Scheduler-based State Estimation Over Multiple Channels Networks

Fuad E. Alsaadi, Zidong Wang, and Khalid H. Alharbi

***Abstract*—We investigate the remote state estimation problem for networked systems over parallel noise-free communication channels. Due to limited network capabilities in practical network environments, communication schedulers are implemented at the transmit side of each subchannel to promote resource efficiency. Specifically, the processed signals are transmitted only when it is necessary to provide the real-time measurements to the remote estimator. The recursive approximate minimum mean-square error (MMSE) estimator is established to restore the state vector of a target plant by utilizing the scheduled transmission signals. All the information coming from the individual subchannels, even if no measurement is sent, will contribute to improve the estimation performance in an analytical form. Finally, a numerical example is given to illustrate the effectiveness of the main results.**

***Index Terms*—State estimation; Event-based communication; Multiple communication channels; Communication rate.**

I. Introduction

In the past decades, with the rapid development of sensing, computing and communication technologies, networked control has become a mainstream research topic receiving much attention from both the control and signal processing communities. A typical networked control system is composed of sensors, controllers, and actuators linked via a wired or wireless shared communication network [8], [25], [45]. To achieve high-quality control performance, state estimate is a necessary part for generating feedback control signals since the state vector of the target plant is extracted from the contaminated partial measurements [3], [4], [13], [32], [33], [40], [46]. The merits of network devices render the remote estimation possible and, in such scenarios, sensor measurements are transmitted to a central unit with sufficient computing resources for further processing [2], [15], [22], [24]. Since the networked environment greatly reduces the costs of installation and maintenance, the remote state estimation has been widely applied in engineering practice such as automated highway systems, battlefield surveillance, and environmental monitoring [18], [31], [47], [48].

This research work was funded by Institutional Fund Projects under grant no. (IFPHI-203-135-2020). Therefore, authors gratefully acknowledge technical and financial support from the Ministry of Education and King Abdulaziz University, DSR, Jeddah, Saudi Arabia. (*Corresponding author: Zidong Wang.*)

F. E. Alsaadi and K. H. Alharbi are with the Department of Electrical and Computer Engineering, Faculty of Engineering, King Abdulaziz University, Jeddah 21589, Saudi Arabia. (Emails: {falsaadi,khalharbi}@kau.edu.sa).

Z. Wang is with the Department of Computer Science, Brunel University London, Uxbridge, Middlesex, UB8 3PH, United Kingdom. (Email: Zidong.Wang@brunel.ac.uk).

Traditionally, the remote state estimation problems have mainly focused on the ideal channel settings, that is, energy supply and available bandwidth for communication networks are inexhaustible, and thus the remote estimator has access to *all* the raw measurements from the sensor, where a Kalman filter algorithm can be employed as an optimal estimator for linear systems with Gaussian noises. However, for some practical applications such as wireless sensor networks, the communication processes are inherently subject to limited bandwidth, and batteries of sensors are driven by restricted energy supply [7], [11], [23], [36]. These adverse factors limit the penetration of remote estimation because too frequent transmissions might not improve the estimation performance but, on the contrary, they could lead to some undesirable phenomena such as network congestion and lifespan reduction. A critical issue is how to utilize the available resources to achieve a satisfactory result efficiently. Notice that the communication process constitutes a major source of energy consumption. For the sake of preserving the bandwidth and prolonging the working hours simultaneously, a feasible scheme is to reduce the number of transmissions as much as possible on the premise of predetermined performance guarantee.

Up to now, a number of resource-efficient scheduling strategies have been extensively investigated, which include power scheduling [35], sensor selection [5], [29], event-based communication [9], [12], [16], [17], [20], [26], [41], [43], self-triggered communication [14], and compressed signals [21], etc. These strategies aim to preserve the system resources from various aspects. To be specific, for power scheduling problems, it is supposed that the transmit side can switch between two different transmission energy levels. A high energy level results in a high packet reception ratio while costing more resources, and *vice versa*. As a result, an optimal transmission power schedule is required for the remote estimator to achieve the optimal estimation performance under prescribed energy constraints. Moreover, for sensor networks with a large number of sensors, it is meaningful to employ an appropriate selection scheme by choosing reliable sensor signals among all the available sources, where the fundamental issues are to find out the optimal set of sensors and design the estimator so as to minimize the error covariance. As for event-based communication, it is essentially a controlled transmission scheduling strategy where the scheduler forwards signals to the remote estimator only when certain events happen. Different from the classical clock-driven mechanism that triggers a transmission at every sampling instant, in such a case, a batch of unnecessary signals can be removed from the transmission sequence to reduce resource consumption.

Self-triggered communication can be regarded as an improved version of the event-based communication. In the mechanism of self-triggered communication, the next signal transmission instant is calculated by a triggering scheduling based on the previous transmitted data and the plant dynamics knowledge. Compared with the event-based communication, the main advantage of the self-triggered communication lies in the fact that mechanism of self-triggered is implemented based on certain "software" rather than the hardware (i.e. event-generator) adopted in event-based communication, and thereby reducing the hardware costs.

Due to its effectiveness in resource saving, the event-based mechanism has received increasing attention in recent years. Some initial works [30], [39] have considered an event-based rule called Send-on-Delta (or Lebesgue sampling) principle. By employing this principle the sensor data will be sent to the estimator when a certain specified threshold is reached. It can be further inferred that, when there is no transmission, the sensor data must lie in the given bound from the previously transmitted value. Therefore, one can utilize the previously transmitted value as the estimator input while keeping in mind that a bounded uncertainty exists. In this case, the exact optimal estimator is hard to obtain, but an alternative is to minimize the upper bound of the error covariance as [19]. Moreover, in [38], the Send-on-Delta principle has been extended to a more general one that is suitable for any type of sampling strategy. A sum of Gaussians approach has been employed to design the approximate optimal estimator for the sake of reducing computational complexity. On the other hand, another communication scheduling policy is based on the values of the real-time innovation as shown in [6], [42], [44]. Since innovations characterize the gap between the predicted and the current measurements, a small innovation implies that the estimator could utilize the predicted value as a quasi-optimal estimate and, in this case, the real-time transmissions are no longer necessary.

Following the existing works, the focus of this paper is on the remote state estimation problem under stringent energy and bandwidth constraints. By co-designing the scheduling policy and the state estimator, a balance between the estimation performance and available resources can be achieved. Furthermore, motivated by the multi-input-multi-output channel technique [10] developed in communication theory, we consider the communication channel to be composed of a set of parallel and independent subchannels, and each subchannel transmits the corresponding entry of the input vector. Since subchannels may own different available resources, the schedulers shall be specifically designed for the subchannels so that each subchannel can work at its desirable working condition. To the best of our knowledge, such a multiple channels setting has *not* yet been taken into account in the design of resource-efficient remote estimators.

The challenge for scheduler-based state estimation over multiple channels networks lies in the fact that the signals from subchannels are correlated and subject to the scheduling strategies. To achieve our objective, the channel input is first reconstructed by a dynamical linear transformation in order to eliminate the correlation between the components of the input vector. Therefore, the remote estimator can utilize the coming information from each subchannel to correct the one-step prediction independently. Furthermore, due to the scheduling process, it is almost impossible to give the exact minimum mean-square error (MMSE) when considering the amount of computation. An alternative way is to utilize a Gaussian assumption of the prior probability density function (PDF) at each step. Throughout this paper, we consider two scheduling strategies characterized by the signals injected to the channel. To be specific, when the pre-assigned conditions are fulfilled, the first one transmits the real-time signals, while the second one condenses the packet of the transmission signal by sending an indicator variable instead.

Summarizing the above discussion, the main contributions of our work can be highlighted as follows. *1) We investigate the remote state estimation problem over multiple communication channels. Under our framework, the average communication rate of each subchannel can be set specifically according to the channel condition; 2) the error covariance of the approximate MMSE estimator is obtained by a recursive algorithm. This covariance sequence turns out to be stochastic, but we can always find its tight upper and lower bounds at each step; 3) a bridge is established between the communication rate and the boundedness of the estimator, which works as a guideline to configure the schedulers.*

The rest of this paper is organized as follows. In Section II, the problem is formulated. Section III presents some preliminary knowledge for preparation. Section IV computes the MMSE under the scheduler-based communication and gives the performance analysis. In Section V, the result is extended to a more compressed scheduling policy. The results are illustrated by a numerical example in Section VI. Section VII concludes this paper.

Notation: Throughout the paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{E}[x]$ stands for the expectation of the stochastic variable x. When the expression for x is long, xWx' is abbreviated as $xW(*)'$. Let the cumulative distribution function of a standard normal distribution be $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})dx$. For any function $g(\cdot)$, its inverse function (if it exists) is denoted as $g^{-1}(\cdot)$.

II. Problem Formulation

Consider a discrete linear time-varying system in the following form:

$$\begin{aligned} x_{k+1} &= A_k x_k + w_k \\ y_k &= H_k x_k + v_k \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system state and $y_k \in \mathbb{R}^m$ is the observed signal. $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^m$ are external disturbances obeying Gaussian distributions with zero mean and covariance matrices $Q_k > 0$ and $R_k > 0$. A_k and H_k are known matrices with appropriate dimensions. The initial state x_0 is a Gaussian random variable with $E[x_0] = \mu_0$ and $\mathrm{Var}(x_0) = \Sigma_0 > 0$. We assume that the initial state x_0, the noises w_k and v_k are mutually independent.

