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Numerical methods for sixth-order boundary-value problems

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NUMERICAL METHODS FOR THE SOLUTION OF SPECIAL AND GENERAL SIXTH-ORDER BOUNDARY-VALUE PROBLEMS, WITH APPLICATIONS TO BÉNARD LAYER EIGENVALUE PROBLEMS.

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ABSTRACT

A family of numerical methods is developed for the solution of special nonlinear sixth-order boundary-value problems. Methods with second-, fourth-, sixth- and eighth-order convergence are contained in the family. Global extrapolation procedures on two and three grids, which increase the order of convergence, are outlined.

A second-order convergent method is discussed for the numerical solution of general nonlinear sixth-order boundary-value problems. This method, with modifications where necessary, is applied to the sixth-order eigenvalue problems associated with the onset of instability in a Bénard layer. Numerical results are compared with asymptotic estimates appearing in the literature.

1. INTRODUCTION

Many mathematical models concerning a Bénard layer assume a uniform steady-state temperature profile and an adiabatic gradient which is constant. Associated calculations reveal that, when a destabilizing temperature gradient exceeds the adiabatic gradient, the whole layer becomes unstable simultaneously (Baldwin, 1987a). Models which assume a non-uniform destabilizing steady-state temperature profile, further assume that convection sets in at a level where the local temperature gradient sufficiently exceeds the adiabatic gradient for the restraining effects of thermal conduction to be controlled. Baldwin (1987b) notes that, if this level is not at a boundary, the motion may be modelled by the sixth order eigenvalue problem

$$(D^2 - A^2)^3 w(x) + RA^2(1 - x^2) w(x) = 0, D \equiv d/dx \quad (1.1)$$

with

$$w(x) \to 0 \text{ as } x \to \pm\infty. \quad (1.2)$$

In this problem, x is a dimensionless boundary layer coordinate, w = w(x) is a dimensionless vertical velocity, R is a Rayleigh number and A is a horizontal wave number. Such problems have applications in astrophysics, as A-type stars are believed to have narrow convecting layers bounded by stable layers (Toomre *et al.* , 1976). Glatzmaier (1985) also notes that dynamo action in some stars may be related to a narrow convecting layer at the base of the convection zone in the critical region between the stable interior and turbulent convection regions. The smallest eigenvalue, RA^2, of (1.1) includes the minimum Rayleigh number R for the onset of stability and the corresponding wave number A. A similar eigenvalue problem discussed by Baldwin (1987a) replaces x^2 by x in the differential equaiton (1.1).

Baldwin (1987a) notes that asymptotic expansions for the solution of sixth order boundary-value problems are difficult to obtain. In a later

paper Baldwin (1987b) expresses the solutions arising as Laplace integrals, the integrands of which involve a function satisfying a second order equation with six transition points. W.K.B. approximations to this function, valid in regions associated with each transition point, are related by using global phase-integral methods. Baldwin then estimates solutions of the sixth-order problem using steepest descent techniques, leading to an eigenvalue condition. The eigenvalue estimates are used for an accurate computation based on the compound matrix method.

The numerical analysis literature on the solution of sixth-order boundary-value problems is sparse. Such problems are contained implicitly in the work of Chawla and Katti (1979), although those authors concentrated on numerical methods for fourth-order boundary-value problems. The book by Agarwal (1986) contains theorems which list the conditions for existence and uniqueness of solutions of sixth-order boundary-value problems, though no numerical methods are contained therein. A low-order numerical method is outlined in Twizell (1988).

Experience in solving second- and fourth-order boundary-value problems has shown that considerable insight may be obtained by solving the special problem first of all, followed by the general problem and the associated eigenvalue problem. To this end, special sixth-order boundary-value probems will be solved in §2 by finite difference methods of orders two, four, six and eight. Global extrapolations on two- and three-grids to increase order of convergence will be given. The general sixth-order boundary-value problem is discussed in §3 and in §4 the sixth-order eigen-value problem (1.1) is solved. The free-free and rigid-rigid cases of the problem discussed by Baldwin (1987a), in which $1-x^2$ in (1.1) is replaced by 1-x, are also solved.

2. THE SPECIAL BOONDAY-VALUE PROBLEM

2.1 A family of numerical methods

Consider the special nonlinear sixth-order boundary-value problem

$$D^6 w(x)=f(x,w), a < x < b;\ a,b,x \in R, w(x) \in C^{15}[a,b], \qquad (2.1)$$

$$\begin{aligned} w(a) &= A_0\ ,\ D^2w(a) = A_2\ ,\ D^4w(a) = A_4\ , \\ w(b) &= B_0\ ,\ D^2w(b) = B_2\ ,\ D^4w(b) = B_4\ . \end{aligned} \qquad (2.2)$$

It is assumed that $f(x,w) \in C^9[a,b]$ is real and that A_0, A_2, A_4, B_0, B_2 and B_4 are real finite constants.

Conside now the mesh G_1 obtained by discretizing the interval $a \le x \le b$ into N+1 subintervals each of width $h = (b-a)/(N+1)$ where $N \ge 5$ is an integer. The solution w(x) will be computed at the points $x_n^{(1)} = a+nh$ (n = 1,2,...,N) of G_1 and the notation $w_n^{(1)}$ will be used to denote the solution of an approximating difference scheme at the grid point $x_n^{(1)}$. Clearly $w_0^{(1)} = A_0$ and $w_{N+1}^{(1)} = B_0$.

A general family of symmetric numerical methods is given by

$$\begin{aligned} &-w_{n-3}^{(1)} + 6w_{n-2}^{(1)} - 15w_{n-1}^{(1)} + 20w_n^{(1)} - 15w_{n+1}^{(1)} + 6w_{n+2}^{(1)} - w_{n+3}^{(1)} \\ &+ h^6[\alpha f_{n-3}^{(1)} + \beta f_{n-2}^{(1)} + \gamma f_{n-1}^{(1)} + (1+2\alpha-2\beta-2\gamma) f_n^{(1)} \\ &+ \gamma f_{n+1}^{(1)} + \beta f_{n+2}^{(1)} + \alpha f_{n+3}^{(1)}] = 0\ , \end{aligned} \qquad (2.3)$$

where $f_n^{(1)} = f(x_n^{(1)}, w_n^{(1)})$ and α, β, γ are parameters chosen to ensure consistency as a minimum requirement. The local truncation error $t_n^{(1)}$ at the point $x_n^{(1)}$ is then given by

$$t_n^{(1)} = c_7 h^7 w^{(vii)}(x_n) + c_8 h^8 w^{(viii)}(x_n) + c_9 h^9 w^{(ix)}(x_n) + c_{10} h^{10} w^{(x)}(x_n) + \ldots\ ; \qquad (2.4)$$

in (2.4) the C_i (i = 7,8,9....) are constants with $C_7 = C_9 = \ldots = 0$ because of symmetry.

Equation (2.3) is applicable only to the N-4 mesh points $x_n^{(1)}$(n = 3,4,…,N-3,N-2) of G_1. In order to be able to implement the global extrapolation procedures to be discussed in §§2.2, 2.3 special formulae are needed for the other mesh points of G_1. These formulae will be assumed to be consistent and to have the forms

$$14w_1^{(1)} - 14w_2^{(1)} + 6w_3^{(1)} - w_4^{(1)} - a_1 w_0 - b_1 h^2 w_0'' - d_1 h^4 w_0^{iv} - d_1 h^6 w_0^{(vi)}$$

$$+ h^6 \left(\alpha_1 f_1^{(1)} + \beta_1 f_2^{(1)} + \gamma_1 f_3^{(1)} + \delta_1 f_4^{(1)} + \varepsilon_1 f_5^{(1)} + \theta_1 f_6^{(1)} + \psi_1 f_7^{(1)} + \tau_1 f_8^{(1)}\right) = 0, \quad (2.5)$$

$$-14w_1^{(1)} + 20w_2^{(1)} - 15w_3^{(1)} + 6w_5^{(1)} - w_5^{(1)} - a_1 w_0 - b_2 h^2 w_0'' - c_2 h^4 w_0^{(iv)} - d_2 h^6 w_0^{(vi)}$$

$$+ h^6(\alpha_2 f_1^{(1)} + \beta_2 f_2^{(1)} + \gamma_2 f_3^{(1)} + \delta_2 f_{41}^{(1)} + \varepsilon_2 f_5^{(1)} + \theta_2 f_6^{(1)} + \psi_2 f_7^{(1)} + \tau_2 f_8^{(1)}) = 0, \quad (2.6)$$

$$- w_{N-4}^{(1)} + 6w_{N-3}^{(1)} - 15w_{N-2}^{(1)} + 20w_{N-1}^{(1)} - 14w_N^{(1)} - a_2 w_{N+1} - b_2 h^2 w_{n+1}'' - c_2 h^4 w_{N+1}^{(iv)} - d_2 h^4 w_{N+1}^{(iv)}$$

$$+ h^6\left(\tau_2 f_{N-7}^{(1)} + \psi_2 f_{N-6}^{(1)} + \theta_2 f_{N-5}^{(1)} + \varepsilon_2 f_{N-4}^{(1)} + \delta_2 f_{N-3}^{(1)} + \gamma_2 f_{N-2}^{(1)} + \beta f_{N-1}^{(1)} + \alpha f_N^{(1)}\right) = 0 \quad (2.7)$$

and

$$- w_{N-3}^{(1)} + 6w_{N-2}^{(1)} - 14w_{N-1}^{(1)} + 14w_N^{(1)} - a_1 w_{N+1} - b_1 h^2 w_{N+1}'' - c_1 h^4 w_{N+1}^{(iv)} - d_1 h^6 w_{N+1}^{(iv)}$$

$$+ h^6\left(\tau_1 f_{N-7}^{(1)} + \psi_1 f_{N-6}^{(1)} + \theta_1 f_{N-5}^{(1)} + \varepsilon_1 f_{N-4}^{(1)} + \delta_1 f_{N-3}^{(1)} + \gamma_1 f_{N-2}^{(1)} + \beta_1 f_{N-1}^{(1)} + \alpha_1 f_N^{(1)}\right) = 0 \quad (2.8)$$

The a_i, b_i, c_i, d_i, α_i, β_i, γ_i, δ_i, ε_i, θ_i, ψ_i and τ_i $(i = 1,2)$ are parameters which must be chosen so that the local truncation errors of (2.5)-(2.8) are identical with (2.3) to the order required in §2.2, 2.3.

