On Finite-Horizon H_∞ State Estimation for Discrete-Time Delayed Memristive Neural Networks under Stochastic Communication Protocol

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Abstract

This paper is concerned with the protocol-based finite-horizon H_∞ estimation problem for discrete-time memristive neural networks (MNNs) subject to time-delays and energy-bounded disturbances. With the purpose of effectively alleviating data collisions and saving energy, the stochastic communication protocol (SCP) is adopted to regulate the data transmission procedure in the sensor-to-estimator communication channel, thereby avoiding unnecessary network congestion. It is our objective to construct an H_∞ estimator ensuring a prescribed disturbance attenuation level over a finite time-horizon for the delayed MNNs under the SCP. By virtue of the Lyapunov-Krasovskii functional in combination with stochastic analysis methods, the delay-dependent criteria are established that guarantee the existence of the desired H_∞ estimator. Subsequently, the estimator gains are computed by resorting to solve a bank of convex optimization problems. Finally, the validity of the designed H_∞ estimator is demonstrated via a numerical example.

Keywords: Memristive neural networks, H_∞ state estimation, finite-horizon estimation, stochastic communication protocol.

1. Introduction

For decades, a recurring research interest has been paid to the recurrent neural networks (RNNs) because of their broad applications in various areas such as prediction and estimation, pattern recognition, intelligent robots as well as automatic control [11, 15, 46, 34, 12, 1, 13, 49, 14, 6, 47, 48]. In practice, RNNs are usually realized via very large scale integration circuits where the connection weights are executed via resistors [43]. However, it is well recognized nowadays that the resistor

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has certain inherent weakness regardless of the usefulness in realizing some specific functions. For instance, in the neural circuit, the volatility of the resistor renders the state information disappear in the absence of voltage and, moreover, the huge amount of resistors leads inevitably to a substantial reduction of the integration degree of the neural circuit [33]. As such, the memristors have been introduced to take place of the resistors in implementing neural networks, and the so-called memristive neural networks (MNNs) have then attracted much attention in various domains such as combinatorial optimization, knowledge acquisition and brain emulation [38, 3, 4, 35].

State estimation (SE) which aims to extract the state information from corrupted measurements is a fundamental yet crucial research focus in physical world. As physical systems are indispensably confronted with all sorts of disturbances, a great deal of research effort has been dedicated to the disturbance-corrupted SE problems. So far, quite a few techniques have been developed including, but are not limited to, the Kalman filtering approach, the H_∞ technique, the l_2-l_∞ method and the set-membership framework, see [18, 16, 19, 28, 39, 48, 47] for some recent publications. Generally speaking, the Kalman filtering method is mainly capable of dealing with Gaussian noises, the H_∞ and l_2-l_∞ methods are put forward to tackle unknown but energy-bounded disturbances, and the set-membership approach primarily has the ability to handle unknown-but-bounded noises. Specifically, the Kalman filtering method provides the optimal solution by minimizing the covariance of the estimation error in the mean square sense, the H_∞ method has the capability to impose a desired disturbance attenuation level on the estimation error [7], the l_2-l_∞ method manages to keep the peak value of the estimation error within an allowable range [25], and the set-membership method is able to confine all possible state estimates within a required specific area containing the true state [8].

The last decades have witnessed an everlasting research enthusiasm on the study of steady-state behaviors (e.g. asymptotic/exponential stability) of MNNs, see [37, 41, 36]. It is notable that, in reality, with ensured *steady-state* behaviors, systems are also required to possess satisfactory *transient* performances (e.g. finite-time convergence) on a finite horizon. Unfortunately, the achievement of the finite-time stability is usually difficult since the system dynamics is required to converge to the equilibrium in a specified yet limited time. As a matter of fact, in comparison to the absolute convergence, the idea that keeps system states below a given level seems much more realistic on a finite horizon in case of exogenous disturbances. In other words, when considering transient performance, it might be more favorable to constrain the state evolution to a desirable level in accordance with engineering practice. As such, much research attention has recently been drawn towards SE problems for MNNs under energy-bounded disturbances with transient behavior requirements and some inspiring results have been published, see e.g. [41].

As the fast development of network technology, nowadays, the SE algorithm of MNNs in practical engineering is sometimes required to be realized at a remote location in a networked environment, which gives rise to the remote SE issue. In such a case, the actual measurement outputs are often transmitted to a remote estimator through a communication medium (e.g. wireless/distributed networks). Because of the big size of MNNs and the high-degree intricacy of the to-do tasks, the data volume of the network output could become considerably high, thereby posing great challenges (e.g. fading measurement and communication delays) onto the transmission networks of limited capacity [10, 40, 45]. To handle these network-induced challenges, an effective measure is to leverage communication protocols that help regulate the data transmission, and some widely deployed protocols include the event-triggering protocol (EEP), Round-Robin protocol (RRP), stochastic communication protocol (SCP) and try-once-discard protocol (TODP), see [41, 42, 5, 44]. For example, the EEP has been successfully applied to the MNNs for the purpose of saving communication resources in [20, 9], and the RRP and SCP have been applied to the traditional RNNs with the view of avoiding data collisions in [17, 32]. Nevertheless, to our best knowledge, very few results have been acquired so far on the finite-time H_∞ SE problem for delayed MNNs, not to mention the case where the energy-bounded disturbances and SCP are both embraced.

In this paper, we aim at developing a finite-time H_∞ estimator for delayed MNNs with energy-bounded disturbances under the SCP. The primary contributions we deliver in this paper are outlined in threefold. 1) The SCP is introduced to orchestrate the data transmission order between the MNNs and the remote estimator, and thereby alleviating data collisions and saving communication resources. 2) A unified H_∞ framework is built for SE issue of MNNs to cope with the mathematical complexities resulting from the time-delays, energy-bounded disturbances and SCP. 3) Sufficient conditions are found for the solvability of the addressed SE problem and the filter is obtained with the help of a bank of recursive linear matrix inequalities (RLMIs) whose solutions are provided by standard software packages.

The subsequent part of this paper is organized as follows. In Section 2, the finite-time H_∞ state estimation problem is formulated for the discrete-time MNNs subject to time-delays and energy-bounded disturbances. Section 3 derives the main results for designing the desired estimator. In Section 4, a simulation example is provided to illustrate the effectiveness of the proposed estimation scheme. Finally, the conclusion is drawn in Section 5.

Notation. $\mathbb{N}$ is the set of $\{1, 2, \ldots, n\}$. I denotes the identity matrix. $\lambda_{\min}(A)$ ($\lambda_{\max}(A)$) is the smallest (largest) eigenvalue of matrix A. $\mathrm{diag}\{\cdots\}$ means a block-diagonal matrix. $\mathbb{E}\{\cdot\}$ is the expectation operator. $\|x\|$ presents the Euclidean norm of x. The symbol $*$ indicates an ellipsis for symmetry-induced terms.

