Consensusability of Discrete-Time Multi-Agent Systems under Binary Encoding with Bit Errors ⋆

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Abstract

This paper is concerned with the consensusability problem for a class of discrete-time multi-agent systems (DT-MASs). A binary encoding scheme (BES) is employed during the data transmission, and only a finite-length binary bit string is transmitted due primarily to the limited network bandwidth in practice. Meanwhile, the binary bit string, transmitted via memoryless binary symmetric channels, might suffer from random bit errors with a certain probability. The purpose of this paper is to derive some consensusability conditions, under which there must exist a distributed controller such that the mean-square bounded consensus is achieved by the considered DT-MASs with BESs subject to random bit errors. To this end, the statistical properties are first revealed for the BES-induced quantization errors and the random bit errors. Then, by resorting to the solvability analysis of a modified Riccati inequality, some sufficient conditions are, respectively, derived to ensure the mean-square bounded consensus under the undirected communication topology in two scenarios: 1) identical bit-error rates (BERs); and 2) non-identical BERs. In particular, a necessary and sufficient condition is established for a single input case with identical BERs. Furthermore, the ultimate upper bound of the consensus errors in the mean-square sense is established to examine the effects of the length of bit string and the BERs. Finally, an illustrative simulation example is provided to validate the effectiveness of the theoretical results.

Key words: Discrete-time multi-agent systems, consensusability, binary encoding scheme, bit-error rate.

1 Introduction

Over the past few decades, the coordination issues of multi-agent systems (MASs) have received unprecedented attention owing mainly to their clear engineering insights in many practical applications (e.g. formation control of mobile-robot systems, distributed optimization, and data privacy preserving), see Chen, Ding, Dong, Wei, & Ge (2021); Guo, Li, & Xie (2020); Mo & Murray (2017); Xie, You, Tempo, Song, & Wu (2018) and the references therein. As one of the most important coordination behaviors of MASs, the consensus has long been a fundamental research problem in the field of control theory and control engineering. Roughly speaking, consensus control aims to drive all the agents to reach an agreement on the variables of interest. Because of the theoretical importance and practical significance, considerable research interest has been devoted to the consensus control problem of MASs, and some representative results have been reported in the literature, see e.g. Ding, Wang, & Han (2021); Gao, Wang, He, & Han (2020); Hu, Zhang, Liu, & Yu (2021); Liu, Lam, Yu, & Chen (2016); Qin, Zhang, Zheng, & Kang (2019); Wen, Zhao, Duan, Yu, & Chen (2016); Xu, Mo, & Xie (2020); Xu, Xiao, & Xie (2016); Xu, Hu, Ho, & Feng (2020); Xu, Wang, & Ho (2018); Zhang, Yue, Dou, Zhao, & Xie (2019).

In order to explore the existence of consensus protocols for MASs, the so-called consensusability of MASs has begun to play a paramount role in the study of consensus problems. More specifically, the consensusability aims to investigate some conditions under which there exists a consensus protocol. So far, some pioneering results have been reported on the consensusability problem of MASs (see e.g. Ma & Zhang (2010); Ren & Beard (2005); Trentelman, Takaba, & Monshizadeh (2013); You & Xie (2011); Yu, Chen, & Cao (2010) and the references therein). For example, the consensusability problem has been addressed in Yu, Chen, & Cao (2010) for second-order continuous-time MASs under the directed graph, which has revealed the importance of the eigenvalues (including both the real and imaginary parts) of the Laplacian matrix in achieving consensus. For discrete-time MASs (DT-MASs), some necessary and sufficient conditions for consensusability have been provided in Ma & Zhang (2010) and You & Xie (2011) by considering the system dynamics and communication topology. It is worth noting that most of the existing results have been obtained with a prerequisite of perfect communication. Nevertheless, in practical engineering, the information exchange over communication channels may suffer from

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the bandwidth constraints (Hu, Cui, Lv, Chen, & Zhan , 2020; Xu, Wang, & Zhang , 2021; Zou, Wang, Hu, & Zhou , 2020), signal-to-noise ratio constraints (Li, Fu, Xie, & Zhang , 2011), packet dropouts (Mao, Sun, Yi, Liu, & Ding , 2021; Xu, Zhang, & Xie , 2019), bit errors (Leung, Seneviratne, & Xu , 2015), and so on. As such, it makes practical sense to investigate the consensusability issue of the MASs over imperfect communication channels, which constitutes the first motivation of our current study.

With rapid technological developments of the communication networks, the digital communication has been becoming an indispensable component of MASs due to its distinctive advantages over the traditional analog one in terms of reliability, flexibility, robustness, security, and cost of cheapness. As one of the most popular schemes in the digital communications, the binary encoding scheme (BES) has attracted a surge of research interest owing to its high reliability and easy implementation (see e.g. Gao, Deng, Ren, & Hu (2019); Li, Fu, Xie, & Zhang (2011); Wang, Wang, Wei, & Alsaadi (2019)). On the other hand, it is often the case, in practical engineering, that the limited communication bandwidth may deteriorate the quality of signal transmissions and further degrade the system performance (Carli, Bullo, & Zampieri , 2010; Liu & Wang , 2021; Zhao, Wang, Wei, & Han , 2020; Zou, Wang, Hu, & Zhou , 2020). Particularly, the inter-agent communication of the MASs based on finite-length binary bit strings is likely to result in more information loss. As a result, it is of particular significance to improve the utilization of limited communication resource by resorting to a suitable BES, and reveal the effect from such a scheme on the consensusability of the DT-MASs.

When it comes to the consensus control issues of MASs over binary symmetric channels (BSCs), it has been implicitly assumed, in most existing results, that the inter-agent data transmission is bit-error free. Such an assumption is, however, a bit too impractical since the binary bits might suffer from the channel-imperfection-induced random bit errors (Aysal & Barner , 2008; Leung, Seneviratne, & Xu , 2015; Wu & Cheng , 2009). The random bit errors, if not well dealt with, may largely deteriorate the communication quality. As mentioned in Aysal & Barner (2008); Wu & Cheng (2009); Zou, Wang, & Zhou (2020), the communication within the IEEE 802.15.4 type wireless sensor networks is regarded as a failure when the bit-error rate (BER) reaches 1%. Although the bit-error problem can be addressed by virtue of some highly sophisticated error control techniques (e.g., error detecting and error correcting codes), the extra communication overhead may be another concern, especially for the MASs. As such, it is more preferable to establish a statistical model for the bit errors, and further analyze the statistical properties of the received signal under the proposed BES subject to random bit errors, thereby mitigating the impact of bit errors on the system performance.

Motivated by the above observations, the main objective of this paper is to establish a comprehensive consensusability analysis framework for a class of DT-MASs under BESs with random bit errors. Three technical difficulties are identified as follows: *1) how to develop a unified mathematical model accounting for DT-MASs with BES? 2) how to establish a theoretical framework on the existence of the common consensus controller for DT-MASs under BES with random occurring bit errors? and 3) how to build a quantitative relationship among the upper bound of consensus errors, BERs and the length of bit strings.* To overcome the above three difficulties, we first focus our attention on the statistical analysis of the received signal over memoryless BSCs. Then, based on the proposed BESs, some sufficient conditions are derived via analyzing the existence of the solution of the modified Riccati inequality (MRI). In addition, by resorting to an iterative calculation method, a tight upper bound of the consensus error is derived to show the relationship between the length of bit strings and the system performance.

