TR/02/84 March 1984

"The best possible estimates for polynomial norms on certain L^P—spaces."

I. Sarantopoulos

ABSTRACT

We disprove a conjecture of Harris [5] by showing that if $\wedge$ is a symmetric m-linecr form on an L_μ^p space and $\hat{\wedge}$ is the associated polynomial then

$$\| \wedge \| \leq \frac{m^{m/p}}{m!} \| \hat{\wedge} \|$$

for $1 \leq p \leq m'$. In general this inequality cannot be improved.

Notation

Throughout this paper K denotes either the field of complex numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$. If the field is not specified the results are valid in both cases, $K = \mathbb{R}$ and $K = \mathbb{C}$.

If $1 \leq p \leq \infty$, we denote the conjugate exponent by p'. Thus

$$\frac{1}{P} + \frac{1}{P'} = 1 .$$

If (X, A, m) is a measure space we shall write L^p_μ for the Banach space of all A-measurable functions $f: X \to K$ for which $\|f\|_p < \infty$ where

$$\| f \|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p} \quad (1 \leq p < \infty)$$

and $\|f\|_\infty$ is the infimum of those non-negative numbers M such that

$$\{x \in X : |f(x)| > M\}$$

is μ-null set.

If μ is the counting measure on a set X, we denote the corresponding L^p_μ -space by ℓ^p if X is countable. An element of ℓ^p may be regarded as a complex sequence $x = (\xi_n)$, and

$$\| x \|_p = \left\{ \sum_{n=1}^{\infty} |\xi_n|^p \right\}^{1/p} .$$

If A is a Banach space a function $f: X \to A$ is strongly measurable if it is Borel measurable and has a separable range. (The range of f is the subset $f(X)$ of A). Of course, a simple function is strongly measurable if and only if it is Borel measurable.

A function $f: X \to A$ is integrable (or Bochner integrable) if it is strongly measurable and the function $x \to \| f(x) \|_A$ is integrable.
By $L^p_\mu(A) = L^P(X, d\mu; A)$ we denote the space of all strongly measurable functions f such that

$$\int_X \| f(x) \|_A \, d\mu(x) < \infty$$

where $1 \leq p < \infty$, We denote by $L^{\infty}_{\mu}(A) - L^{\infty}(X, d\mu; A)$ the completion in the sup-norm of all simple functions

$$s(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x) , \; a_i \in A$$

where X_{E_i} is the characteristic function of the set E_i .

The completion in $L^{\infty}_{\mu}(A)$ of the functions of the above norm with $m(E_i) < \infty$ for every $i = 1, \ldots, n$ is denoted by $L^{\infty}_{\mu,0}(A)$.

1. INTRODUCTION

Let E and F be vector spaces over K. We write E^m for the product $E \times E \times \ldots \times E$ with m factors. An m-linear mapping $\Lambda : E^m \to F$ is said to be symmetric if

$$\Lambda(x_1, \ldots, x_m) = \Lambda(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$$

for any $x_1, \ldots, x_m \in E$ and any permutation σ of the first m natural numbers.

Let $\mathcal{L}_m(E,F)$ ($\mathcal{L}_m^s(E,F)$) denote the space of all (symmetric) m-linear mappings $\Lambda : E^m \to F$ and define

$$\hat{\Lambda}(x) = \Lambda(x, \ldots, x) .$$

A mapping $P : E \to F$ is said to be a homogeneous polynomial of degree m if $P = \hat{\Lambda}$ for some $\Lambda \in \mathcal{L}_m(E,F)$, and it is said to be a polynomial of degree m if

$$P = \sum_{i=0}^{m} P_i , \quad P_m \neq 0$$

where $P_i : E \to F$ is a homogeneous polynomial of degree i for $i = 1, \ldots, m$ and $P_0 : E \to F$ is a constant mapping.

If Λ is a 2-linear $\mathbb{C}$-valued mapping on $\mathbb{C}^m$, $m \in \mathbb{N}$, then there exists an $m \times m$ matrix A such that $\Lambda(x,y = xAy^t$ for all $x = (x_1 \ldots, x_m) \in \mathbb{C}^m$ and all $y = (y_1 \ldots, y_m) \in \mathbb{C}^m$.