2. Problem Formulation and Preliminaries

Consider an MNN with the following structure:

$$\begin{cases} z(s+1) = D(z(s))z(s) + A(z(s))f(z(s)) \\ \qquad\qquad + B(z(s))g(z(s-\tau)) + L(s)v(s) \\ y(s) = C(s)z(s) + M(s)v(s) \\ \bar{z}(s) = N(s)z(s) \end{cases} \tag{1}$$

where

$$\begin{aligned} z(s) &= \begin{bmatrix} z_1(s) & z_2(s) & \cdots & z_n(s) \end{bmatrix}^T, \\ y(s) &= \begin{bmatrix} y_1(s) & y_2(s) & \cdots & y_m(s) \end{bmatrix}^T, \\ D(z(s)) &= \mathrm{diag}\{d_1(z_1(s)), d_2(z_2(s)), \cdots, d_n(z_n(s))\}, \quad d_i(z_i(s)) > 0 \ (i = 1, 2, \ldots, n) \end{aligned}$$

are the neuron state vector, the ideal measurement output and the self-feedback matrix, respectively; $\bar{z}(s) \in \mathbb{R}^p$ is the neural state to be estimated; $v(s) \in l_2[0, \mathrm{N}-1]$ is the disturbance vector; $A(z(s)) = \big(a_{ij}(z_i(s))\big)_{n\times n}$ and $B(z(s)) = \big(b_{ij}(z_i(s))\big)_{n\times n}$ are connection weight matrices with no delays and discrete delays, respectively; the matrices $C(s)$, $L(s)$, $M(s)$ and $N(s)$ are known and with compatible dimensions.

The nonlinear neuron activation functions (NNAFs) $f(z(s))$ and $g(z(s-\tau))$ have the following forms:

$$f(z(s)) = \begin{bmatrix} f_1(z_1(s)) & f_2(z_2(s)) & \cdots & f_n(z_n(s)) \end{bmatrix}^T,$$

$$g(z(s-\tau)) = \begin{bmatrix} g_1(z_1(s-\tau)) & g_2(z_2(s-\tau)) & \cdots & g_n(z_n(s-\tau)) \end{bmatrix}^T$$

where $\tau \in \mathbb{Z}^+$ is a constant delay and the initial condition is $\phi(s) = \begin{bmatrix} \phi_1(s) & \phi_2(s) & \cdots & \phi_n(s) \end{bmatrix}^T$ for $s \in [-\tau, 0]$.

Assumption 1. *The NNAFs $f(z(s))$ and $g(z(s))$ are assumed to be continuous and satisfy*

$$\|f(z(s))\|^2 \leq \vartheta_f(s)\|\Upsilon_f(s)z(s)\|^2, \tag{2}$$

$$\|g(z(s))\|^2 \leq \vartheta_g(s)\|\Upsilon_g(s)z(s)\|^2 \tag{3}$$

for all $s \in [0, \mathbb{N}]$, where $\vartheta_f(s)$ and $\vartheta_g(s)$ are known positive scalars, $\Upsilon_f(s)$ and $\Upsilon_g(s)$ are known matrices, and $f(0) = g(0) = 0$.

By using the technique employed in [20], the MNN given by (1) is rewritten as:

$$\begin{cases} z(s+1) = & (\bar{D} + \Delta D(s))z(s) + (\bar{A} + \Delta A(s))f(z(s)) \\ & + (\bar{B} + \Delta B(s))g(z(s-\tau)) + L(s)v(s) \\ y(s) = & C(s)z(s) + M(s)v(s) \\ \bar{z}(s) = & N(s)z(s) \end{cases} \tag{4}$$

where

$$\bar{D} \triangleq \mathrm{diag}\{d_1, d_2, \ldots, d_n\}, \quad \bar{A} \triangleq [a_{ij}]_{n\times n}, \quad \bar{B} \triangleq [b_{ij}]_{n\times n},$$
$$\Delta D(s) \triangleq \mathcal{H}F_1(s)E_1, \quad \Delta A(s) \triangleq \mathcal{H}F_2(s)E_2, \quad \Delta B(s) \triangleq \mathcal{H}F_3(s)E_3.$$

Here, $\mathcal{H} = \begin{bmatrix} H_1 & H_2 & \cdots & H_n \end{bmatrix}$ and $E_i = \begin{bmatrix} E_{i1}^T & E_{i2}^T & \cdots & E_{in}^T \end{bmatrix}^T$ $(i = 1, 2, 3)$ are known matrices, d_j, a_{ij} and b_{ij} are known positive scalars, and $F_i(s)$ $(i = 1, 2, 3)$ satisfies $F_i^T(s)F_i(s) \leq I$.

Remark 1. *In most existing literature, the norm-bounded conditions are enforced to facilitate the handling of parameter uncertainties. It is worth mentioning that, in the context of MNN, it is exactly the time-varying uncertain terms $\Delta D(s)$, $\Delta A(s)$ and $\Delta B(s)$ that reflect the influence from the memristors. It should be emphasized that, in the current work, our main focus is on examining the effect of the state-dependent switching (towards norm-bounded disturbances) that relies on features of the memristor and the current-voltage.*

For the purpose of reducing data collisions and mitigating network burdens, we adopt the SCP to schedule the data transmission. Let $\vec{y}_s(s)$ be the actually received data from the ith $(i = 1, 2 \ldots, m)$ node. Under the SCP, $\vec{y}_i(s)$ updates as

$$\vec{y}_i(s) = \begin{cases} y_i(s), & \text{if } i = \xi(s) \\ \vec{y}_i(s-1), & \text{otherwise} \end{cases} \tag{5}$$

where $\xi(s) \in \{1, 2, \ldots, m\}$ denotes the selected sensor for data transmissions at time s. $\xi(s)$ is regulated by a Markov chain with the transition probability matrix $\mathcal{P} = \big(p_{ij}\big)_{m\times m}$ where

$$p_{ij} \triangleq \mathrm{Prob}\big\{\xi(s+1) = j|\xi(s) = i\big\}, i, j = 1, 2, \ldots, m. \tag{6}$$

Here $p_{ij} \geq 0$ and $\sum_{j=1}^{m} p_{ij} = 1$. Consequently, this description leads to the following data exchange model between the transmitter and the receiver:

$$\vec{y}(s) = \Psi_{\xi(s)} y(s) + (I - \Psi_{\xi(s)})\vec{y}(s-1) \tag{7}$$

where $\Psi_{\xi(s)} = \mathrm{diag}\{\delta(\xi(s-1)), \delta(\xi(s-2)), \ldots, \delta(\xi(s-m))\}$ is the update matrix, and $\delta(\cdot)$ is the Kronecker delta function.

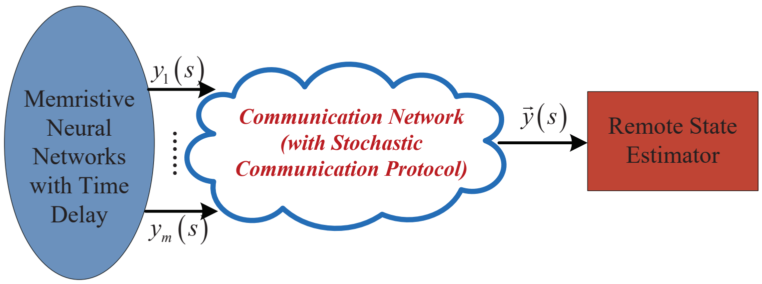

Figure 1: State Estimation with Stochastic Communication Protocols.

Remark 2. *In general, the communication protocols can be classified into two categories, i.e., static and dynamic protocols. Specifically, the RRP is an equal scheduling mechanism that belongs to the category of static protocols, whereas the SCP belongs to dynamic protocols in which the Markov chains are utilized to model the regulation procedure. Regardless of protocol categories, at each time step s, only one sensor is permitted to access the network for data transmission. To make full use of the data, the zero-order holder mechanism is implemented to store information that is not transmitted.*

Based on (1) and (5), the desired estimator is built as

$$\begin{cases} \hat{z}(s+1) = \bar{D}\hat{z}(s) + \bar{A}f(\hat{z}(s)) + \bar{B}g(\hat{z}(s-\tau)) + K(s)(\vec{y}(s) - C(s)\hat{z}(s)) \\ \hat{\bar{z}}(s) = N(s)\hat{z}(s) \end{cases} \tag{8}$$

where $\hat{z}(s)$ and $\hat{\bar{z}}(s)$ are the estimates of $z(s)$ and $\bar{z}(s)$, respectively, and $K(s)$ is the parameter to be designed.