The main contributions of this paper are highlighted as follows: *1) a unified design framework is developed by taking into account both the encoding-decoding technique and bit errors, thereby catering for more general engineering practices; 2) sufficient conditions are obtained, in terms of the system dynamics, BERs, and the communication topology, to ensure the consensusability of DT-MASs with identical and non-identical BERs; 3) a necessary and sufficient condition of the consensusability is further established for a single input case with identical BERs; and 4) a tight relationship concerning the length of bit strings, BERs and the upper bound of consensus errors is derived for DT-MASs with a given controller gain.*

Notation: Throughout this paper, the notation used is fairly standard unless otherwise clarified. $\mathbb{R}$ ($\mathbb{C}$), $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ are, respectively, the space of real (complex) numbers, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. $\|x\|$ and $|x|$ describe, respectively, the Euclidean norm and the modulus of a vector x. In addition, $\mathbb{E}\{x\}$ and $\mathbb{D}\{x\}$ are the expectation and variance of the stochastic variable x. $\mathbb{E}\{x|y\}$ and $\mathbb{D}\{x|y\}$ are conditional expectation and variance. $\mathbb{P}\{\alpha\}$ is the occurrence probability of the event α. $\text{diag}\{\cdots\}$ stands for a block-diagonal matrix, $\text{diag}_M\{A_l\} = \text{diag}\{A_1, A_2, \cdots, A_M\}$, and $\text{col}_N\{x_i\}$ represents $[x_1^T, \cdots, x_N^T]^T$. $\det(A)$ and $\rho(A)$ denote, respectively, the determinant and spectral radius of matrix A.

2 Problem Formulation and Preliminaries

In this paper, the communication topology of the considered MASs is characterized by a fixed graph $\mathscr{G} = (\mathcal{V}, \mathcal{E}, \mathscr{A})$, where $\mathcal{V} = \{1, 2, \cdots, M\}$ is the set of agents, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set of paired agents, and $\mathscr{A} = [l_{ij}]_{M\times M}$ is the adjacency matrix with $l_{ij} > 0$ if the pair $(j, i) \in \mathcal{E}$ and $l_{ij} = 0$ otherwise. The self-edge (i, i) is not allowed, i.e., the pair $(i, i) \notin \mathcal{E}$. Furthermore, the neighborhood set of agent i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$. The graph-related Laplacian matrix L is defined as $L = \mathcal{D} - \mathscr{A}$, where the degree matrix $\mathcal{D}$ is $\mathcal{D} = \text{diag}\{\varsigma_{in}^1, \varsigma_{in}^2, \cdots, \varsigma_{in}^M\}$ with $\varsigma_{in}^i = \sum_{j\in\mathcal{N}_i} l_{ij}$.

Consider a class of DT-MASs with M agents. The dynamics of each agent $i \in \mathcal{V}$ is modeled by the following discrete-time linear system:

$$x_{i,k+1} = Ax_{i,k} + Bu_{i,k}, \tag{1}$$

where $x_{i,k} \in \mathbb{R}^n$ and $u_{i,k} \in \mathbb{R}^p$ are, respectively, the state vector and the control input of agent i. A and B are known constant matrices with compatible dimensions. Without loss of generality, assume that the matrix B is full-column and (A, B) is controllable.

The *ideal consensus protocol* is constructed as follows:

$$u_{i,k} = K \sum_{j\in\mathcal{N}_i} l_{ij}(x_{i,k} - x_{j,k}) \triangleq K\xi_{i,k} \tag{2}$$

where $K \in \mathbb{R}^{p\times n}$ is the controller gain to be determined and $\xi_{i,k} \in \mathbb{R}^n$, regarded as the relative state information, is collected by a smart sensor.

In this paper, the collected information $\xi_{i,k}$ is digitalized before being transmitted to the controller of agent i, and a BES is employed to perform such a digital communication between the smart sensor and the controller. To be more specific, each component of the collected relative state information is first normalized and then encoded into a finite-length binary bit string. Subsequently, these binary bit strings are transmitted to the controller via a memoryless BSC i, where the binary bits may suffer from random bit flipping due probably to the channel imperfection or noises. Finally, the received binary bit strings are decoded for the consensus control.

Denote $\xi^l_{i,k} \in \mathbb{R}$ as the l-th element of vector $\xi_{i,k}$. Generally speaking, such an element $\xi^l_{i,k}$ is usually bounded and belongs to the interval $[-S, S]$ due to the limited operation range of practical systems, where S is a known positive scalar. For convenience, the scalar signal $\xi^l_{i,k}$ is first normalized to the range $[0, 1]$ by the following linear transformation

$$s^l_{i,k} = \frac{1}{2} + \frac{\xi^l_{i,k}}{2S}. \tag{3}$$

Here, $s^l_{i,k} \in [0, 1]$ is a normalized signal, and can be rewritten as a binary expansion:

$$s^l_{i,k} = \sum_{j=1}^{\infty} 2^{-j} b^{i^l}_{j,k} \tag{4}$$

where $b^{i^l}_{j,k} \in \{0, 1\}$ is the j-th binary bit determined by $s^l_{i,k}$.

Because of the limited communication bandwidth, the signal $s^l_{i,k}$ is encoded into an $(m+1)$-bit string. In light of (4), the first m fixed bits are

$$b^{i^l}_{1,k},\ b^{i^l}_{2,k},\ \cdots,\ b^{i^l}_{m,k}$$

and the $(m + 1)$-th bit, denoted as $\alpha^{i^l}_{m+1,k} \in \{0, 1\}$, is randomly generated and has the following statistical properties

$$\mathbb{P}\{\alpha^{i^l}_{m+1,k} = 1\} = \bar{\alpha}^{i^l}_k,\ \mathbb{P}\{\alpha^{i^l}_{m+1,k} = 0\} = 1 - \bar{\alpha}^{i^l}_k, \tag{5}$$

where

$$\bar{\alpha}^{i^l}_k = 2^m\Big(s^l_{i,k} - \sum_{j=1}^{m} 2^{-j} b^{i^l}_{j,k}\Big).$$

Note that $\bar{\alpha}^{i^l}_k$ is known and predetermined by both $s^l_{i,k}$ and the first encoded m bits (named as the most significant m bits).

Now, the element $s^l_{i,k}$ is encoded into the following binary bit string:

$$\Gamma^l_{i,k} = \{b^{i^l}_{1,k},\ b^{i^l}_{2,k},\ \cdots,\ b^{i^l}_{m,k},\ \alpha^{i^l}_{m+1,k}\}, \tag{6}$$

which is equivalent to

$$\psi^l_{i,k} = \sum_{j=1}^{m} 2^{-j} b^{i^l}_{j,k} + 2^{-m}\alpha^{i^l}_{m+1,k} \triangleq \psi_k(s^l_{i,k}, m). \tag{7}$$

In the above equation, $\psi_k(s^l_{i,k}, m)$ is named as a message function, which is utilized to quantize the $s^l_{i,k}$ into a discrete message.

In what follows, the binary bit string $\Gamma^l_{i,k}$ is transmitted via a memoryless BSC i. Due to the effect of channel imperfection or noises, each bit in such a string might be flipped with a small probability (named as the crossover probability). As a result, the *received bit string* is described by

$$\tilde{\Gamma}^l_{i,k} \triangleq \{\tilde{b}^{i^l}_{1,k}, \tilde{b}^{i^l}_{2,k}, \cdots, \tilde{b}^{i^l}_{m,k}, \tilde{\alpha}^{i^l}_{m+1,k}\}, \tag{8}$$

where

$$\begin{aligned}
&\tilde{b}^{i^l}_{j,k} = \beta^{i^l}_{j,k}(1 - b^{i^l}_{j,k}) + (1 - \beta^{i^l}_{j,k}) b^{i^l}_{j,k},\\
&\tilde{\alpha}^{i^l}_{m+1,k} = \beta^{i^l}_{m+1,k}(1 - \alpha^{i^l}_{m+1,k}) + (1 - \beta^{i^l}_{m+1,k})\alpha^{i^l}_{m+1,k}.
\end{aligned}$$

Here, the stochastic variable $\beta^{i^l}_{j,k}$ $(j = 1, 2, \cdots, m+1)$ is characterized by a Bernoulli distributed white sequence satisfying

$$\mathbb{P}\{\beta^{i^l}_{j,k} = 1\} = p_i,\ \mathbb{P}\{\beta^{i^l}_{j,k} = 0\} = 1 - p_i,$$

where $p_i \in [0, 1]$ is the crossover probability (i.e., BER) of the memoryless BSC i. Clearly, the j-th bit is flipped if $\beta^{i^l}_{j,k} = 1$. Furthermore, it is worth mentioning that the crossover probability is assumed to be the same for all bits transmitted by the memoryless BSC i.

