not belonging to any handle.

(v) each handle intersects an odd number (≥ 3) of teeth,

(vi) if a tooth T and a handle H have nonempty intersection, then $H \cap T$ is an *articulation* set of the clique tree.

Figure 9 shows a graphical configuration for generating a clique tree inequality. In their work, the authors showed that this class of inequalities encompasses the subtour elimination, the 2-matching and the comb inequalities as special cases.

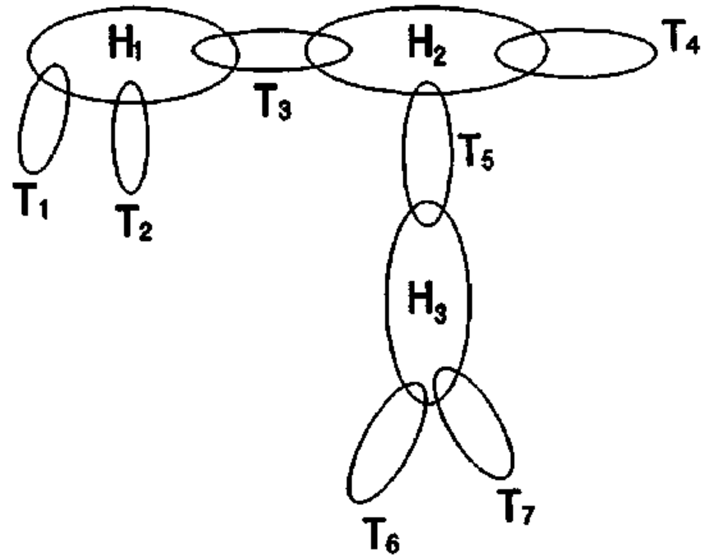


Figure 9

2.4 - Lifting the facets of zero-one polytopes

We have seen in previous sections, that some inequalities are facet-defining for lower dimensional polytope, but may not be facet-defining for the original (possibly higher dimensional) polytope. Some of these facets and valid inequalities for lower dimensional subpolytopes can be raised into the space of the original problem in order to get facets for the possibly higher dimensional polytope. Specifically, let P_I, be the solution set of any zero-one program; that is, P_I is an arbitrary subset of $\{0,1\}^{|N|}$, where $N = \{1, 2,..., n\}$ is the index set for the variables. Also let $P^*_{\mathrm{I}} =$ conv(P_I). For any subset $S \subseteq N$, define:

$$P_I(S) = \{x \in P_I \mid x_{\mathrm{i}} = 0, i \in N \setminus S\} \tag{36}$$
$$P^*_I(S) = conv(P_I(S)).$$

Suppose we have a facet-defining inequality of P_I^* (S):

$$\sum_{j \in S} b_j x_j \leq b_o \tag{37}$$

and we are interested in obtaining a facet-defining inequality for P_I^+ of the form

$$\sum b_j x_j + \sum_{j \in N \setminus S} b_j x_j \leq b_o. \tag{38}$$

In other words, the inequality of (38) is derived by finding suitable coefficients for variables with indices set $N\setminus S$.

A procedure for raising facet-defining inequalities from a lower dimension is called *lifting* and the facet inequalities obtained by this procedure is called *lifted facet-inequalities.* There are several ways one can lift a facet-defining inequality. However, there are two basic approaches mat are used by most researchers. One way is to consider lifting one variable at a time in sequence, where another is to consider several lifting variables at a time. The former procedure is called *sequential lifting* and the latter procedure is called *simultaneous lifting.* The details of these two procedures are presented later in this section. To our knowledge, of these two methods, the sequential lifting procedure is of practical interest and many computational studies [e.g., Crowder et al. (1983) and Hoffman and Padberg (1993)] report experience of applying this approach

Sequential lifting

This lifting procedure was first established by Padberg (1973) for the set packing polytope, men extended this procedure to 0-1 programming polytopes with positive coefficients (Padberg (1975). Wolsey (1976) then extended this procedure for general linear integer programs. The coefficients of a facet obtained by sequential lifting depend on the ordering of $N \backslash S$. That is, the facets obtained depend on the sequence in which new variables are introduced.

Hoffman and Padberg (1991) projected out variables both at zero and at one. Once the most violated minimal cover inequality (over only the fractional variables) is identified, the sequential lifting is applied to it. This is done by first lifting back the remaining fractional variables not in minimal cover, then the variables which are projected at one, and then the variables which were projected out at zero.

Hoffman and Padberg (1991) implemented this approach in order to ensure that the inequalities obtained are valid for the problem and approximate the integer polytope in the area around the fractional linear programming solution. Projecting out at value zero corresponds to the usual lifting procedure which is the Padberg's sequential lifting [Padberg (1973) and (1975)]. Projecting out at value one is the "reverse" lifting [Wolsey (1975)]. By using both type of projection, it is unnecessary to distinguish facets that are generated from minimal covers and from (1, k)-configuration.

The other difference is that of the sequential ordering. The order is determined based on both the first-order lifting coefficient and the reduced cost of the nonbasic variables.

According to Grotschel and Padberg (1985), sequential lifting is also applicable to any of the facet-defining inequalities of the symmetric traveling salesman polytope. However this procedure does not produce any new results.

Sequential lifting of minimal cover and (l,k)-configuration inequalities

In Section 2.1 we have defined, minimal cover inequalities for the zero-one knapsack polytope as in (14), that is, $\sum_{j \in S} x_j \leq |S| - 1$, where S is a minimal cover. The same minimal cover S, may yield as many as $|N \setminus S|!$ facets of the corresponding polytope, though the number of distinct facets are usually much smaller than this number.

For any $S^* \subseteq N$ and any index $t \in (N \setminus S^*)$, the (1, k)-configuration inequality is again defined as in (17), that is,

$$(r - k + 1)\, x_t + \sum_{j \in T(r)} x_j \leq r,$$

where $T(r) \subseteq S^*$ is any subset of cardinality r of S^* and r is any integer satisfying $k \leq r \leq$ $|S^*|$It follows that a (1, k)-cofiguration (the set $S^* \cup \{t\}$) defines $\sum_{r=k}^{p} \binom{p}{r}$ distinct facets of P_{S^*}, where $p = |S^*|$. Using the sequential lifting procedure, yield an even greater number of facets of the knapsack polytope P_I^i.