By defining

$$\begin{aligned} e(s) &\triangleq z(s) - \hat{z}(s), \quad \tilde{\bar{z}}(s) \triangleq \bar{z}(s) - \hat{\bar{z}}(s), \\ \tilde{f}(s) &\triangleq f(z(s)) - f(\hat{z}(s)), \ \tilde{g}(s-\tau) \triangleq g(z(s-\tau)) - g(\hat{z}(s-\tau)), \end{aligned}$$

we acquire the estimation error dynamics from (4), (7) and (8) as follows:

$$\begin{cases} e(s+1) = (\bar{D} - K(s)\Psi_{\xi(s)}C(s))e(s) + (\Delta D(s) - K(s)(\Psi_{\xi(s)} - I)C(s))z(s) \\ \qquad\qquad + \bar{A}\tilde{f}(s) + \Delta A(s)f(z(s)) + \bar{B}\tilde{g}(s-\tau) + \Delta B(s)g(z(s-\tau)) \\ \qquad\qquad - K(s)(I - \Psi_{\xi(s)})\vec{y}(s-1) + (L(s) - K(s)\Psi_{\xi(s)}M(s))v(s) \\ \tilde{\bar{z}}(s) = N(s)e(s). \end{cases} \tag{9}$$

By denoting

$$\eta(s) \triangleq \begin{bmatrix} z^T(s) & e^T(s) & \tilde{y}^T(s-1) \end{bmatrix}^T,$$
$$\vec{f}(\eta(s)) \triangleq \begin{bmatrix} f^T(z(s)) & \tilde{f}^T(s) \end{bmatrix}^T,$$
$$\vec{g}(\eta(s-\tau)) \triangleq \begin{bmatrix} g^T(z(s-\tau)) & \tilde{g}^T(s-\tau) \end{bmatrix}^T,$$

and taking into account (1), (7) and (9), we further have the following augmented system:

$$\begin{cases} \eta(s+1) = \mathcal{D}(s)\eta(s) + \mathcal{A}(s)\vec{f}(\eta(s)) + \mathcal{B}(s)\vec{g}(\eta(s-\tau)) + \mathcal{L}(s)v(s) \\ \quad\tilde{\tilde{z}}(s) = \mathcal{N}(s)\eta(s) \end{cases} \tag{10}$$

where

$$\mathcal{N}(s) \triangleq \begin{bmatrix} 0 & N(s) & 0 \end{bmatrix}, \quad \mathcal{D}(s) \triangleq \bar{\mathcal{D}} + \Delta\mathcal{D}(s), \quad \mathcal{A}(s) \triangleq \bar{\mathcal{A}} + \Delta\mathcal{A}(s), \quad \mathcal{B}(s) \triangleq \bar{\mathcal{B}} + \Delta\mathcal{B}(s),$$
$$\bar{\mathcal{D}} \triangleq \begin{bmatrix} \bar{D} & 0 & 0 \\ \bar{\mathcal{D}}_{21} & \bar{\mathcal{D}}_{22} & \bar{\mathcal{D}}_{23} \\ \Psi_{\xi(s)}C(s) & 0 & I-\Psi_{\xi(s)} \end{bmatrix}, \ \Delta\mathcal{D}(s) \triangleq \begin{bmatrix} \Delta D(s) & 0 & 0 \\ \Delta D(s) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \bar{\mathcal{A}} \triangleq \begin{bmatrix} \bar{A} & 0 \\ 0 & \bar{A} \\ 0 & 0 \end{bmatrix},$$
$$\Delta\mathcal{A}(s) \triangleq \begin{bmatrix} \Delta A(s) & 0 \\ \Delta A(s) & 0 \\ 0 & 0 \end{bmatrix}, \ \bar{\mathcal{B}} \triangleq \begin{bmatrix} \bar{B} & 0 \\ 0 & \bar{B} \\ 0 & 0 \end{bmatrix}, \quad \Delta\mathcal{B}(s) \triangleq \begin{bmatrix} \Delta B(s) & 0 \\ \Delta B(s) & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{L}(s) \triangleq \begin{bmatrix} L(s) \\ L(s) - K(s)\Psi_{\xi(s)}M(s) \\ \Psi_{\xi(s)}M(s), \end{bmatrix},$$
$$\bar{\mathcal{D}}_{21} \triangleq K(s)(I-\Psi_{\xi(s)})C(s), \quad \bar{\mathcal{D}}_{22} \triangleq \bar{D} - K(s)\Psi_{\xi(s)}C(s), \quad \bar{\mathcal{D}}_{23} \triangleq -K(s)(I-\Psi_{\xi(s)}).$$

In this paper, it is our purpose to design estimator of form (8) such that

$$\mathbb{E}\left\{\sum_{s=0}^{N-1} ||\tilde{\tilde{z}}(s)||^2\right\} \le \gamma^2 \mathbb{E}\left\{\sum_{s=0}^{N-1} ||v(s)||^2 + \sum_{s=-\tau}^{0} \eta^T(s)S(s)\eta(s)\right\} \tag{11}$$

where $\gamma > 0$ is a prescribed disturbance attenuation level and $\{S(s)\}_{-\tau\le s\le 0}$ are known positive-definite matrices.

3. Main Results

Before presenting our main results, we first introduce the following lemmas that are helpful in subsequent derivations.

Lemma 1. *(Schur Complement Equivalence [2]) Given constant matrices $\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3$ where $\mathfrak{S}_1 = \mathfrak{S}_1^{\mathrm{T}}$ and $0 < \mathfrak{S}_2 = \mathfrak{S}_2^{\mathrm{T}}$, then $\mathfrak{S}_1 + \mathfrak{S}_3^{\mathrm{T}}\mathfrak{S}_2^{-1}\mathfrak{S}_3 < 0$ if and only if*

$$\begin{bmatrix} \mathfrak{S}_1 & \mathfrak{S}_3^{\mathrm{T}} \\ \mathfrak{S}_3 & -\mathfrak{S}_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\mathfrak{S}_2 & \mathfrak{S}_3 \\ \mathfrak{S}_3^{\mathrm{T}} & \mathfrak{S}_1 \end{bmatrix} < 0. \tag{12}$$

Lemma 2. *[2] Let $\mathfrak{M} = \mathfrak{M}^{\mathrm{T}}$, $\mathfrak{H}$ and $\mathfrak{E}$ be real matrices of appropriate dimensions, and Δ satisfies $\|\Delta\| \le 1$, then*

$$\mathfrak{M} + \mathfrak{H}\Delta\mathfrak{E} + \mathfrak{E}^{\mathrm{T}}\Delta\mathfrak{H}^{\mathrm{T}} \le 0 \tag{13}$$

if and only if there exists a positive scalar ε such that

$$\mathfrak{M} + \varepsilon\mathfrak{H}\mathfrak{H}^{\mathrm{T}} + \varepsilon^{-1}\mathfrak{E}^{\mathrm{T}}\mathfrak{E} \le 0. \tag{14}$$

We are now in the situation to present the sufficient condition under which the required H_∞ performance is guaranteed.