Clearly, the family of numerical methods is described by the set of equations {(2.5),(2.6),(2.3),(2.7),(2.8)} and the solution vector $w^{(1)} = [w_1^{(1)}, w_2^{(1)}, \ldots, w_N^{(1)}]^T$, T denoting transpose, is obtained by solving a nonlinear algebraic system of order N which has the form

$$J_1^3 w^{(1)} + h^6 M_1 f^{(1)}\left(x, w^{(1)}\right) - b^{(1)} = 0 \qquad (2.9)$$

In (2.9) J_1^3 is the cube of the familiar matrix J_1 of order N given by

$$J_1 = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & -1 & 2 & -1 \\ 0 & & & & -1 & 2 \end{bmatrix} \qquad (2.10)$$

for which $\left|J_1^{-1}\right| = \left(N+1\right)^2/8$ (the norm referred to throughout the paper is the L_∞ norm). (The choice of coefficients in the terms in w in {(2.5) - (2.8)} was motivated by the convenience of using J_1^3 in (2.9).) Also in (2.9) the matrix M_1, of order N, is given by

$$M_1 = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \theta_1 & \psi_1 & \tau_1 & & & & & & \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \theta_2 & \psi_2 & \tau_2 & & & & & & \\ \beta & \gamma & \Sigma & \gamma & \beta & \alpha & 0 & 0 & & & 0 & & & \\ \alpha & \beta & \gamma & \Sigma & \gamma & \beta & \alpha & 0 & & & & & & \\ 0 & \alpha & \beta & \gamma & \Sigma & \gamma & \beta & \alpha & & & & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\ & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & & & & \alpha & \beta & \gamma & \Sigma & \gamma & \beta & \alpha & 0 \\ & & & & & & 0 & \alpha & \beta & \beta & \Sigma & \gamma & \beta & \alpha \\ & & 0 & & & & 0 & 0 & \alpha & \alpha & \gamma & \Sigma & \gamma & \beta \\ & & & & & & \tau_2 & \psi_2 & \theta_2 & \varepsilon_2 & \delta_2 & \gamma_2 & \beta_2 & \alpha_2 \\ & & & & & & \tau_1 & \psi_1 & \theta_1 & \varepsilon_1 & \delta_1 & \gamma_1 & \beta_1 & \alpha_1 \end{bmatrix}, \qquad (2.11)$$

in which $\Sigma = 1 - 2\alpha - 2\beta - 2\gamma$. The vector $f^{(1)}$ of order N has the form $f^{(1)} = [f_1^{(1)}, f_1^{(1)}, \ldots, f_N^{(1)}]^T$, the constant vector $b^{(1)}$ is given by

$$b^{(1)} = \begin{bmatrix} a_1A_0 + b_1h^2A_2 + c_1h^4A_4 + d_1h^6w_0^{(vi)} \\ a_2A_0 + b_2h^2A_2 + c_2h^4A_4 + d_2h^6w_0^{(vi)} \\ A_0 - h^6\alpha w_0^{(vi)} \\ 0 \\ . \\ . \\ . \\ . \\ 0 \\ B_0 - h^6\alpha w_{N+1}^{(vi)} \\ a_2B_0 + b_2h^2B_2 + c_2h^4B_4 + d_2h^6w_{N+1}^{(vi)} \\ a_1B_0 + b_1h^2B_2 + c_1h^4B_4 + d_1h^6w_{N+1}^{(vi)} \end{bmatrix}, \qquad (2.1)$$

and 0 is the column zero-vector of order N.

The vector $w^{(1)} = \left[w\left(x_1^{(1)}\right), w\left(x_2^{(1)}\right), \ldots, w\left(x_N^{(1)}\right)\right]^T$ satisfies

$$J_1^3 w^{(1)} + h^6 M_1 f^{(1)}\left(x, w^{(1)}\right) - b^{(1)} - t^{(1)} = 0 \qquad (2.13)$$

Where $t^{(1)} = \left[t_1^{(1)}, t_2^{(1)}, \ldots, t_N^{(1)}\right]^T$ is the vector of local truncation errors and a conventional convergence analysis shows that thenorm of the vector

$$z^{(1)} = w^{(1)} - W^{(1)} \qquad (2.14)$$

Satisfies

$$\left\| z^{(1)} \right\| \le \frac{(b-a)^6}{512 - (b-a)^6 M_1^* F^*} \left\{ |c_8| h^2 v_8 + |c_{10}| h^4 v_{10} + \ldots \right\}$$

where $V_i = \max_{a \le x \le b} \left| d^i w(x)/dx^i \right|$ for $i = 1, 2, \ldots, m^* = \|M_1\|$ and $F^* = \max_{a \le x \le b} \left| \partial\Gamma / dw(x) \right|$, provided the parameters in (2-5) - (2.8) are chosen to ensure that $C_7 = C_9 = 0$. The order of convergence of the numerical method is, thus, p it C_{p+6}, is the first non-vanishing constanton the right hand side of (2.4) and $F^* < 512/[(b-a)^6 M^*]$.

2.2 Global extrapolation on two grids

Suppose, now, that the interval $a \le x \le b$ is subdivided into 2N+2 sub-intervals each of width $\frac{1}{2}h$ giving a finer grid G_2 of interior points named $x_1^{(2)}, x_2^{(2)}, \ldots, x_{2N+1}^{(2)}$. Clearly the points $x_{2i}^{(2)}$ of the fine grid G_2 coincidewith the points $x_i^{(1)}$ of the coarse grid G (i =1,2,...,N).

The finite difference formulae {(2.5),(2.6),(2.3),(2.7),(2.8)} are modified for use on G_2 by replacing h with $\frac{1}{2}h$. They may be written in matrix-vector form as

$$J_2 w^{(2)} + \left(\frac{h}{2}\right)^6 M_2 f^{(2)}\left(x, w^{(2)}\right) - \mathbf{b}^{(2)} = 0 \qquad (2.16)$$

in which J_2 and M_2 are matrices of order 2N+1 which may be written down immediately from (2.10) and (2.11). All vectors in (2.16) have 2N+1 elements; $b^{(2)}$ is obtained from $b^{(1)}$ and $t^{(2)}$ from $t^{(1)}$ by replacing h with $\frac{1}{2}h$, $w^{(2)}$ and $f^{(2)}$ follow in an obvious way from $w^{(1)}$ and $f^{(1)}$, as do $w^{(2)}$ from $w^{(1)}$ and $w^{(1)}$ from $z^{(1)}$.

In the convergence analysis on G_2, $w^{(2)}$ satisfies

$$\left\|z^{(2)}\right\| \le \frac{(b-a)^6}{512-(b-a)^6 M^* F^*}\left\{\left|c_8\right|\left(\frac{1}{2}h\right)^2 v_8 + \left|c_{10}\right|\left(\frac{1}{2}h\right)^4 v_{10} + \ldots\right\} \qquad (2.17)$$

(from (2.15); note $\left\|M_2\right\| = \left\|M_1\right\| = M^*$). Introduce, now, an extrapolation vector $z^{(E)}$ of order N defined by

$$z^{(E)} = qI_{\frac{1}{2}h^2}^{h}(2) + (1-q)z^{(1)} ,$$

where $I_{\frac{1}{2}h}^{h}$ is a fine-to-coarse grid restriction operator with

$$I_{\frac{1}{2}h}^{h} z^{(2)} = [z_2^{(2)}, z_4^{(2)}, \ldots, z_{2N}^{(2)}]^T \text{ and } I_{\frac{1}{2}h}^{h} w^{(2)} = [w_2^{(2)}, w_4^{(2)}, \ldots, w_{2N}^{(2)}]^T .$$

Defining $\left\|I_{\frac{1}{2}h}^{h}\right\|$ to be unity, it follows that

$$\left\|z^{(E)}\right\| \le q\left\|z^{(2)}\right\| + (1+q)\left\|z^{(1)}\right\|$$

and that

(8)

$$\left\| z^{(E)} \right\| = 0\left(h^{p+2}\right)$$

provided

$$q = 2^p/(2^p - 1), \tag{2.18}$$

where p is the order of convergence of the numerical method. The global extrapolation vector

$$w^{(E)} = qI^{h}_{\frac{1}{2}h} w^{(2)} + (1-q)\, w^{(1)} \tag{2.19}$$

is thus of order p+2 also.

