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Chapter 3
Models of option pricing

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CHAPTER OUTLINE:

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Abstract

Starting from humble beginnings, the use of financial options has substantially increased as an important financial tool for both speculation and hedging over the last 50 years. This chapter discusses both the theoretical and practical applications of financial options and related models. While the content is somewhat technical, we provide illustrations of their applications in simple settings. We address particular stylised features of option pricing models.

Keywords

Financial options; Black-Scholes model; Volatility; Hedging

3.1 Introduction

The basic idea regarding financial options has been around for a long-time. The Greeks appear to have been the first to use options to speculate on the price of olive harvests (Abraham, 2019). Recounting the material in Mackay's memoirs of 1841, *Extraordinary Popular Delusions and the Madness of Crowds*, Thompson (2007) indicates that following the 17th century Holland tulip market crash: "The provisions, in effect, converted the futures prices in the original contracts to exercise prices in option contracts. The corresponding option price paid to the planters was only later determined. In particular, after over a year of political negotiation, the legislature of Haarlem, the centre of the tulip-contract trade during the "mania," determined the compensation to the sellers to be only 3½% of the contract price for those contracts made between November 30, 1636 and the spring of 1637." Financial options are traded on many exchanges including the Chicago Board Options Exchange and Euronext. The number of option contracts traded worldwide has grown exponentially. In 2019, 15.23 billion option contracts were traded compared to 9.42 billion in 2013.[1] During February 2021, up to 3 trillion dollars equity options by volume were traded in notional value on U.S. option exchanges. Options are use both to manage financial risk and speculate, although forwards and foreign currency borrowing/lending are preferred compared to currency options when firms hedge their exposures (Joseph, 2000).

This chapter discusses the components of financial options. We distinguish between their use for speculation and for hedging underlying exposure. We also provide illustrations of their use in different settings. We first begin with interest rates as it is important feature of option contracts.

3.2 Interest rate and discounted cash flow

This section discusses the basic approaches to compound interest rates and the concept of present value of a cash flow. Both concepts are important components of option pricing models.

3.2.1 Compound interest

An interest rate is the fee for using money and can be expressed as the amount of interest due per period, as a proportion of the amount borrowed, normally in annual percentage terms. The interest rate is determined by the supply (lenders or savers) and

[1] https://www.statista.com/statistics/377025/global-futures-and-options-volume/ (Accessed February 15, 2021). More forwards and futures are traded than options over the same period. For example, worldwide 19.24 billion futures contracts were traded in 2019 compared to 12.13 billion in 2013 (https://www.statista.com/statistics/377025/global-futures-and-options-volume/). Accessed February 15, 2021.

demand (borrowers or investors) for money, although its level is also determined by macroeconomic policy.

Compound interest arises by reinvesting the earnings of the additional interest on the principal, so that interest in the next period is earned based on the sum of the principal plus the first period and second period interest rates. The first formal documentation of compound of interest rate is by a Florentine merchant Francesco Balducci Pegolotti, in his book *Pratica Della Mercatura* in the 1340s, although the charging of interest rates is much earlier. For example, in Biblical history, the charging of interest rates was considered ungodly following specific instructions to the Hebrews.[2] Pegolotti provides tables of the interest on 100 lire, for rates from 1% to 8%, for up to 20 years (Pegolotti, 1936). The important constant e is discovered by Jacob Bernoulli while thinking about matters of continuous compound interest in 1683 (Reichert, 2019). In the 19th century, modified linear Taylor approximation is used to compute the monthly payment formula by Persian merchants (Milanfar, 1996).

To formally define the compound interest rate, suppose an investor receives a constant and discrete rate of interest ρ per unit of time. Denote $B(t)$ as the value of an investment in the bank at time t. Thus, for the next period, the value of this investment is equal to

$$B(t+1) = B(t)(1+\rho)$$

where the factor $(1+\rho)$ is the growth factor for one-unit time period. Similarly, for the second time period

$$B(t+2) = B(t+1)(1+\rho) = B(t)(1+\rho)^2.$$

It is easy to prove by induction on $h \in \mathbb{Z}$, that

$$B(t+h) = B(t)(1+\rho)^h$$

where the growth factor for time period h is $(1+\rho)^h$. Denote the interest rate for the time period h as $i(h)$. Therefore, $B(t+h) = B(t)(1+i(h))$. One can calculate the interest paid to the investors as

$$i(h) = (1+\rho)^h - 1.$$

If the unit time is one year, ρ is also known as annual rate of return, annual yield or the annual rate of capital growth of the investment. Banks offer products with interest at more frequent intervals, for example, semi-annually, quarterly, or monthly. Denote

[2] "If you lend money to one of my people among you who is needy, do not treat it like a business deal; charge no interest." Exodus 22:25–26.

r as the annual interest rate which has m compounding periods in a year. The future value of the investment after n years equals to

$$B(t+n) = B(t)\left(1+\frac{r}{m}\right)^{mn}.$$

Consider $m \to \infty$. That is, interests are paid almost continuously and the interest is compounded continuously. Then,

$B(t+h) = \lim_{m\to\infty} B(t)\left(1+\frac{r}{m}\right)^{mh}$. Take the nature log on both sides, we obtain

$$\ln\big(B(t+h)\big) = \ln\left(\lim_{m\to\infty} B(t)\left(1+\frac{r}{m}\right)^{mh}\right)$$

$$= \lim_{m\to\infty}\left(\ln\big(B(t)\big) + \ln\left(1+\frac{r}{m}\right)^{mh}\right)$$

$$= \lim_{m\to\infty}\left(\ln\big(B(t)\big) + mh\ln\left(1+\frac{r}{m}\right)\right).$$

As m gets larger, $\frac{r}{m}$ gets smaller, so we could use the log approximation $\ln(1+x)\sim x$, and get

$$\ln\big(B(t+h)\big) = \lim_{m\to\infty}\left(\ln\big(B(t)\big) + mh\frac{r}{m}\right) = \lim_{m\to\infty}\left(\ln\big(B(t)\big) + rh\right)$$

and finally gives that the value of principle with continuous compounding interests is $B(t+h) = B(t)e^{rh}$.

Similarly, as $h \to 0$, $\ln(1+x)\sim x$ and $e^x - 1\sim x$, $\frac{B(t+h)-B(t)}{h} = \frac{B(t)}{h}\left(e^{rh}-1\right) \approx rB(t)$ hence leads to the result $\frac{dB(t)}{dt} = rB(t)$, where r is the continuously compounded interest rate. However, continuous compounding is the mathematical limit that only can be reach theoretically if one can calculate and reinvest interest, continuously. This is not possible in practice. The concept of continuously compounded interest is important in finance and is used extensively for pricing options, forwards and other derivatives.

Both discrete interest rate ρ and its continuously compounded equivalent rate r, are assumed to be constant, as discussed before. In many cases, the interest rate is time dependent, also known as adjustable or floating rate, because it is based on an underlying benchmark interest rate or index that adjusts periodically with fluctuations in market conditions. Therefore, the change in value of an investment with a time dependent interest rate is $\frac{dB(t)}{dt} = r(t)B(t)$.

To solve this differential equation, one can rearrange the formula and integrating each side over the time interval $[t, t+h]$. This gives $\int_{B(t)}^{B(t+h)} \frac{dB}{B} = \int_{t}^{t+h} r(s)ds$.

We can now produce the generalisable form of an investment with continuous compounded interest rate as $B(t+h) = B(t)\exp\left(\int_{t}^{t+h} r(s)ds\right)$.

3.2.2 Discounted cash flow

For any $h > 0$, $B(t+h)$ is the future value of the present value $B(t)$, under the continuously compounded interest rate function r, given as $B(t) = B(t+h)\exp\left(-\int_{t}^{t+h} r(s)ds\right)$. For special cases where r is constant, $B(t) = B(t+h)e^{-rh} = B(t+h)(1+\rho)^{-h}$.

In finance, r is also known as the discount rate. In general, the value of any asset is the present value of the expected cash flows from the asset.