Theorem 1. *Let the attenuation level $\gamma > 0$, the matrix sequence $\{S(s)\}_{-\tau \le s \le 0}$, and parameter $K(s)$ be given. The estimation error $\tilde{\bar{z}}(s)$ satisfies the H_∞ constraint (11) if there exist matrix sequences $\{P(s)\}_{1 \le s \le \mathrm{N}}$ and $\{Q(s)\}_{-\tau+1 \le s \le \mathrm{N}+1}$ and scalar sequences $\{\lambda_1(s)\}_{0 \le s \le \mathrm{N}-1}$ and $\{\lambda_2(s)\}_{0 \le s \le \mathrm{N}-1}$ satisfying*

$$\mathbb{E}\left\{\eta^T(0)P(0)\eta(0) + \sum_{s=-\tau}^{0} \eta^T(s)Q(s+1)\eta(s)\right\} \le \gamma^2 \mathbb{E}\left\{\sum_{s=-\tau}^{0} \eta^T(s)S(s)\eta(s)\right\} \tag{15}$$

and

$$\Omega(s) = \begin{bmatrix} \Theta_{11} & 0 & \Theta_{13} & \Theta_{14} & \Theta_{15} \\ * & \Theta_{22} & 0 & 0 & 0 \\ * & * & \Theta_{33} & \Theta_{34} & \Theta_{35} \\ * & * & * & \Theta_{44} & \Theta_{45} \\ * & * & * & * & \Theta_{55} \end{bmatrix} < 0 \tag{16}$$

where

$$\begin{aligned}
\Theta_{11} &\triangleq \mathcal{D}^T(s)P(s+1)\mathcal{D}(s) + Q(s+1) + \mathcal{N}^T(s)\mathcal{N}(s) + 3\lambda_1(s)\vartheta_f(s)\tilde{\Upsilon}_f^T(s)\tilde{\Upsilon}_f(s) \\
&\quad + 4\lambda_1(s)\vartheta_f(s)\bar{\Upsilon}_f^T(s)\bar{\Upsilon}_f(s) - P(s), \\
\Theta_{22} &\triangleq 3\lambda_2(s)\vartheta_g(s-\tau)\tilde{\Upsilon}_g^T(s-\tau)\tilde{\Upsilon}_g(s-\tau) + 4\lambda_2(s)\vartheta_g(s-\tau)\bar{\Upsilon}_g^T(s-\tau)\bar{\Upsilon}_g(s-\tau) - Q(s+1-\tau), \\
\Theta_{33} &\triangleq \mathcal{A}^T(s)P(s+1)\mathcal{A}(s) - \lambda_1(s)I, \ \Theta_{44} \triangleq \mathcal{B}^T(s)P(s+1)\mathcal{B}(s) - \lambda_2(s)I, \\
\Theta_{55} &\triangleq \mathcal{L}^T(s)P(s+1)\mathcal{L}(s) - \gamma^2 I, \ \Theta_{13} \triangleq \mathcal{D}^T(s)P(s+1)\mathcal{A}(s), \ \Theta_{14} \triangleq \mathcal{D}^T(s)P(s+1)\mathcal{B}(s), \\
\Theta_{15} &\triangleq \mathcal{D}^T(s)P(s+1)\mathcal{L}(s), \ \Theta_{34} \triangleq \mathcal{A}^T(s)P(s+1)\mathcal{B}(s), \ \Theta_{35} \triangleq \mathcal{A}^T(s)P(s+1)\mathcal{L}(s), \\
\Theta_{45} &\triangleq \mathcal{B}^T(s)P(s+1)\mathcal{L}(s)
\end{aligned}$$

with

$$\begin{aligned}
\tilde{\Upsilon}_f(s) &\triangleq \begin{bmatrix} \Upsilon_f(s) & 0 & 0 \end{bmatrix}, \ \bar{\Upsilon}_f(s) \triangleq \begin{bmatrix} \Upsilon_f(s) & 0 & 0 \\ 0 & \Upsilon_f(s) & 0 \end{bmatrix}, \\
\tilde{\Upsilon}_g(s-\tau) &\triangleq \begin{bmatrix} \Upsilon_g(s-\tau) & 0 & 0 \end{bmatrix}, \ \bar{\Upsilon}_g(s-\tau) \triangleq \begin{bmatrix} \Upsilon_g(s-\tau) & 0 & 0 \\ 0 & \Upsilon_g(s-\tau) & 0 \end{bmatrix}.
\end{aligned}$$

Proof. Consider the following Lyapunov-Krasovskii functional:

$$V(\eta(s)) = V_1(\eta(s)) + V_2(\eta(s)) \tag{17}$$

where

$$\begin{aligned}
V_1(\eta(s)) &\triangleq \eta^T(s)P(s)\eta(s), \\
V_2(\eta(s)) &\triangleq \sum_{i=s-\tau}^{s-1} \eta^T(i)Q(i+1)\eta(i).
\end{aligned}$$

Computing $\Delta V(\eta(s))$ based on system (10) and then taking the expectation, we obtain

$$\mathbb{E}\{\Delta V(\eta(s))\} = \mathbb{E}\{\Delta V_1(\eta(s)) + \Delta V_2(\eta(s))\} \tag{18}$$

where

$$\begin{aligned}
\mathbb{E}\{\Delta V_1(\eta(s))\} =& \mathbb{E}\{V_1(\eta(s+1)) - V_1(\eta(s))\} \\
=& \mathbb{E}\{\eta^T(s+1)P(s+1)\eta(s+1) - \eta^T(s)P(s)\eta(s)\} \\
=& \mathbb{E}\{\eta^T(s)[\mathcal{D}^T(s)P(s+1)\mathcal{D}(s) - P(s)]\eta(s) + \vec{f}^T(\eta(s))\mathcal{A}^T(s)P(s+1)\mathcal{A}(s)\vec{f}(\eta(s)) \\
& + \vec{g}^T(\eta(s-\tau))\mathcal{B}^T(s)P(s+1)\mathcal{B}(s)\vec{g}(\eta(s-\tau)) + v^T(s)\mathcal{L}^T(s)P(s+1)\mathcal{L}(s)v(s) \\
& + 2\eta^T(s)\mathcal{D}^T(s)P(s+1)\mathcal{A}(s)\vec{f}(\eta(s)) + 2\eta^T(s)\mathcal{D}^T(s)P(s+1)\mathcal{B}(s)\vec{g}(\eta(s-\tau)) \\
& + 2\eta^T(s)\mathcal{D}^T(s)P(s+1)\mathcal{L}(s)v(s) + 2\vec{f}^T(\eta(s))\mathcal{A}^T(s)P(s+1)\mathcal{B}(s)\vec{g}(\eta(s-\tau)) \\
& + 2\vec{f}^T(\eta(s))\mathcal{A}^T(s)P(s+1)\mathcal{L}(s)v(s) + 2\vec{g}^T(\eta(s-\tau))\mathcal{A}^T(s)P(s+1)\mathcal{L}(s)v(s)\}
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
\mathbb{E}\{\Delta V_2(\eta(s))\} =& \mathbb{E}\{V_2(\eta(s+1)) - V_2(\eta(s))\} \\
=& \mathbb{E}\{\sum_{i=s+1-\tau}^{s} \eta^T(i)Q(i+1)\eta(i) - \sum_{i=s-\tau}^{s-1} \eta^T(i)Q(i+1)\eta(i)\} \\
=& \mathbb{E}\{\eta^T(s)Q(s+1)\eta(s) - \eta^T(s-\tau)Q(s+1-\tau)\eta(s-\tau)\}.
\end{aligned} \tag{20}$$