2.3 Global extrapolation on three grids

Consider, next, a third grid G_3 of step size 1/3h. The interval $a \le x \le b$ is thus divided into 3N+3 subintervals and the interior points of G_3 are named $x^{(3)}_1, x^{(3)}_2, \ldots, x^{(3)}_{3N+1}$. Clearly, the points $x^{(3)}_{3i}$ of G_3 are coincident with the points $x^{(1)}_i$ of G_1 (i=1,2,...,N).

The solution vector $w^{(3)} = [w^{(3)}_1, w^{(3)}_2, \ldots w^{(3)}_{3N+2}]^T$ on G_3 is obtained from the nonlinear algebraic system

$$J_3 w^{(3)} + \left(\frac{h}{3}\right)^6 M_3 f^{(3)}\left(x, w^{(3)}\right) - b^{(3)} = 0 \tag{2.20}$$

in which J_3, M_3, $f^{(3)}$ and $b^{(3)}$ are obtained in an obvious way as in §2.2. In the convergence analysis on G3, $z^{(3)}$) satisfies

$$\left\| z^{(3)} \right\| \le \frac{(b-a)^6}{512 - (b-a)^6 M^* F^*}\left(|c_8|(\tfrac{1}{3}h)^2 v_8 + |c_{10}|(\tfrac{1}{3}h)^4 v_{10} + \ldots\right) \tag{2.21}$$

(from (2.15); note $\left\| M_3 \right\| = M^*$). The extrapolation formula

$$z^{(E)} = rI^{h}_{\frac{1}{3}h} z^{(3)} + sI^{h}_{\frac{1}{3}h} z^{(2)} + (1-r-s)\, z^{(1)},$$

in which the fine-to-coarse grid restriction operator $I^{h}_{\frac{1}{3}h}$ is such that

$$I^{h}_{\frac{1}{3}h} z^{(3)} = [z^{(3)}_3, z^{(3)}_6, \ldots, z^{(3)}_{3N}]^T \text{ and } I^{h}_{\frac{1}{3}h} w^{(3)} = [w^{(3)}_3, w^{(3)}_6, \ldots, w^{(3)}_{3N}]^T,$$

Gives\

$$\left\| z^{(E)} \right\| \le r\left\| z^{(3)} \right\| + s\left\| z^{(2)} \right\| + (1-r-s)\left\| z^{(1)} \right\|$$

(assuming that $\left\| I^{h}_{\frac{1}{3}h} \right\| = 1$) so that

$$\left\| z^{(E)} \right\| = 0\,(h^{p+4})$$

Provided

$$r = 3^{p+3}/(5+3^{p+3}-2^{p+5}) \text{ and } S = -2^{p+5}/(5+3^{p+3}-2^{p+5}). \qquad (2.22)$$

and, thus, $1-r-s = 5/(5+3^{p+3}-2^{p+5})$.

The global extrapolation algorithm

$$w^{(E)} = rI^{h}_{\frac{1}{3}h}w^{(3)} + sI^{h}_{\frac{1}{2}h}w^{(2)} + (1-r-s)\,w^{(1)} \qquad (2.23)$$

is thus of order $p+4$ also, where p is the order of convergence of the numerical method, provided r and s take the values indicated by (2.22).

2.4 Second order methods

Method A Writing $\alpha = \beta = \gamma = o$ in (2.3) gives

$$c_8 = -\frac{1}{4},\ c_{10} = -\frac{1}{240},\ c_{12} = -\frac{2}{945} \qquad (2.24)$$

in (2.4), so that (2.3) is a second order method (Twizell, 1988). To allow global extrapolation on three grids the parameters in the special end - point formulae (2.5)-(2.8) must be chosen so that $C_7 = C_9 = 0$ in (2.4) and so that C_8 and C_{10} in (2.4), with $n = 1,2,N-1$ or N, agree with (2.24). The method of undetermined coefficients reveals that this is achieved provided

$$a_1 = 5,\ b_1 = -2,\ c_1 = \frac{5}{6},\ a_2 = -4,\ b_2 = -1,\ c_2 = \frac{1}{12} \qquad (2.25)$$

together with

$d_1 =$	$717926/d$,	$d_2 =$	0 ,
$\alpha_1 =$	$4026944/d$,	$\alpha_2 =$	$-51467/d$,
$\beta_1 =$	$-439716/d$,	$\beta_2 =$	$3733148/d$,
$\gamma_1 =$	$218144/d$,	$\gamma_2 =$	$-105222/d$,
$\delta_1 =$	$-43286/d$,	$\delta_2 =$	$52868/d$,
		$\varepsilon_2 =$	$-10607/d$,

where

$$d = 3628800 = 10!$$

The parameters ε_1, θ_1, φ_1, τ_1, θ_2, φ_2, τ_2 may then be arbitrarily assigned the value zero.

This set of 24 parameter values gives C_{11} as the first non-zero constant,in (2.4). Global extrapolation on two grids, with p=2 in (2.18), and, on three grids, with p=2 in (2.22), gives the numerical emthods

$$W^{(E)} = \frac{4}{3} I^{h}_{\frac{1}{2}h} W^{(2)} - \frac{1}{3} W(1) \tag{2.26}$$

and

$$W^{(E)} = \frac{81}{40} I^{h}_{1/3h} W^{(3)} - \frac{16}{15} I^{h}_{1/\backslash 2h} W^{(2)} + \frac{1}{24} W^{(1)} \tag{2.27}$$

which are, respectively, $0(h^4)$ and $0(h^5)$ convergent.

Method B Global extrapolation on three grids gives $0(h^6)$ convergence if the parameters in (2.5)-(2.8) are chosen to give $C_7 = C_9 = C_{11} = 0$ as well as C_8 and C_{10} having the values in (2.24). This is achived at minimal cost by the parameters a_1, b_1, c_1, a_2, b_2, c_2 as given in (2.25) with, now,

d_1 = 17590730/d, d_2 = 239881/d ,

α_1 = 98456332/d, α_2 = 70270/d ,

β_1 = -32046202/d, β_2 = 79714751/d ,

γ_1 = 31580488/d, γ_2 = 115316/d ,

δ_1= -18751822/d, δ_2 = -67699/d ,

ε_1 = 6205228/d, ε_2 = 22222/d ,

θ_1= -881774/d, θ_2 = -3139/d ,

and where, now,

$$d = 79833600.$$

The parameters φ_1, τ_1, φ_2, τ_2 may then be arbitrarily assigned the value zero. The parameters of Method B are such that C_{12} also agrees with (2.24) for all n = 1,2,...,N on grid G_1.

The global extrapolation formulae (2.26) and (2.27) are therefore $0(h^4)$

and $0(h^6)$ convergent methods.

2.5 Fourth order methods

Method C Equation (2.3) becomes a fourth order method by choosing $\alpha = \beta = 0$ as before and by writing $\gamma = \frac{1}{4}$. The constants in (2.4) then become

$$C_8 = 0\,,\, C_{10} = -\frac{1}{120}\,,\, C_{12} = -\frac{43}{30240} \qquad (2.28)$$

with $C_7 = C_9 = C_{11} = \ldots = 0$ because of symmetry. Choosing the parameters a_1, b_1, c_1, a_2, b_2, c_2 given in (2.25) with

$d_1 = -1624722/d$, $\qquad d_2 = 118371/d$,

$\alpha_1 = 26624444/d$, $\qquad \alpha_2 = 10004918/d$,

$\beta_1 = 569404/d$, $\qquad \beta_2 = 19922518/d$,

$\gamma_1 = 6972504/d$, $\qquad \gamma_2 = 10005468/d$,

$\delta_1 = -2762606/d$, $\qquad \delta_2 = -10307/d$,

$\varepsilon_1 = 457292/d$, $\qquad \varepsilon_2 = 1694/d$,

where

$$d = 39916800 = 11!\ ,$$

ensures that $C_7 = C_8 = C_9 = C_{11} = 0$ and that $C_{10} = -1/120$ as in (2.28); the parameters θ_1, ψ_1, τ_1, θ_2, ψ_2 and τ_2 can then be arbitrarily assigned the value zero.

The constant C_{12}, however, is different from that in (2.28) and Method C can only be extrapolated on two grids. Writing p=4 in (2.18) leads to the numerical method

$$W^{(E)} = \frac{16}{15} I^{h}_{\frac{1}{2}h} W(2) - \frac{1}{15} W^{(1)} \qquad (2.29)$$

(from (2.19)) which is $0(h^6)$ convergent.

Method D It is possible to extrapolate on three grids if $C_{12} = -43/30240$ for all n = 1,2,...,N. This is achieved for $\alpha = \beta = 0$ and $\gamma = 1/4$ if a_i, b_i, c_i (i = 1,2) are given the values in (2.25) while the other parameters

in (2.5)-(2.8) are given the values

d_1 = -19679504/d , d_2 = 3156504/d ,

α_1 = 838715358/d , α_2 = 260639067/d ,

β_1 = -390245752/d , β_2 = 516574292/d ,

γ_1 = 799053554/d , γ_2 = 262290093/d ,

δ_1 = -632200396/d , δ_2 = -2211872/d ,

ε_1 = 313772290/d , ε_2 = 1087957/d ,

θ_1 = -89164504/d , θ_2 = -307164/d ,

ψ_1 = 11100206/d , ψ_2 = 38051/d ,

where

$$d = 1037836800 \;;$$

t_1 and t_2 may then be arbitrarily assigned the value zero.