If $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le m}}$ then $\Lambda(x, y) = \sum_{i,j=1}^{m} a_{ij} x_i y_j$.

Hence any $\mathbb{C}$—valued homogeneous polynomial P of degree 2, $P : \mathbb{C}^m \to \mathbb{C}$ has the well known form

$$P(x) = \Lambda(x,x) = \sum_{i,j=1}^{m} a_{ij} x_i x_j .$$

This explains the terminology .

If X,Y are normed linear spaces over K we define

$$\|\hat{\Lambda}\| = \sup\{\|\hat{\Lambda}(x)\| : \|x\| \leq 1\}$$

$$\|\Lambda\| - \sup\{\|\Lambda(x_1,\ldots x_m)\| : \|x_j\| \leq 1 \quad (j = 1,\ldots,m)\}$$

for $\Lambda \in \mathcal{L}^s_m(X,Y)$.

Martin [8] proved that

$$\|\hat{\Lambda}\| \leq \|\Lambda\| \leq \frac{m^m}{m!}\|\hat{\Lambda}\| \qquad (1)$$

thus answering a question of Mazur and Orlicz in the Scottish Book [9].

Harris [5] has proved that if X is an L^p_μ space with $1 \leq p \leq \infty$ and m is a power of 2, then

$$\|\Lambda\| \leq \left(\frac{m^m}{m!}\right)^{\frac{|p-2|}{p}} \|\hat{\Lambda}\|$$

(2)

for all $\Lambda \in \mathcal{L}^s_m(X, ₵)$. Harris also conjectured that (2) holds for all positive integers m and that the constant given is best possible when $1 \leq p \leq 2$.

If p=1 then the constant $\frac{m^m}{m!}$ is the best possible [4] . In fact there exists $\Lambda \in \mathcal{L}^s_m(\ell^1, ₵)$ such that

$$\|\Lambda\| = \frac{m^m}{m!}\|\hat{\Lambda}\| \ .$$

If p = 2 inequality (2) gives $\|\Lambda\| = \|\hat{\Lambda}\|$ for every $\Lambda \in \mathcal{L}^s_m(L^2_\mu, ₵)$. This is in fact a result of S. Banach, Banach [1] showed in 1938 that if H is a real Hilbert space and F a real Banach space then $\|\Lambda\| = \|\hat{\Lambda}\|$ for every $\Lambda \in \mathcal{L}^s_m(H,F)$, For expositions see [3] and [5] or [4], Banach's result also holds if H is a complex Hilbert space and F a complex Banach space. Dineen [4] states incorrectly that the problem for complex Hilbert spaces is open. In fact the proof he gives for real Hilbert spaces works just as well for complex Hilbert spaces. Harris [5] proved that if $p = \infty$ then

(3)

for every $\Lambda \in \mathcal{L}^s_m(L^\infty_\mu, ₵)$.

A. Tonge [10] has given another proof of this result and in the case

$m = 2$ he has examples which show that the result cannot be much improved. In this paper we prove that the constant given in (2) is not the best possible when $1 \le p \le 2$ and we give the best possible constant when $1 \le p \le m'$. Our first result is an inequality due to L. Williams [11]. We shall give a simpler proof using an extension of the Riesz-Thorin interpolation theorem.

The n-th Rademacher function $r_n(t)$ is defined on [0,1] by $r_n(t) = \text{sign} \sin 2^n \pi t$. The Rademacher functions $\{r_n\}$ form an orthonormal set in $L^2([0,1], dt)$ where dt denotes Lebesgue measure on [0,1]. The classical Clarkson inequality, which is a generalization of the Hilbert space parallelogram law, asserts that if $f_1, f_2 \in L^p_\mu$ for $1 < p \le 2$ then

$$\| f_1 + f_2 \|_p^{p'} + \| f_1 - f_2 \|_p^{p'} \le 2 \left[\| f_1 \|_p^p + \| f_2 \|_p^p \right]^{p'/p}$$

<u>Theorem 1</u>. (A generalized Clarkson inequality for $1 < p \le 2$).