One way for an investor to raise capital is to sell bonds to the public. A bond is a fixed income instrument that pays bond holders a specific amount of interest, called the coupon payment, at regular intervals, and a final payment, called the face value (or par value or nominal value) of the bond, at maturity or its redemption date. Once a tradable bond has been issued, the bond holder is free to trade in the financial market and the market price of the bond reflects the interest rate and the level of risk attached to the bond issuer. Typical bonds are government issue bonds (also known as treasury bonds, treasuries or Gilts), local government bonds (known as municipal bonds), and corporate bonds.

Assume a n year maturity bond which pays coupons at rate of $c/2$ every 6 months. At maturity, an investor will receive coupon $c/2$ plus the face value F. Suppose that the annual redemption yield equals ρ, and the present value of the first coupon payment is $\frac{c/2}{(1+\rho)^{1/2}}$. In general, the present value at time of purchase of the kth coupon payment is $\frac{c/2}{(1+\rho)^{k/2}}$. Therefore, the bond price, P is the net present value of all future cash flows generated by the bond $P = \sum_{k=1}^{2n} \frac{c/2}{(1+\rho)^{k/2}} + \frac{F}{(1+\rho)^{n}}$,which is called the discounted cash flow formula for bond pricing. Applying the standard formula for the sum of a geometric series, we have, $\sum_{k=1}^{n} a\hat{r}^{(k-1)} = \frac{a(1-\hat{r}^{n})}{(1-\hat{r})}$ where a and $\hat{r}$ are the first term in the series and the common ratio, respectively, and the bond pricing formula is reduced to $P = \frac{\frac{1}{2}c(1-(1+\rho)^{-n})}{(1+\rho)^{1/2}-1} + \frac{F}{(1+\rho)^{n}}$.

The risk-free rate of return is the rate of return that can be obtained where the associated risk is considered to be zero. A risk-free return typically applies to the interest rate on government bonds, on the assumption that a government of a country cannot go bankrupt. A treasury bond is an example of a risk-free investment. A bond holder is exposed to risk arising from changes in interest rates in the economy, if he sells the bond before the bond matures. There is general agreement that most investors are risk-averse. Thus, for risky investments, they require a return above the risk-free nominal interest rate. The differences between the return on the risky investment and the risk-free rate is known as the risk premium.

Another way for an investor or firm to raise capital is to sell equity to the public in the form of shares, via what is called an initial public offering (IPO). In return for their investment, investors obtain dividend plus capital appreciation. Equities are generally much riskier than bonds and can be valued as the total present value of future dividend payments, that is $P = \sum_{n=1}^{\infty} \frac{d_n}{(1+\rho)^n}$,where ρ is the annual rate of return on a share and d_n is the future annual dividend payment after n years. Mathematically, a firm's shares are equivalent to a bond with variable annual coupon payment $d_1, d_2, \ldots$.

3.3 Basics of option pricing

Assets are traded in officially regulated markets such as the London Stock Exchange, which regulates and organizes the trading of shares in public companies in the U.K. Other exchanges include the London International Financial Futures and Options Exchange (LIFFE) and the London Metal Exchange in U.K.[3] According to Futures Industry Association, the National Stock Exchange of India Ltd (NSE) remains the world's largest derivatives exchange in terms of number of contracts traded in 2020 (The Economic Times, 2021). New York Stock Exchange (NYSE), Nasdaq, and Japan Exchange Group are the top 3 largest stock exchanges by value and volume of transactions (Statista, 2021).

The main assets traded in these financial exchanges are: Financial Assets (shares, bonds, currencies), commodities, notional financial assets (interest rates, index numbers). All listed assets are risky in the sense that their market value fluctuates in unpredictable ways. Derivatives are a type of security which are

[3] LIFFE has undergone a series of takeovers. In 2014, it became part of Intercontinental Exchange and was renamed ICE Futures Europe (Scott, 2019).

associated with the above primary assets and derive their value from that of the underlying asset. These financial derivatives can be grouped under two main headings: futures and options. The existence of financial derivatives allows investors to exert some control over the risks inherently associated with the underlying asset.

3.3.1 Futures

A futures contract is a standardised agreement between a trader and a futures exchange either to buy or to sell an asset at a specified future time at a specified price. A trader agrees to buy a specified amount, known as contract size, of the underlying asset at a specified future date (the delivery date) for a specified price, called future price. The trader is then said to open a long contract or take a long position. Alternatively, the trader opens a short contract or take a short position when he/she agrees to sell a contract size of the underlying asset at a delivery date for the future price. The current market price of the asset is known as the spot price. A futures contract is the simplest example of a financial derivative. A forward contract is a non-standardised contract between two parties to buy or sell an asset. If interest rates are certain then futures and forward prices are equal.

Futures exchanges are responsible for organising and regulating the trading of standardised futures contracts. Across the world, the largest exchange, by volume of contracts, is the CME Group, which consists of Chicago Board of Trade (CBOT), Chicago Mercantile Exchange (CME), New York Mercantile Exchange (NYMEX), Commodity Exchange Inc. (COMEX), Kansas City Board of Trade (KCBT), and the NEX Group (Samuelsson, 2021).

In practice, for every long contract opened with the exchange, there will be a counterbalancing short-opened contract, and vice-versa. However, the long and short parties make their contracts with the futures exchanges and are never in direct contact. The futures exchanges and the underlying assets of future contracts are ‘Real’ or ‘Contingent’ depending on the physical holding of the underlying. In theory, it costs nothing to open a futures contract. In practice, the futures exchange charges a deposit known as the initial margin – usually between 5% to 20% of the value of underlying. Once a futures contract is opened, it can be closed at any time up to maturity, by the contract holder or by the exchange. A long (short) contract is closed by opening a short (long) contract, with the same delivery date and contract size. Most futures contracts are closed before the delivery date. Otherwise, the sellers or buyer holding the futures contract at delivery date will have to exercise their rights under the contract, potentially receiving physical delivery of the contract.

Denote the future price of one unit of underlying asset by $F(t,T)$, at time t for delivery at time T. Assuming an investor opens N long (buy) contracts at time t_0 and sells the contract later at time t, the profit is $\big(F(t,T) - F(t_0,T)\big)N$. Similarly, if an investor opens N short (sell) contracts at time t_0 and closes the short contract later at time t, the profit is $\big(F(t_0,T) - F(t,T)\big)N$. The exchange keeps a daily record of every trader's running profit or loss and the process of using the closing futures market prices, to calculate the running profit or loss. This process is known as marking-to-market. The exchange may call for the losing party to make a margin payment, known as a margin call, if losses become too large, and the losing contract may be close by the exchange if no margin payments are received.

Let $S = S(t)$ denote the value of an asset at time t. In our case, $S(t)$ is usually either the price of a share or the value of a financial market traded index. Assuming a constant rate of return λ earned by a non-dividend paying asset, the value of the standard simple futures price, $F(t,T)$, is the present value at time t to maturity T is $\mathrm{F(t,T)} = \mathrm{S(t)}\left(\frac{1+\rho}{1+\lambda}\right)^{\mathrm{T-t}}$ where ρ is the discrete risk-free interest rate. The continuous compounding forward rate $\mathrm{F(t,T)} = \mathrm{S(t)e^{(r-\lambda)(T-t)}}$, where r is the continuously compounded risk-free interest rate. Using the no-arbitrage principle, this means that there is no opportunity to open simultaneous positions in different assets (or the same asset in different markets) without any initial cost or capital outlay that would guarantee a risk-free profit. Assume a strategy for the short party, when $\mathrm{F(t,T)} > \mathrm{S(t)}\left(\frac{1+\rho}{1+\lambda}\right)^{\mathrm{T-t}}$. To make a profit, at time t, investors can borrow $S(t)(1+\lambda)^{-(T-t)}$ at risk-free interest rate ρ to buy the asset with return λ. At the same time, this investor can open a short futures contract to deliver $S(t)$ amount of asset at price $F(t,T)$ at the future time T. At time T, the asset worth $S(t)$, and the investor can deliver to the futures exchange receiving $F(t,T)$ from the long party. Paying back the loan leaves a profit of $\mathrm{F(t,T)} - \mathrm{S(t)}\left(\frac{1+\rho}{1+\lambda}\right)^{\mathrm{T-t}}$ and leave an arbitrage opportunity to the short party. Similarly, assume a strategy for the long party, when $\mathrm{F(t,T)} < \mathrm{S(t)}\left(\frac{1+\rho}{1+\lambda}\right)^{\mathrm{T-t}}$. At time t, the investor opens a long futures contract to buy $S(t)$ amount of asset at price $F(t,T)$ at the future time T. Assume the investor sells $S(t)(1+\lambda)^{-(T-t)}$ assets and saves in a bank the risk-free interest return, which earns interest up to maturity. At time T, the investor withdraws the money from the bank and receives $S(t)(1+\lambda)^{-(T-t)}(1+\rho)^{T-t}$, then buys $S(t)$ amount of asset from the short party at a cost of $F(t,T)$, leaving the profit of $\mathrm{S(t)}\left(\frac{1+\rho}{1+\lambda}\right)^{\mathrm{T-t}}-\mathrm{F(t,T)}$. The original $S(t)(1+\lambda)^{-(T-t)}$ assets are worth $S(t)$ today, which gives an arbitrage opportunity to the long party which is invalid with the no-arbitrage principle, and completes the proof.