Substituting (19) and (20) into (18) leads to

$$\begin{aligned}
\mathbb{E}\{\Delta V(\eta(s))\} =& \mathbb{E}\{\Delta V_1(\eta(s)) + \Delta V_2(\eta(s))\} \\
=& \mathbb{E}\{\eta^T(s+1)P(s+1)\eta(s+1) - \eta^T(s)P(s)\eta(s)\} \\
=& \mathbb{E}\{\eta^T(s)[\mathcal{D}^T(s)P(s+1)\mathcal{D}(s) + Q(s+1) - P(s)]\eta(s) + \vec{f}^T(\eta(s))\mathcal{A}^T(s)P(s+1) \\
& \times \mathcal{A}(s)\vec{f}(\eta(s)) + \vec{g}^T(\eta(s-\tau))\mathcal{B}^T(s)P(s+1)\mathcal{B}(s)\vec{g}(\eta(s-\tau)) + v^T(s)\mathcal{L}^T(s)P(s+1) \\
& \times \mathcal{L}(s)v(s) - \eta^T(s-\tau)Q(s+1-\tau)\eta(s-\tau) + 2\eta^T(s)\mathcal{D}^T(s)P(s+1)\mathcal{A}(s)\vec{f}(\eta(s)) \\
& + 2\eta^T(s)\mathcal{D}^T(s)P(s+1)\mathcal{B}(s)\vec{g}(\eta(s-\tau)) + 2\eta^T(s)\mathcal{D}^T(s)P(s+1)\mathcal{L}(s)v(s) \\
& + 2\vec{f}^T(\eta(s))\mathcal{A}^T(s)P(s+1)\mathcal{B}(s)\vec{g}(\eta(s-\tau)) + 2\vec{f}^T(\eta(s))\mathcal{A}^T(s)P(s+1)\mathcal{L}(s)v(s) \\
& + 2\vec{g}^T(\eta(s-\tau))\mathcal{A}^T(s)P(s+1)\mathcal{L}(s)v(s)\}.
\end{aligned} \tag{21}$$

Next, adding the zero term

$$\tilde{\tilde{z}}^T(s)\tilde{\tilde{z}}(s) - \gamma^2 v^T(s)v(s) - \tilde{\tilde{z}}^T(s)\tilde{\tilde{z}}(s) + \gamma^2 v^T(s)v(s)$$

to both sides of (21) results in

$$\mathbb{E}\{\Delta V(\eta(s))\} \leq \mathbb{E}\left\{\zeta^T(s)\bar{\Omega}(s)\zeta(s) - \tilde{\tilde{z}}^T(s)\tilde{\tilde{z}}(s) + \gamma^2 v^T(s)v(s)\right\} \tag{22}$$

where

$$\bar{\Omega}(s) \triangleq \begin{bmatrix} \bar{\Theta}_{11} & 0 & \Theta_{13} & \Theta_{14} & \Theta_{15} \\ * & \bar{\Theta}_{22} & 0 & 0 & 0 \\ * & * & \bar{\Theta}_{33} & \Theta_{34} & \Theta_{35} \\ * & * & * & \bar{\Theta}_{44} & \Theta_{45} \\ * & * & * & * & \Theta_{55} \end{bmatrix},$$

$$\zeta(s) \triangleq \begin{bmatrix} \eta^T(s) & \eta^T(s-\tau) & \vec{f}^T(\eta(s)) & \vec{g}^T(\eta(s-\tau)) & v^T(s) \end{bmatrix}^T,$$

$$\bar{\Theta}_{11} \triangleq \mathcal{D}^T(s)P(s+1)\mathcal{D}(s) + Q(s+1) + \mathcal{N}(s)^T\mathcal{N}(s) - P(s),$$

$$\bar{\Theta}_{22} \triangleq -Q(s+1-\tau),\ \bar{\Theta}_{33} \triangleq \mathcal{A}^T(s)P(s+1)\mathcal{A}(s),\ \bar{\Theta}_{44} \triangleq \mathcal{B}^T(s)P(s+1)\mathcal{B}(s).$$

Furthermore, we obtain from (2) and (3) that

$$\vec{f}^T(\eta(s))\vec{f}(\eta(s)) \leq 3\vartheta_f(s)\eta^T(s)\tilde{\Upsilon}_f^T(s)\tilde{\Upsilon}_f(s)\eta(s) + 4\vartheta_f(s)\eta^T(s)\bar{\Upsilon}_f^T(s)\bar{\Upsilon}_f(s)\eta(s) \tag{23}$$

and

$$\begin{aligned} \vec{g}^T(\eta(s-\tau))\vec{g}(\eta(s-\tau)) \leq{}& 3\vartheta_g(s-\tau)\eta^T(s-\tau)\tilde{\Upsilon}_g^T(s-\tau)\tilde{\Upsilon}_g(s-\tau)\eta(s-\tau) \\ &+ 4\vartheta_g(s-\tau)\eta^T(s-\tau)\bar{\Upsilon}_g^T(s-\tau)\bar{\Upsilon}_g(s-\tau)\eta(s-\tau). \end{aligned} \tag{24}$$

It is inferred from (22)-(24) that

$$\begin{aligned} \mathbb{E}\{\Delta V(\eta(s))\} \leq{}& \mathbb{E}\{\zeta^T(s)\bar{\Omega}(s)\zeta(s) - \lambda_1(s)(\vec{f}^T(\eta(s))\vec{f}(\eta(s)) - 3\vartheta_f(s)\eta^T(s)\tilde{\Upsilon}_f^T(s)\tilde{\Upsilon}_f(s)\eta(s) \\ &- 4\vartheta_f(s)\eta^T(s)\bar{\Upsilon}_f^T(s)\bar{\Upsilon}_f(s)\eta(s)) - \lambda_2(s)(\vec{g}^T(\eta(s-\tau))\vec{g}(\eta(s-\tau)) \\ &- 3\vartheta_g(s-\tau)\eta^T(s-\tau)\tilde{\Upsilon}_g^T(s-\tau)\tilde{\Upsilon}_g(s-\tau)\eta(s-\tau) \\ &- 4\vartheta_g(s-\tau)\eta^T(s-\tau)\bar{\Upsilon}_g^T(s-\tau)\bar{\Upsilon}_g(s-\tau)\eta(s-\tau)) \\ &- \tilde{\bar{z}}^T(s)\tilde{\bar{z}}(s) + \gamma^2 v^T(s)v(s)\} \\ \leq{}& \mathbb{E}\{\zeta^T(s)\Omega(s)\zeta(s) - \tilde{\bar{z}}^T(s)\tilde{\bar{z}}(s) + \gamma^2 v^T(s)v(s)\}. \end{aligned} \tag{25}$$

Taking the sum on both sides of (25) with respect to s from 0 to $\mathrm{N}-1$ yields

$$\begin{aligned} \sum_{s=0}^{\mathrm{N}-1} \mathbb{E}\left\{\Delta V(\eta(s))\right\} ={}& \mathbb{E}\{\eta^T(\mathrm{N})P(\mathrm{N})\eta(\mathrm{N}) - \eta^T(0)P(0)\eta(0) \\ &+ \sum_{s=\mathrm{N}-\tau}^{\mathrm{N}-1} \eta^T(s)Q(s+1)\eta(s) - \sum_{s=-\tau}^{0} \eta^T(s)Q(s+1)\eta(s)\} \\ \leq{}& \mathbb{E}\left\{\sum_{s=0}^{\mathrm{N}-1} \zeta^T(s)\Omega(s)\zeta(s)\right\} - \mathbb{E}\left\{\sum_{s=0}^{\mathrm{N}-1} \left(||\tilde{\bar{z}}(s)||^2 - \gamma^2||v(s)||^2\right)\right\}. \end{aligned}$$

Consequently, we arrive at

$$\mathbb{E}\left\{\sum_{s=0}^{\mathrm{N}-1} ||\tilde{\bar{z}}(s)||^2\right\} - \gamma^2\mathbb{E}\left\{\sum_{s=0}^{\mathrm{N}-1} ||v(s)||^2 + \sum_{s=-\tau}^{0} \eta^T(s)S(s)\eta(s)\right\}$$

$$\leq \mathbb{E}\left\{\sum_{s=0}^{\mathrm{N}-1}\zeta^T(s)\Omega(s)\zeta(s)\right\} - \eta^T(\mathrm{N})P(\mathrm{N})\eta(\mathrm{N}) - \sum_{s=\mathrm{N}-\tau}^{\mathrm{N}-1}\eta^T(s)Q(s+1)\eta(s)$$

$$+ \eta^T(0)P(0)\eta(0) + \sum_{s=-\tau}^{0}\eta^T(s)Q(s+1)\eta(s) - \gamma^2\sum_{s=-\tau}^{0}\eta^T(s)S(s)\eta(s).$$

Noting that $\Omega(s) < 0$ and inequality (15), the H_∞ performance (11) is ensured, which completes the proof.