Padberg's sequential lifting procedure for facets of zero-one knapsack polytope, is as follows:

Initialisation step:

For a minimal cover, set

$$\begin{cases} \beta_j = 1 & for\ all\ j \in S \\ \beta_j = 0 & for\ all\ j \in N \setminus S \\ \beta_0 = |S| - 1 \end{cases} \tag{39}$$

and for some index t and some integer number k of a (1, k)-configuration, set

$$\begin{cases} \beta_j = 1 \ \ for \ \ all \ \ S^* \\ \beta_j = 0 \ \ for \ \ all \ \ j \in N \setminus S^* \\ \beta_t = (r - k + 1) \\ \beta_0 = r \\ S = S \bigcup t \end{cases} \tag{40}$$

Iterative step:

Let $l \in N \setminus S$. Determine

$$\overline{Z}_i = \max \{ \sum_{j \in S} \beta_j x_j \mid \sum_{j \in S} a_j x_j \le a_0 - a_r \ x_j = 0 \ \ or \ 1$$

$$for \ all \, j \in S \}. \tag{41}$$

Define $\beta_l = \beta_0 - \overline{z}_l$*:* if the coefficient β_l is positive, merge variable l into the set S according to its ratio $\beta_l / \ |a_l|$. Redefine S to be $S \cup \{l\}$ and repeat until $N \setminus S$ is empty. The resulting inequality $\beta x \le \beta_0$ defines a facet for the polytope P_I^i associated with (12), that is, $\sum_{j \in N} a_j x_j \le a_0$.

Therefore when lifting is applied to (14), one gets inequalities of the form

$$\sum_{j \in S} x_j + \sum_{j \in N \setminus S} \beta_j x_j \le |S| - 1 \tag{42}$$

and when it is applied to (1, k)-configuration (17) the lifted inequalities are of the form

$$(r-k+1)x_i \;+\; \sum_{j \notin T(r)} x_j + \sum_{j\, \in N \setminus S^* \setminus \{t\}} \beta_j x_j \le r \tag{43}$$

This sequential lifting procedure requires the solution of a sequence of 0-1 knapsack problems. Since the 0-1 knapsack problem is known to be NP-*hard* one usually relaxes (41) to a linear program, to get an approximate lifting procedure, and thus efficiently produce 'almost' facet-defining inequalities for the knapsack polytope.

<u>Example 2.4</u>:

Consider a problem with $N = \{1, 2, ..., 5)$, and let

$$P_{\mathrm{I}} = \{x \in \{0, 1\}^5 \!: 15x_1 + 13x_2 + 13x_3 + 12x_4 + 10x_5 \le 30\}.$$

A minimal cover is $S = \{1, 2, 3\}$ with a corresponding valid inequality

$$x_1 + x_2 + x_3 \le 2.$$

This inequality is facet defining for 3-dimensional polytope

$$P_I(S) = \{x \in \{0, 1\}^5 \mid 15x_1 + 13x_2 + 13x_3 + 12x_4 + 10x_5 \le 30,\, x_4 = x_5 = 0\},$$

but may not necessarily define a facet for polytope P_I. Thus, we would like to lift and find an inequality that is facet-defining for P_I . Lifting the inequality using the ordering of indices $l \in N \setminus S = \{4, 5\}$, and solving the LP-relaxation of (41) at each iteration step, we have as follows:

<u>Initialisation step</u>: Set $\beta_j = 1$ for j = 1, 2, 3., $\beta_0 = |\,3\,| - 1 = 2$.

<u>*Iterative step*</u>:

Iteration 1: $l = 4$, $a_4 = 12$, $S = \{1, 2, 3\}$, solve

$$\overline{Z}_4 = \{\max\ x_1 + x_2 + x_3 : 15x_1 + 13x_2 + 13x_3 \leq 30 - a_4 = 18,$$
$$0 \leq x_j \leq 1, \textit{for } j = 1, 2, 3\}.$$

Using Dantzig's method [Martello and Toth (1990)] we have,

$$\overline{x}_1 = 1, \overline{x}_2 = \tfrac{3}{13},\ \overline{x}_3 = 0 \textit{ and } \overline{z}_4 = 1\tfrac{3}{13}$$

and thus rounding, we get $z_4^* = 1$ and yields a lifting coefficient $\beta_4 = \beta_0 - z_4^* = 1$ for x_4.

Iteration 2: With $l = 5$, $a_5 = 10$ and $S = S \cup \{4\}$ we solve,

$$\overline{Z}_5 = \{\max\ x_1 + x_2 + x_3 + x_4 : 15x_1 + 13x_2 + 12x_4 \leq 30 - a_5 = 20,$$
$$0 \leq x_j \leq 1, \textit{for } j = 1, 2, 3, 4\}.$$

Using Dantzig's method we have

$$\overline{x}_1 = 1,\ \overline{x}_2 = \tfrac{5}{13},\ \overline{x}_4 = 0 \textit{ and } \overline{z}_5 = 1\tfrac{5}{13},$$

and rounding $z_5^* = 1$ and the lifting coefficient for x_5 is $\beta_5 = \beta_0 - z_5^* = 1$.

Hence, the lifted minimal cover inequality is given by

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 2.$$

Again, because of the relaxation of (41) at the iteration step, the resulting inequality is not guaranteed to be facet-defining, but can be expected to be very strong. On this small example, it is easy to check that the lifted inequality is facet-defining for the polytope $P_I(S)$. (That is, by solving (41) exactly, gives $\bar{z}_4 = 1$ and $\bar{z}_5 = 1$ which implies that the coefficients of x_4 and x_5 are 1).

Sequential lifting of odd hole inequalities

We next look at facet-defining inequalities of the set packing polytope. As in Section 2.2, let the set packing polytope P_I be defined as (23). Unlike clique inequalities, the odd hole inequalities generally do not provide facets for P_I. In order to obtain facets for P_I we need the rifted odd hole inequalities. This can be done by applying the sequential lifting procedure only since odd holes and odd anti-holes are *strongly facet-producing*. A graph G is called strongly facet-producing if the polytope P_G has a facet which cannot be obtained by simultaneously lifting a facet of lower dimensional polytope. The lifted odd hole inequalities can be obtained by lifting the odd hole facet inequalities (26) to give

$$\sum_{j \in N} x_j + \sum_{j \in N \backslash V} \beta_j x_j \leq \tfrac{(n-1)}{2} \tag{44}$$

a facet of set packing polytope P_I. Padberg (1973) has shown that there always exists at least one lifted inequality (44) with the coefficients $\{\beta_j\}_{j \in N \backslash V}$ that are all integer.