3.3.2 Options

An option is a contract which provides the holder with the right, not obligation, to buy or sell an underlying asset at or before a pre-determined time (maturity date) in the future at a fixed price (strike price or exercise price). There are two basic types of options: the call option and the put option. The opening transaction on an option exchange can be either to sell the option, also known as writing an option, or to buy the option. Hence there are four option positions that may be opened, namely the long call, the short call, the long put and the short put as in Figure 3. 1. Each of these transactions will be discussed in detail below, for option contracts with finite lifetime and expirations.

[Insert Figure 3. 1 about here]

The owner of an option may trade the option in a secondary market, in either an over-the-counter transaction or on an option market exchange. The cash outlay on the options is called option premium; the risk of loss is limited to the premium. Each type of underlying asset gives rise to an option contract with a specific contract size. For example, a stock or equity option contract is usually for options of 1000 underlying shares. An American style option is one that can be exercised at any time up to an including to the expiry date, whereas a European style option is one that can only be exercised on the expiry date.

A trader who expects an asset price to increase can buy a call option to purchase the asset at a fixed price at a later date, rather than purchase the asset now. The opening transaction on the option exchange is 'buy to open' one call option contract. The buying price of call option is often referred to as the call option premium. In purchasing this contract, investors are acquiring the right to buy the underlying derivative at any time before the expiry date with pre-determined exercise price. In general, if the market price of the underlying asset increases, then the market price of a call option on that asset will also increase.

A long call contract is different to a long futures contract. The long party to a futures contract is legally bound to buy the asset at the forward price agreed in the contract. The same as in the forward contract. The long party to a call option contract has the right not the obligation to buy the asset at the agreed exercise price, i.e., the long party to an option contract may exercise the right to buy or may choose not to exercise.

The long party to a call contract has three closing transactions available, as follows:

- Exercise the option rights, i.e., buy the contract size of the underlying asset at the exercise price.
- Trade on the option market. Opening a short call contract with the same exercise price and expiry date as the original long one. The option exchange considers the position to be closed with a short and a long contract of equal amounts.
- Do nothing. The option contract expires on the expiry date and is worthless after the expiry date, the loss of the long party is the option premium paid to open the contract.

Denote $C = C(S, t)$ as the market price of a call option with underlying asset valuing $S = S(t)$ at time t. Option price is $C \geq 0$ and $C \leq S$ because investors will never pay for the right to buy something more than they actually paid to buy it outright. The value of an option is the profit on the closing transaction. Denote the strike price as E. When $S > E$, the payoff of a long call holder is $S - E$, by exercise the option paying E, and sell the asset at price S. When $S \leq E$, the option is worthless and should not be exercised. Therefore, the market price of the call option at expiry is

$$\mathrm{C(S, T) = max(S - E, 0)}. \quad (3.1)$$

The net profit takes into consideration the premium that the long party paid up front is equal to $C(S, T) - premium = max(S - E, 0) - premium$. Thus, in order to gain a profit from a long call contract, the underlying asset price must rise. Figure 3. 2 illustrates the payoff and profit on a long call option at expiration.

[Insert Figure 3. 2 about here]

A trader who expects an asset price to decrease can sell, or "write", a call option. The opening transaction is a call option contract. The seller of the contract is the short party to the contract and is said to have written the option and receives the premium paid by the long party. If the long party is exercising the rights, the short party is obliged to sell the contract size at the exercise price.

Having opened a short call contract, the short party may experience one of three possible closing transactions, as follows:

- The contract is exercised, i.e., the long party exercises the right to buy, while the short party is obliged to sell at the exercise price.
- Buy a long contract with the same exercise price and expiry date as the original short one. The option exchange considers the position to be closed with a long and a short contract, and the closing transaction is a buy contract.
- Do nothing. The short party keeps the option premium received which is the ideal situation for the short party, if the long party does not exercise its rights.

The short party of a call option is in the opposite position of the long party of the same call option, with payoffs at expiry date. That is, $min(S-E,0)$, profit equals to $premium - min(S-E,0)$. Thus, to profit from a short call contract, the underlying asset price must not rise, and the potential loss is unlimited. Figure 3. 3 illustrates the payoff and profit on a short call option at expiration.

[Insert Figure 3. 3 about here]

A trader who expects an asset price to decrease can buy a put option to sell the asset at a fixed price at a later date, rather than sell the asset now. Here the opening transaction is to buy (buy to open) a put option contract. The long party is buying the right to sell the contract size of the underlying asset at the exercise price at any time (American style assumed) before the expiry date. The market price of a put option will increase (decrease) when the price of the underlying share decreases (increases).

Having opened a long put contract, there are three possible closing transactions:

- Exercise the option, i.e., sell the contract size of the underlying asset at the exercise price.
- Opening a short put contract with the same exercise price and expiry date as the original long one. The option exchange considers the position to be closed with a long and a short contract, and the transaction is closed.
- Do Nothing. The option will be worthless after the expiry date, the loss of the long party is the option premium paid to open the contract.

Denote $P = P(S,t)$ as the market price of a call option with underlying asset valuing $S = S(t)$ at time t. The put price increases as the underlying value decreases.

The option price $P \geq 0$ and $P \leq E$ because the put option is most valuable when asset is worthless, and investors can sell put option for E. The value of an option is the profit on the closing transaction. Denote the strike price as E. When $S \leq E$, the payoff of a long put holder is $E - S$, by exercise the option receiving E, and buy the asset at price S. When $S > E$, the option is worthless and should not exercise. Therefore, the market price of the call option at expiry is

$$\mathrm{P(S,T) = max(E - S, 0).} \tag{3.2}$$

The net profit is the premium that long party paid up front equals to $P(S,T) - premium = max(E - S, 0) - premium$. Thus, in order to profit from a long put contract, the underlying asset price must fall. Figure 3. 4 illustrates the payoff and profit on a long put option at expiration.

[Insert Figure 3. 4 about here]

A trader who expects an asset's price to increase can sell, or "write", a put option. The party who opens a short put contract is on the opposite side of the option transaction to the long put party, the opening transaction is to sell (sell to open) a put option contract. The short party receives the premium paid by the long party and in return is obliged to buy the contract size of the underlying asset at the exercise price, if called upon to do so by the long party.

Having opened a short put contract, there are three possible closing transactions, as follows:

- Be exercised against, i.e., the long party exercises the right to sell, while the short party is obliged to buy at the exercise price.
- Buy a long contract with the same exercise price and expiry date as the original short one. The option exchange considers the position to be closed with a long and a short contract, and the closing transaction is a buy to close.
- Do nothing. The short party keeps the option premium received which is the ideal situation for the short party, if the long party does not exercise his/her rights.

The payoffs at expiry date $min(E - S, 0)$, profit equals to $premium - min(E - S, 0)$. Thus, to profit from a short put contract, the underlying asset price must not fall, and the potential loss is unlimited. Figure 3. 5 illustrates the payoff and profit on a short put option at expiration.