In this paper, we consider the remote estimation problem as shown in Fig. 1. The processed measurements are transmitted

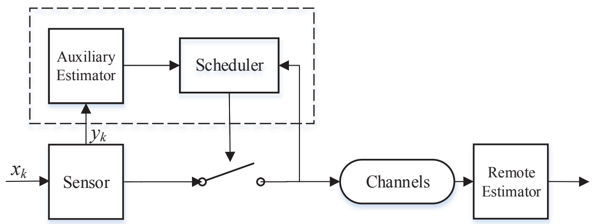

Fig. 1. Scheduler-based remote estimator

to the remote estimation center for further signal processing. For the sake of improving the utilization efficiency of network resources, a set of schedulers are installed at the smart sensor to prevent unnecessary transmissions. Moreover, it is worth pointing out that the channel here is characterized by the multiple-input-multiple-output model, see Fig. 2, where each scalar component of the input vector is transmitted only under the permission of schedulers in the corresponding subchannels. In this research, it is assumed that the signal transmissions over these subchannels are free from the packet losses and channel fading effects. In particular, we denote the scheduling policy for the channel as follows

$$\mathcal{S} = [\mathcal{S}_1, \cdots, \mathcal{S}_m]$$

where $\mathcal{S}_i$ represents the scheduling policy for ith subchannel that will be specifically clarified later on.

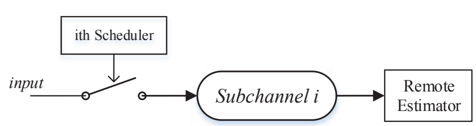

Fig. 2. Multiple communication channels

Under the scheduler-based communication, we define the information set provided by the ith subchannel for the estimator at step k as $l_{k,i}$. For brevity, we introduce a bounded set by stacking $l_{0,i}$ until $l_{k,i}$, denoted by $l_{0:k,i}$. As a result, at instant k, the available information for the estimator is

$$\mathcal{H}_k = \bigcup_{i=1}^{m} l_{0:k,i}, \tag{2}$$

Likewise, we have $l_{k,1:m}$ by stacking $l_{k,1}$ until $l_{k,m}$, which stands for the information gathered from all the subchannels at step k. Thus, it is obvious that $\mathcal{H}_k = \mathcal{H}_{k-1} \cup l_{k,1:m}$. For the remote optimal estimator, we take the minimum variance estimator given by the conditional expectation

$$\hat{x}_{k|k-1} = \mathbb{E}[x_k|\mathcal{H}_{k-1}], \quad \hat{x}_{k|k} = \mathbb{E}[x_k|\mathcal{H}_k]$$

and the corresponding error covariance matrices

$$P_{k|k-1} = \mathbb{E}[(x_k - \hat{x}_{k|k-1})(*)'|\mathcal{H}_{k-1}]$$
$$P_{k|k} = \mathbb{E}[(x_k - \hat{x}_{k|k})(*)'|\mathcal{H}_k]$$

where $\hat{x}_{0|-1} = \mu_0$, $P_{0|-1} = \Sigma_0$ and $\mathcal{H}_{-1} = \emptyset$. Here, $\emptyset$ represents the empty set.

Noting that $R_k > 0$, we know $H_k P_{k|k-1} H_k' + R_k > 0$. Therefore, there exists an orthogonal matrix such that $U_k'(H_k P_{k|k-1} H_k' + R_k)U_k = \Lambda_k$, where Λ_k is a diagonal matrix of eigenvalues of the matrix $H_k P_{k|k-1} H_k' + R_k$. By letting

$$F_k = U_k \Lambda_k^{-1/2} \tag{3}$$

it is obvious that $F_k F_k' = (H_k P_{k|k-1} H_k' + R_k)^{-1}$. Furthermore, we denote

$$b_k = F_k'(y_k - H_k \hat{x}_{k|k-1}) \tag{4}$$

where $b_k = [b_{k,1}, \cdots, b_{k,m}]'$ is a column vector with m components, and each component is the input of individual channel. Moreover, the decision of when the communication occurs and what the data is transmitted depends completely on the underlying scheduling strategy $\mathcal{S}_i$.

Throughout this paper, we aim to develop MMSE estimators $\hat{x}_{k|k}$ for system (1) under the given scheduling schemes, and then carry out performance analysis on the proposed estimators. Note that the desired MMSE estimator can be calculated by the conditional mean

$$\hat{x}_{k|k} = \mathbb{E}[x_k|\mathcal{H}_k] = \int x_k f(x_k|\mathcal{H}_k) dx_k$$

Unfortunately, the closed-form expression for $f(x_k|\mathcal{H}_k)$ is difficult to obtain due to the scheduler process, and we then use an alternative approach to calculate the approximate MMSE by exploiting Gaussian approximation of the prior pdf $f(x_k|\mathcal{H}_{k-1})$ at each step.

Remark 1: Rather than adopting raw measurements y_k to be channel input as the classical Kalman filter, we utilize b_k in our paper. A distinguished advantage of doing so is that the transformation (4) removes the correlation between signals injected into different subchannels, and therefore the estimator can update the estimate based on the arrivals of each signal from the subchannels independently.

Remark 2: To perform scheduler-based communication, the one-step prediction $\hat{x}_{k|k-1}$ and error covariance $P_{k|k-1}$ are indispensable for generating the channel input b_k. One can provide this information by letting the remote estimator send the real-time estimates back to the sensor, but this will inevitably increase the transmissions and lead to the additional consumption of network resources. Noticing that the smart sensor owns access and full control of the signals transmitted to the remote estimator, an alternative is to make the sensor run an identical copy of the estimator as the remote one.

III. PRELIMINARY

In this section, some preliminary knowledge, which is necessary for the solution of the estimation problem, is presented for preparation. We will show some properties of conditional distributions of Gaussian random variables, and derive the preliminary MMSE estimator with any type of scheduling strategy $\mathcal{S}$.

Lemma 1: Given the condition $x \in \Omega$, the posteriori distribution of the random variable x can be determined as follows:

$$f(x|x \in \Omega) = \begin{cases} \frac{1}{\Pr(x\in\Omega)} f(x), & x \in \Omega \\ 0, & \text{otherwise} \end{cases}$$

Lemma 2: Let X, Y and Z be random vectors with a jointly Guassian distribution. Assume that Y and Z are mutually independent. Then, the conditional distribution of X given Y, Z is with the expectation

$$\mathbb{E}[X|Y,Z] = \bar{X} + \Sigma_{XY}\Sigma_Y^{-1}(Y-\bar{Y}) + \Sigma_{XZ}\Sigma_Z^{-1}(Z-\bar{Z})$$

and covariance

$$\begin{aligned}&\mathbb{E}\left[(X-\mathbb{E}[X|Y,Z])(*)'|Y,Z\right] \\ &\quad = \Sigma_X - \Sigma_{XY}\Sigma_Y^{-1}\Sigma_{YX} - \Sigma_{XZ}\Sigma_Z^{-1}\Sigma_{ZX}\end{aligned}$$

where $\bar{\chi} = \mathbb{E}[\chi]$, $\Sigma_\chi = \mathbb{E}[(\chi-\bar{\chi})(*)']$, and $\Sigma_{\chi_1\chi_2}\mathbb{E}[(\chi_1 - \bar{\chi_1})(\chi_2-\bar{\chi_2})']$.

Given the scheduling strategy $\mathcal{S}$, the approximate MMSE can be obtained by using the following two-step procedure.

Prediction Step. Given the previous estimation $\hat{x}_{k-1|k-1}$ and the corresponding covariance $P_{k-1|k-1}$, the linearity of the expectation operator yields that

$$\begin{aligned}\hat{x}_{k|k-1} &= A_k\hat{x}_{k-1|k-1} \\ P_{k|k-1} &= A_kP_{k-1|k-1}A_k' + Q_k\end{aligned} \tag{5}$$

Correction Step: By adopting the Gaussian approximation $f(x_k|\mathcal{H}_{k-1}) = \mathcal{N}(\hat{x}_{k|k-1}, P_{k|k-1})$ and exploiting Lemma 2, it is straightforward to show that

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + P_{k|k-1}H_k'F_k\mathbb{E}[b_k|\mathcal{H}_k] \tag{6}$$

$$\begin{aligned}P_{k|k} &= P_{k|k-1} - P_{k|k-1}H_k'(H_kP_{k|k-1}H_k'+R_k)^{-1}H_kP_{k|k-1} \\ &\quad + P_{k|k-1}H_k'F_k\mathbb{E}[(b_k-\mathbb{E}[b_k|\mathcal{H}_k])(*)'|\mathcal{H}_k]F_k'H_kP_{k|k-1}\end{aligned} \tag{7}$$

IV. Scheduler-based Communication

In this section, we will give the approximate MMSE estimator with multiple communication channels under the scheduler-based communication.