Finally, the received bit string $\tilde{\Gamma}^l_{i,k}$ is decoded according to the following relationship

$$\tilde{\psi}^l_{i,k} = \sum_{j=1}^{m} 2^{-j} \tilde{b}^{i^l}_{j,k} + 2^{-m}\tilde{\alpha}^{i^l}_{m+1,k} \triangleq \tilde{\psi}_k(s^l_{i,k}, m), \tag{9}$$

and the *received relative information* $\check{\xi}^l_{i,k}$ is described by

$$\check{\xi}^l_{i,k} = 2S\tilde{\psi}^l_{i,k} - S. \tag{10}$$

For brevity, denote

$$\begin{aligned}
&\check{\xi}_{i,k} = \mathrm{col}_n\{\check{\xi}^j_{i,k}\}, \quad \check{\xi}_k = \mathrm{col}_M\{\check{\xi}^T_{i,k}\},\\
&u_k = \mathrm{col}_M\{\check{\xi}^T_{i,k}\}, \quad x_k = \mathrm{col}_M\{x_{i,k}\}.
\end{aligned}$$

Due to the implementation of the proposed BES, only the decoded information $\check{\xi}_{i,k}$ is available at the receiver side, and thus the *actual consensus protocol* for agent i is constructed as follows:

$$u_{i,k} = K\check{\xi}_{i,k}. \tag{11}$$

Then, the augmented form of the considered DT-MASs (1) with the control law (11) can be written as

$$\begin{aligned} x_{k+1} &= (I_M \otimes A)x_k + (I_M \otimes B)u_k \\ &= (I_M \otimes A)x_k + (I_M \otimes BK)\check{\xi}_k. \end{aligned} \tag{12}$$

Before proceeding further, the following assumption and definition are introduced.

Assumption 1 *All the eigenvalues of A are either on or outside the unit disk.*

Definition 1 *Under an undirected network topology $\mathscr{G}$, the considered DT-MASs (1) with the control law (11) are able to achieve mean-square bounded consensus under any known initial condition x_0, if there exists a consensus gain K such that the consensus error satisfies*

$$\lim_{k\to\infty} \mathbb{E}\{\|x_k - 1/M(\boldsymbol{11}^T \otimes I_n)x_k\|^2\} \leq \theta(m, p_i, S) \tag{13}$$

where $\theta(m, p_i, S) > 0$ is a predetermined bounded constant that is related to the length of bit string m, crossover probability p_i, and signal amplitude S.

The objectives of this paper are threefold: 1) analyze the statistical properties of quantization errors and bit errors; 2) disclose the consensusability condition in terms of the BER, the system dynamics and the connection topology; and 3) calculate the ultimate upper bound of the consensus error in the mean-square sense.

3 Main Results

3.1 Statistical analysis of BES

Let us first analyze the statistical property of quantization errors. To this end, define the quantization error $\varrho^l_{i,k}$ as

$$\begin{aligned} \varrho^l_{i,k} &= (2S\psi^l_{i,k} - S) - \xi^l_{i,k} \\ &= (2S\psi^l_{i,k} - S) - (2Ss^l_{i,k} - S) \\ &= 2S(2^{-m}\alpha^{i^l}_{m+1,k} - 2^{-m}\bar{\alpha}^{i^l}_k) \\ &\triangleq 2Sq^l_{i,k}. \end{aligned} \tag{14}$$

Recalling (5), one has

$$\mathbb{E}\{\alpha^{i^l}_{m+1,k} | s^l_{i,k}\} = \bar{\alpha}^{i^l}_k. \tag{15}$$

Then, it is not difficult to verify that

$$\begin{aligned} \mathbb{E}\{\varrho^l_{i,k}|s^l_{i,k}\} &= 2S \times \mathbb{E}\{2^{-m}\alpha^{i^l}_{m+1,k} - 2^{-m}\bar{\alpha}^{i^l}_k\} = 0, \\ \mathbb{D}\{\varrho^l_{i,k}|s^l_{i,k}\} &= 4S^2 \times \mathbb{E}\{(2^{-m}\alpha^{i^l}_{m+1,k} - 2^{-m}\bar{\alpha}^{i^l}_k)^2\} \\ &= 4S^2 \times 2^{-2m}\bar{\alpha}^{i^l}_k(1 - \bar{\alpha}^{i^l}_k) \\ &\leq 2^{-2m}S^2. \end{aligned} \tag{16}$$

The following lemma presents some significant statistical properties of the bit errors.

Lemma 1 *Assume that the signal $\xi^l_{i,k}$ is first encoded into the signal $\psi^l_{i,k}$ (corresponding to a binary bit string $\Gamma^l_{i,k}$) and then sent via a memoryless BSC i with crossover probability p_i. The received signal $\tilde{\psi}^l_{i,k}$ (corresponding to the bit string $\tilde{\Gamma}^l_{i,k}$) is with the mean and variance determined by*

$$\begin{cases} \mathbb{E}\{\tilde{\psi}^l_{i,k}|\psi^l_{i,k}\} = (1-2p_i)\psi^l_{i,k} + p_i, \\ \mathbb{D}\{\tilde{\psi}^l_{i,k}|\psi^l_{i,k}\} = \dfrac{1}{3}p_i(1-p_i)(1+2^{-2m+1}). \end{cases} \tag{17}$$

Moreover, the mean and variance of the decoded signal $\check{\xi}^l_{i,k}$ given $\xi^l_{i,k}$ satisfy

$$\begin{cases} \mathbb{E}\{\check{\xi}^l_{i,k}|\xi^l_{i,k}\} = (1-2p_i)\xi^l_{i,k}, \\ \mathbb{D}\{\check{\xi}^l_{i,k}|\xi^l_{i,k}\} \leq \dfrac{4}{3}p_i(1-p_i)\Big(1+2^{-2m+1}\Big)S^2 \\ \qquad\qquad + 2^{-2m}(1-2p_i)^2S^2. \end{cases} \tag{18}$$

proof: Let us calculate the conditional expectation of the signal $\tilde{\psi}^l_{i,k}$ given $\psi^l_{i,k}$ as follows:

$$\begin{aligned} &\mathbb{E}\left\{\tilde{\psi}^l_{i,k}|\psi^l_{i,k}\right\} \\ &= \mathbb{E}\Big\{\sum_{j=1}^m 2^{-j}\tilde{b}^{i^l}_{j,k} + 2^{-m}\tilde{\alpha}^{i^l}_{m+1,k}\Big|\psi^l_{i,k}\Big\} \\ &= \mathbb{E}\Big\{\sum_{j=1}^m 2^{-j}\big(\beta^{i^l}_{j,k}(1-b^{i^l}_{j,k}) + (1-\beta^{i^l}_{j,k})b^{i^l}_{j,k}\big) \\ &\quad + 2^{-m}\big(\beta^{i^l}_{m+1,k}(1-\alpha^{i^l}_{m+1,k}) \\ &\quad + (1-\beta^{i^l}_{m+1,k})\alpha^{i^l}_{m+1,k}\big)\Big\} \\ &= (1-2p_i)\psi^l_{i,k} + p_i. \end{aligned}$$