Equation (2.29) gives the extrapolation of the $0(h^4)$ convergent Method D on two grids to $0(h^6)$ convergence, while putting p=4 in (2.22) gives the numerical method

$$W^{(E)} = \frac{729}{560} I^{h}_{1/3h} W^{(3)} - \frac{32}{105} I^{h}_{\frac{1}{2}h} W^{(2)} + \frac{1}{336} W^{(1)} \qquad (2.30)$$

(from (2.23) which is $0(h^8)$ convergent. This higher order convergence is obtained at the cost of increasing the number of non-zero diagonals in the matrix M_1 given by (2.11).

2.6 Sixth order methods

Method E Equation (2.3) attains sixth order by writing $\alpha = 0$ as before and then by choosing $\beta = \frac{1}{120}$ and $\gamma = \frac{13}{60}$, so that $1 - 2\beta - 2\gamma = \frac{11}{20}$. The constants in (2.4) become

$$C_8 = C_{10} = 0\,,\, C_{12} = -\frac{1}{30240}\,,\, C_{14} = \frac{11}{1209600} \qquad (2.31)$$

with C_7= C_9= C_9= C_{13} =...- 0 because of symmetry. Choosing the parameters. a_1, b_1, c_1, a_2, b_2, c_2 as given in (2.25) with

$d_1 = -54274064/d$, $d_2 = -5492136/d$,

$\alpha_1 = 648445278/d$, $\alpha_2 = 226044507/d$,

$\beta_1 = 59483528/d$, $\beta_2 = 568466132/d$,

$\gamma_1 = 202297394/d$, $\gamma_2 = 227695533/d$,

$\delta_1 = -147957056/d$, $\delta_2 = 6436768/d$,

$\varepsilon_1 = 71610370/d$, $\varepsilon_2 = 1087957/d$,

$\theta_1 = -19975384/d$, $\theta_2 = -307164/d$,

$\psi_1 = 2451566/d$, $\psi_2 = 38051/d$,

where, now,

$$d = 1037836800 ,$$

ensures that $C_{12} = -\dfrac{1}{30240}$ is the first non-zero constant in $t_n^{(1)}$ given by (2.4) and that $C_{13} = 0$ also (for all n = 1,2,...,N). The parameters T_1 and T_2 may then be assigned the value zero. The constant C_{14} does not, however, have the value given in (2.31) for n = 1,2,N-1,N and the global extrapolation of Method E can consequently be carried out on two grids only.

Writing p=6 in (2.18) leads to the numerical method

$$W^{(E)} = \frac{64}{63} I^{h}_{\frac{1}{2}h} W^{(2)} - \frac{1}{63} W^{(1)} \qquad (2.32)$$

(from (2.19)) which is $0(h^8)$ convergent.

Method F Ninth order convergence may be obtained by extrapolation on three grids by increasing the number of non-zero diagonals in M_1 given by (2.11). This is achieved for the same values of α, β, γ used in Method E, and for the values of a_i, b_i, c_i (i = 1,2) given in (2.25), by changing the remaining parameters in (2.5)-(2.8) to the following values:

d_1 = -5473830536/d , d_2 = -572925812/d ,

α_1 = 69886323662/d , α_2 = 23764660979/d ,

β_1 = -52722712/d , β_2 = 59583986756/d ,

γ_1 = 33838212674/d , γ_2 = 24117945173/d ,

δ_1 = -31281723760/d , δ_2 = 413467880/d ,

ε_1 = 20116075154/d , ε_2 = 324149693/d ,

θ_1 = -0395908472/d , θ_2 = -137209324/d ,

φ_1 = 2056983902/d , φ_2 = 33983099/d ,

τ_1 = 224946184/d , τ_2 = -3748468/d ,

where

$$d = 108972864000 .$$

Equation (2.32) gives the extrapolation of Method F from $0(h^6)$ to $0(h^8)$ convergence, while putting p=6 in (2.22) gives the numerical method

$$W^{(E)} = \frac{2187}{1960} I_{1/3h}{}^{h} W^{(3)} - \frac{256}{2205} I^{h}_{\frac{1}{2}h} W^{(2)} + \frac{1}{3528} W^{(1)} \tag{2.33}$$

(from (2.23)) which is $0(h^9)$ convergent.

2.7 An eight order method

Method G writing $\alpha = \frac{1}{30240}, \beta = \frac{41}{5040}$ and $\gamma = \frac{2189}{10080}$, so that $1 - 2\alpha - 2\beta - 2\gamma = \frac{4153}{7560}$, gives the unique eighth order method of the family (2.3) for n =3,4,…,N-2. The constants in (2.4) become

$$C_8 = C_{10} = C_{12} = 0 \, , C_{14} = \frac{1}{57600} \tag{2.34}$$

with $C_7 = C_9 = C_{11} = C_{13} = C_{15} = \ldots = 0$ because of symmetry.

The same values of C_i (i = 7,8,...,14) can be attained for the end points n= 1,2,N-1,N by choosing a_i, b_i, c_i (i = 1,2) as given by (2.25) and by choosing the following values of the remaining parameters in (2.5)-(2.8):

$d_1 = -5495452136/d$, $d_2 = -583736612/d$,

$\alpha_1 = 69727765262/d$, $\alpha_2 = 23688985379/d$,

$\beta_1 = 455384888/d$, $\beta_2 = 59814617156/d$,

$\gamma_1 = 32908483874/d$, $\gamma_2 = 23717945573/d$,

$\delta_1 = -30218661760/d$, $\delta_2 = 845899880/d$,

$\varepsilon_1 = 19337697554/d$, $\varepsilon_2 = 25050893/d$,

$\theta_1 = -8039152072/d$, $\theta_2 = -7479724/d$,

$\varphi_1 = 1963290302/d$, $\varphi_2 = 1550699/d$,

$\tau_1 = -214135384/d$, $\tau_2 = -144868/d$,

where

$$d = 108972864000 .$$

These parameter values give $C_{15} \neq 0$ for n = 1,2,N-1,N and so extrapolation of Method G can be carried out on two grids only. Writing p=8 in (2.18) leads, from (2.19), to the numerical method

$$W^{(E)} = \frac{256}{255} I^{h}_{\frac{1}{2}h} W(2) - \frac{1}{255} W(1) \tag{2.35}$$

which is $0(h^9)$ convergent.

Equation (2.3) does not yield a numerical method of order higher than Method G.

2.8 Numerical results

The numerical methods outlined in §§2.4-2.8 were tested on the following problem.

Problem 2.1

$$D^6 w(x) = 20\exp[-36w(x) - 40(1+x)^{-6}, 0 < x < 1$$

With boundary conditions

$$w(0) = 0, D^2w(0) = -\frac{1}{6}, D^4w(0) = -1, w(1) = \frac{1}{6}\lambda n2, D^2w(1) = \frac{1}{24}, D^4w(1) = -\frac{1}{16}$$

for which the theoretical solution is

$$w(x) = \frac{1}{6}\ln(1+x).$$

The interval $0 \le x \le 1$ was divided into N+1 equal subintervals each of width $h = 2^{-m}$ with m = 3,4,5 so that N = 7,15,31 respectively.

The value of $\|w - W\|$, where W is some numerical solution, was computed for each value of N. The results for all second, fourth, fifth, sixth, eighth and ninth order methods are given in Tables 2.1, 2.2, 2.3, 2.4, 2.5 and 2.6, respectively. These tables include results for the global extrapolation algorithms (the notation EXT(A,2,5) is used, for example, to denote the extrapolation of Method A which is second order convergent to achieve fifth order convergence) as well as for Methods A-G.

Tables 2.1-2.6 here

The two second order methods give very similar results and, as Method B has more non-zero off-diagonal elements in the matrix M_1, it is more expensive to implement than Method A. It does however give a higher order of convergence than Method A when extrapolated using three grids.

The global extrapolation of Method A on two grids (equation (2.26)), which gives fourth order convergence, gives slightly more accurate results than the similar extrapolation of Method B. Each gives better results than Method C which, in turn, gives higher accuracy than Method D. Methods C and D, however, are cheaper to implement than the two extrapolation formulations, especially Method C which has fewer non-zero off-diagonal elements in matrix M_1 (see (2.11)) than Method D.

The global extrapolation of Method A on three grids (equation(2.27)) is the only method with fifth order convergence. Generally, as is expected, results relating to it are intermediate to those of fourth and sixth order methods.

No sixth order method is significantly better than any other sixth order method though Method F did give better results on the two fine grids.

Also in its favour, Method F is cheaper to implement than any of the extrapolation methods, especially the extrapolation of Method B on three grids which gives poor results for small values of h.

Similar observations can be made regarding the four eighth-order methods tested, though on the finest grid (N=31) Method G gave better results, at significantly less cost, than any of the three extrapolation algorithms.

The global extrapolation on three grids of Method F (formula (2.33)), using the smallest values of h, gave more accurate results than the extrapolation on two grids (formula (2.35)) of Method G. However, the former is the more expensive of the two ninth-order methods and, to the engineer or scientist, the gain in accuracy may not warrant the extra cost.

Overall, there is evidence in Tables 2.1 - 2.6 that decreasing the grid size does not necessarily produce the desired effect of a considerable improvement in accuracy when using the higher order methods. This is due to the small value of h, raised to a large power, having little bearing on the calculation. This observation is also applicable to the extrapolation procedures which use fine grids.

3. THE GENERAL BOTWDARY-VALUE PROBLEM

The general nonlinear sixth-order boundary-value problem consists of a differential equation of the form

$$D^6 w(x) = g(x, w, w', w'', w''', w^{(iv)}, w^{(v)}), \; a < x < b \qquad (3.1)$$

with given associated boundary conditions. The book by Agarwal (1986) gives theorems on existence and uniqueness relating to this problem.