The Padberg's sequential lifting for minimal cover or (1, k)-configuration inequalities for the zero-one knapsack polytope can be applied to the odd hole inequalities of the set packing polytope. Thus the lifting procedure for an odd hole inequality is as follows:

Let $H \subseteq V$ be the index set of any set of nodes of the intersection graph G which define an odd hole in the graph G. Denote A_G be the edge-node matrix of the associated intersection graph. Again let a^j be the jth column of A_G, Let the sequence of the variables be $j_l,...,j_t$ in $N \setminus V$ where $t = |N \setminus V|$.

Initialisation step:

Define the set $V \setminus H = T^q = T^{q-1} \cup \{j_q\}$ for $j_q \in T \setminus T^{q-1}$ and $q = 1, \ldots, Q = |T|$, with $T^0 = 0$.

Iterative step:

Solve the problem (M_q):

$$\max z_q = \sum_{j \in H} x_j + \sum_{j \in T^{q-1}} \beta_j x_j \mid \sum_{j \in H \cup T^{q-1}} a^j x_j \le e_G - a^{j_q}, \tag{45}$$

$$x_j = 0 \; or \; 1 \;\; for \, j \in H \cup T^{q-1}.$$

where β_j, are defined recursively by $\beta_{j_q} = s - \bar{z}_q$, [$s = \frac{1}{2}(n - 1)$ and $\bar{z}_q$ is the optimal value of the objective function of problem (Mq)].

Example 2.5:

Consider an odd hole $H = \{3, 4, 5, 6, 7\}$ and the corresponding inequality obtained in the Example 2.3,

$$x_3 + x_4 + x_5 + x_6 + x_7 \le 2$$

which is a facet-defining for the 5-dimensional polytope

$$P_I(H) = \{x \in \{0,1\}^7 \mid A_G x \le e,\ x_1 = x_2 = 0\}$$

but may not be a facet for the set packing polytope P_I and therefore sequential lifting is needed. The set index $V \setminus H = \{1, 2\}$.

Iteration 1: $q = 1$, $T_l = 1$, we have to solve

$$\begin{aligned} z_1 &= \textit{Maximise}\ \ x_3 + x_4 + x_5 + x_6 + x_7 \\ \textit{such that}\quad & a^3 x_3 + a^4 x_4 + a^5 x_5 + a^6 x_6 + a^7 x_7 \le e_G - a^{j1} \end{aligned}$$

That is, by solving

$$z_1 = \max\ x_3 + x_4 + x_5 + x_6 + x_7\ \ \textit{such that}$$

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{vmatrix} \le \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} - \begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

we get

$$x_3 = x_4 = x_6 = 0 \text{ and } x_5 = x_7 = 1$$

which implies that $\bar{z}_1 = 2$, $\beta_1 = 1$, that is, the coefficient of $x_1 = 1$.

Iteration 2: $q = 2$, $T_2 = \{1\} \cup \{2\}$, we have to solve

$$z_2 = \textit{Maximise } x_1 + x_3 + x_4 + x_5 + x_6 + x_7$$
$$\textit{such that} \quad a^1x_1 + a^3x_3 + a^4x_4 + a^5x_s + a^6x_6 + a^7x_7 - a^{j2}$$

That is, by solving

$$z_2 = \max x_1 + x_3 + x_4 + x_5 + x_6 + x_7 \ \textit{such that}$$

$$\begin{vmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&1&1&0&0 \\ 1&0&0&0&0 \\ 1&1&0&0&0 \\ 0&0&1&0&0 \\ 0&1&0&1&0 \\ 0&0&0&1&1 \\ 0&0&1&1&0 \\ 0&1&1&0&0 \\ 0&0&0&0&1 \end{vmatrix} \begin{vmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{vmatrix} \leq \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} - \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}$$

we have,

$$x_1 = x_3 = x_4 = x_6 = 0 \text{ and } x_5 = x_7 = 1$$

which implies that $\bar{z}_2 = 2$, $\beta_2 = 1$, that is, the coefficient of $x_2 = 1$.

Hence the lifted odd hole inequality is given by

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 2.$$

The lifting procedure for the odd hole inequalities, that is, the procedure to calculate β_j of (44) consists of solving a sequence of set packing problems, one for each $j \in N \setminus V$, in the variables $j \in V \cup V'$, where V' is the index set for the coefficients already computed. As in the lifting of minimal cover inequalities, the sequence in which the coefficients β_j are computed does matter, and different sequences may produce different facets.

Simultaneous lifting

The sequential approach was generalized by Zemel (1978) and Balas and Zemel (1978) and they called this procedure simultaneous lifting since the variables are introduced in groups.

In Padberg's sequential lifting procedure, the lifting coefficients are computed one by one. For instance, in the lifting of the minimal cover inequalities the computation of each coefficent β_j requires that a certain 0-1 knapsack problem of size between $| S |$ and n be solved to optimality. The coefficients obtained in this way depend on the sequence in which they are calculated and, in general there may be an exponential number of sequences yielding distinct facets of P_I^i. Moreover, there may exist facets of P_I^i which are liftings from S, but which cannot be obtained by Padberg's algorithm for any sequence of $N \setminus S$. To overcome this problem Balas and Zemel (1978) and Zemel (1978) proposed a simultaneous lifting procedure where variables in $N \setminus S$ are lifted in group rather than one by one. They showed that the sequentially lifted facets are precisely those corresponding to integer vertices, whereas facets that are obtained by simultaneous liftings are associated with the fractional vertices. This procedure can be viewed as a generalization of Padberg's procedure for obtaining the sequentially lifted facets.

Gottlieb and Rao (1988) have studied a class of facets of the knapsack polytope containing fractional coefficients; these facets can be derived from disjoint and overlapping minimal covers and (1, k)-configurations. For such class, they have given necessary and sufficient conditions which can be verified without the use of the simultaneous lifting procedure.

Simultaneous lifting procedure [Zemel (1978)]

Consider an arbitrary fixed subset $S \subseteq N$. For a subset $M \subseteq N \backslash S$ let

$$P_I(S,M) = \{x \in P_I \mid x_j = 1, j \in M, x_j = 0, j \in N \backslash (M \cup S)\} \quad (46)$$

Denote by $\mathscr{F}$ the family of those subsets of $N \setminus S$, M, for which $P_I(S,M) \neq 0$, that is,

$$\mathscr{F} = \{M \subset N \backslash S \mid PI(S,M) \neq 0\}. \quad (47)$$

The members of $\mathscr{F}$ induce a partition of the vertices of P_I, such that every vertex of P_I belong to exactly one subset $P_I(S,M)$.