Theorem 2. *Let $\gamma > 0$ and $\{S(s)\}_{-\tau\leq s\leq 0}$ be given. The concerned H_∞ SE issue of MNN (1) is solved if there exist matrix sequences $\{P(s) > 0\}_{1\leq s\leq \mathrm{N}}$ ($P(s) = \mathrm{diag}\{P_1(s), P_2(s), P_3(s)\}$), $\{Q(s) > 0\}_{-\tau+1\leq s\leq \mathrm{N}+1}$ and $\{Y(s)\}_{0\leq s\leq \mathrm{N}-1}$, scalar sequences $\{\lambda_1(s)\}_{0\leq s\leq \mathrm{N}-1}$, $\{\lambda_2(s)\}_{0\leq s\leq \mathrm{N}-1}$ and $\{\kappa(s)\}_{0\leq s\leq \mathrm{N}-1}$ satisfying (15) and the following RLMIs*

$$\breve{\Omega}(s) = \begin{bmatrix} \tilde{\Omega}(s) & \tilde{\mathcal{H}} & \kappa(s)\tilde{\mathcal{E}}^T \\ * & -\kappa(s)I & 0 \\ * & * & -\kappa(s)I \end{bmatrix} < 0 \tag{26}$$

for $0 \leq s \leq \mathrm{N}$, where

$$\tilde{\Omega}(s) \triangleq \begin{bmatrix} -P(s+1) & 0 & 0 & 0 & 0 & 0 & \bar{\Pi}_1 \\ 0 & -I & 0 & 0 & 0 & 0 & \Pi_2 \\ * & * & -\frac{1}{\mu_1(s)}I & 0 & 0 & 0 & \Pi_3 \\ * & * & * & -\frac{1}{\mu_2(s)}I & 0 & 0 & \Pi_4 \\ * & * & * & * & -\frac{1}{\mu_3(s)}I & 0 & \Pi_5 \\ * & * & * & * & * & -\frac{1}{\mu_4(s)}I & \Pi_6 \\ * & * & * & * & * & * & \Pi_7 \end{bmatrix},\ \check{\mathcal{E}} \triangleq \begin{bmatrix} \mathcal{E}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{E}_2 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{E}_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{E}_1 \triangleq \begin{bmatrix} E_1 & 0 & 0 \\ E_1 & 0 & 0 \end{bmatrix},\ \mathcal{E}_2 \triangleq \begin{bmatrix} E_2 & 0 \\ E_2 & 0 \end{bmatrix},\ \mathcal{E}_3 \triangleq \begin{bmatrix} E_3 & 0 \\ E_3 & 0 \end{bmatrix},\ \tilde{\mathcal{E}} \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \check{\mathcal{E}} \end{bmatrix},$$

$$\tilde{\mathcal{H}} \triangleq \begin{bmatrix} \check{\mathcal{H}}^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,\ \check{\mathcal{H}} \triangleq \begin{bmatrix} P(s+1)\mathcal{H} & 0 & P(s+1)\mathcal{H} & P(s+1)\mathcal{H} & 0 \end{bmatrix},\ \mathcal{H} \triangleq \begin{bmatrix} H & 0 \\ H & 0 \\ 0 & 0 \end{bmatrix},$$

$$\bar{\Pi}_1 \triangleq \begin{bmatrix} \Theta_{11} & 0 & P(s+1)\bar{\mathcal{A}} & P(s+1)\bar{\mathcal{B}} & P(s+1)\mathcal{L}(s) \end{bmatrix},\ \Pi_2 \triangleq \begin{bmatrix} \mathcal{N} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Pi_3 \triangleq \begin{bmatrix} \tilde{\Upsilon}_f(s) & 0 & 0 & 0 & 0 \end{bmatrix},\ \Pi_4 \triangleq \begin{bmatrix} \bar{\Upsilon}_f(s) & 0 & 0 & 0 & 0 \end{bmatrix},\ \Pi_5 \triangleq \begin{bmatrix} \tilde{\Upsilon}_g(s-\tau) & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Pi_6 \triangleq \begin{bmatrix} \bar{\Upsilon}_g(s-\tau) & 0 & 0 & 0 & 0 \end{bmatrix},\ \Pi_7 \triangleq \mathrm{diag}\left\{Q(s+1)-P(s), -Q(s+1-\tau), -\lambda_1(s)I, -\lambda_2(s)I, -\gamma^2 I\right\},$$

$$\Theta_{11} \triangleq \begin{bmatrix} P_1(s+1)\bar{D} & 0 & 0 \\ \tilde{\mathcal{D}}_{21} & \tilde{\mathcal{D}}_{22} & \tilde{\mathcal{D}}_{22} \\ P_3(s+1)\Psi_{\xi(s)} & 0 & P_3(s+1)(I-\Psi_{\xi(s)}) \end{bmatrix}$$

with

$$\tilde{\mathcal{D}}_{21} \triangleq -Y(s)(\Psi_{\xi(s)}-I)C(s),\ \tilde{\mathcal{D}}_{22} \triangleq P_2(s+1)\bar{D} - Y(s)C(s),\ \tilde{\mathcal{D}}_{23} \triangleq -Y(s)(I-\Psi_{\xi(s)}),$$

$$\mu_1(s) \triangleq 3\lambda_1(s)\vartheta_f(s),\quad \mu_2(s) \triangleq 4\lambda_1(s)\vartheta_f(s),\quad \mu_3(s) \triangleq 3\lambda_2(s)\vartheta_g(s),\quad \mu_4(s) \triangleq 4\lambda_2(s)\vartheta_g(s).$$

Furthermore, if inequalities (16) and (26) hold, then $K(s) = P_2^{-1}(s+1)Y(s)$.

Proof. To remove uncertainty effects in (16), we denote

$$\Pi_1 \triangleq \begin{bmatrix} P(s+1)\mathcal{D}(s) & 0 & P(s+1)\mathcal{A}(s) & P(s+1)\mathcal{B}(s) & P(s+1)\mathcal{L}(s) \end{bmatrix}. \tag{27}$$

By using Schur Complement Equivalence Lemma (Lemma 1), it can be deduced from (16) that

$$\tilde{\Omega}(s) + (\tilde{\mathcal{H}}\mathcal{F}(s)\tilde{\mathcal{E}}) + (\tilde{\mathcal{H}}\mathcal{F}(s)\tilde{\mathcal{E}})^T < 0, \tag{28}$$

where $\tilde{\Omega}(s)$ is defined in (26), $\mathcal{F}(s) \triangleq \mathrm{diag}\{\mathcal{F}_1(s), 0, \mathcal{F}_2(s), \mathcal{F}_3(s), 0\}$ with $\mathcal{F}_1(s) \triangleq \mathrm{diag}\{F_1(s), F_1(s)\}$, $\mathcal{F}_2(s) \triangleq \mathrm{diag}\{F_2(s), F_2(s)\}$ and $\mathcal{F}_3(s) \triangleq \mathrm{diag}\{F_3(s), F_3(s)\}$. Based on this fact, it follows from $K(s) = P_2^{-1}(s+1)Y(s)$ and Lemma 2 that (28) holds if (26) is true. The proof is now complete.