A particular form of the differential equation (3.1) is given by

$$-(D^2-A^2)^3 w(x) - RA^2(1-x^2) w(x) + f(x, w(x)) = 0, \; 0 < x < x, \qquad (3.2)$$

with the boundary conditions

$$w(0) - A_0 \ , \ D^2W(0) = A_2 \ , \ D^4 \, w(0) = A_4 \ ,$$
$$w(X) = B_0 \ , \ D^2w(X) = B_2 \ , \ D^4w(X) = B_4 \tag{3.3}$$

specified; it is assumed that $w \in C^{10}[0,X]$ and that A_0, A_2, A_4, B_0, B_2, B_4 are real finite constants. Other forms of boundary conditions will be considered in §§4.2, 4.4. The physical situation associated with (3.2) was discussed in §1.

The interval $0 \leq x \leq X$ will be divided into N+1 subintervals $(N \geq 5)$ each of width h, so that $(N+1)h = X$, giving a grid G of points $x_n = nh$ $(n = 0,1,\ldots,N,N+1)$ including the boundary points $x_0 = 0$ and $x_{N+1} = X$. The notations introduced in §2.1 may thus be used. However, as extrapolation will not be considered in this section, the superscripts will not be used.

In order to use powers of the matrix J_1 (see (2.10)) in the convergence analysis, the derivatives in (3.2) will be approximated by the finite difference replacements

$$w^{(vi)}(x_n) = h^{-6}(w_{n-3} - 6w_{n-2} + 15w_{n-1} - 20w_n + 15w_{n+1} - 6w_{n+2} + w_{n+3} + o(h^2) \ , \tag{3.4}$$

$$w^{(iv)}(x_n) = h^{-4}(w_{n-2} - 4w_{n-1} + 6w_n - 4w_{n+1} + w_{n+2}) + o(h^2) \ , \tag{3.5}$$

and

$$w''(x_n) = h^{-2}(w_{n-1} - 2w_n + w_{n+1}) + 0(h^2) \tag{3.6}$$

Substituting (3.4), (3.5) and (3.6) into (3.2) leads to the numerical method

$$-w_{n-3} = 3(2 + A^2h^2w_{n-2} - 3(5 + 4A^2h^2 + A^4h^4)w_{n-1}$$
$$= [20 + 18A^2h^2 + 6A^4h^4 + A^6h^6 - RA^2h^6(1 - x^2)]w_n - 3(5 + 4A^2h^2 + A^4h^4)w_{n+1}$$
$$= 3(2 + \Lambda^2h^2)w_{n+2} - w_{n+3} + h^6f_n = 0 \tag{3.7}$$

(Twizell, 1988) which has local truncation error given by

$$t_n = h^8[-\tfrac{1}{4} w^{(viii)}(x_n) + \tfrac{1}{2} A^2 w^{(vi)}(x_n) - \tfrac{1}{4} A^4 w^{(iv)}(x_n)] + 0(h^{10}) . \qquad (3.8)$$

It is noted that, when A=0, the differential equation (3.2) becomes the differential equation (2.1),the method (3.7) becomes Method A of §2.4, and t_n in (3.8) becomes $t_n,^{(1)}$ associated with Method A.

The numerical method (3.7) may be applied for n = 3,...,N-2 only; for n = 1,2,N-1 and N special approximations to $w^{(vi)}(x_n)$, and for n=l and N special approximations to $w^{(iv)}(x_n)$, must be used. Assume they are of the forms

$$-w^{(vi)}(x_1) = h^{-6}(\alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 + \alpha_4 w_4 + \alpha_5 w_5$$

$$-\gamma_1 w_0 - h^2 \gamma_2 w_0'' - \gamma_3 h^4 w_0^{(iv)} - \gamma_8 h^6 w_0^{(vi)} \qquad (3.9)$$

$$w^{(iv)}(x_1) = h^{-4}(5w_1 - 4w_2 + w_3 + \gamma_5 w_0 + \gamma_6 h^2 w_0'' + \gamma_7 h^4 w_0^{(iv)} + \gamma_8 h^6 w_0^{(vi)}, \qquad (3.10)$$

$$-w^{(vi)}(x_2) = h^{-6}(\beta_1 w_1 + \beta_2 w_2 + \beta_3 w_3 + \beta_4 w_4 + \beta_5 w_5$$

$$-\delta_1 w_0 - \delta_2 h^2 w_o'' - \delta_3 h^4 w_o^{(iv)} - \delta_4 h^6 w_o^{(vi)}, \qquad (3.11)$$

$$w^{(iv)}(x_2) = h^{-4}(-4w_1 + 6w_2 - 4w_3 + w_4 + \delta_5 w_0 + \delta_6 h^2 w_0'' + \delta_7 h^4 w_0^{(iv)} + \delta_8 h^6 w_0^{(vi)}), \qquad (3.12)$$

$$-w^{(vi)}(x_{n-1}) = h^{-6}(\beta_5 w_{N-4} \beta_4 w_{N-3} + \beta_3 w_{N-2} + \beta_2 w_{N-1} + \beta_1 w_N$$

$$-\delta_1 w_{N+1} - \delta_2 h^2 w_{N+1}'' - \delta_3 h^4 w_{N+1}^{(iv)} - \delta_4 h^6 w_{N+1}^{(vi)}, \qquad (3.13)$$

$$w^{(iv)}(x_{N-1}) = h^{-4}(w_{N-3} - 4w_{N-2} + 6w_{N-1} - 4w_N + \delta_5 w_{n+1} + \delta_6 h^2 w_{N+1}'' + \delta_7 h^4 w_{N+1}^{(iv)} + \delta_8 h^4 w_{N+1}^{(vi)}), \qquad (3.14)$$

$$-w^{(vi)}(x_n) = h^{-6}(\alpha_5 w_{N-4} + \alpha_4 w_{N-3} + \alpha_3 w_{N-2} + \alpha_2 w_{N-1} + \alpha_1 w_N$$

$$-\gamma_1 w_{N+1} - \gamma_2 h^2 w_{N+1}'' - \gamma_3 h^4 w_{N+1}^{(iv)} - \gamma_4 h^6 w_{N+1}^{(vi)}) \qquad (3.15)$$

And

$$w^{(iv)}(x_N) = h^{-4}(w_{n-2} - 4w_{N-1} + 5w_N + \gamma_5 w_{n+1} + \gamma_6 h^2 w''_{N+1} + \gamma_7 h^4 w^{(iv)}_{N+1} + \gamma_8 h^6 w^{(vi)}_{N+1}), \quad (3.16)$$

then (3.9)-(3.16) are substituted into (3.2) to give finite difference methods for n = 1,2,N-1,N.

The 26 parameters α_i, β_i (i = 1,2,…,5), and γ_i, δ_i (i = 1,2,...,8), which have different values to those in §2, are chosen to give local truncation error

$$t_n = h^8[-\tfrac{1}{4}w^{(vii)}(x_n) + \tfrac{1}{2}A^2 w^{(vi)}(x_n) - \tfrac{1}{4}A^4 w^{(iv)}(x_n)] + 0(h^9) \quad (3.17)$$

for n = 1,2,N-1,N. To achieve (3.8) for n = 1,2,N-1,N also, requires more parameters and consequently produces a method which is more expensive to implement. The method of undetermined coefficients gives

$\alpha_1 = 14 - 10500/d$, $\beta_1 = -14 - 42/d$,

$\alpha_2 = -14 + 12000/d$, $\beta_2 = 20 + 48/d$,

$\alpha_3 = 6 - 6750/d$, $\beta_3 = -15 - 27/d$,

$\alpha_4 = -1 + 2000/d$, $\beta_4 = 6 + 8/d$,

$\alpha_5 ,= -250/d$, $\beta_5 = -1 - 1/d$,

$\gamma_1 = 5-3500/d$, $\gamma_5 = -2$,

$\gamma_2 = -2 + 1250/d$, $\gamma_6 = 1$,

$\gamma_3 = \frac{5}{6} - 2375/d$, $\gamma_7 = 1/12$,

$\gamma_4 = \frac{29}{180} + 6125/36d$, $\gamma_8 = 1/360$,

$$\begin{aligned}
\delta_1 &= -4 - 14/d, \\
\delta_2 &= 1 + 5/d, \\
\delta_3 &= \frac{1}{12} - 19/d, \\
\delta_4 &= \frac{1}{360}\, 49/72d, \\
\delta_5 &= 1, \\
\delta_6 &= \delta_7 = \delta_8 = 0,
\end{aligned} \quad (3.18)$$

where d = 15619 (writing the parameter values in the above forms is motivated by the convenience of using powers of the matrix J_1).