Every set $M \subseteq N \backslash S$ is associated with integer program IP_M, whose variables are those of S and whose feasible set is $P_I(S,M)$:

$$(IP_M): \qquad \overline{Z}_M = \max \sum_{j \in S} b_j x_j$$
$$s.t. \;\; x \in P_I(S,M) \quad (48)$$

where $\overline{Z}_M = -\infty$ if $P_I(S,M) = \emptyset$. Let $\overline{\pi}_M = b_0 - \overline{Z}_M$ and let PL be the polyhedral set:

$$PL = \{b \in R^{N\setminus S} \mid \sum b_j \leq \overline{\pi}_M \ \ for\ every\ M \in \mathscr{F}\}. \qquad (49)$$

It was shown by Zemel (1978) that if $S \subseteq N$ and (37) is a valid inequality for $P_I(S)$, then (38) is a valid inequality for P_I if and only if $b \in PL$.

Zemel (1978) generalised the fact which was stated by Hammer et al. (1975) for monotone polytopes and for nonhomogeneous facet that if $| N \setminus S| \leq 2$, the only lifted facets of a given lower dimensional facet are sequentially lifted facet.

3 - Facet identification for the zero-one polytopes

In the previous section we were concerned with the problem of *describing* facet inequalities for the zero-one knapsack polytope, set-packing polytope and symmetric travelling salesman polytope. This section deals with the problem that one encounters if one wants to use the theoretical results of the preceding section in a cutting plane (constraint generation) algorithm for the zero-one problems. That is, the problem of algorithmically *finding* facet inequalities that are violated by a solution of the current LP-relaxation in a constraint generation technique. As we are considering general zero-one problems and since the symmetric travelling salesman is a very specific problem, the facet identification procedures for this polytope are not presented. However, for readers interested in identification procedures, we recommend Padberg and Rinaldi (1990).

3.1 - The Facet identification problem

Consider a family of linear inequalities $\mathcal{L}$ of a zero-one polytope. Let $\mathcal{L}_0$ be a known subfamily of the facet-inducing inequalities $\mathcal{L}$. It is known that $\mathcal{L}' \subseteq \mathcal{L}_0 \subset \mathcal{L}$ where $\mathcal{L}'$ is the inequalities that have been identified. For most combinatorial optimization problems, $\mathcal{L}_0 \subset \mathcal{L}$. An example of a problem where $\mathcal{L}_0 = \mathcal{L}$ is the *matching* polytope where Edmonds (1965) defined completely the integer hull of the matching polytope. A constraint generation algorithm adapted from Padberg and Rinaldi (1990) for solving a certain zero-one IP problem is as follows:

Procedure 3.1:

Step 1: Set $\mathcal{L}' = 0$.

Step 2: Solve the associated LP of the 0-1 problem and let $\bar{x}$ be its solution.

Step 3: Find one or more inequalities in $\mathcal{L}_0$ violated by $\bar{x}$.

Step 4: If none is found, stop. Otherwise add the violated inequalities to $\mathcal{L}'$ and go to Step 2.

Since $\mathcal{L}_0$ is finite, the Procedure 3.1 stops after a finite number of steps. The core of the procedure is Step 3 which is called the *identification problem or separation problem.* It is formally stated as follows:

***Facet identification problem*:**

Given a point $\bar{x} \in \mathrm{R}^n$ *and a family* $\mathcal{L}_0$ *of inequalities* $\in \mathrm{R}^n$*,*

identify one or more inequalities in $\mathcal{L}_0$ *violated by* $\bar{x}$ (50)

or prove that no such inequality exists.

In actual computation, especially for solving a large scale problem it is impossible to generate a priori all possible valid inequalities or facet-defining inequalities for a polytope associated with a given class of problem. Although L_0 is finite, it is exponential in number. By using (50) as a routine in a solver, one can generate violated facet inequalities "on the fly". In other words generate them in the course of computation as they are needed.

Grötschel, Lovász and Schrijver (1981) have shown that a combinatorial optimization problem can be solved polynomially if and only if there exists a polynomial algorithm for the identification problem (or separation problem) (50). Another way of looking at this is to note that IP optimization problems are **NP**-*hard* if and only if the identification problem for these IPs is **NP**-*hard*. Given a family of inequalities L, a procedure is called *exact* if it solves problem (50) and *heuristic* if it sometimes identifies violated inequalities, but does not guarantee the solution of (50). By using fast approximate methods or heuristics, it is frequently possible to solve the facet identification problem (50) quickly. An example of a heuristic procedure was discussed in Section 2.4 where finding exact liftings was relaxed. In this section, some exact and heuristic facet identification procedures for zero-one polytopes that were discussed in the previous section and implemented in various computational studies are discussed.

3.2 - Facet identification procedures for the zero-one knapsack polytope

In the course of computation, it is common to restrict one's search to not only finding a violated inequality but to find the *most* violated inequality. The motivation for doing this is that most classes of known facet inequalities are exponential in size and therefore generating them would simply explode the memory required to store all such constraints. An example of such constraints would be (1, *k*)-configurations for a zero-one knapsack problem

$\sum_{j \in N} a_j x_j \leq a_0$ which is exponential in the number of variables [see Crowder et al. (1983)].

Finding the most violated inequality can be accomplished by solving the *constraint identification problem* in approximation to the facet identification problem (50) and is stated as follows:

Constraint identification problem:

Given $\bar{x}$*, find a minimal cover inequality*

$$\sum_{j \in S} x_j \leq |S| - 1$$

or a (l, k)-configuration inequality (51)

$$(r - k + 1)x_t + \sum_{j \in T(r)} x_j \leq r$$

that chops off $\bar{x}$*, if such an inequality exists.*

Identification of minimal cover inequalities

One of the procedures that solves the constraint identification for a most violated minimal cover inequality was developed by Crowder et al. (1983). To solve problem (51), one needs to solve the zero-one knapsack problem

$$\zeta = \min \{ \textstyle\sum_{j \in N} (1 - \bar{x}_j) s_j \mid \sum_{j \in N} a_j s_j > a_0, S_j = 0 \text{ or } 1 \quad \text{for all } j \in N \}. \tag{52}$$

It follows that there is a minimal cover inequality $\sum_{j \in S} x_j \leq |S| - 1$ that chops off $\bar{x}$ if and only if the optimal objective function value ζ of (52) is less than one. This can be shown as follows. Suppose that there is a minimal cover $S \subseteq N$ that chops off $\bar{x}$. By letting $s_j = 1$ for all $j \in S$ and $S_j = 0$ for all $j \in N \setminus S$ we get an objective function value less than one. Conversely, observe that $0 \leq \bar{x}_j \leq 1$ for ally $j = 1, ..., n$ implies that the objective function coefficients in (52) are nonnegative, and among the optimal solutions to (52), at least one defines a minimal cover. Let S be the set of variables with value one in such a solution. If the optimum value of the objective function value of (52) is less than one, then the corresponding inequality chops off $\bar{x}$.