A. MMSE estimator design

Consider the following scheduling strategies for the respective subchannels:

$$\mathcal{S}_i: \quad \gamma_{k,i} = \begin{cases} 1, & \text{if } |b_{k,i}| \geq \Delta_i \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

where $\Delta_i \in \mathbb{R}$ is a nonnegative real number. The scheduler at each subchannel decides whether the measurement $b_{k,i}$ shall be sent or not at every step k. Specifically, when $\gamma_{k,i}=1$, the scheduler allows the transmission of $b_{k,i}$, otherwise, it does not. In this case, the information set contains the index $\gamma_{k,i}$ and transmitted data $b_{k,i}$ as follows:

$$l_{k,i} = \{\gamma_{k,i},\ \gamma_{k,i}b_{k,i}\}.$$

According to the definitions of $l_{k,i}$ and $\mathcal{H}_k$, it is easy to see that the available information $\mathcal{H}_k$ for the estimator is largely dependent on the scheduling strategies $\mathcal{S}_i$, which implies that the resultant estimation performance is affected by the scheduling strategies. Furthermore, letting $F_k = [f_{k,1},\cdots,f_{k,m}]$ with $f_{k,i}$ being the ith columns, and introducing the following notations

$$\hat{x}_{k|k}^* := \hat{x}_{k|k-1} + \sum_{i=1}^m P_{k|k-1}H_k'f_{k,i}b_{k,i} \tag{9}$$

we have the following results.

Theorem 1: Suppose that the prediction $\hat{x}_{k|k-1}$ and the corresponding covariance $P_{k|k-1}$ are known, and the prior PDF is given by $f(x_k|\mathcal{H}_{k-1}) = \mathcal{N}(\hat{x}_{k|k-1}, P_{k|k-1})$. Given the scheduling strategy $\mathcal{S}_i$ in (8), the approximate MMSE can be computed recursively as follows

$$\begin{aligned}\hat{x}_{k|k} &= \hat{x}_{k|k-1} + \sum_{i=1}^m \gamma_{k,i}P_{k|k-1}H_k'f_{k,i}b_{k,i} \\ P_{k|k} &= P_{k|k-1} - \sum_{i=1}^m \nu_{k,i}P_{k|k-1}H_k'f_{k,i}f_{k,i}'H_kP_{k|k-1}\end{aligned} \tag{10}$$

where

$$\begin{aligned}\nu_{k,i} &= \gamma_{k,i} + (1-\gamma_{k,i})\psi(\Delta_i) \\ \psi(\Delta_i) &= \sqrt{\frac{2}{\pi}}(2\Phi(\Delta_i)-1)^{-1}\Delta_i\exp\left(-\Delta_i^2/2\right)\end{aligned}$$

Proof: Given the past observation $\mathcal{H}_{k-1}$, b_k obeys the standard Gaussian distribution $\mathcal{N}(0, I_{m\times m})$. In the sequel, it is known that $b_{k,i}$ is independent with $b_{k,j}$, for $i \neq j$, and therefore we have the following equality

$$\mathbb{E}[b_{k,i}|\mathcal{H}_{k-1}, l_{k,1:m}] = \mathbb{E}[b_{k,i}|\mathcal{H}_{k-1}, l_{k,i}]$$

Partition the set $\mathfrak{N} = \{1,2,\cdots,m\}$ into two complement subsects as follows

$$\mathcal{M}_k = \{i\in\mathfrak{N}|\gamma_{k,i}=1\}, \quad \mathcal{M}_k^C = \{i\in\mathfrak{N}|\gamma_{k,i}=0\}$$

Obviously, there are two possible statuses for the ith subchannel:

- $i \in \mathcal{M}_k$: There exists communication.
- $i \in \mathcal{M}_k^C$: No communication occurs.

In the following analysis, we will discuss the above cases separately.

Case I: $i \in \mathcal{M}_k$. Since the subchannel transmits the current observation $b_{k,i}$ accurately, it is trivial to see that $l_{k,i} = \{1,\ b_{k,i}\}$ and the conditional expectation becomes

$$\mathbb{E}[b_{k,i}|\mathcal{H}_{k-1}, b_{k,i}] = b_{k,i} \tag{11}$$

$$\mathbb{E}[b_{k,i}^2|\mathcal{H}_{k-1}, b_{k,i}] = b_{k,i}^2 \tag{12}$$

Case II: $i \in \mathcal{M}_k^C$. Although no communication occurs, the predefined scheduling rule still sheds some lights on the state estimation by implying the fact that $|b_{k,i}| < \Delta_i$. Therefore, we have

$$\mathbb{E}[b_{k,i}|\mathcal{H}_k] = \mathbb{E}[b_{k,i}|\mathcal{H}_{k-1}, |b_{k,i}|<\Delta_i]$$

In view of Lemma 1, it can be derived that

$$\begin{aligned}&\mathbb{E}[b_{k,i}|\mathcal{H}_{k-1}, |b_{k,i}|<\Delta_i] \\ &\quad = \frac{1}{\Pr(|b_{k,i}|<\Delta_i|\mathcal{H}_{k-1})}\int_{-\Delta_i}^{\Delta_i} x f_{b_{k,i}}(x|\mathcal{H}_{k-1})dx\end{aligned}$$

According to the definition of $b_{k,i}$, we can see $f_{b_{k,i}}(x|\mathcal{H}_{k-1}) = \mathcal{N}(0,1)$, which implies that

$$\mathbb{E}[b_{k,i}|\mathcal{H}_{k-1}, |b_{k,i}| < \Delta_i] = 0 \tag{13}$$

In addition, one has

$$\begin{aligned}&\mathbb{E}[b_{k,i}^2|\mathcal{H}_{k-1}, |b_{k,i}| < \Delta_i] \\ &= \frac{1}{\Pr(|b_{k,i}| < \Delta_i|\mathcal{H}_{k-1})} \int_{-\Delta_i}^{\Delta_i} x^2 f_{b_{k,i}}(x|\mathcal{H}_{k-1}) dx\end{aligned} \tag{14}$$

The PDF of the random variable $b_{k,i}$ conditioned on $\mathcal{H}_k$ is

$$f_{b_{k,i}}(x|\mathcal{H}_{k-1}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

Therefore, one has

$$\Pr(|b_{k,i}| < \Delta_i|\mathcal{H}_{k-1}) = 2\Phi(\Delta_i) - 1 \tag{15}$$

and

$$\begin{aligned}&\int_{-\Delta_i}^{\Delta_i} x^2 f_{b_{k,i}}(x|\mathcal{H}_{k-1}) dx \\ &= \int_{-\Delta_i}^{\Delta_i} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \\ &= -\frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\Big|_{-\Delta_i}^{\Delta_i} + \int_{-\Delta_i}^{\Delta_i} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \\ &= 2\Phi(\Delta_i) - 1 - \frac{2\Delta_i}{\sqrt{2\pi}} \exp\left(-\frac{\Delta_i^2}{2}\right)\end{aligned} \tag{16}$$

Substituting the above inequalities into (14) yields that

$$\begin{aligned}&\mathbb{E}[b_{k,i}^2|\mathcal{H}_{k-1}, |b_{k,i}| < \Delta_i] \\ &= 1 - \frac{\Delta_i}{\sqrt{2\pi}} (2\Phi(\Delta_i) - 1)^{-1} \exp\left(-\frac{\Delta_i^2}{2}\right)\end{aligned} \tag{17}$$

Now, from (6), we have

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \sum_{i\in\mathcal{M}_k} P_{k|k-1} H_k' f_{k,i} b_{k,i} \tag{18}$$

In order to calculate the covariance $P_{k|k}$ in (7), by noting that definition of $\hat{x}_{k|k}^*$ in (9), one can derive that

$$\begin{aligned}&P_{k|k-1} H_k' F_k \mathbb{E}[(b_k - \mathbb{E}[b_k|\mathcal{H}_k])(*)'|\mathcal{H}_k] F_k' H_k P_{k|k-1} \\ &= \mathbb{E}[(\hat{x}_{k|k} - \hat{x}_{k|k}^*)(*)'|\mathcal{H}_{k-1}, b_k]\end{aligned}$$

which yields that

$$\begin{aligned}&\mathbb{E}[(\hat{x}_{k|k} - \hat{x}_{k|k}^*)(*)'|\mathcal{H}_{k-1}, b_k] \\ &= \mathbb{E}\Big[\Big(\sum_{i\in\mathcal{M}_k^C} P_{k|k-1} H_k' f_{k,i} b_{k,i}\Big)\Big(*\Big)'\Big|\mathcal{H}_{k-1}, b_k\Big]\end{aligned}$$

To this end, we have

$$\begin{aligned}&\mathbb{E}[(\hat{x}_{k|k} - \hat{x}_{k|k}^*)(*)'|\mathcal{H}_k] \\ &= \mathbb{E}\Big[\mathbb{E}[(\hat{x}_{k|k} - \hat{x}_{k|k}^*)(*)'|\mathcal{H}_{k-1}, b_k]\Big|\mathcal{H}_{k-1}, l_{k,1:m}\Big] \\ &= \sum_{i\in\mathcal{M}_k^C} P_{k|k-1} H_k' f_{k,i} \mathbb{E}[b_{k,i}^2|\mathcal{H}_{k-1}, l_{k,i}] f_{k,i}' H_k P_{k|k-1}\end{aligned}$$

which, together with (17), leads to the main result presented in this theorem, and the proof is complete. ■

Remark 3: A major difference between the scheduler-based communication and the packet losses (e.g., [37]) is that the former gives up transmissions actively according to the predetermined guideline, while the latter drops transmissions stochastically based on the condition of the network. Therefore, when the signal $b_{k,i}$ is dropped because of the packet loss, the remote estimator can only run the time update process. By contrast, for the scheduler-based communication, even if the remote estimator does not receive the signals $b_{k,i}$, the implementation of the measurement update process is still possible since the estimator is aware of the fact $|b_{i,k}| < \Delta_i$.