Let $f_1, \dots, f_m \in L^p_\mu$ for $1 < p \le 2$. Then

$$\left(\int_0^1 \| r_1(t)\, f_1 + \dots + r_m(t)\, f_m \|_p^{p'} \, dt \right)^{1/p} \le \left(\sum_{i=1}^{m} \| f_i \|_p^p \right)^{1/p} \tag{4}$$

where $r_i(t)$, $i = 1, \dots, m$ is the i-th Rademacher function.

Observe that when $m = 2$ we recover Clarkson's original inequality in a slightly disguised form. The second topic of this paper involves a polarization formula.

<u>Theorem 2</u>. (Polarization formula)

If X and Y are vector spaces over K, $\Lambda \in \mathcal{L}^s_m(X,Y)$ and $x_1, \dots, x_m \in X$ then

$$\Lambda(x, \dots, x_m) = \frac{1}{m!} \int_0^1 r_1(t) \dots t_m(t)\, \hat{\Lambda}\,(r_1(t)\, x_1 + \dots + r_m(t)\, x_m)\, dt$$

(5)

where $r_i(t)$, $i = 1, \dots, m$ is the i-th Rademacher function.

The main result of this paper is the following theorem.

Theorem 3

Let X be an L^p_μ space with $1 \le p \le m'$. Then

$$\| \Lambda \| \le \frac{m^{m/p}}{m!} \| \hat{\Lambda} \| \qquad (6)$$

for all $\Lambda \in \mathcal{L}^s_m(X,K)$.

The following example shows that the constant given in (6) cannot be improved. It is based on an argument in Dineen's book [4].

Example

Consider the real or complex sequence space ℓ^p where the norm of $x = (x_i)$ is given by

$$\| x \|_p = \left\{ \sum_{i=1}^{\infty} | x_i |^p \right\}^{1/p} < \infty .$$

Let $\Lambda \in \mathcal{L}^s_m(\ell^p, K)$ be defined by

$$\Lambda(x^1, \dots, x^m) = \frac{1}{m!} \sum_{\sigma \in S_m} x_1^{\sigma(1)} \dots x_m^{\sigma(m)}$$

(7)

where $x_i = \left(x^i_n\right)_{n=1}^{\infty}$ for $i = 1, \dots, m$ and S_m is the set of permulations of the first m natural number , If e^i $\left(\delta^i_k\right)_{k=1}^{\infty}$, $i = 1, \dots, m$ where

$$\delta^i_k = \begin{cases} 1 , & i = k \\ 0 , & \text{otherwise} \end{cases}$$

then $e^i \in \ell^p$ and

$$\Lambda(e^1, \dots, e^m) = \frac{1}{m!}$$

and so $\| \Lambda \| \ge \dfrac{1}{m!}$.

on the other hand $|\hat{\Lambda}(x)| = |\Lambda(x, \ldots, x)| = |x_1 \ldots x_m|$

$$= \left\{ (|x_1|^p \ldots |x_m|^p > ^{1/m} \right\}^{m/p} \leq \left(\frac{|x_1|^p \ldots + |x_m|^p}{m} \right)^{m/p}$$ by the familiar

inequality between the arithmetic and geometric means of m positive numbers.

so $$\|\hat{\Lambda}\| = \sup_{\|x\|_p \leq 1} |\Lambda(x)| \leq \frac{1}{m^{m/p}} .$$

Thus for the symmetric m-linear form A defined by (7) we have

$$\|\Lambda\| \leq \frac{m^{m/p}}{m!} \|\hat{\Lambda}\| .$$

2. THE PROOFS

Proof of theorem 1

We shall write ℓ_m^p for the vector space of all m-tuples $x = (x_1 \ldots, x_m)$ equipped with the norm

$$\|x\|_p = (|x_1|^p + \ldots + |x_m|^p)^{1/p} \qquad (1 \leq p < \infty) .$$

consider the linear operator $T : \ell_m^2 (L_\mu^2) \to L_{dt}^2 (L_\mu^2)$ defined by

$$T : f = (f_1, \ldots f_m) \longrightarrow r_1(t) f_1 + \ldots + r_m(t) f_m \qquad (*)$$

where $f_i \in L_\mu^2$, $i = 1, \ldots, m$ and $r_i(t)$, $i = 1, \ldots, m$ is the i-th Rademacher function.