Crowder et al. (1983) claimed that the constraint identification problem (52) is constructed in such a manner that its solution finds a most violated minimal cover inequality. One might note that (52) is itself a (0,1) IP problem which is also NP-hard. The authors also conjectured that by using a different approach than the one they had, a violated minimal cover inequality can be identified by a polynomially bounded algorithm. To get around the problem of solving (52) exactly, they solve its LP-relaxation in the following procedure:

Procedure 3.2:

Step 1: Solve the associated LP-relaxation of the zero-one knapsack problem. Let $\bar{x}$ be the current solution.

Step 2: If $\bar{x}$ is a zero-one solution - terminate; $\bar{x}$ solves the problem. Otherwise go to Step 3.

Step 3: Solve the associated LP-relaxation of the constraint identification (52).

Step 4: If the objective function value $\zeta < 1$, then the corresponding minimal cover inequality is violated by $\bar{x}$. Otherwise look for a (1, *k*)-configuration inequality using Procedure 3.3

Example 3.1:

Let

$$\boldsymbol{K} = \{12x_1 + 13x_2 + 13\,x_3 + 12x_4 + 9x_5 + 10x_6 + 9x_7 + 11x_8 \leq 39\ \}\ \ \textit{and}$$

$$\bar{x} = (1, 1, 1, \frac{11}{12}, 0, 0, 0, 0).$$

To check whether there is a violated minimal cover, we solve the constraint identification problem

$$\zeta = \min\ \{0s_1 + 0s_2 + 0s3 + \frac{11}{12}s_4 + s_5 + s_6 + s_7 + s_8 \mid$$
$$12s_1\ + 13s_2\ + 13s_3\ + 12s_4\ + 9s_5 + 10s_6 + 9s_7 + 11s_8 \geq 40$$
$$s_i = 0\ or\ 1\quad for\ i = 1, 2\ldots,\ 8\}.$$

Note: The data of the constraint are integer and thus >39 is replaced by $\geq$ 40. (For rational data replace >39 by $\geq 39+\in$ where $\in$ is a small positive number.)

The optimal solution is

$$s_1 = s_2 = s_3 = s_4 = 1, s_5 = s_6 = s_7 = s_8 = 0 \text{ and } \zeta = \frac{11}{12}.$$

As $\zeta < 1$, the cover inequality

$$x_1 + x_2 + x_3 + x_4 \leq 3$$

is violated by $\bar{x}$. ■

Identification for (l,k)-configuration inequalities

The procedure to identify a most violated (l-k)-configuration inequality is invoked only if Procedure 3.2 does not generate a most violated minimal cover inequality. Crowder et al. (1983) do not have a formulation of the facet identification problem in a tractable form such as (52). However, for a large scale zero-one problem they proposed an ad-hoc procedure to identify these inequalities.

Procedure 3.3:

Step 1: Start with a minimal cover S that is not violated by $\bar{x}$ (obtained through Procedure 3.2) and define the index t of the (1, k)-configuration to be the index with the largest $|a_{ij}|$ for all $j \in S$. If this does not yield a unique index, take the next knapsack constraint. Otherwise, set $S^* = S \setminus \{t\}$ and $k = |S| - 1$.

Step 2: Scan the indices in $K \setminus S^* \setminus \{t\}$ one by one. Let $i \in K \setminus S^* \setminus \{t\}$ be the current index and check whether or not (16) is valid with S^* replace by $S^* \cup \{i\}$. If yes, replace S^* by $S^* \cup \{i\}$. Otherwise scan the next index.

Step 3: Check whether or not the inequality (17) chops off the solution $\bar{x}$. If yes, save S^* for further use. Otherwise scan the next index.

Step 4: Extend the inequality found to the other variables in $N \setminus S^* \setminus \{t\}$ by sequential lifting (as discussed in Section 2.4).

3.3 - Procedures for solving a general zero-one problem

We now consider how facet identification procedures for a 0-1 knapsack problem can be used for solving a general zero-one problem. Crowder et al. (1983) proposed a procedure that is based on the constraint identification for a most violated minimal cover inequality or a most violated (1, *k*)-configuration for each individual constraint of the zero-one problem. (To minimise the actual computational effort, one needs to set up auxiliary data structures by scanning each constraint.) Assume that all constraints are of the form (11), and let the current solution to the LP-relaxation of the general (0,1) IP problem be $\bar{x}$.

Procedure 3.4:

Let *i* be the current row.

Step 1: Express row *i* into the a knapsack constraint with all nonnegative coefficients (if necessary use the substitution $x_j^1 = 1 - x_j$).

Step 2: Let the solution vector be $\bar{y}$ (due to the substitution).

Step 3: Define

$$K_1 = \{j \in N \mid \bar{y}_j > 0,\ a_{ij} \neq 0\},$$
$$K_0 = \{j \in N \mid \bar{y}_j = 0,\ a_{ij} \neq 0\}$$

where $N = \{1, 2..., n\}$.

Step 4: If $K_1 = \emptyset$, or if $j \in K_1$ implies $\bar{y}_j = 1$, or if $j \in K_1$ implies $|a_{ij}| = 1$, then process the next constraint (go to Step 1). Otherwise go to Step 5.

Step 5: Let $N = K_1$ and $\bar{x} = \bar{y}$. Solve the associated LP-relaxation of the problem (52).

> *Step 5.1*: Obtain a cover by rounding the fraction variables to one.
>
> *Step 5.2*: By dropping some of the variables to zero, change a cover obtain in Step 5.1 into a minimal cover S where S is the index set of the variables in the minimal cover.
>
> *Step 5.3*: If the objective function $\zeta \leq 1$, then the corresponding minimal cover inequality is violated by $\bar{y}$. Otherwise go to Step 8.

Step 6 : Sort the variable in S by increasing order of magnitude of the coefficients $|a_{ij}|$. If $K_1 \setminus S \neq \emptyset$ then lift the variables using the sequential lifting procedure of Section 2.4.