The corresponding variance is calculated by

$$\begin{aligned} &\mathbb{D}\{\tilde{\psi}^l_{i,k}|\psi^l_{i,k}\} \\ &= \mathbb{E}\{(\tilde{\psi}^l_{i,k}|\psi^l_{i,k})^2\} - \big(\mathbb{E}\{\tilde{\psi}^l_{i,k}|\psi^l_{i,k}\}\big)^2 \\ &= \mathbb{E}\Big\{\big(\sum_{j=1}^m 2^{-j}\tilde{b}^{i^l}_{j,k} + 2^{-m}\tilde{\alpha}^{i^l}_{m+1,k}\big)^2\Big|\psi^l_{i,k}\Big\} \\ &\quad - \big(\mathbb{E}\{\tilde{\psi}^l_{i,k}|\psi^l_{i,k}\}\big)^2 \\ &= \mathbb{E}\Big\{\Big(\sum_{j=1}^m 2^{-j}\big(\tilde{b}^{i^l}_{j,k} - \mathbb{E}\{\tilde{b}^{i^l}_{j,k}\}\big) + 2^{-m}\big(\tilde{\alpha}^{i^l}_{m+1,k} \\ &\quad - \mathbb{E}\{\tilde{\alpha}^{i^l}_{m+1,k}\}\big)\Big)^2\Big\} \\ &= \sum_{j=1}^m 2^{-2j}\Big(\mathbb{E}\{\big(\tilde{b}^{i^l}_{j,k}\big)^2\} - \big(\mathbb{E}\{(\tilde{b}^{i^l}_{j,k})\}\big)^2\Big) \\ &\quad + 2^{-2m}\Big(\mathbb{E}\{\big(\tilde{\alpha}^{i^l}_{m+1,k}\big)^2\} - \big(\mathbb{E}\{(\tilde{\alpha}^{i^l}_{m+1,k})\}\big)^2\Big) \\ &= p_i(1-p_i)\Big(\frac{1-2^{-2m}}{3}\Big) + p_i(1-p_i)2^{-2m} \\ &= \frac{1}{3}p_i(1-p_i)\Big(1+2^{-2m+1}\Big). \end{aligned}$$

Furthermore, the conditional expectation of signal $\check{\xi}^l_{i,k}$ given signal $\xi^l_{i,k}$ can be written as

$$\mathbb{E}\{\check{\xi}^l_{i,k}|\xi^l_{i,k}\} = \mathbb{E}\{\mathbb{E}\{\check{\xi}^l_{i,k}|\psi^l_{i,k}\}|\xi^l_{i,k}\}$$

$$
\begin{aligned}
&= \mathbb{E}\{2S\big((1-2p_i)\psi_{i,k}^l + p_i\big) - S|\xi_{i,k}^l\} \\
&= \mathbb{E}\{(1-2p_i)(2S\psi_{i,k}^l - S)|\xi_{i,k}^l\} \\
&= (1-2p_i)\xi_{i,k}^l. \qquad (19)
\end{aligned}
$$

Bearing in mind the property of variance (i.e., $\mathbb{D}\{aX+b\} = a^2\mathbb{D}\{X\}$ and $\mathbb{D}\{X\} = \mathbb{D}\big\{\mathbb{E}\{X|Y\}\big\}+\mathbb{E}\big\{\mathbb{D}\{X|Y\}\big\}$), we have

$$
\begin{aligned}
&\mathbb{D}\{\check{\xi}_{i,k}^l|\xi_{i,k}^l\} \\
&= \mathbb{D}\{2S\tilde{\psi}_{i,k}^l - S|\xi_{i,k}^l\} \\
&= 4S^2\mathbb{D}\{\tilde{\psi}_{i,k}^l|\xi_{i,k}^l\} \\
&= 4S^2\Big(\mathbb{E}\big\{\mathbb{D}\{\tilde{\psi}_{i,k}^l|\psi_{i,k}^l\}|\xi_{i,k}^l\big\} + \mathbb{D}\big\{\mathbb{E}\{\tilde{\psi}_{i,k}^l|\psi_{i,k}^l\}|\xi_{i,k}^l\big\}\Big) \\
&\le \frac{4}{3}p_i(1-p_i)\Big(1+2^{-2m+1}\Big)S^2 + 2^{-2m}(1-2p_i)^2S^2 \qquad (20)
\end{aligned}
$$

which ends the proof.

Remark 1 *Up to now, the statistical properties of the quantization errors and the random bit errors have been analyzed for a scalar signal. It is not difficult to see that the developed results can be easily generalized to the vector case since all elements are mutually uncorrelated. Specifically, the mean and variance of the vector signal* $\check{\xi}_{i,k}$ *satisfy*

$$
\begin{cases}
\mathbb{E}\{\check{\xi}_{i,k}|\xi_{i,k}\} = (1-2p_i)\xi_{i,k}, \\
\mathbb{D}\{\check{\xi}_{i,k}|\xi_{i,k}\} \le \iota_i I_n
\end{cases} \qquad (21)
$$

where $\iota_i = \frac{4}{3}p_i(1-p_i)\big(1+2^{-2m+1}\big)S^2 + 2^{-2m}(1-2p_i)^2S^2$.

3.2 DT-MASs under BESs with identical BER

In this subsection, we aim to discuss the consensusability problem of the considered DT-MASs (1) in the case that all BERs are identical. Before proceeding further, let us introduce the following assumption.

Assumption 2 *The crossover probability is identical in all memoryless BSCs, i.e.,* $p_i = \bar{p}$ *for* $\forall i \in \{1, 2, \cdots, M\}$.

Let us rewrite (12) as

$$
\begin{aligned}
x_{k+1} &= (I_M \otimes A)x_k + (I_M \otimes BK)\check{\xi}_k \\
&= (I_M \otimes A)x_k + (1-2\bar{p})(L \otimes BK)x_k \\
&\quad + (I_M \otimes BK)\varpi_k, \qquad (22)
\end{aligned}
$$

where $\varpi_k = \big(\check{\xi}_k - (1-2\bar{p})\xi_k\big)$. According to Lemma 1, it is clear that ϖ_k is with zero mean and bounded variance, i.e., $\mathbb{E}\{\varpi_k\} = 0$ and $\mathbb{D}\{\varpi_k\} \le \mathrm{diag}_{Mn}\{\iota\}$ with $\iota = \frac{4}{3}\bar{p}(1-\bar{p})(1+2^{-2m+1})S^2 + 2^{-2m}(1-2\bar{p})^2S^2$.

Next, define the average state information for all agents as follows:

$$
\bar{x}_k = \frac{1}{M}\sum_{i=1}^{M} x_{i,k} = \frac{1}{M}(\mathbf{1}^T \otimes I)x_k.
$$

Let the consensus error be $\tilde{x}_{i,k} \triangleq x_{i,k} - \bar{x}_k$, and the corresponding augmented vector be $\tilde{x}_k = \begin{bmatrix} \tilde{x}_{1,k} & \tilde{x}_{2,k} & \cdots & \tilde{x}_{M,k} \end{bmatrix}$.

Then, the relationship between the consensus error and the system states can be described by

$$
\tilde{x}_{k+1} = x_{k+1} - (\mathbf{1} \otimes I_n)\bar{x}_{k+1} = (H \otimes I_n)x_{k+1} \qquad (23)
$$

where $H = I_M - \frac{1}{M}\mathbf{1}\mathbf{1}^T$.