Then $$\|Tf\| = \left(\int_0^1 \|r_1(t) f_1 + \ldots + r_m(t) f_m\|_2^2 \, dt \right)^{\frac{1}{2}}$$

$$= \left\{ \int_0^1 \left(\int_X |r_1(t) f_1(x) + \ldots + r_m(t) f_m(x)|^2 du(x) \right) dt \right\}^{\frac{1}{2}}$$

$$= \left\{ \int_X \left(\int_0^1 |r_1(t) f_1(x) + \ldots + r_m(t) f_m(x)|^2 dt \right) du(x) \right\}^{\frac{1}{2}}$$

by Fubini 's therem

$$= \left\{ \int_X \left(|f_1(x)|^2 + \ldots + |f_m(x)|^2 \right) du(x) \right)^{\frac{1}{2}}$$

by the orthonormality of the Rademacher functions

$$= \left(\sum_{i=1}^{m} \| f_i \|_2^2 \right)^{\frac{1}{2}} .$$

so $\quad \left(\int_0^1 \| r_1(t)\, f_1 + \dots + r_m(t)\, f_m \|_2^2 dt \right)^{\frac{1}{2}} = M_0 \left(\sum_{i=1}^{m} \| f_i \|_2^2 \right)^{\frac{1}{2}}$

where $M_0 = 1$. Now we consider the linear operator

$$T : \ell_m^1 (L_\mu^1) \to L_{dt,0}^\infty (L_\mu^1)$$

defined as in (*) where $f_i \in L_\mu^1$.

Then $\quad \| Tf \| = \sup_t \| r_1(t)\, f_1 + \dots + r_m(t)\, f_m \|_1$

$$\leq \sup_t \left\{ | r_1(t) | \, \| f_1 \|_1 + \dots + | r_m(t) | \, \| f_m \|_1 \right\} = \sum_{i=1}^{m} \| f_i \|_1 .$$

so $\quad \sup_t \| r_1(t)\, f_1 + \dots + r_m(t) f_m \|_1 \leq M_1 \sum_{i=1}^{m} \| f_i \|_1$

where $M_1 = 1$.

Thus from theorems 4.1.2, 5.1.1, and 5.1.2 of [2] we conclude that

$$T : \ell_m^p (L_\mu^q) \to L_{dt}^r (L_\mu^s)$$

where $\quad \frac{1}{p} = \frac{1-t}{2} + \frac{t}{1}, \ \frac{1}{q} = \frac{1-t}{2} + \frac{t}{1},$

$\frac{1}{r} = \frac{1-t}{2}, \ \frac{1}{s} = \frac{1-t}{2} + \frac{t}{1}$ for $0 < t < 1$.

Hence if $\quad \frac{1}{p} = \frac{1-t}{2} + t = \frac{1+t}{2}, \ 0 < t < 1$ then $q = s = p$, $r = p'$.

consequent ly the linear operator $T : \ell_m^p (L_\mu^p) \to L_{dt}^{p'} (L_\mu^p)$ has norm $M \leq M_0^{1-t} M_1^t = 1$ i.e. $\| Tf \| \leq \| f \|$ which implies (4) .

<u>Proof of theorem 2</u>

we have $\int_0^1 r_1(t) \dots r_m(t)\ \hat{\Lambda}(r_1(t)\, x_1 + \dots + r_m(t)\, x_m)\, dt$

$$\int_0^1 r_1(t) \dots r_m(t)\ \Lambda\, (r_1(t)\, x_1 + \dots + \underbrace{r_m(t)\, x_m, \dots, r_1(t)\, x_1}_{m} + \dots + r_m(t)\, x_m)\, dt$$

since $\int_0^1 r_1^{k_1}(t) \cdot r_2^{k_2}(t) \ldots r_m^{k_m}(t)\,dt$ is zero unless all the $\{k_i\}_{i=1}^{m}$ are even, in which case the integral is 1 , we have

$$\int_0^1 r_1(t)\ldots r_m(t)\,\hat{\Lambda}(r_1(t)x_1 + \ldots + r_m(t)x_m)\,dt$$