Step 7 : Check if the lifted inequality cuts off $\bar{y}$. If it does lift the variables in K_0. Otherwise go to Step 8.

Step 8 : Look for a $(1,k)$-configuration inequality using Procedure 3.3.

Nemhauser and Wolsey (1988) proposed a procedure for the identification of violated inequalities that is similar to Procedure 3.4 but without looking for (1,k)-configurations inequality.

Constraint generation algorithm for solving a general large scale zero-one programming

By combining the procedures above, we can now expand the constraint generation Procedure 3.1 to solve a general large scale zero-one IP problem as follows:

Procedure 3.5:

Step 1: Solve the associated LP-relaxation of the zero-one problem, intialise $i = 0$.

Step 2: If the optimal solution is a zero-one solution, terminate - it solves the zero-one problem. Otherwise go to Step 3.

Step 3: Let $i = i + 1$ (repeat until $i = n$) and solve for the ith constraint the identification problem using Procedure 3.4 to obtain the most violated minimal cover inequality. If a most violated inequality is found, process the next row (i+1). Otherwise go to Step 4.

Step 4: Apply Procedure 3.3 to identify the most violated (1, k)-configuration inequality. If a violated inequality is found, lift it and go to Step 3.

Step 5: Append all the lifted inequalities to the LP of the associated 0-1 IP problem and goto Step 1.

This procedure is repeated until one of the following conditions occur:

(i) a zero-one solution is found;

(ii) no more constraints are found;

(iii) the gain in the objective function value becomes too small e.g., is less than one unit in terms of the objective function.

Crowder et al (1983) implemented this algorithm to solve pure 0-1 problems within their hybrid algorithm which comprise of preprocessing, constraint generation and clever branch and bound. Later Hoffman and Padberg (1991) and Padberg and Rinaldi (1991) implemented this procedure in their branch-and-cut algorithm to solve large-scale zero-one programming and traveling salesman problem respectively.

3.4 - Facet identification procedures for the set-packing polytope

There are few studies made on procedures to identify violated minimal covers in the knapsack polytope but only limited studies on how to identify violated clique or odd hole inequalities for the set packing polytope. Procedures we are going to discuss in this section, for identification of violated clique inequalities and odd hole inequalities are proposed in computational studies by Hoffman and Padberg (1993) and Nemhauser and Sigismondi (1992). We adapt these procedures to develop a constraint generation algorithm for solving the set packing problem. Again, we work with the same graph structures defined in Section 2.2 (that is, the cliques and the odd holes of the intersection graph G).

Identification of clique inequalities

In order to find a violated clique inequality Nemhauser and Sigismondi (1992) defined node weights equal to the values of the variables in the optimal solution of the associated LP-relaxation of the problem. The following is the constraint identification problem for a clique inequality:

$$\text{Given } \bar{x}\text{, find a clique } C \text{ such that } \sum_{j \in C} x_j > 1 \text{ holds}$$
$$\text{or prove that no such inequality exists.} \tag{53}$$

The authors use a heuristic procedure to find a clique *C* of large weight. If the weight of *C* is greater than one, then a violated clique inequality is found. This process is repeated until optimality is proven or no more violated clique inequalities are found. In the latter case, the procedure attempts to find odd hole inequalities.

Nemhauser and Sigismondi (1992) use the following fast greedy procedure that tries to find violated clique inequalities. Since it is a heuristic procedure, it may not always identify violated inequalities when they exist.

Procedure 3.6:

Let star(v) = $v \cup N(v)$ where $N(v)$ are neighbours of v. Starring a graph means choosing a node v, deleting all of the nodes not in star(v) and then marking node v. Repeat this process until all the nodes in the remaining graph are marked. Since *star* (v) contains every clique that contains v, the set of marked nodes is a clique. This procedure uses the two following heuristics for selecting the star node at each iteration.

(i) choose a variable that maximises $\{x_v : x_v < 1\}$.

(ii) choose a variable that minimizes $\{|x_v\text{-}1/2| : 0 < x_v < 1\}$.

Hoffman and Padberg (1993) proposed three procedures to identify violated clique inequalities. In all these procedures for detecting violated inequalities, it suffices to investigate the subgraph induced by nodes of G that are in F (where F is the index sets of variables that have fractional values of $\bar{x}$) and edges E_F, with both endpoints in $\boldsymbol{F}$. Such a subgraph will henceforth be denoted $G_F = (F, E_F)$. For all the following procedures, let $F = \{j \in V : 0 < \bar{x}_j < 1\}$ where $\bar{x}$ R^n is the solution vector of current LP. The first procedure offered is given as follows:

Procedure 3.7:

Step 1: Scan every row of A that has a nonempty intersection with the columns in F.

Step 2: Let $M_r^F = \{j \in F : a_{rj} = 1\}$ where r is the current row.

Step 3: Find the set $K \subseteq F \setminus M_r^F$ of columns in F that are nonorthogonal to all columns in M_r^F.

Step 4: If $K \neq \emptyset$ then some or all the columns form cliques with columns of M_r^F.

Step 5: Identify a most violated inequality and lift it.

The efficiency of this procedure is evidently dependent upon the storage of the A matrix. The data structures used in the implementation of Hoffman and Padberg (1993) is discussed in Section 4.

The second procedure of clique identification uses the fact that small problems can be solved quickly by enumeration. This procedure is similar to that used by Nemhauser and Sigismondi (1992).

Procedure 3.8:

Step 1: Let d(v) denote the degree of node $v \in F$. Select a node v of minimum d(v) of G_F.

Step 2: Let *star*(v) denote the set of nodes of G_F that contains v and is itself a subset of the neighbours of v. Note that $d(v) = |\ star(v)\ |$.

Step 3: If v is a *pending* node, (that is, $d(v) = 1$) then go to step 5.

Step 4: If v is a *simplicial* node, that is, $\{v\} \cup star(v)$ forms a clique, check if the corresponding clique constraint is violated. If yes, lift the constraint and store it and go to step 5. Otherwise go to step 5.

Step 5: Delete node v and select next node and go to step 1.

If d(v) $\leq$ 16, (that is, if the degree of node v is less or equal to 16), then one simply enumerates all possibilities by looking at the complement graph of the subgraph induced by the nodes in *star*(v). Any clique in G defines a stable (independent) set in the complement graph and vice versa. Prior to enumerating, however a greedy routine on the complement graph is called to find a violated clique quickly. If one is found, there is no need to enumerate. Otherwise we do. If all degrees of the input graph are less than 16, then the above Procedure 3.8 is exact: it is guaranteed to find a violated clique inequality if one exists. Consequently, one either finds a violated clique inequality that contains v or one has proven that no such inequality exists. In the first case, one lifts the corresponding inequality and stores it. In either case, delete the node and repeat.