According to Theorem 2, the H_∞ SE approach is summarized in Algorithm 1 as follows.

Algorithm 1 : The H_∞ SE Algorithm.

Step 1. Let $s = 0$, γ, N, $\{S(s)\}_{-\tau \le s \le 0}$ and $\{Q(s)\}_{-\tau+1 \le s \le 0}$, $\{\phi(s)\}_{-\tau \le s \le 0}$ and $\{P_1(0), P_2(0), Q(0)\}$ be given.

Step 2. Solve the RLMI (26) for $\{P_1(s+1), P_2(s+1), P_3(s+1), Q(s+1), Y(s)\}$. Then, the desired estimator gain $K(s)$ can be obtained by $K(s) = P_2^{-1}(s+1)Y(s)$.

Step 3. Set $s = s+1$. If $s <$ N, go to Step 2. Otherwise, stop.

Remark 3. *Up to now, the protocol-based finite-time SE problem has been well settled for the concerned delayed MNNs. In Theorem 2, the sufficient condition that ensures the existence of the required estimator* (8) *has been established in terms of the solvability of the feasibility of a bank of RLMIs. Noticing that the system complexity is caused jointly by the SCP protocol, state-dependent parameters and external disturbances, the resulting impacts are obviously reflected in the preceding analysis, and the developed estimation scheme is able to achieve a good compromise between the system performance and resource consumption. Our results could stand out from the rich body of literature for mainly two reasons: 1) the SCP is, for the first time, introduced to schedule the data traffic between the MNNs and the remote estimator for the sake of alleviating data collisions; and 2) a new unified H_∞ estimation framework is established to account for time-delays, energy-bounded disturbances and SCP.*

Remark 4. *In Theorem 2, the design of the discussed protocol-based finite-horizon H_∞ state estimator is successfully converted into the solution to a convex optimization problem via the semi-definite programme method. The computational complexity of the proposed LMI-based algorithm depends polynomially on the dimensions of the system state and the measured output vector, and the number of neurons in the MNN.*

4. An Illustrative Example

Consider (1) with parameters:

$$d_1(z_1(\cdot)) = \begin{cases} 0.48, & |z_1(\cdot)| > 0.01, \\ 0.48, & |z_1(\cdot)| \le 0.01, \end{cases} \quad d_2(z_2(\cdot)) = \begin{cases} 0.64, & |z_2(\cdot)| > 0.01, \\ 0.64, & |z_2(\cdot)| \le 0.01, \end{cases}$$

$$d_3(z_3(\cdot)) = \begin{cases} 0.56, & |z_3(\cdot)| > 0.01, \\ 0.56, & |z_3(\cdot)| \le 0.01, \end{cases} \quad a_{11}(z_1(\cdot)) = \begin{cases} 0.12, & |z_1(\cdot)| > 0.01, \\ 0.10, & |z_1(\cdot)| \le 0.01, \end{cases}$$

$$a_{12}(z_1(\cdot)) = \begin{cases} -0.66, & |z_1(\cdot)| > 0.01, \\ -0.70, & |z_1(\cdot)| \le 0.01, \end{cases} \quad a_{13}(z_1(\cdot)) = \begin{cases} 0.10, & |z_1(\cdot)| > 0.01, \\ 0.07, & |z_1(\cdot)| \le 0.01, \end{cases}$$

$$a_{21}(z_2(\cdot)) = \begin{cases} -0.40, & |z_2(\cdot)| > 0.01, \\ -0.46, & |z_2(\cdot)| \le 0.01, \end{cases} \quad a_{22}(z_2(\cdot)) = \begin{cases} 0.12, & |z_2(\cdot)| > 0.01, \\ 0.08, & |z_2(\cdot)| \le 0.01, \end{cases}$$

$$a_{23}(z_2(\cdot)) = \begin{cases} 0.11, & |z_2(\cdot)| > 0.01, \\ 0.09, & |z_2(\cdot)| \le 0.01, \end{cases} \quad a_{31}(z_3(\cdot)) = \begin{cases} 0.10, & |z_3(\cdot)| > 0.01, \\ 0.08, & |z_3(\cdot)| \le 0.01, \end{cases}$$

$$a_{32}(z_3(\cdot)) = \begin{cases} -0.36, & |z_3(\cdot)| > 0.01, \\ -0.16, & |z_3(\cdot)| \le 0.01, \end{cases} \quad a_{33}(z_3(\cdot)) = \begin{cases} 0.12, & |z_3(\cdot)| > 0.01, \\ 0.10, & |z_3(\cdot)| \le 0.01, \end{cases}$$

$$b_{11}(z_1(\cdot)) = \begin{cases} -0.15, & |z_1(\cdot)| > 0.01, \\ -0.15, & |z_1(\cdot)| \le 0.01, \end{cases} \quad b_{12}(z_1(\cdot)) = \begin{cases} 0.05, & |z_1(\cdot)| > 0.01, \\ 0.05, & |z_1(\cdot)| \le 0.01, \end{cases}$$

$$b_{13}(z_1(\cdot)) = \begin{cases} 0.10, & |z_1(\cdot)| > 0.01, \\ 0.10, & |z_1(\cdot)| \le 0.01, \end{cases} \quad b_{21}(z_2(\cdot)) = \begin{cases} -0.01, & |z_2(\cdot)| > 0.01, \\ -0.01, & |z_2(\cdot)| \le 0.01, \end{cases}$$

$$b_{22}(z_2(\cdot)) = \begin{cases} 0.20, & |z_2(\cdot)| > 0.01, \\ 0.20, & |z_2(\cdot)| \le 0.01, \end{cases} \quad b_{23}(z_2(\cdot)) = \begin{cases} 0.15, & |z_2(\cdot)| > 0.01, \\ 0.15, & |z_2(\cdot)| \le 0.01, \end{cases}$$

$$b_{31}(z_3(\cdot)) = \begin{cases} -0.01, & |z_3(\cdot)| > 0.01, \\ -0.01, & |z_3(\cdot)| \le 0.01, \end{cases} \quad b_{32}(z_3(\cdot)) = \begin{cases} 0.065, & |z_3(\cdot)| > 0.01, \\ 0.065, & |z_3(\cdot)| \le 0.01, \end{cases}$$

$$b_{33}(z_3(\cdot)) = \begin{cases} 0.075, & |z_3(\cdot)| > 0.01, \\ 0.075, & |z_3(\cdot)| \le 0.01. \end{cases}$$

Other parameters of system (1) are given by

$$C(s) = \begin{bmatrix} 0.30\sin(2s) & 0.30 & 0.25 \\ -0.20 & 0.25 & 0.10 \end{bmatrix}, \ L(s) = \begin{bmatrix} 0.01 & -0.01\sin(s) & 0 \end{bmatrix}^T,$$
$$M(s) = \begin{bmatrix} -0.01 & 0.02\sin(3s) \end{bmatrix}^T, \ N(s) = \begin{bmatrix} 0.10\sin(s) & -0.10 & 0.10 \end{bmatrix}.$$