After substitution of (3.6) and (3.9)-(3.16) with (3.18) into (3.2), and using (3.7), it is seen that the solution vector W may be found by solving the nonlinear algebraic system

$$(J + 3A^2h^2J_1^2 + 3A^4h^4J_1 + A^6h^6I - RA^2h^6G)w + h^6f(x,w) = b \qquad (3.19)$$

in which J_1 is given in (2.11), I is the identity matrix of order N, $G = G(x) = \mathrm{diag}\{(1-X\frac{2}{n})\}$, $f = [f_1, f_2, \ldots, f_n]^T$, and $b = [b_1, b_2, \ldots, b_N]^T$ with

$$\begin{aligned} b_1 = {} & (\gamma_1 - 3\gamma_5A^2h^2 + 3A^4h^4 + \gamma_4A^6h^6 - \gamma_4RA^2h^6 - 3\gamma_8A^8h^8 + 3\gamma_8RA^4h^8)A_0 \\ & + h^2(\gamma_2 - 3\gamma_6A^2h^2 - 3\gamma_4A^4h^4 + 9\gamma_8A^6h^6)A_2 \\ & + h^4(\gamma_3 - 3\gamma_7A^2h^2 + 3\gamma_4A^2h^2 - 9\gamma_8A^4h^4)A_4 + h^6(\gamma_4 - 3\gamma_8A^2h^2)f(0,A_0), \end{aligned} \qquad (3.20)$$

$$\begin{aligned} b_2 = {} & (\delta_1 - 3A^2h^2 + \delta_4A^6h^6 - \delta_4RA^2h^6)A_0 + h^2(\delta_2 - 3\delta_4A^4h^4)A_2 \\ & + h^4(\delta_3 + 3\delta_4A^2h^2)A_4 + h^6\delta_4f(0,A_0), \end{aligned} \qquad (3.21)$$

$$b_3 = A_0 , \qquad (3.22)$$

$$b_{N-2} = B_0 , \qquad (3.23)$$

$$\begin{aligned} b_{N-1} = {} & [\delta_1 - 3A^2h^2 + \delta_4A^6h^6 - \delta_4RA^2h^6(1-X^2)]B_0 + h^2(\delta_2 - 3\delta_4A^4h^4)B_2 \\ & + h^4(\delta_3 + 3\delta_4A^2h^2)B_4 + h^4\delta_4f(X,B_0), \end{aligned} \qquad (3.24)$$

$$\begin{aligned} b_N = {} & [\gamma_1 + 6A^2h^2 + 3A^4h^4 + \gamma_4A^6h^6 - \gamma_4RA^2h^2(1-X^2) - \frac{1}{120}A^8h^8 + \frac{1}{120}RA^4h^8(1-X^2)B_o \\ & + h^2(\gamma_2 - 3A^2h^2 - 3\gamma_4A^4h^4 - \frac{1}{40}A^6h^6)B_2 \\ & + h^4[\gamma_3 - (\tfrac{1}{4} - 3\gamma_4)Ah^2 - \frac{1}{40}A^4h^4]B_4 + h^6(\gamma_4 - \frac{1}{120}A^2h^2)f(X,B_0) \end{aligned} \qquad (3.25)$$

and $b_n = 0$ for $n = 4,5,...,N-3$. The matrix J is given by

$$J = J_1^3 + P \tag{3.26}$$

where

$$p = d^{-1}\begin{bmatrix} -10500 & 12000 & -6750 & 2000 & -250 & & & \\ -42 & 48 & -27 & 8 & -1 & & 0 & \\ 0 & 0 & 0 & 0 & 0 & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & 0 & 0 & 0 & 0 & 0 \\ & 0 & & -1 & 8 & -27 & 48 & -42 \\ & & & -250 & 2000 & --6750 & 12000 & -10500 \end{bmatrix}. \tag{3.27}$$

Now, the matrix P can also be written in the form

$$P = QJ_1^3 \tag{3.28}$$

where

$$Q = d^{-1}\begin{bmatrix} -500 & 250 & & & & \\ -2 & 1 & & & 0 & \\ 0 & 0 & 0 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & 0 & 0 & 0 \\ & 0 & & & -1 & -2 \\ & & & & 250 & -500 \end{bmatrix}. \tag{3.29}$$

So that

$$Q^* = \|Q\| = \frac{750}{15619} \tag{3.30}$$

A standard convergence analysis then verifies that (3.19) is second-order convergent if

$$192A^2X^2 + 24A^4X^4 + A^6X^6 + RA^6X^6G^* + X^6F^* + 512Q^* < 512 \tag{3.31}$$

where $G^* = \max_n \left|1 - x_n^2\right|$ and $F^* = \max_{0\le x\le x} \left|\partial f/\partial w(x)\right|$.

4. SIXTH-ORDER EIGENVALUE PROBLEMS

The numerical methods developed in §3 for the boundary value problem {(3.2),(3.3)} may be adopted to solve the following Bénard layer boundary value problems in Baldwin (1987a,1987b).

Problem 4.1 Baldwin considers the integration of the differential equation (1.1) over the interval [0,10], that is to say

$$(D^2 - A^2)^3 w(x) + RA^2(1 - x^2) w(x) = 0, \quad 0 < x < 10, \tag{4.1}$$

with the even-mode boundary conditions

$$\begin{gathered} w(0) = D^2\, w(0) = D^4\, w(0) = 0, \\ w(10) = D^2\, w(10) = D^4\, w(10) = 0. \end{gathered} \tag{4.2}$$

The eigenvalue problem {(4.1),(4.2)} is obtained from (3.2) with f = 0 and X = 10, and from (3.3) with $A_0 = A_2 = A_4 = B_0 = B_2 = B_4 = 0$. Therefore, f = 0 and b = 0 in (3.19) and the eigenvalues, RA^2, of {(4.1), (4.2)} may be obtained from the algebraic eigenvalue problem

$$A^{-2}h^{-6}G^{-1}(J_1^3 + p + 3A^2h^2J_1^2 + A^6h^6I)W = RW \tag{4.3}$$

in which the matrices J_1, G and P are defined in §§2.1,3.

Taking $h = 0.02$ (N=499), the eigenvalues were obtained using the NAG (Numerical Algorithms Group) library package F02AFF in an iterative technique. First of all, two values of A, say $A^{(1)}$ and $A^{(2)}$ are chosen arbitrarily and corresponding values of R, say $R = R(A^{(1)})$ and $R = R(A^{(2)})$, are determined from (4.3); let $R(\bar{A})$, be the smaller of $R^{(1)}$ and $R^{(2)}$. Next, choose a small number $\varepsilon > 0$ and find the value of $R = R(\bar{A}+\varepsilon)$ corresponding to the use of $A = \bar{A}+\varepsilon$ in (4.3); if $R(\bar{A}+\varepsilon)$ is smaller than $R(\bar{A})$ then refine ε and iterate again, otherwise compare R with $R(\bar{A}-\varepsilon)$, refine ε, and iterate again. This procedure, which is used to find the eigenvalue-pairs required, is repeated until the sequence of iterates converges.

The first three even-mode critical values of R and A are given in

Table 4.1, which includes the equivalent results of Baldwin (1987b, p.303). The results of Table 4.1 show that the computed results are smaller than the results of Baldwin, indicating lower minimum values of R and A for the onset of instability in a Bénard layer. Further experiments with smaller and larger values of h produce computed results which approach and recede from, respectively, the results of Baldwin (1987b). Refining the grid, and thus increasing N, is an expensive adjustment which could only be justified in situations demanding accuracy to the high number of significant figures claimed for the results in Baldwin (1987b).

Table 4.1 here

Proble 4.2 This eigenvalue problem consists of the differential equation (4.1) and the odd-mode boundary conditions

$$Dw(0) = D^3w(0) = D^5w(0) = 0$$
$$Dw(10) = D^3w(10) = D^5w(10) = 0 \qquad (4.4)$$

for which the method of §3 requires modification.

The finite difference method (3.7) may be applied for n = 4,5,...,N-3 but, in (4.1), special approximations to $w''(x_1)$, $w''(x_N)$, $w^{(iv)}(x_n)$, n=1,2,N-1,N and $w^{(Vi)}(x_1)$, n = 1,2,3,N-2 ,N-1 ,N utilizing (4.4) instead of (3.2) / (4.2) must be determined. They are assumed to have the forms

$$-w^{(vi)}(x_1) = h^{-6}(\alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 + \alpha_4 w_4 + \alpha_5 w_5 + \alpha_6 w_6 - \gamma_1 h w_0' - \gamma_2 h^3 w_0''' - \gamma_3 h^5 w_0^{(v)}), \qquad (4.5)$$

$$w^{(iv)}(x_1) = h^{-4}(\alpha_7 w_1 + \alpha_8 w_2 + \alpha_9 w_3 + \alpha_{10} w_4 + \alpha_{11} w_5 + \gamma_4 h w_0' + \gamma_5 h^3 w_0'''), \qquad (4.6)$$

$$-w''(x_1) = h^{-2}(\alpha_{12} w_1 + \alpha_{13} w_2 + \alpha_{14} w_3 + \alpha_{15} w_4 - \gamma_6 h w_0'), \qquad (4.7)$$

$$-w^{(vi)}(x_2) = h^{-6}(\beta_1 w_1 + \beta_2 w_2 + \beta_3 w_3 + \beta_4 w_4 + \beta_5 w_5 + \beta_6 w_6$$

$$-\delta_1 h w_0' - \delta_2 h^3 w_0'' - \delta_3 h^5 w_0^{(v)}) , \qquad (4.8)$$

$$w^{(iv)}(x_2) = h^{-4}(\beta_7 w_1 + \beta_8 w_2 + \beta_9 w_3 + \beta_{10} w_4 + \beta_{11} w_5 + \delta_4 h w_0' + \delta_5 h^3 w_0''), \qquad (4.9)$$

$$-w^{(vi)}(x_3) = h^{-6}(\varepsilon_1 w_1 + \varepsilon_2 w_2 + \varepsilon_3 w_3 + \varepsilon_4 w_4 + \varepsilon_5 w_5 + \varepsilon_6 w_6$$