However, if the minimum degree node of the graph has a degree greater than 16, then determine a most violated inequlity in *star*(v) greedily as before, if applicable lift and store it, and delete the node and repeat. For solving large scale problems and after various graph reductions have been carried out, the minimum degree node frequently has a degree less or equal to 16.

The third procedure for clique detection is invoked only if the graph G_F used as input for the second procedure is dense; that is, if its total number of edges exceeds 50% of the number of edges for a complete graph on the nodeset F of G_F.

Procedure 3.9:

Step 1: Invoke procedure 3.8.

Step 2: Set up the complement graph.

Step 3: For every node of the complement graph, determine a maximum weight (given by the values $\overline{x}_j$ for $j \in F$) stable set containing that node.

Step 4: If the weight exceeds one, then a violated clique is found in the original graph.

Step 5: Lift and store the violated inequality.

It is possible that the three procedures find the same violated clique constraint. However, since Hoffman and Padberg (1993) use a randomization of the lifting sequence, these procedures tend to identify distinct clique constraints.

Identification of odd hole inequalities

By solving no more than $|V|$ shortest path problems, one either identifies a violated odd hole inequality or shows that none exists. (One way of solving shortest path is to use Dijkstra's algorithm [see for instance, Aho et al. (1983)]). If a violated odd hole inequality is found, then a lifting procedure is applied. Nemhauser and Sigismondi (1992) implemented the following procedure in their computational studies. This procedure was given in

Grotschel et al. (1988) to identify odd hole inequalities violated by the current fractional optimal solution $\bar{x}$.

Procedure 3.10: Approach 1

From the graph $G_F = (F, E_F)$ and the weight vector $\bar{x}$, construct a bipartite graph $K_F = (F_1, F_2, E^{'})$ where $F_1 = F_2 = F$. For every edge $(u,v) \in E_F$, introduce a pair of edges (u,v), $(v,u^{'}) \in E'$ where $u',v' \in F_2$ are the duplicates of $u, v \in F_1$. Assign to them identical edge-weights $C_{uv} = 1 - \bar{x}_u - \bar{x}_v \geq 0$. For every node pair $(v,v^{'})$ in K_F find a shortest path between them. If the weight of the path is less than 1, a violated odd hole inequality is identified. Otherwise each node pair of the form $(u,u^{'})$ is considered in turn until a violated inequality is found.

Hoffman and Padberg (1993) claimed that although this algorithm is easy to implement, it will not produce the desired result. If there is no violated odd hole inequality in G_F, it may return an odd circuit. Secondly, in general, it may be that after a few rounds, none of the odd holes generated gives rise to a violated odd hole inequality. Further, since odd hole C with $|C| \geq 5$ is needed to start the lifting of violated lifted odd hole inequality, Hoffman and Padberg (1993) implemented a procedure that is a modification of this approach.

Before one can apply Procedure 3.12 to identify violated odd hole inequalities, one has to construct the "layered" graph as follows:

Procedure 3.11: Construction of layered graph

Pick a node $v \in F$, call it the root, and build a layered graph starting from node v. Each level of the layered graph is defined by the *edge distance* that its nodes have from the root. All neighbours of v are on level 1, the neighbours of the neigbours (except v and those nodes that have already been assigned to level 1) form level 2, and so forth. In general, the shortest path from level k, say, to the root level contains k edges.

Procedure 3.12: Approach 2

Step 1: Construct the layered graph level by level.

Step 2: To all edges $(u,w) \in E_F$ that are in the layered graph, assign edgeweight 1 - $\bar{x}_u - \overline{x_w}$

Step 3: At level $k \geq 2$, let u and w be any two nodes on level k such that $(u, w) \in E_F$.

Step 4: Determine a shortest path from u to the root.

Step 5: "block" in the graph, all neighbours of the nodes in the path (except v) by assigning the corresponding edges a very large weight M.

Step 6: For the remaining graph and for all nodes on the level that are smaller than k, determine the shortest path from w to v.

Step 7: If a shortest path of length less than M exists, then an odd hole inequality containing u, v and w is detected.

Step 8: If none exists, take another edge on level k until they are exhausted.

Step 9: Construct the next level of the layered graph and go to step 2.

Identification of an odd anti-hole inequality

The other facet-producing configuration of the intersection graph mentioned in Section 2.2 is the complement of the odd hole, the odd anti-hole. Let $\overline{C}$ be the node set of the odd anti-hole in G_F. The node set of every odd hole in the complement graph $\overline{G}_F$ of the G_F defines such a node set $\overline{C}$ and vice versa. To find a violated odd anti-hole inequality, one has to produce the complement graph and get the node set *C*; run Procedures 3.7, 3.8 and 3.9. The corresponding inequality has to be lifted in order to have the inequality that defines a facet for the set packing poly tope.

The above procedure for finding violated odd anti-hole inequalities was proposed by Hoffman and Padberg (1993) while Nemhauser and Sigismondi (1992) claimed that they have not found an efficient procedures to find odd anti-hole inequalities.

4 – Conclusion

This report concerns the facial structures of the zero-one problems for which good understanding is needed to develop an efficient cutting plane method. We present valid inequalities, in particular the facet-defining inequalities that are derive from the zero-one knapsack, set packing and the travelling salesman polytopes, and in turn are used to eliminate nonintegral solutions. Procedures in detecting most violated inequalities are also discussed.

We consider two classes of valid inequalities that described the zero-one knapsack polytope. These are the minimal cover inequalities and the (1, k)-configuration inequalities. Established procedures for identifying these inequalities are discussed as well as the development of the constraint generation algorithm for solving a large scale zero-one program.

The solution procedures to the set packing problem use intersection graph to derive valid inequalities. The graph structures that give rise to facet-defining inequalities include cliques, odd holes and odd anti-holes. Procedures to detect these graphs and to construct the associated valid inequalties are presented.

For the travelling salesman problem, four classes of valid inequalities, namely the subtour elimination, the 2-matching, the comb and the clique tree inequalities, are presented. These inequalities partially define the convex hull of the integer points for the solution space.

Some of the above mentioned valid inequalities are facet-defining for lower dimensional polytopes but may not be facet-defining for the original polytopes. Lifting procedures have also been considered to obtain a facet of a possibly higher dimension.