The NNAFs are chosen as

$$f(z(s)) = (1.2 + 0.12\sin(s))^{\frac{1}{2}} \begin{bmatrix} \tanh(0.5z_1(s)) \\ \tanh(0.7z_2(s)) \\ \tanh(0.6z_3(s)) \end{bmatrix}, \ g(z(s)) = (1.2 - 0.10\sin(s))^{\frac{1}{2}} \begin{bmatrix} \tanh(0.5z_1(s)) \\ \tanh(0.7z_2(s)) \\ \tanh(0.6z_3(s)) \end{bmatrix},$$

which satisfy the constraints (2) and (3) with $\vartheta_f(s) = (1.2+0.12\sin(s))^{\frac{1}{2}}$, $\vartheta_g(s) = (1.2-0.10\sin(s))^{\frac{1}{2}}$, and

$$\Upsilon_f(s) = \Upsilon_g(s) = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.49 & 0 \\ 0 & 0 & 0.36 \end{bmatrix}.$$

In the example, the time-delay is chosen as $\tau = 3$, the attenuation value $\gamma = 1.05$, $\bar{z}(0) = \begin{bmatrix} -0.7 & 0.5 & 0.4 \end{bmatrix}^T$, $\hat{\bar{z}}(0) = \begin{bmatrix} -0.5 & 0.3 & 0 \end{bmatrix}^T$, and the simulation run length is set as N = 21.

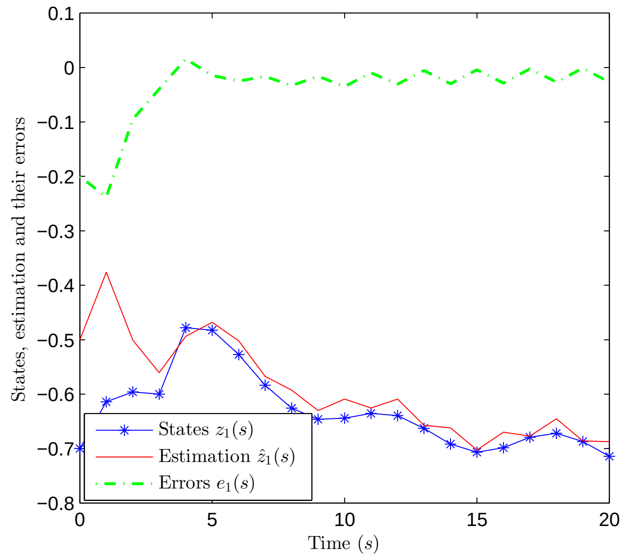

Figure 2: $z_1(s)$ and $\hat{z}_1(s)$.

Based on the proposed protocol-based H_∞ algorithm, the RLMIs in Theorem 2 are solved recursively under prescribed initial conditions, and the corresponding demonstration results are given in Figs. 2–5. Specifically, Figs. 2–4 show the neuron states, estimates and estimation errors $e_i(s)(i = 1, 2, 3)$, respectively, while Fig. 5 plots the output estimation errors $\tilde{\bar{z}}(s)$. From the simulation results, we see clearly that our proposed estimation scheme is indeed effective.

5. Conclusions

The finite-time H_∞ SE problem has been investigated in this paper for delayed MNNs under the SCP. To reduce communication burdens in case of large-scale data exchange in the sensor-estimator channel, the SCP (modeled as a Markov chain) has been used to regulate the data transmission process. First, a theoretical framework has been established for the addressed MNNs to analyze the finite-time H_∞ performance. Within such a framework, sufficient conditions have been obtained for the existence of the desired remote estimator. Subsequently, the required estimator gains have been obtained by resorting to solve certain RLMIs. Finally, an illustrative example has been provided to verify the validity of our estimation scheme. Moreover, it is also our interest to apply the developed algorithm to MNNs under other protocols, for instance, the RRP [32] and TODP [24] in the near future. Also, the methodology proposed in this paper can be utilized to deal with more complicated systems with more comprehensive network-induced phenomena [21, 26, 27, 29, 30, 31, 22, 23].

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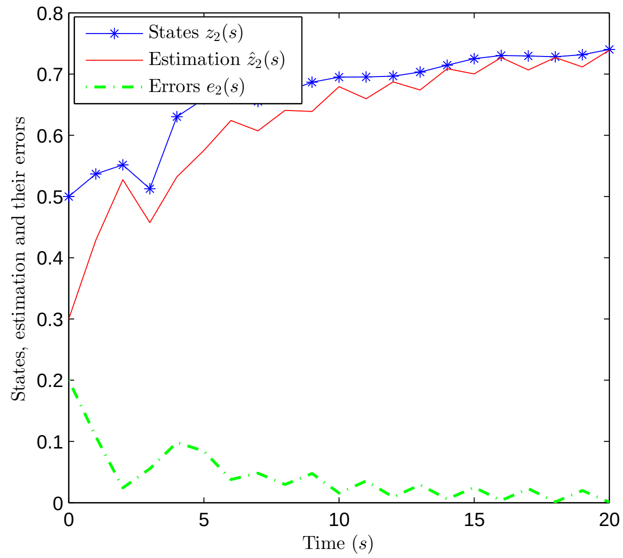

Figure 3: $z_2(s)$ and $\hat{z}_2(s)$.

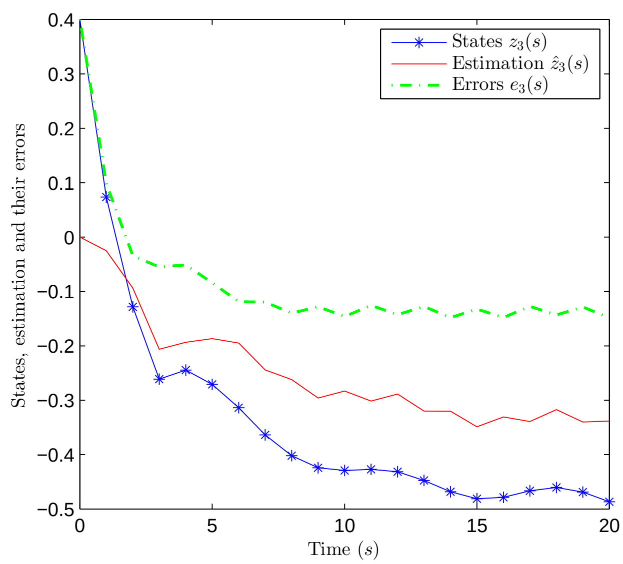

Figure 4: $z_3(s)$ and $\hat{z}_3(s)$.

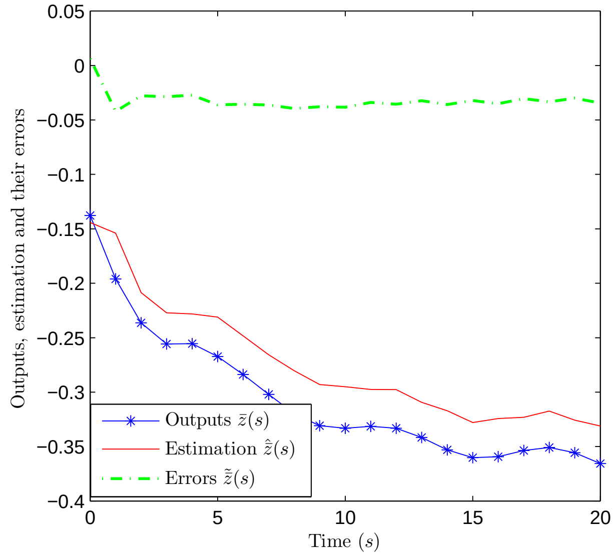

Figure 5: $\bar{z}(s)$ and $\hat{\bar{z}}(s)$.

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