$$-\theta_1 h w_0' - \theta_2 h^3 w_0'' - \theta_3 h w_0^{(v)}), \qquad (4.10)$$

$$-w^{(vi)}(x_{N-2}) = h^{-6}(\varepsilon_6 w_{N-5} + \varepsilon_5 w_{N-4} + \varepsilon_4 w_{N-3} + \varepsilon_3 w_{N-2} + \varepsilon_2 w_{N-1} + \varepsilon_1 w_N$$

$$-\theta_1 h w_{N+1}' - \theta_2 h^3 w_{N+1}'' - \theta_5 h^5 w_{N+1}^{(v)}) , \qquad (4.11)$$

$$-w^{(vi)}(x_{N-1}) = h^{-4}(\beta_6 w_{N-5} + \beta_5 w_{N-4} + \beta_4 w_{N-3} + \beta_3 w_{N-2} + \beta_2 w_{N-1} + \beta_1 w_N$$

$$-\delta_1 h w_{N+1}' - \delta_2 h^3 w_{N+1}'' - \delta_5 h^5 w_{N+1}^{(v)}) , \qquad (4.12)$$

$$w^{(vi)}(x_{N-1}) = h^{-4}(\beta_{11} w_{N-4} + \beta_{10} w_{N-3} + \beta_9 w_{N-2} + \beta_8 w_{N-1} + \beta_7 w_N$$

$$+\delta_4 h w_{N+1}' + \delta_5 h^3 w_{N+1}'', \qquad (4.13)$$

$$-w^{(iv)}(x_N) = h^{-6}(\alpha_6 w_{N-5} + \alpha_5 w_{N-4} + \alpha_4 w_{N-3} + \alpha_3 w_{N-2} + \alpha_2 w_{N-1} + \alpha_1 w_N$$

$$-\gamma_1 h w_{N+1}' - \gamma_2 h^3 w_{N+1}'' - \gamma_3 h^5 w_{N+1}^{(v)}), \qquad (4.14)$$

$$w^{(iv)}(x_N) = h^{-4}(\alpha_{11} w_{N-4} + \alpha_{10} w_{N-3} + \alpha_9 w_{N-2} + \alpha_8 w_{N-1} + \alpha_7 w_N$$

$$+\gamma_5 h w_{N+1}' + \gamma_5 h^3 w_{N+1}'') \qquad (4.15)$$

and

$$-w''(x_N) = h^{-2}(\alpha_{15} w_{N-3} + \alpha_{14} w_{N-2} + \alpha_{13} w_{N-1} + \alpha_{12} w_N - \gamma_6 h w_{N+1}' \qquad (4.16)$$

The 46 parameters α_i ($i = 1,\ldots,15$), β_i ($i = 1,\ldots,11$), γ_i ($i = 1,\ldots,6$),

δ_i (i = 1,…,5), ε_i (i = 1,….,6) and θ_i (i = 1,2,3) are chosen to give local truncation error (3.17) for n=1,2,3,N-2,N-1,N. The metod of undetermined coefficients givens

$$
\begin{array}{ll}
\alpha_1=14-751920/d_1, & \beta_1=-14+49220/d_2, \\
\alpha_2=-14+735065/d_1, & \beta_2=20-42150/d_2, \\
\alpha_3=6-336860/d_1, & \beta_3=-15+21925/d_2, \\
\alpha_4=-1+24870/d_1, & \beta_4=6-7918/d_2, \\
\alpha_5=-1+24870/d_1, & \beta_5=-1+1753/d_2, \\
\alpha_6=715/d_1, & \beta_6=-178/d_2, \\
& \\
\alpha_7=5-5896/d_3, & \beta_7=-4+616/d_4, \\
\alpha_8=-4+7888/d_3, & \beta_8=6-428/d_4, \\
\alpha_9=-1-3593/d_3, & \beta_9=-4+208/d_4, \\
\alpha_{10}=996/d_3, & \beta_{10}=1-61/d_4, \\
\alpha_{11}=-125/d_3, & \beta_{11}=8/d_4, \\
& \\
\alpha_{12}=2-48/d_5, & \gamma_1=15540/d_1, \\
\alpha_{13}=1+36/d_5, & \gamma_2=-16600/d_1, \\
\alpha_{14}=-16/d_5, & \gamma_3=34352/d_1, \\
\alpha_{15}=-3/d_5, & \gamma_4=240/d_3, \\
& \gamma_5=-936/d3, \\
\delta_1= -6720/d_2, & \gamma_6=12/d_5, \\
\delta_2= 6380/d_2, & \\
\delta_3= -118/d_2, & \varepsilon_1=6-60480/d_6, \\
\delta_4= -180/d_4, & \varepsilon_2=15+41985/d_6, \\
\delta_5= -16/d_4, & \varepsilon_3=20-21760/d_6, \\
& \\
\theta_1=18060/d_6, & \varepsilon_4=-15+7830/d_6, \\
\theta_2=-1800/d_6, & \varepsilon_5=6-1728/d_6, \\
\theta_3=-216/d_6, & \varepsilon_6=-1+175/d_6,
\end{array}
\tag{4.17}
$$

where

$$d_1 = 56630,\ d_2 = 5663,\ d_3 = 1715,\ d_4 = 343,\ d_5,= 25,\ d_6 = 33978\ , \quad (4.18)$$

and it follows that the eigenvalues of {(4.1),(4.4)} are obtained by solving the algebraic eigenvalue problem

$$A^{-2}h^{-6}G^{-1}[J_1^3 + P_3 + 3A^2h^2(J_1^2 + P_2) + 3A^4h^4(J_1 + P_1+)A^6h^6]W = RW. \quad (4.19)$$

It is seen from (4.5)-(4.18) that the matrices P_1, P_2 and P_3 of order N are given by

$$p_1 = d_5^{-1}\begin{bmatrix} -48 & 36 & -16 & 3 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & \cdot & \\ & & & 0 & 0 & 0 & 0 \\ & 0 & & 3 & -16 & 36 & -48 \end{bmatrix}$$

$$P_2 = d_3^{-1}\begin{bmatrix} -8596 & 7888 & -3593 & 996 & -125 & & & \\ 3080 & 2140 & 1040 & 305 & 40 & 0 & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 40 & 305 & 1040 & 2140 & 3080 \\ & & & -125 & 996 & -3593 & 7888 & -8596 \end{bmatrix}$$

and

$$P_3 = d_7^{-7}\begin{bmatrix} -2255760 & 2205195 & -1010580 & 74610 & -38100 & 2145 & & \\ 1476600 & -1264500 & 657750 & -237540 & 52590 & -5340 & & \\ -302400 & 209925 & -108800 & 39150 & -8640 & 875 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & & \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 875 & -8640 & 39150 & -108800 & 209925 & -302400 \\ & 0 & & & & & & \\ & & -5340 & 52590 & -237540 & 657750 & -1264500 & 1476600 \\ & & 2145 & -38100 & 74610 & -1010580 & 2205195 & -2255760 \end{bmatrix}$$

where d_7 = 169890, and G = diag{$(1-x_n^2)$} as in Problem 4.1.

Taking h = 0.02 as, before, the routine, outlined for Problem 4.1 was used to obtain the eigenvalues. The first three odd-mode critical values of R and A are given in Table 4.2, which includes the equivalent results of Baldwin (1987b, p.303). The computed results are lower than those of Baldwin (1987b): choosing a smaller value of h would narrow the gaps between the two sets of results.

Table 4.2 here

Problem 4.3 The differential equation here is given by

$$(D^2-A^2)^3\, w(x)+RA^2\,(1-x)w(x)=0,\quad 0 < x < 10 \qquad (4.20)$$

with the ***free-free*** boundary conditions (4.2) (Baldwin, 1987a).

This eigenvalue problem is very similar to that of Problem 4.1 and clearly (4.3) may be used to obtained the eigenvalues: in (4.3), now, G = diag{$(1-x_n)$}.

Taking h = 0.02 once again and using the computational routine outlined for Problem 4.1, yields the critical values of R and A, the first four of which are given in Table 4.3. This table includes the equivalent results of Baldwin (1987a, p. 152). The difference between the results may again be explained by the use of a low-order numerical method: the numerical results; reported are, again, lower than the estimates of Baldwin (1987a).

Table 4.3 here

Problem 4.4 The differential equation in this eigenvalue problem is that in (4.20) while the boundary conditions are given by

$$w(0) = Dw(0) = w(10) = Dw(10) = 0\ , \qquad (4.21)$$

$$(D^2-A^2)^2w(0) = (D^2-A^2)^2w(10) = 0 , \qquad (4.22)$$

(the ***rigid-rigid*** boundary conditions, Baldwin (1987a)).

These boundary conditions do not permit the use of the numerical method Developed in §3. Instead the following second-order "splitting" approach, on the same discretizaiton of the interval $0 \le X \le 10$ is proposed.