Computational studies have indicated that a combination of problem preprocessing, cutting planes and branch and bound (or branch-and-cut) techniques can be extremely efficient in obtaining exact solutions of large scale zero-one IP problems. This motivates further investigation on the development of these cutting plane procedures within a branch-and-cut algorithm. As a follow up to this review, the implementation of the procedures discussed will be reported.

5 – References

Abdul-Hamid, F., K. Darby-Dowman, G. Mitra (1993). Cutting plane methods for general integer programming. *Department of Mathematics and Statistics Report*, Brunel University.

Aho, A.V., J.E. Hopcroft, J.D. Ullman (1983). *Data Structures and Algorithm.* Addison Wesley.

Balas, E. (1975). Facets of the knapsack polytope. *Mathematical Programming* 8 ,146-164.

Balas, E., Jeroslow (1972). Canonical cuts on the unit hypercube. *SIAM journal of Applied Mathematics* 23, 61-69.

Balas, E., M.W. Padberg (1976). Set partitioning: A survey. *SIAM Review* 18 (4), 710-760.

Balas, E., E. Zemel (1978). Facets of knapsack polytope from minimal covers. *SIAM Journal of Applied Mathematics* 34 (1), 119-148.

Chvatal, V. (1973). Edmonds polytopes and weakly Hamiltonian graphs. *Mathematical Programming* 5, 29-40.

Chvatal, H. (1975). On certain polytopes associated with groups. *Journal of Combinatorial Theory* B 18, 138-154.

Crowder, H., M.W. Padberg (1980). Solving large-scale symmetric travelling salesman problems to optimality. *Management Science* 26, 495-509.

Crowder, H., E.L. Johnson, M.W.Padberg (1983). Solving large-scale zero-one linear programming problems. *Operations Research* 3, (5), 803-834.

Dantzig, G.B., D.R. Fulkerson, S.M. Johnson (1954). Solution of large scale travelling salesman problem. *Operations Research* 2 (4), 393-410.

Dantzig, G.B., D.R. Fulkerson, S.M. Johnson (1959). On a linear-programming, combinatorial approach to the travelling-salesman problem. *Operations Research* 7, 58-66.

Darby-Dowman, K., G. Mitra (1985). An extension of set partitioning with application to scheduling problems. *European Journal of Operational Research* 21, 200-205.

Edmonds, J. (1965). Maximum matching and a polyhedron with 0-1 vertices. *Journal of Research of the National Bureau of Standards* 69B, 125-130.

Garey, M.R., D.S. Johnson (1979). *Computers and intractability*: *A guide to the theory of NP-completeness.* Freeman, New York.

Gottlieb, E.S., M.R. Rao (1988). Facets of the knapsack polytope derived from disjoint overlapping index configurations. *Operations Research Letters* 7, 95-100.

Grotschel, M., L. Lovász, A. Schrijver (1981). The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica* 1, 70-89.

Grötschel, M., L. Lovász, A. Schrijver (1981). *Geometric Algorithms and Combinatorial Optimization*. Springer Verlag, Berlin.

Grotschel, M., M.W. Padberg (1979a). On the symmetric travelling salesman I: Inequalities. *Mathematical Programming* 16, 265-280.

Grötschel, M., M.W. Padberg (1979b). On the symmetric travelling salesman II: Lifting theorems and Facets. *Mathematical Programming* 16, 281-302.

Grötschel, M., M.W. Padberg (1985). Polyhedral theory, in Lawler, Lenstra et al., 251-302.

Grötschel, M., W.R. Pulleyblank (1986). Clique tree inequalities and the symmetric traveling salesman problem. *Mathematics of Operations Research* 11, 537-569.

Hammer, P.L., E.L. Johnson, U.N. Peled (1975). Facets of regular 0-1 polytopes. *Mathematical Programming* 8, 179-206.

Hoffman, K.L., M.W. Padberg (1985/86). LP-based combinatorial problem solving. *Annals of Operations Research* 4, 145-194.

Hoffman,K.L., M.W. Padberg (1991) Improving LP-representations of zero-one linear programs for branch-and-cut. *ORSA Journal on Computing* 3, (2), 121-134.

Hoffman, K.L., M.W. Padberg (1993). Solving airline crew scheduling problems by branch-and-cut. *Management Science* 39, (6), 657-682.

Lawler, E.L., J.K. Lenstra, A.H.G. Rinnooy Kan, D.B. Shmoys (1985). *The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization*, Wiley.

Martello, S., P. Toth (1990). *Knapsack problems - Algorithms and computer implementations*. Wiley.

Nemhauser, G.L., G. Sigismondi (1992). A strong cutting plane/branch-and-bound algorithm for node packing. *Journal of Operational research society* 43, (5), 443-457.

Nemhauser, G.L., L.E. Trotter (1974). Properties of vertex packing and independence system polyhedra. *Mathematical Programming* 6, 48-61.

Nemhauser, G.L., L.A. Wolsey (1988). *Integer and Combinatorial Optimization*. Wiley).

Padberg, M.W. (1973). On the facial structure of set packing polyhedra.. *Mathematical Programming* 5, 199-215.

Padberg, M.W. (1975). A note on zero-one programming *Operations Research* 23, 833-837.

Padberg, M.W. (1979). Covering, packing and knapsack problems. *Annals of Discrete Mathematics* 4, 265-287.

Padberg, M.W. (1980). (1,k)-configurations and facets for packing problems. *Mathematical Programming* 18, 94-99.

Padberg, M., G. Rinaldi (1991) .A branch-and-cut algorithm for the resolution of large-scale symmetric traveling salesman problems. *SIAM REVIEW* 33, 60-100.

Parker, R.G., R.L. Rardin (1988). *Discrete Optimization*, Academic Press.

Trotter, L.E. (1975). A class of facets producing graphs for vertex packing polyhedra, *Discrete Mathematics* 12, 373-388.

Weyl, H. (1935). Elementare theorie der konvexen polyeder. *Comm.Math.Helv*., 7,290-306. English transl., in Contributions to the theory of games, 1, *Annals of Mathematics Studies* 24,Princeton, N.J., 1950, 3-18.

Wolsey, L.A. (1975). Faces for a linear inequality in 0-1 variables. *Mathematical Programming* 8, 165-178.

Wolsey, L.A. (1976). Facets and strong valid inequalities for integer programs. *Operations Research* 24, (2), 367-372.

Zemel, E. (1978). Lifting the facets of 0-1 polytopes. *Mathematical Programming* 15, (3), 268-277.