$$= \int_0^1 r_1(t)\ldots r_m(t)\,\Lambda(r_1(t)x_1, r_2(t)x_2, \ldots, r_m(t)x_m)\,dt + \ldots$$

$$+ \int_0^1 r_1(t)\ldots r_m(t)\,\Lambda(r_m(t)x_m, r_{m-1}(t)x_{m-1}, \ldots, r_1(t)x_1)\,dt$$

$$= m! \int_0^1 r_1^2(t)\ldots r_m^2(t)\,\Lambda(x_1, \ldots, x_m)\,dt$$

$$= m!\,\Lambda(x_1, \ldots, x_m) .$$

We have used the fact that the m-linear mapping Λ is symmetric.

<u>Proof of theorem 3</u>

For $\Lambda \in \mathcal{L}_m^s(X,K)$ theorem 2 gives

$$\Lambda(x_1, \ldots, x_m) = \frac{1}{m!}\int_0^1 r_1(t)\ldots r_m(t)\,\hat{\Lambda}\,(r_1(t)\,x_1 + \ldots + r_m(t)\,x_m)\,dt .$$

Since X is an L_μ^p space we have

$$|\,\Lambda(x_1, \ldots, x_m)\,|\ \frac{1}{m!}\int_0^1 |\hat{\Lambda}\,(r_1(t)x_1 + \ldots + r_m(t)\,x_m|\,dt$$

$$\leq \frac{1}{m!}\|\hat{\Lambda}\|\int_0^1 \|r_1(t)x_1 + \ldots + r_m(t)\,x_m\|_p^m\,dt \tag{8}$$

for $x_1, \ldots, x_m \in L_\mu^p$. But $m' \leq 2$ since $m \geq 2$ and so fort $1 < p \leq m'$ (4) holds .

Now $1 < p \leq m'$ implies $p' \geq m$ and thus Holder's inequality gives

$$\int_0^1 \| r_1(t)\, x_1 + ... + r_m(t)\, x_m \|_p^m \, dt \leq \left\{ \int_0^1 \| r_1(t)\, x_1 + ... + r_m(t)\, x_m \|_p^{p'} \, dt \right\}^{m/p'} \qquad (9)$$

Now applying (4) we have from (8) and (9) that

$$| \Lambda (x_1 , ... , x_m) | \leq \frac{1}{m!} \| \hat{\Lambda} \| \left(\sum_{i=1}^{m} \| x_i \|_p^p \right)^{m/p}$$

This inequality proves (6).

REFERENCES

1. S. Banach, Über homogene polynome in (L^2), Studia Math. 7 (1938).

2. J, Bergh, J. Löfström, Interpolation Spaces, An Introduction. Springer (Grundlehren der Math. Band 223). Berlin 1976.

3. J. Bochnak and J. Siciak, Polynomials and multilinear mappings in topological vector spaces. Studia Math., 39, 1971,pp. 59-76.

4. S. Dineen, Complex Analysis in Locally Convex Spaces. North Holland (Mathematics Studies, Vol. 57) Amsterdam 1981.

5. L.A. Harris, Bounds on the derivatives of holomorphic functions of vectors, Colloque d'analyse, Rio de Janeiro, 1972. (Ed, L. Nachbin) Herman Paris, Act. Sc. et. Ind., 1367 (1975), pp. 145-163,

6. T.L. Hayden and J.H. Wells, On the extension of Lipschitz-Hölder maps of order β. J.Math.Anal. Appl. 33, 1971, pp.627-640.

7. J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces I, Springer (Ergebrisse der Math. Band 92), Berlin 1977.

8. R.S.Martin, Thesis Cal, Inst, of Tech., 1932.

9. The Scottish Book (Mathematics from the Scottish Café). Ed. R.D. Mauldin, Birkhäuser, Boston, 1981

10. A.M. Tonge, Polarisation and the complex Grothendieck inequality, Math. Proc. Cambridge Philosophical Society, to appear March 1984

11. L. Williams and J. Wells, L^p inequalities, J.Math.Anal.Appl. 64, 1978, pp. 518-529.