Remark 4: It is worth pointing out that, when we set $\Delta_i = 0$ for all the subchannels, the inequalities $|b_{k,i}| \geq \Delta_i$ are always fulfilled. According to the scheduling strategies (8), all the signals are transmitted to the estimator at every step, and hence the results in the above theorem will reduce to the traditional Kalman filter.

B. Performance analysis

The average communication rate of the ith subchannel is defined by

$$\bar{\gamma}_i := \lim_{N\to\infty} \sup \frac{1}{N+1} \sum_{k=0}^{N} \gamma_{k,i} \tag{19}$$

Corollary 1: Given the estimation algorithm proposed in Theorem 1, the average communication rate for the ith subchannel can be computed as follows

$$\bar{\gamma}_i = 2 - 2\Phi(\Delta_i), \text{ for } i = 1, \cdots, m \tag{20}$$

Proof: Since the random variables $b_{k,i}$ satisfy the standard Gaussian distributions, it can be verified that

$$\mathbb{E}[\gamma_{k,i}] = \Pr(\gamma_{k,i} = 1) = \Pr(|b_{k,i}| > \Delta_i|\mathcal{H}_{k-1}) = 2 - 2\Phi(\Delta_i)$$

where the first equality is from the scheduling rule (8), and the last equality comes directly from (15). For the stationary variable $b_{k,i}$, it is obvious that $\bar{\gamma}_i = \mathbb{E}[\gamma_{k,i}]$. ■

The iteration equation of the MMSE estimator in (10) is stochastic because of the randomness of the indicator $\gamma_{k,i}$. For the sake of evaluating the effect of the thresholds on estimation performance, it is necessary to investigate the average covariance reduction per correction step as follows:

$$\delta P_k := \mathbb{E}[P_{k|k-1} - P_{k|k}] \tag{21}$$

Corollary 2: At each step, the average covariance reduction (21) is monotonically decreasing with respect to the threshold.

Proof: First, one has from (10) that

$$\delta P_k = \sum_{i=1}^{m} \mathbb{E}[\nu_{k,i}] P_{k|k-1} H_k' f_{k,i} f_{k,i}' H_k P_{k|k-1} \tag{22}$$

Notice that $\nu_{k,i} = \gamma_{k,i} + (1 - \gamma_{k,i})\psi(\Delta_i)$ is a stochastic scalar at each step k, and its average can be calculated by

$$\begin{aligned}\hbar(\Delta_i) &:= \mathbb{E}[\nu_{k,i}] \\ &= \Pr(\gamma_{k,i} = 1) + \Pr(\gamma_{k,i} = 0)\psi(\Delta_i) \\ &= 2 - 2\Phi(\Delta_i) + \frac{2\Delta_i}{\sqrt{2\pi}} \exp\left(-\frac{\Delta_i^2}{2}\right)\end{aligned}$$

Taking derivative to the both sides of such an equality with respect to x gives rise to

$$\frac{\partial \hbar(x)}{\partial x} = -2\frac{x^2}{\sqrt{2\pi}}\exp(-\frac{x^2}{2}) \leq 0$$

where the equality is achieved only when $x = 0$. Therefore, $\mathbb{E}[\nu_{k,i}]$ is a monotonically decreasing function in $\Delta_i > 0$. ■

As for the *time-invariant* systems, we would like to establish a relationship revealing how the selection of thresholds influences boundedness of the proposed remote estimator. For stochastic matrix sequence $\{P_k\}_{k\in\mathbb{N}}$, the boundedness of P_k is usually investigated in the mean sense, such as [37]. Note however that even if $\sup_{k\in\mathbb{N}}\mathbb{E}[P_k] < +\infty$, we still cannot assert that P_k is bounded for every single sample path. Therefore, it is necessary to evaluate the evolution properties of P_k in the rest of this subsection.

To facilitate the following analysis, it is assumed that all the thresholds are identical, i.e.,

$$\Delta_1 = \cdots = \Delta_m = \Delta > 0$$

Introduce a set of m dimension column vectors as $\boldsymbol{\nu}_k = [\nu_{k,1}, \cdots, \nu_{k,m}]'$, $\boldsymbol{\psi}(\Delta) = [\psi(\Delta), \cdots, \psi(\Delta)]'$, and $\mathbf{1} = [1, \cdots, 1]'$. Moreover, let us denote a simplified notation $P_k := P_{k|k-1}$ and define the evolution equation of the one-step prediction error covariance by using the modified algebraic Riccati equation (MARE) as follows:

$$\begin{aligned} P_{k+1} &= \mathcal{G}(\boldsymbol{\nu}_k, P_k) \\ &:= AP_kA' + Q - \sum_{i=1}^{m}\nu_{k,i}AP_kC'f_{k,i}f'_{k,i}CP_kA' \end{aligned}$$

Now, we can give the bounds of the stochastic covariance P_k in the following corollary.

Corollary 3: The upper and lower bounds of the one-step prediction error covariance P_k are given by

$$\underline{X}_k \leq P_k \leq \overline{X}_k \tag{23}$$

where $\underline{X}_k$ and $\overline{X}_k$ are positive definite matrices satisfying the following respective MAREs

$$\underline{X}_k = \mathcal{G}(\mathbf{1}, \underline{X}_{k-1}), \quad \overline{X}_k = \mathcal{G}(\boldsymbol{\psi}(\Delta), \overline{X}_{k-1})$$

with the initial values $\underline{X}_0 = \overline{X}_0 = P_0$.

Proof: From the definition of $\nu_{k,i}$ in Theorem 1, one has $\psi(\Delta) \leq \nu_{k,i} \leq 1, \ \forall k \in \mathbb{N}$. Obviously,

$$\mathcal{G}(\mathbf{1}, \underline{X}_0) \leq \mathcal{G}(\boldsymbol{\nu}_0, P_0) \leq \mathcal{G}(\boldsymbol{\psi}(\Delta), \overline{X}_0)$$

Assume, inductively, that the inequalities $\underline{X}_k \leq P_k \leq \overline{X}_k$ are fulfilled at the step k. Utilizing the properties of the MARE, we have

$$\mathcal{G}(\boldsymbol{\nu}_k, P_k) \leq \mathcal{G}(\boldsymbol{\psi}(\Delta), P_k) \leq \mathcal{G}(\boldsymbol{\psi}(\Delta), \overline{X}_k)$$

which indicates that $P_{k+1} \leq \overline{X}_{k+1}$. In analogy to the above procedure, we can show that the inequality $\underline{X}_{k+1} \leq P_{k+1}$ also holds. The inductive hypothesis implies that the inequalities (23) are always true. ■

Introduce a scalar $\bar{\nu} > 0$, which can be obtained by solving the following optimization problem as [37]

$$\begin{aligned} \bar{\nu} = \arg\inf_{\nu}\{\exists X > 0 | X > AXA' + Q \\ - \nu AXC'(CXC' + R)^{-1}CXA'\} \end{aligned} \tag{24}$$

Subsequently, the following corollary sheds light on how to set the communication rate to guarantee the boundedness of the covariance matrix P_k.

Corollary 4: Suppose that (A, C) is observable, $(A, Q^{\frac{1}{2}})$ is controllable, and A is unstable. Then, the following hold:

1) Let $\widetilde{\Delta} = \psi^{-1}(\bar{\nu})$. The stochastic covariance sequence is bounded (i.e., $\sup_{k\in\mathbb{N}} P_k < +\infty$) if and only if the average sensor-to-estimator communication rate satisfied $\bar{\gamma}_i > \bar{\gamma}_i^c$, where $\bar{\gamma}_i^c$ is the critical value defined as

$$\bar{\gamma}_i^c = 2 - 2\Phi(\widetilde{\Delta})$$

2) If $\bar{\gamma}_i > \bar{\gamma}_i^c$, when time tends to infinity, the error covariance are bounded by

$$\underline{X} \leq \lim_{k\to\infty} P_k \leq \overline{X}$$

where two positive definite matrices $\underline{X}$ and $\overline{X}$ are the solutions of the respective MAREs as follows

$$\underline{X} = \mathcal{G}(\mathbf{1}, \underline{X}), \quad \overline{X} = \mathcal{G}(\boldsymbol{\psi}(\Delta), \overline{X}).$$

Proof: Since the lower bound $\underline{X}_{k+1}$ equals to the prediction error covariance of the Kalman filter, by the observability and controllability of the system, it is not hard to verify that $\lim_{k\to\infty}\underline{X}_k = \underline{X}$. On the other hand, according to (24), we conclude that the upper bound is convergent if and only if $\psi(\Delta) > \bar{\nu}$. In the proof of Theorem 1, it can be seen that

$$\psi(\Delta) = 1 - \frac{1}{2\Phi(\Delta) - 1}\int_{-\Delta}^{\Delta}\frac{x^2}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx$$

Keeping in mind that both $\Phi(\Delta)$ and $\int_{-\Delta}^{\Delta}\frac{x^2}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx$ are increasing in Δ, we have that $\psi(\Delta)$ is a decreasing function in Δ with the largest threshold ensuring the convergence of MARE as $\widetilde{\Delta} = \psi^{-1}(\bar{\nu})$, which ends the proof. ■

It is important to remark that, compared to the Kalman filter, the covariance sequence $\{P_k\}$ will not converge to a fixed point, and it still shows the time-varying properties even when time trends to infinity. That is because such a sequence depends heavily on the stochastic variable $\gamma_{k,i}$. Fortunately, if the communication rate is larger than the critical value given in above theorem, the limit of the covariance P_k will be always bounded by two positive definite matrices $\underline{X}$ and $\overline{X}$.