[Insert Figure 3. 5 about here]

3.3.3 Put-Call Parity

Assume that the underlying share X pays no dividends. At time t, investor A buys one share X costing $S = S(t)$. At the same time, the investor writes one European style call option for underlying share with exercise price E, expiry time T. The income from the call option is $C_e(S,t)$. The total expenditure by investor A at time t is $S - C_e(S,t)$. Assume that investor B invests $Ee^{-r(T-t)}$ at time t in the risk-free market, e.g., bank, and earns continuously compounded interest rate r. In addition, the investor writes one European style put option on share X with exercise price E and expiry time T. The income from the put option is $P_e(S,t)$. The total expenditure by the investor B at time t is $Ee^{-r(T-t)} - P_e(S,t)$.

The investors hold both investments until expiry time T, with $S_T = S(T)$. The final value of investor A is $S_T - C_e(S_T,T) = S_T - max(S_T - E, 0)$, and the final value for investor B is $Ee^{-r(T-t)}e^{r(T-t)} - P_e(S_T,t) = E - max(E - S_T, 0)$. When $S_T \leq E$, both investors have value equals to S_T. However, if $S_T > E$, the value is E. That is, that the final value of investors A and B are identical, and these values are guaranteed and risk-free. If two risk-free investments have the same final value at time T, then the no-arbitrage principal implies and they must have the same value at all times $t < T$. Otherwise, the investors can benefit from an arbitrage opportunity arising from selling the initially more expensive investment short, and buying the cheaper one. Thus, the expenditure at time t are the same, $S - C_e(S,t) = Ee^{-r(T-t)} - P_e(S,t)$, or, as it is normally written,

$$S + P_e(S,t) = Ee^{-r(T-t)} + C_e(S,t). \tag{3.3}$$

This relationship between the prices of European style call and put option prices is known as the law of put-call parity, and the relationship holds only in the case where the underlying share does not pay a dividend.

From the law of put-call parity, if $P_e(S,t) \geq 0$, it follows that $C_e(S,t) \geq S - Ee^{-r(T-t)}$. Furthermore, since $C_e(S,t) \geq 0$, then $C_e(S,t) \geq \max\left(S - Ee^{-r(T-t)}, 0\right)$. Similarly, $P_e(S,t) \geq \max\left(Ee^{-r(T-t)} - S, 0\right)$.

3.3.4 Speculation

A hedge is a transaction undertaken by an investor to protect an exposure from adverse price movements. A speculator may undertake a similar transaction either to partially hedge or simply execute a derivative transaction without have an underlying exposure. In this sense, both a partial hedger and what we call a *true* speculator may be betting on future price movements of the underlying asset. Partial hedging is common among firms. Indeed, there is some agreement in the literature that partial hedging is speculative (Géczy et al., 2007). Overtime, the idea of firms speculating on the direction of financial prices to hedge exposures has become more common in risk management settings. Indeed, the U.S. participants in the Jilling (1978) study were adamant that they do not speculate.[4] However, more recently, Géczy et al. (2007) report that 40% of U.S. firms use derivatives for speculative purposes in the sense that they take a market view; at least 7% of them do so frequently. The finding that firms with exposure speculate has been reported for other countries (Hakkarainen et al., 1998). Therefore, we consider speculation as an appropriate context to evaluate the role of option pricing.

For options, investors gain profit when the underlying asset price falls and they hold a short call or a long put contract. Both option types may be used to hedge against falling prices of an asset. Similarly, investors gain profit when the underlying asset price rises and they hold a long call or a short put contract. These option types may be used to hedge against rising asset prices. More details are discussed in Section 3.7.

A speculator may have a view about future price movement of the underlying asset. He/she may then invest in an option without having any underlying asset to protect. The profit/loss characteristics of the basic option types may be: long call, short call, long put and short put.

Suppose the speculator is long on a call contract at time t_0 with initial cost of

$\alpha = C(S_0, t_0)$ per option. At time period $(t_0, T]$, the profit per option from trading the contract is $C(S, t) - \alpha$ and expiry profit per option is

$$\max(S - E, 0) - \alpha. \qquad (3.4)$$

The short party's profit is the long party's loss. Thus, opening a short call contract produces an initial income of α per option at time t_0, the subsequent profit from trading the contract is $\alpha - C(S, t)$ and the expiry profit per option is

$$\alpha - \max(S - E, 0). \qquad (3.5)$$

[4] Jilling (1978, p. 144) indicates that: "... more than 93 per cent of the respondents point out that they are not in business to "speculate" on foreign exchange movements ...".

Suppose the speculator opens a long put option at time t_0 with initial cost of $\beta = P(S_0, t_0)$ per option. At time period $(t_0, T]$, the profit per option from trading the contract is $P(S,t) - \beta$ and at expiry time, the profit per option is

$$\max(E - S, 0) - \beta. \quad (3.6)$$

Again, the short party's profit is the long party's loss. Thus, opening a short put contract produces an initial income of β per option at time t_0, the subsequent profit from trading the contract is $\beta - P(S,t)$ and the expiry profit per option is

$$\beta - \max(E - S, 0). \quad (3.7)$$

Many more interesting and complex speculative strategies can be designed to take advantage of various underlying price behaviour, such as long strangles, short strangles, long straddles, short straddles, butterflies, condors, ratio spreads, calendar spreads. The speculator can sustain consumption level by bearing more risks, adding liquidity to the market, and therefore promote an efficient market.

3.4 Option pricing models

Previous sections only give explicit formulae for the prices of call and put options at expiry time $t = T$, when $S = S(T)$ and $C = \max(S - E, 0)$, $P = \max(E - S, 0)$. At the present time t, where $t < T$, the future expiry price $S = S(T)$ of the underlying asset is unknown and generally unpredictable. Therefore, to evaluate the current value of options, the first task is to model the asset price process.

The Figure 3. 6 shows the FTSE100 daily index covering 2020[5]. The FTSE100 index has a sudden drop in March due to the COVID19 pandemic and a recovery afterwards. The detailed day-to-day behaviour is highly erratic and unpredictable. Financial prices of major exchanges exhibit random day-to-day price changes, especially at high frequency. Thus, some type of statistical model involving random day-to-day price movements would be appropriate for modelling the behaviour of this type of asset.

[Insert Figure 3. 6 about here]

Suppose that underlying asset price $S(T)$ process is a random variable with probability density function $f(\cdot)$. In particular, since asset prices are essentially positive, $\mathbb{P}(0 \le S(T) \le \infty) = \int_0^\infty f(x)dx = 1$. In general, if $u = u(S(T))$ is any function

[5] Chart from www.londonstockexchange.com/

of the expiry asset price $S(T)$, then the expected value of u is $\mathbb{E}(\mathrm{u}(\cdot)) \coloneqq \int_0^\infty \mathrm{u(x)f(x)dx}$. This means that at expiry $t = T$, the call option is equals to

$$\bar{C}(T) = \mathbb{E}(C(\cdot, T)) = \int_0^\infty C(x, T) f(x) dx$$

$$= \int_0^\infty \max(x - E, 0) f(x) dx$$

$= \int_E^\infty (x - E) f(x) dx$. The put option

$$\bar{P}(T) = \mathbb{E}(P(\cdot, T)) = \int_0^\infty P(x, T) f(x) dx$$

$$= \int_0^\infty \max(E - x, 0) f(x) dx$$

$$= \int_0^E (E - x) f(x) dx.$$

Denote the present values by $\widehat{C}(S, t)$ and $\widehat{P}(S, t)$. Therefore, the theoretical formulae of present values at time t for option prices are

$$\widehat{C}(S, t) = e^{-r(T-t)} \bar{C}(T) = e^{-r(T-t)} \int_E^\infty (x - E) f(x) dx \tag{3.8}$$

$$\widehat{P}(S, t) = e^{-r(T-t)} \bar{P}(T) = e^{-r(T-t)} \int_0^E (E - x) f(x) dx. \tag{3.9}$$

This section addresses the following three questions:

1. What is a reasonable assumption to make about the probability density $f(\cdot)$ of an asset price $S(T)$ at the option expiry time T?
2. For a given $f(\cdot)$, how to evaluate the integrals?
3. Are the resulting formulae consistent with the no-arbitrage principle?