Introduce an "intermediate function" v(x) defined by

$$v(x) = (D^2-A^2)^2w(x) . \qquad (4.23)$$

Then, from (4.22),

$$v(0) = v(10) = 0 \qquad (4.24)$$

and w(x) may be determined by solving the fourth-order boundary-value problem {(4.23),(4.21)}. To this end, the second-order approximants to $D^4w(x)$ and $D^2w(x)$, given by (3.5) and (3.6), are used to replace the derivatives in (4.23) at the general mesh point $x_n = nh$ $(n = 2,3,... ,N-1)$. This gives, from (4.23)

$$\left[\frac{w_{n-2} - 4w_{n-1} + 6w_n - 4w_{n+1} + w_{n+2}}{h^4}\right] - 2A^2\left[\frac{w_{n-1} - 2w_n + w_{n+1}}{h^2}\right] + A^4w_n - v_n = 0, \quad (4.25)$$

for which the local truncation error is

$$t_n = \frac{1}{6}h^2\left[- A^2w^{(iv)}(x_n) + w^{(vi)}(x_n)\right] + 0(h^4) \qquad (4.26)$$

In order to use the matrix J_1^2 , special formulae, which use the elements of the first and last rows of J_1^2, must be constructed. To achieve this, equation (4.23) is approximated by the equation

$$\left[\frac{5w_1 - 4w_2 + w_3}{h^4}\right] - \left[\frac{\alpha w_1 + \beta w_2 + \gamma w_3 + \delta w_4 + \in w_5 + \varphi w_0 + \varphi_0'}{h^4}\right]$$

$$- 2A^2\left[\frac{w_0 - 2w_1 + w_2}{h^2}\right] + A^4w_1 - v_1 = 0 \quad (4.27)$$

for n=1 and by the equation

$$\left[\frac{w_{N-2} - 1w_{N-1} + 5w_N}{h^4}\right] - \left[\frac{\in w_{N-4} + \delta w_{N-3} + \gamma w_{N-2} + \beta w_{N-1} + \alpha w_N + \varphi w_{N+1} - \varphi w'_{N+1}}{h^4}\right]$$

$$- 2A^2\left[\frac{w_{N-1} - 2w_N + w_{N-1}}{h^2}\right] + A^4 w_N - v_N = 0 \qquad (4.28)$$

for n N, where a, α, β, γ, δ, ε, φ and ψ are parameters (with different values to those jn earlier sections of the paper).

The method of undetermined coefficients verifies that, choosing the values

$$\alpha = -\frac{65}{12}, \beta = \frac{10}{3}, \gamma = \frac{10}{9}, \delta = \frac{1}{4}, \varepsilon = -\frac{1}{30}, \varphi = \frac{224}{45}, \psi = \frac{13}{3}, \qquad (4.29)$$

achieves the aim of involving J_1^2 and ensures that t_1, and t_N, the local truncation errors at x_1 and x_N, have principal parts as indicatedin (4.26).

Equations. (4.27), (4.25), (4.28) maybe written in matrix-vector form as

$$J_1^2 W + 2A^2h^2J_1W + h^4A^4W - MW - h^4V + b = 0 , \qquad (4.30)$$

Where $W=[w_1, w_2, ..., w_n]^T$, $V = [v_1, v_2, ..., v_N]^T$ J_1 in given by(2.10),

$$M = \begin{bmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & 0 & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & \varepsilon & \delta & \gamma & \beta & \alpha \end{bmatrix} \qquad (4.31)$$

and b Θ 0 from 4.21)). Equation 4.30) then gives

$$V = h^{-4}(J_1^2 + 2A^2h^2J_1 + h^4A^4I - M)W. \qquad (4.32)$$

Returning now to equations 4.20) and (4.23) it follows that

$$(D^2 - \wedge^2)v(x) + RA^2(1-x)w(x) = 0 , \quad 0 < x < 10 \qquad (4.33)$$

and the boundary value problem {(4.33),(4.24)} may be solved using the second order method

$$-\left[\frac{-v_{n-1}+2v_n-v_{n+1}}{h^2}\right]-A^2v_n+RA^2(1-x_n)w_n=0 \qquad (4.34)$$

in which n = 1,2,...,N (note v_0, = v_{N+1} = 0 from (4.24)). The local truncation error t_n at the point $x = x_n$, (n = 1,2,...,N) is given by

$$t_n = \frac{1}{12}h^2v^n(x_n) + 0(h^4) \qquad (4.35)$$

Written in matrix-vector form, equation (4.34) becomes

$$-(J_1+A^2h^2I)v + RA^2GW = 0 \ , \qquad (4.36)$$

in which G = diag{(l-x_n)} as in Problem 4.3. Substituting for the intermediate vector V from (4.32), equation (4.36) becomes

$$h^{-6}G^{-1}(J_1+A^2h^2I)\left[(J_1+A^2h^2I)^2-M\right]W = RA^2W \qquad (4.37)$$

and it follows that the eigenvalues of the boundary-value problem {(4.20),(4.21),(4.22)} coincide with the eigenvalues of the matrix

$$h^{-6}G^{-1}(J_1+A^2h^2I)\,[\,(J_1+A^2h^2I)-M]\,. \qquad (4.38)$$

Writing (4.37) as

$$A^{-2}h^{-6}G^{-1}(J_1+A^2h^2I)\,[\,(J_1+A^2h^2I)-M]W = RW \qquad (4.39)$$

the computational routine outlined for Problem 4.1, using h = 0.02 once more, gives the critical values of R and A. The first four of these pairs are given in Table 4.4 which also contains the corresponding values calculated by Baldwin (1987a, p.153) .

Table 4.4 here

As in Problems 4.1, 4.2 and 4.3 the results yielded by the numerical method are all lower than the corresponding values of Baldwin (1987a). The numerical methods therefore predict that the onset of instability in a Bénard layer occurs for lower minimum values of the Rayleigh number, R, and associated horizontal wave number, A, than is predicted by the global phase-integral methods used by Baldwin (1987a,1987b). The use of a finer discretization does, however, increase the predictions of the numerical method, nearer to those of Baldwin (1987a,1987b).

REFERENCES

Agarwal, R.P. 1986 *Boundary value problems for higher order differential equations*. Singapore: World Scientific.

Baldwin, p. 1987a *Applicable Anal.* 24, 117-156.

Baldwin, P. 1987b *Phil. Trans. R. soc. London.* A 322, 281-305.

Chawla, M.M. &, Katti, C.P. 1979 *BIT* 19, 27-33.

Glatzmaier, G.A. 1985 *Geophys. and Astrophys. Fluid Dyn.* 31, 137-150.

Toomre, J., Zahn, J.R., Latour, J. & Spiegel, E.A. 1976 *Astrophys. J.* 207, 545-563.

Twizell, K.H. 1988 *International series of numerical mathematics,* 86. Basel: Birkäiiuser Verlag (Agarwal, R.P., Chow, Y.M. & Wilson, S.J., eds.).

Table 2.1 Error norms for second-order methods

N	Method A	Method B
7	0.432E-3	0.435E-3
15	0.105E-3	0.105E-3
31	0.259E-4	0.259E-3

(The theoretical solution is in the interval $0 \leq x \leq$ 0.116 approximately for $o \leq x \leq 1$.

Table 2.2 Error norms for fourth-order methods

N	Method C	Method D	EXT(A,2,4)	EXT(B,2,4)
7	0.844E-5	0.997E-5	0.448E-5	0.550E-5
15	0.625E-6	0.651E-6	0.332E-6	0.357E-6
31	0.393E-7	0.394E-7	0.196E-7	0.206E-7

Table 2.3 Error norms for the fifth order extrapolation of Method A

N	EXT(A,2,5)
7	0.947E-7
15	0.369E-8
31	0.522E-7

Table 2.4 Error norms for sixth-order methods

N	Method E	Method F	EXT(B,2,6)	EXT(C,4,6)	EXT(D,4,6)
7	0.241E-6	0.496E-5	0.251E-7	0.105E-6	0.296E-7
15	0.756E-9	0.135E-10	0.808E-9	0.906E-9	0.566E-9
31	0.225E-7	0.123E-10	0.152E-7	0.439E-8	0.176E-8

Table 2.5 Error norms for eight-order methods

N	Method G	EXT(D,4,8)	EXT(E,6,8)	EXT(F,6,8)
7	0.463E-5	0.273E-8	0.306E-8	0.787E-7
15	0.720E-9	0.219E-6	0.349E-10	0.123E-10
31	0.975E-11	0.238E-8	0.613E-8	0.113E-10

Table 2.6 Error norms for ninth-order methods

N	EXT(F,6,9)	EXT(G,8,9)
7	0.135E-8	0.174E-7
15	0.572E-9	0.126E-10
31	0.171E-7	0.924E-9

Table 4.1 First three even-mode (n=2,4,6) critical values for Problem 4.1 with h = 0.02

	Baldwin (1987b)		Computed results	
n	R	A	R	A
2	411.720155	1.6791	411.515421	1.6790
4	11382.695328	3.8130	11356.557010	3.8112
6	68778.117	5.971	68397.491	5.965

Table 4.2 First three odd-mode (n=1,3,5) critical values for Problem 4.2 with h = 0.02

	Baldwin (1987b)		Computed results	
n	R	A	R	A
1	9.78136567	0.72605	9.77836945	0.72603
3	3006.709534	2.7379	3003.053226	2.7374
5	30916.2534	4.8916	30800.6998	4.8882

Table 4.3 First four critical values (n=1,2,3,4) for Problem 4.3 with h = 0.02

	Baldwin (1987a)		Computed results	
n	R	A	R	A
1	550.790984	1.5928	550.539887	1.5925
2	16380.4958	3.7529	16342.5918	3.7513
3	99807.1956	5.9031	99239.9841	5.8980
4	344966.91	8.051	341332.66	8.036

Table 4.4. First four critical values (n=1,2,3,4) for Problem 4.4 with h = 0.02

	Baldwin (1987a)		Computed results	
n	R	A	R	A
1	1178.594406	2.0337	1178.183739	2.0335
2	2893.6831	4.1829	22846.6806	4.1811
3	123586.84	6.322	122930.96	6.314
4	403656.60	8.466	399600.86	8.449