V. Extension to Single Bit Communication

In this section, we would like to further condense the information set available to the remote estimator by employing new scheduling strategies

$$\mathcal{S}'_i: \quad \gamma_{k,i} = \begin{cases} 1, & \text{if } b_{k,i} \in \Omega_i \\ 0, & \text{otherwise} \end{cases} \tag{25}$$

where the interval $\Omega_i = (\underline{\Delta}_i, \overline{\Delta}_i)$ is with the extreme points $\underline{\Delta}_i \in \mathbb{R}$ and $\overline{\Delta}_i \in \mathbb{R}$. When $\gamma_{k,i} = 1$, the sensor sends a single bit packet to inform the remote estimator, otherwise, it will not. Therefore, the information set is described by

$$l_{k,i} = \{\gamma_{k,i}\}$$

Compared with the scheduling strategies (8), under such strategies, there is no need to provide $b_{k,i}$ for the estimator when $\gamma_{k,i} = 1$, and therefore the network traffic can be greatly reduced. However, the problem now is more complicated due to the fact that the estimator cannot access to the exact observation data any more. Fortunately, we could still use the technique developed previously to obtain the following MMSE.

Theorem 2: Suppose that the prediction $\hat{x}_{k|k-1}$ and the corresponding covariance $P_{k|k-1}$ are known, and the prior PDF is $f(x_k|\mathcal{H}_{k-1}) = \mathcal{N}(\hat{x}_{k|k-1}, P_{k|k-1})$. Given the scheduling strategy $\mathcal{S}_i'$ in (25), the approximate MMSE can be computed recursively as follows

$$\begin{aligned} \hat{x}_{k|k} &= \hat{x}_{k|k-1} + \sum_{i=1}^{m} \alpha_{k,i} P_{k|k-1} H_k' f_{k,i} \\ P_{k|k} &= P_{k|k-1} - \sum_{i=1}^{m} \beta_{k,i} P_{k|k-1} H_k' f_{k,i} f_{k,i}' H_k P_{k|k-1} \end{aligned} \tag{26}$$

where

$$\alpha_{k,i} = \frac{1}{\sqrt{2\pi}} \frac{h(\gamma_{k,i})(\exp(-\frac{\underline{\Delta}_i^2}{2}) - \exp(-\frac{\overline{\Delta}_i^2}{2}))}{1 - \gamma_{k,i} + h(\gamma_{k,i})(\Phi(\overline{\Delta}_i) - \Phi(\underline{\Delta}_i))}$$

$$\beta_{k,i} = \alpha_{k,i}^2 + \frac{1}{\sqrt{2\pi}} \frac{h(\gamma_{k,i})(\overline{\Delta}_i \exp(-\frac{\overline{\Delta}_i^2}{2}) - \underline{\Delta}_i \exp(-\frac{\underline{\Delta}_i^2}{2}))}{1 - \gamma_{k,i} + h(\gamma_{k,i})(\Phi(\overline{\Delta}_i) - \Phi(\underline{\Delta}_i))}$$

and

$$h(\gamma_{k,i}) = \begin{cases} 1, & \text{if } \gamma_{k,i} = 1 \\ -1, & \text{otherwise} \end{cases}$$

Proof: Partitioning the set $\mathfrak{N} = \{1, 2, \cdots, m\}$ into two complement subsects

$$\mathcal{M}_k = \{i \in \mathfrak{N} | \gamma_{k,i} = 1\}, \quad \mathcal{M}_k^C = \{i \in \mathfrak{N} | \gamma_{k,i} = 0\},$$

we discuss the individual cases as follows.

Case I: For $i \in \mathcal{M}_k$, it indicates $b_{k,i} \in \Omega_i$, so we have the equality

$$\mathbb{E}[b_{k,i}|\mathcal{H}_{k-1}, l_{k,i}] = \mathbb{E}[b_{k,i}|\mathcal{H}_{k-1}, b_{k,i} \in \Omega_i]$$

According to Lemma 1, an explicit expression of the conditional expectation can be calculated as follows

$$\begin{aligned} &\mathbb{E}[b_{k,i}|\mathcal{H}_{k-1}, b_{k,i} \in \Omega_i] \\ &= \frac{1}{\Pr(b_{k,i} \in \Omega_i|\mathcal{H}_{k-1})} \int_{x \in \Omega_i} x f_{b_{k,i}}(x|\mathcal{H}_{k-1}) dx \\ &= \frac{1}{\Phi(\overline{\Delta}_i) - \Phi(\underline{\Delta}_i)} \int_{\underline{\Delta}_i}^{\overline{\Delta}_i} \frac{1}{\sqrt{2\pi}} x \exp(-\frac{x^2}{2}) dx = \alpha_{k,i} \end{aligned} \tag{27}$$

where the last equality can be obtained directly by computing the integral term $\int x \exp(-x^2/2) dx$. Likewise, we can deal with the conditional expectation of the quadratic term, and have

$$\begin{aligned} &\mathbb{E}[b_{k,i}^2|\mathcal{H}_{k-1}, b_{k,i} \in \Omega_i] \\ &= \frac{1}{\Pr(b_{k,i} \in \Omega_i|\mathcal{H}_{k-1})} \int_{\underline{\Delta}_i}^{\overline{\Delta}_i} x^2 f_{b_{k,i}}(x|\mathcal{H}_{k-1}) dx \\ &= 1 - \left(\beta_{k,i} - \alpha_{k,i}^2\right) \end{aligned} \tag{28}$$

Case II: For $i \in \mathcal{M}_k^C$, we understand $b_{k,i} \notin \Omega_i$. Therefore, it can be seen that

$$\mathbb{E}[b_{k,i}|\mathcal{H}_{k-1}, l_{k,i}] = \mathbb{E}[b_{k,i}|\mathcal{H}_{k-1}, b_{k,i} \notin \Omega_i]$$

Along the procedure of Case I, we have

$$\begin{aligned} &\mathbb{E}[b_{k,i}|\mathcal{H}_{k-1}, b_{k,i} \notin \Omega_i] \\ &= \frac{1}{1 - (\Phi(\overline{\Delta}_i) - \Phi(\underline{\Delta}_i))} \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\underline{\Delta}_i} x \exp(-x^2/2) \right. \\ &\quad \left. + \int_{\overline{\Delta}_i}^{\infty} x \exp(-x^2/2) \right) = \alpha_{k,i} \end{aligned} \tag{29}$$

and

$$\begin{aligned} &\mathbb{E}[b_{k,i}^2|\mathcal{H}_{k-1}, b_{k,i} \notin \Omega_i] \\ &= \frac{1}{\Pr(b_{k,i} \notin \Omega_i|\mathcal{H}_{k-1})} \int_{x \notin \Omega_i} x^2 f_{b_{k,i}}(x|\mathcal{H}_{k-1}) dx \\ &= 1 - \left(\beta_{k,i} - \alpha_{k,i}^2\right) \end{aligned} \tag{30}$$

Substituting (27) and (29) into (6), we have the propagation of $\hat{x}_{k|k}$ as shown in Theorem 2. Furthermore, let $\tilde{\alpha}_i^{(1)} = \alpha_{k,i}$, $\forall i \in \mathcal{M}_k$, and $\tilde{\alpha}_i^{(0)} = \alpha_{k,i}$, $\forall i \in \mathcal{M}_k^C$. Combining (6) and (9) yields

$$\begin{aligned} \hat{x}_{k|k}^* - \hat{x}_{k|k} = &\sum_{i \in \mathcal{M}_k} P_{k|k-1} H_k' f_{k,i} (b_{k,i} - \tilde{\alpha}_i^{(1)}) \\ &+ \sum_{i \in \mathcal{M}_k^C} P_{k|k-1} H_k' f_{k,i} (b_{k,i} - \tilde{\alpha}_i^{(0)}) \end{aligned}$$

Here, we can partition $\mathbb{E}[(\hat{x}_{k|k} - \hat{x}_{k|k}^*)(*)'|\mathcal{H}_{k-1}, l_{k,1:m}]$ into four terms, namely,

$$\mathbb{E}[(\hat{x}_{k|k} - \hat{x}_{k|k}^*)(*)'|\mathcal{H}_{k-1}, l_{k,1:m}] = \mathfrak{B}_{11} + \mathfrak{B}_{12} + \mathfrak{B}_{12}' + \mathfrak{B}_{22} \tag{31}$$

where

$$\begin{aligned} \mathfrak{B}_{11} = &\mathbb{E}\left[\left(\sum_{i \in \mathcal{M}_k} P_{k|k-1} H_k' f_{k,i} (b_{k,i} - \tilde{\alpha}_i^{(1)}) \right)(*)' \middle| \mathcal{H}_{k-1}, l_{k,1:m} \right] \\ \mathfrak{B}_{12} = &\mathbb{E}\left[\left(\sum_{i \in \mathcal{M}_k} P_{k|k-1} H_k' f_{k,i} (b_{k,i} - \tilde{\alpha}_i^{(1)}) \right) \right. \\ &\left. \times \left(\sum_{i \in \mathcal{M}_k^C} P_{k|k-1} H_k' f_{k,i} (b_{k,i} - \tilde{\alpha}_i^{(0)}) \right)' \middle| \mathcal{H}_{k-1}, l_{k,1:m} \right] \\ \mathfrak{B}_{22} = &\mathbb{E}\left[\left(\sum_{i \in \mathcal{M}_k^C} P_{k|k-1} H_k' f_{k,i} (b_{k,i} - \tilde{\alpha}_i^{(0)}) \right)(*)' \middle| \mathcal{H}_{k-1}, l_{k,1:m} \right] \end{aligned}$$