3.4.1 Random walks and Brownian motion

A random walk describes the process by which randomly moving objects wander away from where they started. The term random walk was first introduced by Karl Pearson (Pearson, 1905) and is often used to model shares prices in financial economics. The simplest random walk example is the 1-dimentional random walk on the integer number line, which starts at origin and moves forward or backwards at each step with equal probability. This moving patten can be used to describe the behaviour of a financial index over the time, similar to Figure 3.6. To compute a random walk model, denote the modelling value as $W = W(t)$ over the time interval $[t_0, T]$. Suppose $W(t_0) = W_0$ has a known value at some initial time t_0 and divide $[t_0, T]$

into n equal sub-intervals of width $\delta t = \frac{T-t_0}{n}$. It is easy to get that $t_1 = t_0 + \delta t, t_2 = t_0 + 2\delta t, \ldots, t_n = t_0 + n\delta t = T$.

Suppose that at time $t = t_1$, $W_1 = W(t_1) = W_0 + \delta W_0$ where the increment δW_0 is chosen at random from a normal distribution. Similarly,

$$W_2 = W(t_2) = W_1 + \delta W_1$$

$$W_3 = W(t_3) = W_2 + \delta W_2$$

$$\ldots$$

$$W_n = W(t_n) = W_{n-1} + \delta W_{n-1}$$

where the increments $\delta W_1, \delta W_2, \ldots, \delta W_{n-1}$ are chosen independently at random from the same normal distribution as δW_0 . The equations used to compute $W_1, W_2, \ldots, W_n$ are known as a random walk model. It is easy to show that $W(T) = W_0 + \sum_{i=0}^{n-1} \delta W_i$.

Theorem 3. 1

Under the assumptions above, $W(T)$ is normally distributed, with expect value $W_0 + n\mathbb{E}(\delta W)$ and variance $nVar(\delta W)$.

Proof:

Since the sequence of random numbers $\delta W_0, \delta W_1, \delta W_2, \ldots, \delta W_{n-1}$ are independent and normally distributed, $\sum_{i=0}^{n-1} \delta W_i$, and hence $W(T)$ are normally distribution random variables. The expected value of the random variable is equal to

$$\mathbb{E}\big(W(T)\big) = \mathbb{E}\left(W_0 + \sum_{i=0}^{n-1} \delta W_i\right)$$

$$= W_0 + \sum_{i=0}^{n-1} \mathbb{E}(\delta W_i)$$

$$= W_0 + \sum_{i=0}^{n-1} \mathbb{E}(\delta W)$$

$$= W_0 + n\mathbb{E}(\delta W)$$

with variance

$$\mathrm{Var}\big(W(T)\big) = \mathrm{Var}\left(W_0 + \sum_{i=0}^{n-1} \delta W_i\right)$$

$$= \sum_{i=0}^{n-1} \mathrm{Var}(\delta W_i)$$

$$= \sum_{i=0}^{n-1} \mathrm{Var}(\delta W)$$

$$= n\mathrm{Var}(\delta W)$$

which completes the proof.

While simple random walk is a discrete space (integers) and time model, Brownian Motion is a continuous space time model, which is motivated by the simple random walk. The Brownian motion is a mathematical model used to describe the random fluctuations of particles. It was named after the Scottish botanist Robert Brown (1773-1858) who first discovered in 1827 that chaotic movements of pollen are suspended in water. The Brownian motion was wildly used by physicists to describe the diffusion movements of particles, in particular, by Albert Einstein (Einstein, 1905). Louis Bachelier, the father of modern option pricing theory, use the Brownian motion for the first time to pricing of options in his PhD thesis (Bachelier, 1900). In mathematics, the Brownian motion is also known as the Wiener process in honour of American mathematician Norbert Wiener (1894-1964), which is a real value continuous-time stochastic process.

The stochastic process W is called Wiener process if the following conditions hold:

1. $W(t_0) = 0$.
2. W has stationary, independent increments: for every $t > 0$, future increments $W(t+s) - W(t)$ are independent of the past.
3. W has Gaussian increments: increment $W(t+s) - W(t)$ is normally distributed with mean 0 and variance s.
4. W has continuous paths: the function $W(t)$ is continuous in t.

The standardized Wiener process, usually denoted by X, is the special Wiener process satisfies

$$\delta X = \sqrt{\delta t}\mathcal{N}(0,1) \tag{3.10}$$

with solution

$$X(T) = \sqrt{T - t_0}\mathcal{N}(0,1). \tag{3.11}$$

Define $\mathcal{N}$ to be the cumulative distribution function for the standard normal probability density $\mathcal{N}(0,1)$ with mean 0 and variance 1, then

$$\mathcal{N}(x) = \mathbb{P}(z \le x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\, dt \tag{3.12}$$

with its derivative

$$\mathcal{N}'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}. \qquad (3.13)$$

And hence the general Wiener process may be written as

$$\delta W = \mu\delta t + \sigma\delta X \qquad (3.14)$$

with solution

$$W(T) = W_0 + \mu(T - t_0) + \sigma X(T). \qquad (3.15)$$

3.4.2 Binomial model

The binomial option pricing model uses an iterative procedure. Assuming in any one time period, the underlying asset value can move to one of two possible prices, up or down. Figure 3. 7 shows the general formulation of a two-period stock price path process, which follows a Binomial model. For a multi-periods model, $S(T)$ follows a binomial distribution, which is a simple 1-dimentional random walk. The initial stock price at $t = 0$ equals to S_0, at the first time period $t = 1$, price moves up to uS_0 with probability p and moves down to dS_0 with probability $1 - p$. When $t \geq 2$, stock price moves up by u with probability p and moves down by d with probability $1 - p$. One can easily observe the stock price path with combinations of 'up, down' and 'down, up' leads to the same final price udS_0 with probability $2p(1 - p)$.

[Insert Figure 3. 7 about here]

The major advantage of the binomial option pricing model is that it is mathematically simple, when calculating the option value along the range of possible paths for each period. The basic method of valuing the binomial option model is by creating a replicating portfolio until the option expires. Assume that an investor borrows capital B from a bank at a rate of ρ per period, to buy Δ_0 units of the underlying asset at $t = 0$, in order to replicate the payoffs of a call option. Note that $d < (1 + \rho) < u$ to ensure no arbitrage opportunity. At $t = 1$, C_u is the value of the call option if the underlying asset increases in value to uS_0. C_d is the value of the call option if the underlying asset's value decreases to dS_0. The payoffs of a call option equal to corresponding replicating portfolio

$$C_u = \Delta_0 uS_0 - B(1 + \rho)$$

$$C_d = \Delta_0 dS_0 - B(1 + \rho).$$

Solve the equations and can obtain that

$$\Delta_0 = \frac{C_u - C_d}{uS_0 - dS_0} \qquad (3.16)$$

$\mathrm{B} = \frac{\mathrm{uC_d - dC_u}}{(1+\rho)(\mathrm{u-d})}$. (3.17)

(3.9)

The valuation for a multiperiod process can be proceed iteratively from the final period to the current time $t = 0$, composed of delta shares of the underlying asset, ignoring risk-free borrowing. So, the option value at current time is

$$\hat{\mathrm{C}}(\mathrm{S}, 0) = \mathrm{C} = \Delta_0 \mathrm{S}_0 + \mathrm{B}$$

$$= \frac{\mathrm{C_u - C_d}}{\mathrm{uS_0 - dS_0}} \mathrm{S}_0 + \frac{\mathrm{uC_d - dC_u}}{(1+\rho)(\mathrm{u-d})}$$

$$= \frac{1}{(1+\rho)} \left[\frac{(1+\rho) - \mathrm{d}}{\mathrm{u-d}} \mathrm{C_u} + \frac{\mathrm{u} - (1+\rho)}{\mathrm{u-d}} \mathrm{C_d} \right]$$

$$= \frac{1}{(1+\rho)} [\mathrm{qC_u} + (1-\mathrm{q})\mathrm{C_d}] \quad (3.18)$$

where $q = \frac{(1+\rho)-d}{u-d}$ is the probability that the stock goes up to C_u under the risk-neutral assumption.

For illustrative purposes, consider a call option with strike price 100 and expiration two time periods ahead. The current value of the underlying stock price is 100, for each period. The price has 50% chance of moving up by 10% and 50% chance of moving down by 10%. That is, $u = 1.1, d = 0.9$, shown in Figure 3. 8. Assume interest rate is ρ=5%, at the end of the second period. The call option will only be exercised if the underlying stock price increases in both periods with value $21 (= 121 - 100)$.