Based on the fact that $HH = H$ and $HL = LH = L$, the dynamics of consensus error can be written as

$$
\begin{aligned}
\tilde{x}_{k+1} = (I_M \otimes A)\tilde{x}_k &+ (1-2\bar{p})(L \otimes BK)\tilde{x}_k \\
&+ (H \otimes BK)\varpi_k.
\end{aligned} \qquad (24)
$$

By resorting to the property of Laplacian matrix, the closed-loop system (24) is decomposed into two parts. Specifically, select $\varphi_i \in \mathbb{R}^M$ satisfying $L\varphi_i = \lambda_i\varphi_i$ and then construct the unitary matrix $U \triangleq [1/\sqrt{M}\mathbf{1}_M, \varphi_2, \cdots, \varphi_M]$ such that $U^TLU = \mathrm{diag}\{0, \lambda_2, \cdots, \lambda_M\} \triangleq \Lambda$. Without loss of generality, it is assumed that $0 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_M$. According to the linear transformation $\phi_k \triangleq (U^T \otimes I)\tilde{x}_k$, one has

$$
\begin{aligned}
\phi_{k+1} = (I_M \otimes A)\phi_k &+ (1-2\bar{p})(\Lambda \otimes BK)\phi_k \\
&+ (U^THU \otimes BK)\tilde{\varpi}_k
\end{aligned} \qquad (25)
$$

where $\tilde{\varpi}_k \triangleq (U^T \otimes I)\varpi_k$.

Due to the boundedness of the variance of ϖ_k, the mean-square consensus of (25) omitting the effect of $\tilde{\varpi}_k$ is equivalent to the stability of the following system

$$
\phi_{k+1} = (I_M \otimes A)\phi_k + (1-2\bar{p})(\Lambda \otimes BK)\phi_k. \qquad (26)
$$

Subsequently, decompose ϕ_k into $[\phi_{1,k}^T \ \phi_{2,k}^T \ \cdots \ \phi_{M,k}^T]^T$ with $\phi_{i,k} \in \mathbb{R}^n$. Note that $\phi_{1,k} \equiv 0$, the above system can be decoupled as

$$
\phi_{i,k+1} = A\phi_{i,k} + (1-2\bar{p})\lambda_i BK\phi_{i,k}, \forall i = 2, 3, \cdots, M. \qquad (27)
$$

In what follows, we are ready to derive some consensusability conditions to reveal the relationship among the crossover probability, the communication topology and the system dynamics. To this end, an important lemma on the solvability of a MRI is first introduced to facilitate the consensusability analysis of the DT-MASs described by (27).

Lemma 2 *(Schenato, Sinopoli, Franceschetti, Poolla, & Sastry , 2007; Sinopoli, Schenato, Franceschetti, Poolla, Jordan, & Sastry , 2004; Zheng, Xu, Xie, & You , 2019) Assume that* (A, B) *is controllable. There exists a positive definite matrix* P *such that the following MRI is true*

$$
P > A^TPA - \gamma A^TPB(B^TPB)^{-1}B^TPA, \qquad (28)
$$

if γ *is greater than a critical value* $\gamma_0 \in (0, 1]$, *where* γ_0 *satisfies* $\underline{\gamma} \le \gamma_0 \le \overline{\gamma}$ *with* $\underline{\gamma} = 1 - 1/\max_i|\lambda_i(A)|^2$ *and* $\overline{\gamma} = 1 - 1/det(A)^2$. *Specially,* $\gamma_0 = \overline{\gamma}$ *if rank*$(B) = 1$ *and* $\gamma_0 = \underline{\gamma}$ *if* B *is non-singular.*

It is worth mentioning that Lemma 2 is a simplified version of Lemma 5.4 in Schenato, Sinopoli, Franceschetti, Poolla, & Sastry (2007), and its critical value γ_0 plays an important role to obtain the consensusability condition.

Theorem 1 *Under Assumption 2, the considered DT-MASs (1) are able to achieve mean-square consensus with the control law (11) under an undirected graph $\mathscr{G}$, if (A, B) is controllable and*

$$\gamma_1 = 1 - \Big(\frac{\lambda_M - \lambda_2}{\lambda_M + \lambda_2}\Big)^2 > \gamma_0 \tag{29}$$

where γ_0 is given in Lemma 2. Moreover, under the condition (29), there exists a solution $P > 0$ satisfying the MRI (28) with $\gamma = \gamma_1$, and the admissible controller gain is given by

$$K = -\frac{2}{(\lambda_2 + \lambda_M)(1 - 2\bar{p})}(B^T PB)^{-1} B^T PA, \tag{30}$$

where $\bar{p}$ ($\bar{p} \neq 0.5$) is a known crossover probability.

proof: Construct the common Lyapunov function as follows:

$$V_{i,k} = \phi_{i,k}^T P \phi_{i,k}, \ \forall i = 2, 3, \cdots, M.$$

Then, the difference along (27) can be calculated by

$$\begin{aligned} \Delta V_{i,k} &= \phi_{i,k+1}^T P \phi_{i,k+1} - \phi_{i,k}^T P \phi_{i,k} \\ &= \phi_{i,k}^T (A + (1 - 2\bar{p})\lambda_i BK)^T P \\ &\quad \times (A + (1 - 2\bar{p})\lambda_i BK)\phi_{i,k} - \phi_{i,k}^T P \phi_{i,k} \\ &= \phi_{i,k}^T \big(A^T PA + (1 - 2\bar{p})\lambda_i K^T B^T PA \\ &\quad + (1 - 2\bar{p})\lambda_i A^T PBK - P \\ &\quad + (1 - 2\bar{p})^2 \lambda_i^2 K^T B^T PBK\big)\phi_{i,k}. \end{aligned} \tag{31}$$

Substituting the admissible controller gain (30) into (31) leads to

$$\begin{aligned} \Delta V_{i,k} =& \phi_{i,k}^T \big(A^T PA - P \\ & - \gamma_i A^T PB(B^T PB)^{-1} B^T PA\big)\phi_{i,k} \end{aligned} \tag{32}$$

where $\gamma_i = \frac{4(\lambda_i(\lambda_2 + \lambda_M) - \lambda_i^2)}{(\lambda_2 + \lambda_M)^2}$.

Noting that $(\lambda_2 - \lambda_i)(\lambda_i - \lambda_M) \geq 0$ ($\forall\, i = 2, 3, \cdots, M$) and recalling the inequality condition (29), one has

$$\gamma_i = \gamma_1 + \frac{4(\lambda_2 - \lambda_i)(\lambda_i - \lambda_M)}{(\lambda_2 + \lambda_M)^2} \geq \gamma_1 > \gamma_0.$$

Then, it follows from Lemma 2 that there exists a positive $P > 0$ such that $\Delta V_{i,k} < 0$ for $\forall i = 2, 3, \cdots, M$, i.e., the decoupled systems (27) are stable. In other words, the considered DT-MASs (1) are able to achieve mean-square consensus under the condition (29) and the controllability of (A, B), which completes the proof.

The above theorem gives a sufficient condition on consensusability of DT-MASs with BES. In what follows, we aim to present a theorem to show that the proposed sufficient condition is also necessary for DT-MASs with single input, i.e., $u_{i,k} \in \mathbb{R}$.