Since $b_{k,i}$ are mutually independent, and the scalars $\tilde{\alpha}_i^{(1)}$ and $\tilde{\alpha}_i^{(0)}$ are deterministic, recalling (27) and (29), it is not hard to verify that $\mathfrak{B}_{12} = 0$,

$$\mathfrak{B}_{11} = \sum_{i \in \mathcal{M}_k} P_{k|k-1} H_k' f_{k,i} f_{k,i}' H_k P_{k|k-1} \times \mathbb{E}\left[(b_{k,i} - \tilde{\alpha}_i^{(1)})^2 | \mathcal{H}_{k-1}, l_{k,1:m}\right] \quad (32)$$

and

$$\mathfrak{B}_{22} = \sum_{i \in \mathcal{M}_k^C} P_{k|k-1} H_k' f_{k,i} f_{k,i}' H_k P_{k|k-1} \times \mathbb{E}\left[(b_{k,i} - \tilde{\alpha}_i^{(0)})^2 | \mathcal{H}_{k-1}, l_{k,1:m}\right] \quad (33)$$

By combining (27)-(30), (32), and (33), one can compute (31). Moreover, we substitute (31) into (7), and eventually have the result of $P_{k|k}$, which ends the proof. ■

Now, the average covariance reduction per correction step is described by

$$\delta P_k = \sum_{i=1}^{m} \mathbb{E}[\beta_{k,i}] P_{k|k-1} H_k' f_{k,i} f_{k,i}' H_k P_{k|k-1} \quad (34)$$

Obviously,

$$\mathbb{E}[\beta_{k,i}] = \frac{1}{2\pi} \frac{\left(\exp(-\underline{\Delta}_i^2/2) - \exp(-\overline{\Delta}_i^2/2)\right)^2}{\left(\Phi(\overline{\Delta}_i) - \Phi(\underline{\Delta}_i)\right)\left(1 - \left(\Phi(\overline{\Delta}_i) - \Phi(\underline{\Delta}_i)\right)\right)}$$

To maximization the average correction, it is necessary to choose the optimal interval Ω_i by considering the following optimization problem

$$\Omega_i^* = (\underline{\Delta}_i^*,\ \overline{\Delta}_i^*) = \arg \sup_{\underline{\Delta}_i,\ \overline{\Delta}_i} \mathbb{E}[\beta_{k,i}]$$

whose numerical solutions can be easily computed by utilizing MATLAB as $\underline{\Delta}_i^* = 0$, and $\overline{\Delta}_i^* = \infty$. Substituting these values into (26), one has $\alpha_{k,i} \equiv h(\gamma_{k,i})\sqrt{2/\pi}$ and $\beta_{k,i} \equiv 2/\pi$. For the scalar measurement case, i.e, $m = 1$, the proposed MMSE estimator with Ω^* shares the same form as SOI-KF in [34] indicating that making decisions based on the sign of the innovation sequence is indeed the best strategy.

Remark 5: The above result is interesting from a practical point of view, since it explicitly gives a clue on how to implement the modified Kalman filter according to the indicators from the subchannels. For vector measurements, the iteratively quantized Kalman filter (IQKF) has been proposed to compute MMSE with single bit transmissions in [28]. However, the IQKF needs to iterate m times so as to give the final estimate at each step. In contrast, our method does not need multi-step iteration at each step, and hence is more efficient.

Remark 6: An intriguing observation of Theorems 1-2 reveals that the proposed estimator owns the merits of high flexibility. Communication schedulers implemented in subchannels can choose strategies with different thresholds according to their available channel resources. Additionally, mixture strategies containing both $\mathcal{S}_i$ and $\mathcal{S}_i'$ among subchannels can be also employed. To obtain the corresponding MMSE, the only thing we need to do is simply revising the update from the specific channel in the iteration of $\hat{x}_{k|k}$ and $P_{k|k}$.

Remark 7: In this paper, the plant under consideration is a time-varying system, where the measurement model is a delay-free model. Nevertheless, our proposed method can be extended to the scheduler-based state estimation problem subject to measurement delays by using the measurement reorganization method [27].

VI. NUMERICAL STUDIES

In this section, we present the simulation results to validate the proposed scheduler-based remote estimation algorithms.

The linear time-varying system under consideration in (1) is with the transition matrix

$$A_k = \begin{bmatrix} 0.9 + 0.2\cos(0.2k) & 0.21 & 0 \\ 0 & 0.9 & 0.5 \\ 0 & 0 & 0.98 + 0.1\sin(0.3k) \end{bmatrix}$$

and the measurement matrix

$$H_k = \begin{bmatrix} 2 + 0.5\sin(0.2k) & 3 & 1 \\ 1 & 0 & 0.98 + 0.25\cos(0.31k) \end{bmatrix}$$

The covariances of the process noise and measurement noise are given by $Q = 0.2I_{3\times3}$ and $R = 0.2I_{2\times2}$, respectively. The initial value x_0 is a Gaussian random vector with mean $[1; 1; 1]$ and covariance $4I_{3\times3}$.

From Theorems 1-2, it can be found that $\{P_{k|k}\}$ is in fact a stochastic sequence due to the randomness of the indicators $\gamma_{k,i}$ in the MAREs (10) and (26). To numerically compute the expectation $\mathbb{E}[(P_{k|k})]$, we conduct the Monte Carlo simulations with $1,000$ repeated trials. The simulation results are depicted in Fig. 3, where the evolutionary trajectories of $\mathbb{E}[\mathrm{tr}(P_{k|k})]$ under two strategies $\mathcal{S}_i$ and $\mathcal{S}_i'$ are compared. Additionally, the variance of the Kalman filter without communication scheduling is presented as a benchmark. We can see that the estimation error covariance of the proposed strategies, i.e., $\mathbb{E}[\mathrm{tr}(P_{k|k})]$, is always larger than the optimal one $P_{k|k}^{KF}$ from the Kalman filter. For the single-trial tracking performance, the actual states $x_k = [x_{k,1}, x_{k,2}, x_{k,3}]$ and the corresponding estimates are given in Fig. 4, which implies that the scheduler-based estimator performs well to estimate the system states. The average communication rate $\bar{\gamma}_i$ and its empirical value under the strategies $\mathcal{S}_i$ are shown in Fig. 5, which indicates that the two values fit quite well when the threshold Δ is small. With the increase of Δ, the Gaussian approximation becomes inaccurate, and thus a gap appears. We finally consider the influence of the thresholds $\Delta_1 = \Delta_2 = \Delta$ on the performance of the remote estimator under strategies (8) in Fig. 6. As $\Delta \to 0$, the gap between $\mathbb{E}[\mathrm{tr}(P_{k|k})]$ and $\mathrm{tr}(P_{k|k}^{KF})$ narrows, and, eventually, it has $\mathbb{E}[\mathrm{tr}(P_{k|k})] = \mathrm{tr}(P_{k|k}^{KF})$ when $\Delta = 0$.

VII. CONCLUSION

This paper has investigated the remote estimation problem for linear time-varying systems under constrained network resources. The communication network is composed of a set of parallel and independent subchannels, which transmit the vector input in a componentwise manner. Schedulers are implemented in every subchannel to help reducing the communication rate. We aim to design the MMSE estimator which

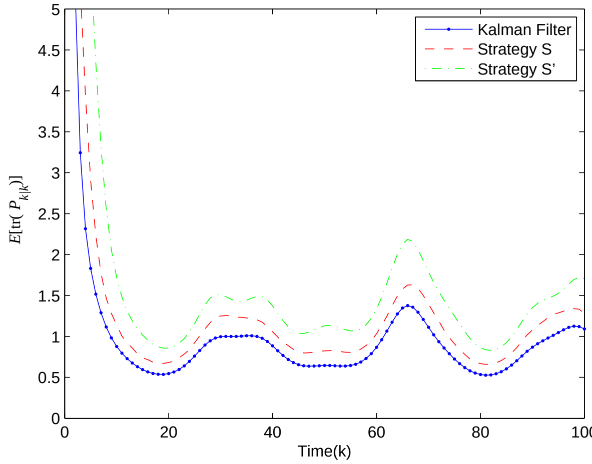

Fig. 3. Expectation of the variances of the remote estimators with different scheduling strategies $\mathcal{S}_i$ and $\mathcal{S}_i'$. Kalman Filter is presented as a benchmark. The expectations are obtained by averaging 1,000 Monte Carlo simulations. ($\Delta_1 = \Delta_2 = 1$ and $\Omega_1 = \Omega_2 = [0, 10]$)