[Insert Figure 3. 8 about here]

Consider the top end nodes when $S_1 = 110$ at $t = 1$. The replicating portfolios with pricing up or down at $t = 2$ are $(121 \times \Delta) - 1.05 \times \mathrm{B} = 21$ and $(99 \times \Delta) - 1.05 \times \mathrm{B} = 0$.

Solving the equations, the units of the underlying asset are $\Delta = \frac{21-0}{121-99} = 0.955$ and $\mathrm{B} = -90$.

If the stock price is 110 at $t = 1$, borrowing 90 capital from the risk-free market and buying on 0.955 share of the stock will give the same cash flows as buying the call. The value of the call option at $t = 1$ is $110 \times \Delta + \mathrm{B} = 15$.

The other leg of the tree is:

$\Delta = \frac{0-0}{99-81} = 0$ with $B = 0$. At $t = 1$ if the stock price is 90, then the call is worthless. Moving back to the one period earlier and create a replicating portfolio at $t = 0$ are

$(110 \times \Delta) - 1.05 \times \mathrm{B} = 15$

$(90 \times \Delta) - 1.05 \times \mathrm{B} = 0.$

Solving the equations generates $\Delta = 0.75$ and $B = -64.286$. The value of the call option at $t = 0$ equals to $100 \times \Delta + \mathrm{B} = 10.714$.

The binomial model provides insight into numerical methods to determine option value under the assumption of no-arbitrage principle with discreate time model. The value of an option reflects expectations at a future date and is determined by the current price of the asset deriving from the value of the replicating portfolio. If the replicate portfolio costs less than the corresponding call option in the market, an investor could sell the call and buy the replicating portfolio at the same time, with a guaranteed risk-free profit. A rational market will move accordingly, and call option will reduce to the level of the replicating portfolio, to be consistent with the no-arbitrage principle. The value of a call option increases as the time to maturity, asset price volatility (u and d), and interest rate increase.

Although the methodology of evaluating option pricing with binomial model avoid the integral calculation and is more intuitive, it requires a large number of inputs to calculate the expected future prices recursively at each node, which means it is especially computational expensive in the multiperiod models. Assume that as the time periods in the binomial model get shorter, the price changes become smaller and infinitesimally approaching zero. The option pricing models could be evaluated while underlying asset with a continuous price process. The chapter will not discuss the case under the assumption of price changes staying large with shorter period, i.e., a jump price process.

3.4.3 Black-Scholes Model

The Black-Scholes and Black-Scholes-Merton model provide a closed form-theoretical estimate of the value of a option using a small number of inputs, (Black and Scholes, 1973; Merton, 1973). Merton and Scholes received the 1997 Nobel Memorial Prize in Economic Sciences for their breakthrough work that separates the option from the risk of the underlying security using the risk neutral dynamic. Black–Scholes model underpin similar assumptions as the binomial model, while binomial model assumes discrete time process and the underlying approximately follows a binomial distribution. Black–Scholes model assumes a continuous process underlying, while the binomial distribution approaches the lognormal distribution. The value estimated from binomial model converges on the Black–Scholes formula value as the number of periods n increases and goes to infinity. The most significant contribution of the Black–Scholes pricing is the formula depends only on the market observable inputs.

3.4.3.1 Stochastic differential equations

Recall that in ordinary calculus, for a differentiable function $W = f(t)$, there is no difference between a differential and an increment for the independent variable, i.e., $dt = \delta t$. However, the differential of the dependent variable, i.e., $dW = f'(t)dt$, is not equal to the increment for W

$$\delta W = f(t + \delta t) - f(t)$$
$$= f'(t)\delta t + \frac{1}{2} f''(\eta)(\delta t)^2$$
$$= dW + \mathcal{O}(\delta t^2)$$

where $\mathcal{O}(\delta t^2)$ is a term which is no greater than the multiple of $|\delta t|^2$. However, $dW \approx \delta W$ if δt is infinitesimally small. In the case where increments and differentials are stochastic and δt is infinitesimally small, define

$$dt = \delta t, dX = \delta X, dW = \delta W \quad (3.19)$$

where X denote the standardized Wiener process. For more discussion of stochastic calculus in an economic context, see Merton (1975), and Fischer (1975).

Denote $S = S(t)$ as the value of an asset at time t, dS as the change in value over the next infinitesimal time interval dt, and dS/S is the corresponding relative change in value over this time interval dt. Relative change dS/S is a financially more meaningful quantity to model than the absolute change dS and suppose asset prices satisfy the following stochastic differential equation

$$\frac{dS}{S} = \mu dt + \sigma dX, \quad S(t_0) = S_0 \quad (3.20)$$

where μ and σ are constants. If S satisfies the above stochastic differential equation, then it is said to follow a *geometric Brownian motion*. The parameters μ and σ are usually known as the *drift* and the *volatility* of the asset price respectively.

To solve the above equation and determine the probability density for the random variable S, the first step is to integrate both sides over the interval $[t_0, T]$ and obtain

$$\int_{t_0}^{T} \frac{dS}{S} = \mu(T - t_0) + \sigma X(T).$$

It is tempting to suppose that

$$\frac{d(\ln(S))}{dS} = \frac{1}{S} \tag{3.21}$$

in which case gives the result of the integral

$$\int_{t_0}^{T} \frac{dS}{S} = \int_{t_0}^{T} d(\ln(S)) = \ln(S)|_{t_0}^{T} = \ln\left(\frac{S(T)}{S_0}\right).$$

However, for a stochastic variable, S the above results do not hold. In particular, for a stochastic variable S, $d(\ln(S)) \neq \frac{dS}{S}$.

One should apply Itô's lemma (Itô, 1944; 1951) to solve the stochastic differential equation. Itô's lemma is the chain rule for stochastic calculus and is an identity used to calculate the differential of a function that depends on a stochastic variable (a stochastic process). Assume that the asset price variable S is described by the stochastic differential equation

$$dS = A(S, t)dX + B(S, t)dt. \tag{3.22}$$

To calculate the differential of a function

$$U = f(S, t) \tag{3.23}$$

where $f(\cdot)$ is some given function, X is the standard Wiener process and A, B are given functions. Ito's lemma states how dU is related to dS and dt. In other words, it determined the stochastic differential equation that is satisfied by U.

Lemma 3. 1

Itô's lemma states that

$$dU = \frac{\partial U}{\partial S} dS + \frac{\partial U}{\partial t} dt + \frac{1}{2} \frac{\partial^2 U}{\partial S^2} A^2 dt. \tag{3.24}$$

The first two terms on the right-hand side of Lemma 3. 1 are just what one would expect from ordinary non-stochastic calculus. The third term is new and arises only when S is a stochastic variable. S, therefore satisfies an ordinary non-stochastic differential equation if and only if $A \equiv 0$. Substituting dS and obtain the stochastic differential equation that U satisfies as

$$dU = \frac{\partial U}{\partial S} A dX + \left(\frac{\partial U}{\partial S} B + \frac{\partial U}{\partial t} + \frac{1}{2} \frac{\partial^2 U}{\partial S^2} A^2\right) dt. \tag{3.25}$$

This solution plays a key role in the Black-Scholes option price derivation. This section provides a sketch of proof by expanding a Taylor series without getting into too much of details of the limit of a sequence of random variables.

Proof:

The Taylor expansion gives

$\delta U = f(S + dS, t + dt) - f(S, t)$

$= f_S dS + f_t dt + \frac{1}{2}(f_{SS} dS^2 + 2f_{St} dS dt + f_{tt} dt^2 + \cdots)$ (3.26)

where subscripts denote partial differentiation with respect to S or t. As $dt \to 0$ so $\delta U \to dU$; that is, dU is the dominant contribution to δU when dt is small. Define dU informally as the sum of dominant random term with zero mean in δU as $dt \to 0$ and dominant non-random term in δU as $dt \to 0$ which have both a random and a non-random component. Then, exam each term in the Taylor expansion of δU

1. Substitute dS into the first two terms

   $f_S dS + f_t dt = f_S(AdX + Bdt) + f_t dt$

   To simplify the notation, $dX = \mathcal{N}(0, dt) = \sqrt{dt}\mathcal{N}(0,1) = z\sqrt{dt}$, where z is a standardized normal random variable, with $\mathbb{E}(z) = 0$ and $\mathbb{E}(z^2) = 1$. Therefore, $dS = Az\sqrt{dt} + Bdt$, and

   $f_S dS + f_t dt = f_S Az\sqrt{dt} + (Bf_S + f_t)dt.$

   Since $\mathbb{E}(z) = 0$, $f_S Az\sqrt{dt}$ is a random variable with zero mean and $(Bf_S + f_t)dt$ is a non-random term.