Theorem 2 *Under Assumption 2, the considered DT-MASs (1) with rank$(B) = 1$ are able to achieve mean-square consensus with the control law (11) under an undirected graph $\mathscr{G}$, if and only if (A, B) is controllable and*

$$\gamma_1 \triangleq 1 - \Big(\frac{\lambda_M - \lambda_2}{\lambda_M + \lambda_2}\Big)^2 > \gamma_0 \tag{33}$$

where $\gamma_0 = 1 - 1/det(A)^2$. Moreover, under the condition (33), there exists a solution $P > 0$ satisfying the MRI (28) with $\gamma = \gamma_1$, and the admissible controller gain is given by

$$K = -\frac{2}{(1 - 2\bar{p})(\lambda_2 + \lambda_M)}(B^T PB)^{-1} B^T PA \tag{34}$$

where $\bar{p}$ ($\bar{p} \neq 0.5$) is a known crossover probability.

proof: The sufficiency can be easily checked by Theorem 1, and therefore we only prove the necessity.

The considered DT-MASs (1) are consensusable, which means that there exists a common controller gain K such that all subsystems in (27) are stable. In other words, there exist a positive matrix $P > 0$ and a gain K such that the difference $\Delta V_{i,k} < 0$ holds for $\forall i = 2, 3, \cdots, M$, which means that

$$\begin{aligned} & P - \big(A + (1 - 2\bar{p})\lambda_i BK\big)^T P \big(A + (1 - 2\bar{p})\lambda_i BK\big) \\ =& P - A^T PA - (1 - 2\bar{p})\lambda_i K^T B^T PA \\ & - (1 - 2\bar{p})\lambda_i A^T PBK - (1 - 2\bar{p})^2 \lambda_i^2 K^T B^T PBK \\ >& 0, \end{aligned} \tag{35}$$

and further results in

$$\begin{aligned} & P - A^T PA + A^T PB(B^T PB)^{-1} B^T PA \\ & - (1 - 2\bar{p})^2 \lambda_i^2 \big(K + \frac{1}{(1 - 2\bar{p})\lambda_i}(B^T PB)^{-1} B^T PA\big)^T \\ & \times B^T PB \big(K + \frac{1}{(1 - 2\bar{p})\lambda_i}(B^T PB)^{-1} B^T PA\big) \\ >& 0. \end{aligned} \tag{36}$$

On the other hand, in light of Lemma 5 in Xu, Mo, & Xie (2020), one has

$$\frac{B^T (A^{-1})^T PA^{-1} B}{B^T PB} \leq \frac{1}{\det(A)^2}.$$

Then, pre- and post-multiplying (36) by $B^T (A^T)^{-1}$ and its transpose, and then dividing by $B^T PB$ lead to

$$\begin{aligned} & \big((1 - 2\bar{p})\lambda_i KA^{-1}B + 1\big)^2 \\ <& \frac{B^T (A^T)^{-1} PA^{-1} B}{B^T PB} - 1 + 1 \\ \leq& \frac{1}{\det(A)^2}, \end{aligned} \tag{37}$$

which implies

$$\underline{\varepsilon}_i < KA^{-1}B < \overline{\varepsilon}_i \tag{38}$$

with

$$\underline{\varepsilon}_i = \frac{-\frac{1}{|\det(A)|} - 1}{\lambda_i(1 - 2\bar{p})}, \ \ \overline{\varepsilon}_i = \frac{\frac{1}{|\det(A)|} - 1}{\lambda_i(1 - 2\bar{p})}. \tag{39}$$

Since there exists a common $KA^{-1}B$ such that the inequalities (38) hold for all $i = 2, 3, \cdots, M$, we must have $\cap_i(\underline{\varepsilon}_i, \overline{\varepsilon}_i) \neq \emptyset$, which implies $\overline{\varepsilon}_2 > \underline{\varepsilon}_M$ with $\bar{p} < 0.5$ or $\underline{\varepsilon}_2 < \overline{\varepsilon}_M$ with $\bar{p} > 0.5$. It is not difficult to find that the above two cases can both lead to

$$\frac{\lambda_M - \lambda_2}{\lambda_M + \lambda_2} < \frac{1}{|\det(A)|},$$

which is equivalent to (33). The proof of the necessity is now complete.

3.3 DT-MASs under BESs with non-identical BERs

In this subsection, the consensusability problem is discussed under the case of non-identical BERs. A sufficient condition is derived to guarantee the existence of desired controller with known crossover probability. To this end, let us introduce the following assumption.

Assumption 3 *The crossover probabilities are non-identical in the memoryless BSCs, i.e., $p_i \neq p_j$, $\exists\, i \neq j$ and $i, j \in \{1, 2, \cdots, M\}$.*

The augmented form of the DT-MASs (1) with the control law (11) subject to non-identical BERs can be written as

$$\begin{aligned} x_{k+1} &= (I_M \otimes A)x_k + (I_M \otimes BK)\check{\xi}_k \\ &= (I_M \otimes A)x_k + (I_M \otimes BK)\big(\check{\xi}_k + (\Theta \otimes I_n)\xi_k \\ &\quad - (\Theta \otimes I_n)\xi_k\big) \\ &= (I_M \otimes A)x_k + (\Theta L \otimes BK)x_k + (I_M \otimes BK)\vec{\varpi}_k \end{aligned}$$

where $\Theta = \mathrm{diag}_M\{1 - 2p_i\}$, and $\vec{\varpi}_k = \check{\xi}_k - (\Theta \otimes I_n)\xi_k$. It is not difficult to find that $\vec{\varpi}_k$ is with zero mean and bounded variance, that is, $\mathbb{E}\{\vec{\varpi}_k\} = 0$ and $\mathbb{D}\{\vec{\varpi}_k\} \leq \mathrm{diag}_M\{\iota_i I_n\} \triangleq R^*$ with ι_i defined in (21).

By resorting to Lemma 3.3 in Ren & Beard (2005) as well as the similar manipulation in Subsection 3.2, the consensus problem is transformed into the stability problem of the following decoupled systems

$$\bar{\phi}_{i,k+1} = A\bar{\phi}_{i,k} + \bar{\lambda}_i BK\bar{\phi}_{i,k},\ \forall i = 2, 3, \cdots, M, \quad (40)$$

where $\bar{\lambda}_i \in \mathbb{C}$ is the non-zero eigenvalue of the matrix ΘL.

Theorem 3 *Under Assumption 3, the considered DT-MASs (1) are able to achieve mean-square consensus with the control law (11) under an undirected graph $\mathscr{G}$, if (A, B) is controllable and*

$$\gamma_3 = 1 - \Big(\frac{|\bar{\lambda}_M| - |\bar{\lambda}_2|}{|\bar{\lambda}_M| + |\bar{\lambda}_2|}\Big)^2 > \gamma_0 \quad (41)$$

where γ_0 is given in Lemma 2. Moreover, under the condition (41), there exists a solution $P > 0$ satisfying the MRI (28) with $\gamma = \gamma_3$, and the admissible controller gain is given by

$$K = -\frac{2}{(|\bar{\lambda}_2| + |\bar{\lambda}_M|)}(B^T PB)^{-1}B^T PA \quad (42)$$

where p_i ($p_i \neq 0.5$) for $\forall i = 2, 3, \cdots, M$ is a known crossover probability.

Proof: Bearing in mind the fact that

$$\begin{aligned} &(|\bar{\lambda}_2| - |\bar{\lambda}_i|)(|\bar{\lambda}_i| - |\bar{\lambda}_M|) \geq 0, \\ &|1 + \pi|\bar{\lambda}_i|| - |1 + \pi\bar{\lambda}_i| \geq 0,\ \ \forall \pi \in \mathbb{R} \end{aligned}$$

one has

$$\gamma_3 \leq 1 - (|\kappa|\bar{\lambda}_i| + 1|)^2 \leq 1 - (|1 + \kappa\bar{\lambda}_i|)^2 \quad (43)$$

where $\kappa = -2/(|\bar{\lambda}_2| + |\bar{\lambda}_M|)$.