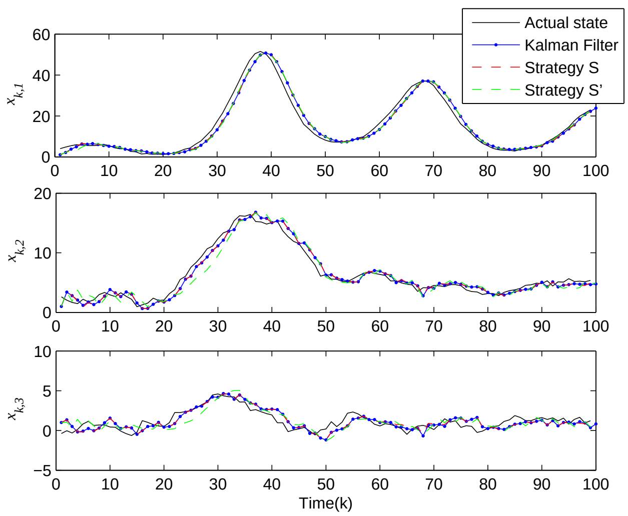

Fig. 4. Tracking performance of the remote estimator with different the scheduling strategies $\mathcal{S}_i$ and $\mathcal{S}_i'$ in a single-trial experiment. Kalman Filter is presented as a benchmark. ($\Delta_1 = \Delta_2 = 1$ and $\Omega_1 = \Omega_2 = [0, 10]$)

utilizes the scheduled signals from each subchannel under the predetermined scheduling strategies. The high computational cost inherent to handling the scheduling process motivates an approximate MMSE as an alternative. We have made a Gaussian assumption on the prior PDF of the prediction so as to obtain the approximate MMSE in a linear recursive form. Two scheduling policies have been taken into consideration, wherein the first one transmits the signals when it is necessary, and the second one further condenses the signals to be transmitted into a single bit variable. Finally, the effectiveness of the proposed algorithms has been validated by a numerical example. It should be pointed out that the estimation method developed in this paper is inapplicable to the scheduler-based distributed state estimation problem over sensor networks. The main challenge of such an issue lines in the fact that the signal

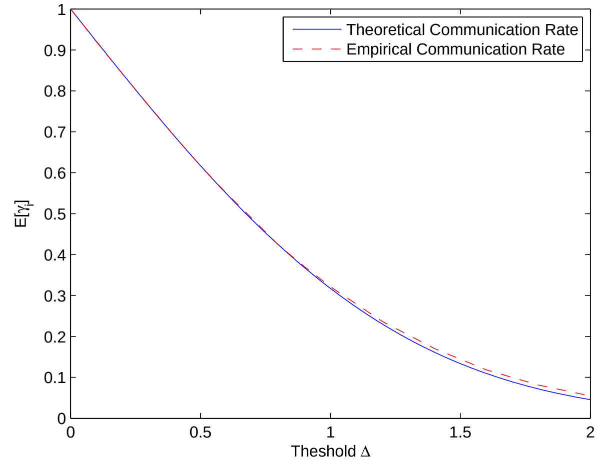

Fig. 5. Compare the theoretical communication rate $\bar{\gamma}_i$ with its empirical value obtained by averaging 1,000 Monte Carlo simulations.

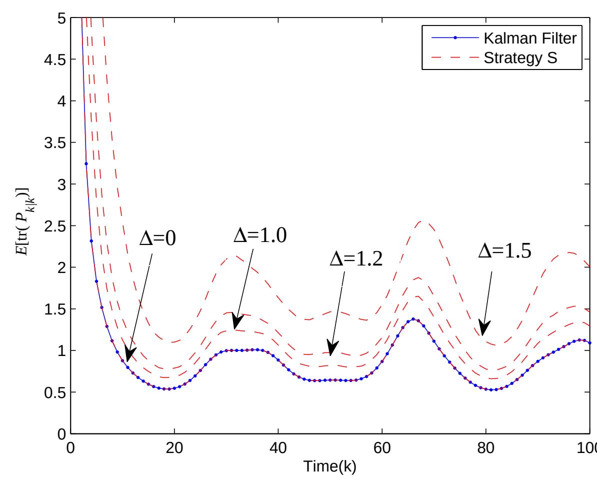

Fig. 6. Expectation of the variances of the remote estimators with the scheduling strategies $\mathcal{S}_i$ under different thresholds $\Delta_1 = \Delta_2$ $(= 0, 1.0, 1.2, 1.50)$. The expectations are obtained by averaging 1,000 Monte Carlo simulations.

transmission behaviors of distributed state estimation are more complicated as compared with the non-distributed case. The corresponding estimator structure and scheduling strategies are quite distinguished from those proposed in this paper. One of our future research topics is to study the scheduler-based distributed state estimation problem over sensor networks.

APPENDIX

A. Proof of Lemma 1

Proof: Using Bayes' rules, one has

$$f(x|x \in \Omega) = \frac{\Pr(x \in \Omega|x)}{\Pr(x \in \Omega)} f(x) \tag{35}$$

Moreover, it can be seen that

$$\Pr(x \in \Omega|x) = \begin{cases} 1 & x \in \Omega \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

which yields the results presented in this Lemma. ■

B. Proof of Lemma 2

Proof: Let $\vec{Y} = \begin{bmatrix} Y^T & Z^T \end{bmatrix}^T$ and $\check{Y} = \begin{bmatrix} \bar{Y}^T & \bar{Z}^T \end{bmatrix}^T$ where $\bar{Y}$ and $\bar{Z}$ represent the expectations of Y and Z, respectively. Then, according to the results in [1], it is easy to conclude that

$$\begin{cases} \mathbb{E}[X|\vec{Y}] = \bar{X} + \Sigma_{X\vec{Y}}\Sigma_{\vec{Y}}^{-1}(\vec{Y} - \check{Y}) \\ \mathbb{E}[(X - \mathbb{E}[X|\vec{Y}])(*)'|\vec{Y}] = \Sigma_X - \Sigma_{X\vec{Y}}\Sigma_{\vec{Y}}^{-1}\Sigma_{\vec{Y}X} \end{cases} \tag{37}$$

Noting that

$$\Sigma_{X\vec{Y}} = \begin{bmatrix} \Sigma_{XY} & \Sigma_{XZ} \end{bmatrix}, \ \Sigma_{\vec{Y}} = \begin{bmatrix} \Sigma_Y & 0 \\ 0 & \Sigma_Z \end{bmatrix},$$

it follows from (37) that

$$\begin{cases} \mathbb{E}[X|\vec{Y}] = \bar{X} + \Sigma_{XY}\Sigma_Y^{-1}(Y - \bar{Y}) + \Sigma_{XZ}\Sigma_Z^{-1}(Z - \bar{Z}) \\ \mathbb{E}[(X - \mathbb{E}[X|\vec{Y}])(*)'|\vec{Y}] = \Sigma_X - \Sigma_{XY}\Sigma_Y^{-1}\Sigma_{YX} \\ \qquad\qquad - \Sigma_{XZ}\Sigma_Z^{-1}\Sigma_{ZX} \end{cases}, \tag{38}$$

which yields the results presented in Lemma 2. ■

References

[1] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*, Courier Corporation, 2012.

[2] G. Battistelli, B. Alessio, and C. Luigi, State estimation with remote sensors and intermittent transmissions, *Systems & Control Letters*, vol. 61, no. 1, pp. 155–164, 2012.

[3] R. Caballero-Águila, A. Hermoso-Carazo, and J. Linares-Peŕez, Optimal state estimation for networked systems with random parameter matrices,correlated noises and delayed measurements, *Int. J. Gen. Syst.*, vol. 44, no. 2, pp. 142–154, 2015.

[4] R. Caballero-Águila, A. Hermoso-Carazo and J. Linares-Pérez, Networked distributed fusion estimation under uncertain outputs with random transmission delays, packet losses and multi-packet processing, *Signal Processing*, vol. 156, pp. 71-83, Mar. 2019.

[5] V. Gupta, T. H. Chung, B. Hassibi and R. M. Murray, On a stochastic sensor selection algorithm with applications in sensor scheduling and sensor coverage, *Automatica*, vol. 42, no. 2, pp. 251–260, 2006.

[6] D. Han, Y. Mo, J. Wu, S. Weerakkody, B. Sinopoli and L. Shi, Stochastic event-triggered sensor schedule for remote state estimation, *IEEE Trans. Automatic Control*, available online, 2014.

[7] F. Han, Z. Wang, H. Dong and H. Liu, Partial-nodes-based scalable H_∞-consensus filtering with censored measurements over sensor networks, *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 3, pp. 1892–1903, 2021.

[8] J. Hu, H. Zhang, H. Liu and X. Yu, A survey on sliding mode control for networked control systems, *International Journal of Systems Science*, vol. 52, no. 6, pp. 1129–1147, 2021.

[9] S. Hu, D. Yue, X. Yin, X. Xie and Y. Ma, Adaptive event-triggered control for nonlinear discrete-time systems, *International Journal of Robust and Nonlinear Control*, vol. 26, no. 18, pp. 4104–4125, 2016.

[10] C. Komninakis, C. Fragouli, A. Sayed and R. Wesel, Multi-input multi-output fading channel tracking and equalization using Kalman estimation, *IEEE Trans. Signal Processing*, vol. 50, no. 5, pp. 1065–1076, 2002.