2. The third term is

   $$f_{SS} dS^2 = f_{SS}\left(Az\sqrt{dt} + Bdt\right)^2 = f_{SS}\left(A^2 z^2 dt + 2ABz(dt)^{3/2} + B^2 dt^2\right)$$

   As $dt \to 0$, terms with higher order dt negligible. Hence $f_{SS} dS^2 \approx f_{SS} A^2 z^2 dt$. This random term has a non-zero expected value $\mathbb{E}(f_{SS} dS^2) \approx f_{SS} A^2 \mathbb{E}(z^2) dt = f_{SS} A^2 dt$.

3. The fourth term is

   $2f_{St} dS dt = 2f_{St}\left(Az\sqrt{dt} + Bdt\right)dt = 2f_{St}\left(Az(dt)^{3/2} + B^2 dt^2\right)$

   is also negligible as $dt \to 0$.

4. The fifth term on the right Taylor expansion δU, $f_{tt} dt^2$ negligible as $dt \to 0$.

   Hence ignoring the negligible terms, as $dt \to 0$,

   $\delta U \approx f_S dS + f_t dt + \frac{1}{2} f_{SS} A^2 dt$

   That is

   $$dU = f_S dS + f_t dt + \frac{1}{2} f_{SS} A^2 dt$$

$$= \frac{\partial U}{\partial S}(AdX + Bdt) + \frac{\partial U}{\partial t}dt + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}A^2 dt$$

which completes the proof.

3.4.3.2 Log-normal distribution

Consider the question in the previous subsection where S follows the geometric Brownian motion. So, $A(S,t) = \sigma S$ and $B(S,t) = \mu S$. Assume the function $U = ln(S)$, and easily get

$$\frac{\partial U}{\partial S} = \frac{1}{S}, \ \frac{\partial U}{\partial t} = 0, \ \frac{\partial^2 U}{\partial S^2} = -\frac{1}{S^2}$$

and hence Itô's lemma yields

$$d(\ln(S)) \equiv dU = \frac{dS}{S} + 0 + \frac{1}{2}\left(-\frac{1}{S^2}\right)(\sigma S)^2 dt$$

$$= \frac{dS}{S} - \frac{\sigma^2}{2}dt$$

which provides the correct expression for $d(ln(S))$ in the case where S follows a geometric random walk. Taking the integration gives

$$\int_{t_0}^{T}\frac{dS}{S} = \int_{t_0}^{T} d(\ln(S)) + \frac{\sigma^2}{2}\int_{t_0}^{T} dt$$

$$= (\ln(S))_{t_0}^{T} + \frac{\sigma^2}{2}(T - t_0)$$

$$= \ln\left(\frac{S(T)}{S_0}\right) + \frac{\sigma^2}{2}(T - t_0).$$

Therefore,

$$\int_{t_0}^{T}\frac{dS}{S} = \mu(T - t_0) + \sigma X(T) = \ln\left(\frac{S(T)}{S_0}\right) + \frac{\sigma^2}{2}(T - t_0)$$

rearranging and we get

$$\ln\left(\frac{S(T)}{S_0}\right) = \left(\mu - \frac{\sigma^2}{2}\right)(T - t_0) + \sigma X(T)$$

$$\ln(S(T)) = \ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)(T - t_0) + \sigma X(T)$$

where the standardised Winer process $X(T) = \sqrt{T - t_0}\mathcal{N}(0,1)$. Therefore, $ln(S(T))$ is normally distributed with

$$\mathbb{E}[\ln(S(T))] = \ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)(T - t_0) \tag{3.27}$$

and

$$\mathrm{Var}[\ln(S(T))] = \sigma^2(T - t_0). \tag{3.28}$$

If the logarithm of a random variable is normally distributed, then we say that the variable itself is log-normally distributed. Equivalently, whereas the probability density function for $ln(S(T))$ is given by the normal distribution. Assume that $S = e^Z$, then denote

$$Z \sim \mathcal{N}(\mu, \sigma^2) \Longrightarrow S = e^Z \sim \mathcal{LN}(\mu, \sigma^2).$$

Therefore

$$Z = \ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)(T - t_0) + \sigma X(T) \sim \mathcal{N}\left(\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)(T - t_0), \sigma^2(T - t_0)\right)$$

$$S = e^Z \sim \mathcal{LN}\left(\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)(T - t_0), \sigma^2(T - t_0)\right).$$

The probability density function of $Z \sim \mathcal{N}(\mu, \sigma^2)$ is $f_Z(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$, and the probability density function of the log-normal distribution $\mathcal{LN}\left(ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)(T - t_0), \sigma^2(T - t_0)\right)$, given $\frac{dz}{dS} = \frac{d}{dS} lnS = \frac{1}{S}$, is

$$f_S(S) = f_Z(\ln S)\left|\frac{dz}{dS}\right| = \frac{1}{S\sigma\sqrt{2\pi(T - t_0)}} e^{-\frac{\left(\ln\left(\frac{S}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)(T - t_0)\right)^2}{2\sigma^2(T - t_0)}}. \qquad (3.29)$$

In finance, a reasonable assumption regarding the underlying asset price is that $S(T)$ follows a log-normal distribution. Given that $\mathbb{E}[e^Z] = e^{\mathbb{E}[Z] + \frac{1}{2}Var[Z]}$, $Var[S] = \mathbb{E}[S^2] - \mathbb{E}[S]^2 = \mathbb{E}[e^{2Z}] - \mathbb{E}[e^Z]^2$, and $Cov[S(T_1), S(T_2)] = \mathbb{E}[S(T_1), S(T_2)] - \mathbb{E}[S(T_1)]\mathbb{E}[S(T_2)]$, the solution $S(T)$ is a log-normally distributed random variable with expected value, variance and covariance given by

$$\mathbb{E}[S(T)] = \mathbb{E}\left[e^{\ln(S(T))}\right] = e^{\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)T + \frac{1}{2}\sigma^2 T} = S_0 e^{\mu T}$$

$$Var[S(T)] = e^{2\left(\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)T + \frac{1}{2}\sigma^2 T\right) + 2\sigma^2 T} - S_0^2 e^{2\mu T} = S_0^2 e^{2\mu T}\left(e^{\sigma^2 T} - 1\right)$$

$$Cov[S(T_1), S(T_2)] = \mathbb{E}\left[S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T_1 + \sigma X(T_1)} S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T_2 + \sigma X(T_2)}\right] - S_0 e^{\mu T_1} S_0 e^{\mu T_2}$$

$$= S_0^2 e^{\mu(T_1 + T_2) - 0.5\sigma^2 T_1 - 0.5\sigma^2 T_2} \mathbb{E}\left[e^{\sigma X(T_1) + \sigma X(T_2)}\right] - S_0^2 e^{\mu(T_1 + T_2)}$$

$$= S_0^2 e^{\mu(T_1 + T_2) - 0.5\sigma^2 T_1 - 0.5\sigma^2 T_2} \mathbb{E}\left[e^{2\sigma X(T_2)}\right]\mathbb{E}\left[e^{\sigma\left(X(T_1) - X(T_2)\right)}\right] - S_0^2 e^{\mu(T_1 + T_2)}$$

$$= S_0^2 e^{\mu(T_1 + T_2) - 0.5\sigma^2 T_1 - 0.5\sigma^2 T_2} e^{0 + \frac{1}{2}\left(4\sigma^2 T_2\right)} e^{0 + \frac{1}{2}\sigma^2(T_1 - T_2)} - S_0^2 e^{\mu(T_1 + T_2)}$$

$$= S_0^2 e^{\mu(T_1 + T_2) + \sigma^2 T_2} - S_0^2 e^{\mu(T_1 + T_2)} = S_0^2 e^{\mu(T_1 + T_2)}\left(e^{\sigma^2 T_2} - 1\right).$$