Furthermore, along the similar line of proof of Theorem 1 (i.e., substituting the admissible controller gain (42) into (31)), we can easily find that there exists a positive definite matrix P such that $\Delta V_{i,k} < 0$ holds for all $i \in \{2, 3, \cdots, M\}$ if the condition (41) is satisfied, which ends the proof.

Remark 2 *It is worth mentioning that the obtained consensusability condition (41) is closely related to the BERs, the Laplacian matrix, and the system dynamics. To be specific, the effects of the non-identical BERs and the Laplacian matrix are indirectly reflected in the maximum and minimum non-zero eigenvalues of the matrix ΘL. It should also be pointed out that the derived sufficient condition in Theorem 3 is degraded into the results in Theorem 1 when $p_i = p_j$ for $\forall i, j \in \{1, 2, \cdots, M\}$. Furthermore, the proposed consensusability condition is degenerated to the standard results in You & Xie (2011) when $p_i = 0$ for $\forall i \in \{1, 2, \cdots, M\}$.*

3.4 Boundedness analysis

Due to the boundedness of the variance of ϖ_k, the considered DT-MASs (1) under control law (11) can achieve mean-square bounded consensus if the corresponding ϖ_k-free systems with the adopted controller gain K can achieve consensus. In this subsection, we aim to analyze the effect of ϖ_k on consensus error bound. Let us first discuss the case that the BER is identical. For ease of representation, denote

$$\begin{aligned} &\mathcal{A} = I_M \otimes A + (1 - 2\bar{p})\Lambda \otimes BK, \\ &\bar{\mathcal{A}} = I_{M-1} \otimes A + (1 - 2\bar{p})\bar{\Lambda} \otimes BK, \\ &\bar{\Lambda} = \mathrm{diag}\{\lambda_2, \lambda_3, \cdots, \lambda_M\}, \\ &\mathcal{B} = U^T H \otimes BK, \\ &\mathcal{I} = \begin{bmatrix} 0_{M-1} & I_{M-1} \end{bmatrix} \otimes I_n, \end{aligned}$$

where K is the consensus gain obtained by (33).

Define the average consensus error covariance as

$$\mathcal{P}_k \triangleq \mathbb{E}\{\tilde{x}_k \tilde{x}_k^T\} = (U \otimes I_n)\Phi_k(U^T \otimes I_n)$$

where $\Phi_k = \mathbb{E}\{\phi_k \phi_k^T\}$.

In what follows, the recursion of Φ_k can be expressed by

$$\begin{aligned} \Phi_{k+1} &= \mathcal{A}\Phi_k\mathcal{A}^T + \mathcal{B}R_k\mathcal{B}^T \\ &= \mathcal{A}(\mathcal{A}\Phi_{k-1}\mathcal{A}^T + \mathcal{B}R_{k-1}\mathcal{B}^T)\mathcal{A}^T + \mathcal{B}R_k\mathcal{B}^T \\ &= \mathcal{A}^{k+1}\Phi_0(\mathcal{A}^T)^{k+1} + \mathcal{A}^k\mathcal{B}R_0\mathcal{B}^T(\mathcal{A}^T)^k + \cdots \\ &\quad + \mathcal{A}\mathcal{B}R_{k-1}\mathcal{B}^T\mathcal{A}^T + \mathcal{B}R_k\mathcal{B}^T \end{aligned} \quad (44)$$

where R_k reflects the variance of quantization errors and bit errors satisfying $R_k \leq R = \mathrm{diag}_{Mn}\{\iota\}, (\forall k \geq 0)$ with

$$\iota = \frac{4}{3}\bar{p}(1 - \bar{p})\Big(1 + 2^{-2m+1}\Big)S^2 + 2^{-2m}(1 - 2\bar{p})^2 S^2.$$

Furthermore, it follows from (44) that

$$\begin{aligned} \Phi_{k+1} \leq\ &\mathcal{A}^{k+1}\Phi_0(\mathcal{A}^T)^{k+1} + \mathcal{A}^k\mathcal{B}R\mathcal{B}^T(\mathcal{A}^T)^k + \cdots \\ &+ \mathcal{A}\mathcal{B}R\mathcal{B}^T\mathcal{A}^T + \mathcal{B}R\mathcal{B}^T. \end{aligned} \quad (45)$$

Recalling the structure of ϕ_k in (26) where $\phi_{1,k} \equiv 0$, the matrix Φ_k can be further decomposed and its nonzero block is extracted as $\Phi_{2,k} = \mathcal{I}\Phi_k\mathcal{I}^T$. Furthermore, such a matrix has the following recursion relationship

$$\begin{aligned} \Phi_{2,k+1} \leq\ &\bar{\mathcal{A}}^{k+1}\Phi_{2,0}(\bar{\mathcal{A}}^T)^{k+1} + \bar{\mathcal{A}}^k\Upsilon(\bar{\mathcal{A}}^T)^k \\ &+ \cdots + \bar{\mathcal{A}}\Upsilon\bar{\mathcal{A}}^T + \Upsilon \end{aligned} \quad (46)$$

where $\Upsilon = \mathcal{I}\mathcal{B}R\mathcal{B}^T\mathcal{I}^T$.

According to the properties of matrix trace, it is obtained from (46) that

$$\begin{aligned}\mathrm{Tr}\{\Phi_{2,k+1}\} &\le \mathrm{Tr}\{\bar{\mathcal{A}}^{k+1}\Phi_{2,0}(\bar{\mathcal{A}}^T)^{k+1} + \bar{\mathcal{A}}^k\Upsilon(\bar{\mathcal{A}}^T)^k \\ &\quad + \cdots + \bar{\mathcal{A}}\Upsilon\bar{\mathcal{A}}^T + \Upsilon\} \\ &= \mathrm{Tr}\{\bar{\mathcal{A}}^{k+1}\Phi_{2,0}(\bar{\mathcal{A}}^T)^{k+1} + \Upsilon(\bar{\mathcal{A}}\bar{\mathcal{A}}^T)^k \\ &\quad + \cdots + \Upsilon\bar{\mathcal{A}}\bar{\mathcal{A}}^T + \Upsilon\} \\ &= \mathrm{Tr}\{\Phi_{2,0}\bar{\mathcal{A}}^{k+1}(\bar{\mathcal{A}}^T)^{k+1} + \Upsilon\mathscr{A}_k\}\end{aligned}$$

where $\mathscr{A}_k \triangleq (\bar{\mathcal{A}}\bar{\mathcal{A}}^T)^k + (\bar{\mathcal{A}}\bar{\mathcal{A}}^T)^{k-1} + \cdots + I$.

Due to the fact that $\rho(\bar{\mathcal{A}}) < 1$, we have $\lim_{k\to\infty}\bar{\mathcal{A}}^k = 0$ and $\lim_{k\to\infty}\mathscr{A}_k = (I - \bar{\mathcal{A}}\bar{\mathcal{A}}^T)^{-1}$. Thus, one has

$$\lim_{k\to\infty}\mathrm{Tr}\{\Phi_{2,k}\} \le \mathrm{Tr}\{\Upsilon(I - \bar{\mathcal{A}}\bar{\mathcal{A}}^T)^{-1}\}. \tag{47}$$

Noting that $\mathrm{Tr}\{\mathcal{P}_k\} = \mathrm{Tr}\{\Phi_k\} = \mathrm{Tr}\{\Phi_{2,k}\}$, one can conclude that the ultimate upper bound of the consensus error in the mean-square sense can be expressed by

$$\theta(m, p_i, S) = \mathrm{Tr}\{\Upsilon(I - \bar{\mathcal{A}}\bar{\mathcal{A}}^T)^{-1}\}.$$

Similar to the above steps, we can obtain that, for the case of the non-identical BERs, the ultimate upper bound of consensus error is

$$\theta(m, p_i, S) = \mathrm{Tr}\{\Upsilon^*(I - \bar{\mathcal{A}}^*(\bar{\mathcal{A}}^*)^T)^{-1}\}$$

where

$$\begin{aligned}&\bar{\mathcal{A}}^* = I_{M-1}\otimes A + \mathrm{diag}\{\bar{\lambda}_2, \bar{\lambda}_3, \cdots, \bar{\lambda}_M\}\otimes BK^*, \\ &\Upsilon^* = \mathcal{I}\mathcal{B}^*R^*(\mathcal{B}^*)^T\mathcal{I}^T,\ \mathcal{B}^* = (U^*)^T H \otimes BK^*.\end{aligned}$$

Here, K^* is the consensus gain obtained by (42), and U^* is a unitary matrix satisfying $(U^*)^T\Theta L U^* = \mathrm{diag}\{0, \bar{\lambda}_2, \cdots, \bar{\lambda}_M\}$.