[11] Q. Li, B. Shen, Z. Wang and W. Sheng, Recursive distributed filtering over sensor networks on Gilbert-Elliott channels: A dynamic event-triggered approach, *Automatica*, vol. 113, art. no. 108681, 2020.

[12] Q. Li, Z. Wang, W. Sheng, F. E. Alsaadi and F. E. Alsaadi, Dynamic event-triggered mechanism for H_∞ non-Fragile state estimation of complex networks under randomly occurring sensor saturations, *Information Sciences*, vol. 509, pp. 304–316, 2020.

[13] X. Li, F. Han, N. Hou, H. Dong and H. Liu, Set-membership filtering for piecewise linear systems with censored measurements under Round-Robin protocol, *International Journal of Systems Science*, vol. 51, no. 9, pp. 1578–1588, 2020.

[14] Y. Li, L. Liu, C. Hua and G. Feng, Event-triggered/self-triggered leader-following control of stochastic nonlinear multiagent systems using high-gain method, *IEEE Transactions on Cybernetics*, vol. 51, no. 6, pp. 2969–2978, Jun. 2021.

[15] D. Liu, Y. Liu and F. E. Alsaadi, Recursive state estimation based-on the outputs of partial nodes for discrete-time stochastic complex networks with switched topology, *Journal of the Franklin Institute*, vol. 355, no. 11, pp. 4686-4707, Jul. 2018.

[16] J. Liu, Y. Gu, L. Zha, Y. Liu and J. Cao, Event-triggered H_∞ load frequency control for multiarea power systems under hybrid cyber attacks, *IEEE Transactions on Systems Man Cybernetics: Systems*, vol. 49, no. 8, pp. 1665–1678, Aug. 2019.

[17] J. Liu, M. Yang, E. Tian, J. Cao and S. Fei, Event-based security controller design for state-dependent uncertain systems under hybrid-attacks and its application to electronic circuits, *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 66, no. 12, pp. 4817–4828, Dec. 2019.

[18] L. Liu, L. Ma, J. Zhang and Y. Bo, Distributed non-fragile set-membership filtering for nonlinear systems under fading channels and bias injection attacks, *International Journal of Systems Science*, vol. 52, no. 6, pp. 1192–1205, 2021.

[19] Q. Liu, Z. Wang, X. He, D. H. Zhou, Event-based recursive distributed filtering over wireless sensor networks, *IEEE Trans. Automatic Control*, vol. 60, no. 9, pp. 2470–2475, Sept. 2015.

[20] Q. Liu, Z. Wang, X. He, D. H. Zhou, Event-based distributed filtering over Markovian switching topologies, *IEEE Trans. Automatic Control*, vol. 64, no. 4, pp. 1595–1602, 2019.

[21] Q. Liu and Z. Wang, Moving-horizon estimation for linear dynamic networks with binary encoding schemes, *IEEE Trans. Automatic Control*, vol. 66, no. 4, pp. 1763–1770, May 2020.

[22] Q. Liu, Z. Wang, Q. L. Han, and C. Jiang, Quadratic estimation for discrete time-varying non-Gaussian systems with multiplicative noises and quantization effects, *Automatica*, vol. 113, art. no. 108714, Mar. 2020.

[23] S. Liu, Z. Wang, G. Wei and M. Li, Distributed set-membership filtering for multi-rate systems under the Round-Robin scheduling over sensor networks, *IEEE Transactions on Cybernetics*, vol. 50, no. 5, pp. 1910–1920, 2020.

[24] J. Mao, D. Ding, G. Wei and H. Liu, Networked recursive filtering for time-delayed nonlinear stochastic systems with uniform quantisation under Round-Robin protocol, *International Journal of Systems Science*, vol. 50, no. 4, pp. 871-884, Mar. 2019.

[25] J. Mao, Y. Sun, X. Yi, H. Liu, and D. Ding, Recursive filtering of networked nonlinear systems: a survey, *International Journal of Systems Science*, vol. 52, no. 6, pp. 1110–1128, 2021.

[26] A. Le and R. McCann, Event-based measurement updating Kalman filter in network control systems. *Proceedings of the IEEE region 5 technical conference, Fayetteville, AR*, pp. 138–141, 2007.

[27] X. Lu, H. Zhang, W. Wang and K.-L. Teo, Kalman filtering for multiple time-delay systems, *Automatica*, vol. 41, no. 8, pp. 1455–1461, 2005.

[28] E. Msechu, S. Roumeliotis, A. Ribeiro and G. Giannakis, Decentralized quantized Kalman filtering with scalable communication cost, *IEEE Trans. Signal Processing*, vol. 56, no. 8, pp. 3727–3741, 2008.

[29] Y. Mo, R. Ambrosino and B. Sinopoli, Sensor selection strategies for state estimation in energy constrained wireless sensor networks, *Automatica*, vol. 47, no. 7, pp. 1330–1338, 2011.

[30] M. Miskowicz, Send-on-delta concept: An event-based data reporting strategy, *Sensors*, vol. 6, no. 1, pp. 49–63, 2006.

[31] C. Peng, Q.-L. Han and D. Yue, Communication-delay-distribution-dependent decentralized control for large-scale systems with IP-based communication networks, *IEEE Transactions on Control Systems Technology*, vol. 21, no. 3, pp. 820–830, 2012.

[32] W. Qian, Y. Li, Y. Chen, and W. Liu, L_2-L_∞ filtering for stochastic delayed systems with randomly occurring nonlinearities and sensor saturation, *International Journal of Systems Science*, vol. 51, no. 13, pp. 2360–2377, 2020.

[33] W. Qian, Y. Li, Y. Zhao, and Y. Chen, New optimal method for L_2-L_∞ state estimation of delayed neural networks, *Neurocomputing*, vol. 415, pp. 258–265, 2020.

[34] A. Ribeiro, G. Giannakis, S. Roumeliotis, SOI-KF: Distributed Kalman filtering with low-cost communications using the sign of innovations, *IEEE Trans. Signal Processing*, vol. 54, no. 12, pp. 4782–4795, 2006.

[35] Z. Ren, P. Cheng, J. Chen, L. Shi and H. Zhang, Dynamic sensor transmission power scheduling for remote state estimation, *Automatica*, vol. 50, no. 4, pp. 1235–1242, 2014.

[36] B. Shen, Z. Wang, D. Wang and H. Liu, Distributed state-saturated recursive filtering over sensor networks under Round-Robin protocol,

IEEE Transactions on Cybernetics, vol. 50, no. 8, pp. 3605–3615, Aug. 2020.

[37] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. Jordanand S. Sastry, Kalman filtering with intermittent observations, *IEEE Trans. Automatic Control*, vol. 49, no. 9, pp. 1453–1464, 2004.

[38] J. Sijs and M. Lazar, Event based state estimation with time synchronous updates, *IEEE Trans. Automatic Control*, vol. 57, no. 10, pp. 2650–2655, 2012.

[39] Y. S. Suh, V. H. Nguyen and Y. S. Ro, Modified Kalman filter for networked monitoring systems employing a send-on-delta method, *Automatica*, vol. 43, no. 2, pp. 332-338, Feb. 2007.

[40] H. Tan, B. Shen, K. Peng and H. Liu, Robust recursive filtering for uncertain stochastic systems with amplify-and-forward relays, *International Journal of Systems Science*, vol. 51, no. 7, pp. 1188–1199, 2020.

[41] E. Tian and D. Yue, Decentralized control of network-based interconnected systems: A state-dependent triggering method, *International Journal of Robust and Nonlinear Control*, vol. 25, no. 8, pp. 1126–1144, 2015.

[42] J. Wu, Q. Jia, K. Jonhansson and L. Shi, Event-based sensor data scheduling: trade-off between communication rate and estimation quality, *IEEE Trans. Automatic Control*, vol. 58, no. 4, 2013.

[43] H. Yang, Z. Wang, Y. Shen, F. E. Alsaadi and F. E. Alsaadi, Event-triggered state estimation for Markovian jumping neural networks: On mode-dependent delays and uncertain transition probabilities, *Neurocomputing*, vol. 424, pp. 226–235, 2021.

[44] K. You and L. Xie, Kalman filtering with scheduled measurements, *IEEE Trans. Signal Processing*, vol. 61, no. 6, pp. 1520–1530, 2013

[45] W. Yue, Z. Wang, W. Liu, B. Tian, S. Lauria and X. Liu, An optimally weighted user- and item-based collaborative filtering approach to predicting baseline data for Friedreich's Ataxia patients, *Neurocomputing*, vol. 419, pp. 287–294, 2021.

[46] Z. Zhao, Z. Wang, L. Zou and J. Guo, Set-membership filtering for time-varying complex networks with uniform quantisations over randomly delayed redundant channels, *International Journal of Systems Science*, vol. 51, no. 16, pp. 3364–3377, 2020.

[47] L. Zou, Z. Wang, Q.-L. Han and D. H. Zhou, Full information estimation for time-varying systems subject to Round-Robin scheduling: A recursive filter approach, *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 3, pp. 1904–1916, 2021.

[48] L. Zou, Z. Wang, J. Hu, and D. Zhou, Moving horizon estimation with unknown inputs under dynamic quantization effects, *IEEE Transactions on Automatic Control*, vol. 65, no. 12, pp. 5368–5375, 2020.