Note that such a bound is dependent of the length of the bit string as well as the BER, which implies that there exists a trade-off between the length of bit string and control performance. Furthermore, it should be pointed out that, according to the established statistical relationship in Lemma 1, the control performance (reflected by the upper bound) is improved exponentially as the length of bit string m increases.

Remark 3 *So far, we have thoroughly investigated the consensusability issue for a class of DT-MASs under the BESs with random bit errors. In contrast to existing results, our results presented in this paper stand out in the following three aspects: 1) the encoding-decoding scheme and random bit errors are both taken into consideration, thereby better reflecting the engineering reality; 2) some consensusability conditions are, for the first time, established for the DT-MASs under the BES with identical and non-identical BERs; 3) the relationship among the length of the bit string, the BERs and the upper bound of consensus error is derived to reveal the effects of quantization errors and bit errors on system performance.*

Table 1
The instants of bit errors

Bit strings	The i-th bit								
	1	2	3	4	5	6	7	8	9
$\psi^1_{1,k}$	*	*	*	*	*	*	*	*	*
$\psi^2_{1,k}$	*	*	*	*	133	*	*	*	48
$\psi^1_{2,k}$	107	*	*	*	*	*	*	84	*
$\psi^2_{2,k}$	*	*	*	*	188	*	*	*	*
$\psi^1_{3,k}$	*	131	161	*	*	*	*	*	*
$\psi^2_{3,k}$	*	*	*	*	*	*	*	*	*
$\psi^1_{4,k}$	*	*	*	*	14	*	*	*	39
$\psi^2_{4,k}$	*	131	*	*	*	6	149	*	13

4 Simulation Example

In this section, a numerical simulation is given to verify the effectiveness of the obtained theoretical results.

Consider an MAS composed of four agents. The communication topology is described by an undirected graph $\mathcal{G}$ with the adjacency matrix $\mathcal{A} = [l_{ij}]_{M\times M}$, where $l_{12} = l_{21} = l_{13} = l_{31} = l_{23} = l_{32} = l_{34} = l_{43} = 1$ and other elements are 0. The maximum and minimum non-zero eigenvalues of the corresponding Laplacian matrix L are 2 and 4.

The system matrices in (1) are given by

$$A = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix}.$$

For each agent, the initial state is selected as

$$\begin{aligned}x_{1,0} &= \begin{bmatrix} 1.5231 & 2.2723 \end{bmatrix}^T, x_{2,0} = \begin{bmatrix} 5.0238 & 3.3522 \end{bmatrix}^T, \\ x_{3,0} &= \begin{bmatrix} 3.6322 & 4.2234 \end{bmatrix}^T, x_{4,0} = \begin{bmatrix} 2.1623 & 5.2423 \end{bmatrix}^T.\end{aligned}$$

The identical BER is set as $\bar{p} = 0.001$, the length of the fixed bit string is chosen as $m = 8$ and the signal bound is set as $S = 6$. In light of Theorem 1, the corresponding positive definite matrix P and controller gain K can be obtained as

$$P = \begin{bmatrix} 0.6598 & -0.3153 \\ -0.3153 & 0.3712 \end{bmatrix},\ K = \begin{bmatrix} -0.5895 \\ -0.3889 \end{bmatrix}^T.$$

Simulation results are plotted in Figs. 1-3, where Figs. 1 and 2 display, respectively, the state trajectories of all agents without and with the distributed control law. Furthermore, the consensus errors are plotted in Fig. 3. TABLE 1 presents the instants and positions for all bit strings, where symbol '*' represents the whole time-horizon without any bit error. It is observed that the bit errors, occurring in the front of the bit string, have a more serious influence on the consensus performance.

5 Conclusions

In this paper, we have investigated the consensusability problem of DT-MASs under the BESs with random

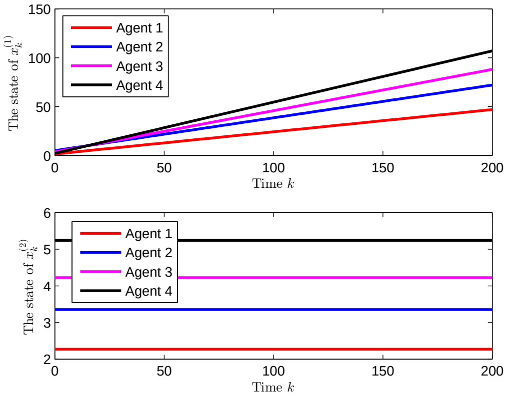

Fig. 1. State trajectories of agent i without control protocol.

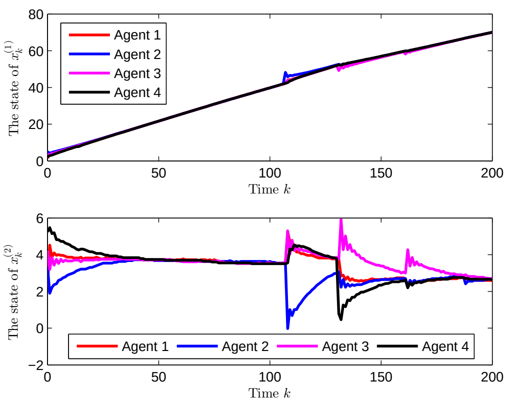

Fig. 2. State trajectories of agent i with control protocol.

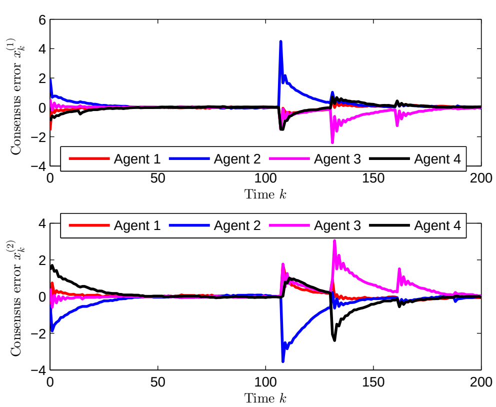

Fig. 3. The consensus error.

bit errors. The statistical properties have been first explored for the quantization errors and flipped bit errors. Under the undirected communication graph, some sufficient conditions have been derived, respectively, for the mean-square bounded consensus of DT-MASs with identical and non-identical BERs. In particular, the obtained sufficient condition is also necessary for the single-input systems with identical BERs. Furthermore, a sufficient condition has also been derived for the stochastically ultimate boundedness of the consensus error, which establishes a non-trivial relationship between the upper bound and the length of the bit string. Finally, a numerical example has been presented to show the effectiveness of the obtained results. Future research topics would include the extensions of the present results to more sophisticated cases including dynamic quantization (Zou, Wang, Hu, & Zhou , 2020), adaptive quantization (Niu & Ho , 2014) as well as saturation constraints (Yang, Meng, Dimarogona, & Johansson , 2014).

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