3.4.3.3 Expected values of option prices at expiry

In the beginning of this section, we derived the expected value of a call option at the expiry time T, given that $f(\cdot)$ is the probability density function for the share price $S(T)$ at maturity

$$\bar{\mathrm{C}}\,(\mathrm{T}) = \int_0^\infty \max(\mathrm{x}-\mathrm{E})\mathrm{f}(\mathrm{x})\mathrm{dx} = \int_{\mathrm{E}}^\infty \mathrm{xf}(\mathrm{x})\mathrm{dx} - \mathrm{E}\int_{\mathrm{E}}^\infty \mathrm{f}(\mathrm{x})\mathrm{dx}$$

$$\bar{\mathrm{P}}\,(\mathrm{T}) = \int_0^\infty \max(\mathrm{E}-\mathrm{x})\mathrm{f}(\mathrm{x})\mathrm{dx} = \mathrm{E}\int_0^{\mathrm{E}} \mathrm{f}(\mathrm{x})\mathrm{dx} - \int_0^{\mathrm{E}} \mathrm{xf}(\mathrm{x})\mathrm{dx}\,.$$

Assume that $S(T)$ follows the geometric random walk with a log-normal density function given in the previous sub section $\mathrm{f}(\mathrm{x}) = \frac{1}{\mathrm{x}\sigma\sqrt{2\pi(\mathrm{T}-\mathrm{t}_0)}}\mathrm{e}^{-\frac{(\ln(\mathrm{x}/\mathrm{S}_0)-\lambda(\mathrm{T}-\mathrm{t}_0))^2}{2\sigma^2(\mathrm{T}-\mathrm{t}_0)}}$ where $\lambda = \mu - \frac{\sigma^2}{2}$. To evaluate $\bar{C}\,(T)$and $\bar{P}\,(T)$, the key is to solve the integrals in the equation. Recall the cumulative distribution function of the standard normal distribution $\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt$. Due to the symmetrical shape of the normal density about its mean value 0 , $\mathbb{P}(z \le -x) = \mathbb{P}(z \ge x) = 1 - \mathbb{P}(z \le x)$, hence $\mathcal{N}(-x) = 1 - \mathcal{N}(x)$. The following two lemmas aim to calculate the integrals in the formulae of share price at maturity.

Lemma 3. 2

If $f(\cdot)$ denotes the log-normal density, then

$$\int_{\mathrm{E}}^\infty \mathrm{f}(\mathrm{x})\mathrm{dx} = \mathcal{N}(\delta_2), \int_0^{\mathrm{E}} \mathrm{f}(\mathrm{x})\mathrm{dx} = \mathcal{N}(-\delta_2) \qquad (3.30)$$

where

$$\delta_2 = \frac{ln(S_0/E)+\left(\mu-\frac{1}{2}\sigma^2\right)(T-t_0)}{\sigma\sqrt{T-t_0}}. \qquad (3.31)$$

Proof:

$$\int_{\mathrm{E}}^\infty \mathrm{f}(\mathrm{x})\mathrm{dx} = \mathbb{P}\big(\mathrm{E} \le \mathrm{S}(\mathrm{T})\big)$$

$$= \mathbb{P}\left(\ln\mathrm{E} \le \ln\big(\mathrm{S}(\mathrm{T})\big)\right)$$

$$= \mathbb{P}\left(\frac{ln(E/S_0) - \lambda(T-t_0)}{\sigma\sqrt{T-t_0}} \le z\right)$$

$$= \mathbb{P}(-\delta_2 \le z) = \mathbb{P}(z \le \delta_2) = \mathcal{N}(\delta_2)$$

where $\lambda = \mu - \frac{\sigma^2}{2}$ and z denotes the standardised normal variable, which proves the first half of the lemma. For the second half, use the following relationship

$$\int_0^{\mathrm{E}} \mathrm{f}(\mathrm{x})\mathrm{dx} + \int_{\mathrm{E}}^\infty \mathrm{f}(\mathrm{x})\mathrm{dx} = \int_0^\infty \mathrm{f}(\mathrm{x})\mathrm{dx} = 1$$

and obtain

$$\int_0^E f(x)dx = 1 - \mathcal{N}(\delta_2) = \mathcal{N}(-\delta_2)$$

which completes the proof.

Lemma 3. 3

If $f(\cdot)$ denotes the log-normal density, then

$$\int_E^\infty xf(x)dx = S_0 e^{\mu(T-t_0)} \mathcal{N}(\delta_1), \int_0^E xf(x)dx = S_0 e^{\mu(T-t_0)} \mathcal{N}(-\delta_1) \quad (3.32)$$

where

$$\delta_1 = \delta_2 + \sigma\sqrt{T-t_0} = \frac{\ln(S_0/E) + \left(\mu + \frac{1}{2}\sigma^2\right)(T-t_0)}{\sigma\sqrt{T-t_0}} \quad (3.33)$$

$$\delta_2 = \frac{\ln(S_0/E) + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t_0)}{\sigma\sqrt{T-t_0}}. \quad (3.34)$$

Proof:

$$\int_E^\infty xf(x)dx = \frac{1}{\sigma\sqrt{2\pi(T-t_0)}} \int_E^\infty e^{-\frac{\left(\ln\left(\frac{x}{S_0}\right) - \left(\mu + \frac{1}{2}\sigma^2\right)(T-t_0)\right)^2}{2\sigma^2(T-t_0)}} dx.$$

Denote $\lambda = \mu - \frac{\sigma^2}{2}$ and changing the variable of integration to

$z = \frac{\ln(x/S_0) - \lambda(T-t_0)}{\sigma\sqrt{T-t_0}}$.

Rearranging the formula and denote $x = S_0 exp\left(\lambda(T-t_0) + z\sigma\sqrt{T-t_0}\right)$. Taking the derivative of x, we then get $dx = S_0\sigma\sqrt{T-t_0} exp\left(\lambda(T-t_0) + z\sigma\sqrt{T-t_0}\right)dz$. Consider the limit, when $x = E$,

$$z = \frac{\ln(E/S_0) - \lambda(T-t_0)}{\sigma\sqrt{T-t_0}} = -\delta_2$$

and $z \to \infty$ as $x \to \infty$. Therefore, changing the variable to z

$$\int_E^\infty xf(x)dx = \frac{1}{\sigma\sqrt{2\pi(T-t_0)}} \int_{-\delta_2}^\infty e^{-\frac{z^2}{2}} S_0\sigma\sqrt{T-t_0} e^{\lambda(T-t_0) + z\sigma\sqrt{T-t_0}} dz$$

$$= \frac{S_0}{\sqrt{2\pi}} \int_{-\delta_2}^\infty e^{-\frac{z^2}{2} + \lambda(T-t_0) + z\sigma\sqrt{T-t_0}} dz$$

$$= \frac{S_0}{\sqrt{2\pi}} \int_{-\delta_2}^\infty e^{-\frac{1}{2}\left(z - \sigma\sqrt{T-t_0}\right)^2 + \mu(T-t_0)} dz$$

$$= \frac{S_0 e^{\mu(T-t_0)}}{\sqrt{2\pi}} \int_{-\delta_2}^\infty e^{-\frac{1}{2}\left(z - \sigma\sqrt{T-t_0}\right)^2} dz.$$

Change the variable from z to $w = z - \sigma\sqrt{T - t_0}$, and $dw = dz$. Now the lower limit $z = -\delta_2 = -(\delta_2 - \sigma\sqrt{T - t_0}) = -\delta_1$.

$$\int_E^{\infty} xf(x)dx = \frac{S_0 e^{\mu(T-t_0)}}{\sqrt{2\pi}} \int_{-\delta_1}^{\infty} e^{-\frac{1}{2}w^2} dw$$

$$= S_0 e^{\mu(T-t_0)} \left(1 - \mathcal{N}(-\delta_1)\right)$$

$$= S_0 e^{\mu(T-t_0)} \mathcal{N}(\delta_1)$$

which proves the first half of the lemma.

The second half using the relationship below gives,

$$\int_0^E xf(x)dx + \int_E^{\infty} xf(x)dx = \int_0^{\infty} xf(x)dx$$

when $E \to 0$, so $-lnE \to +\infty$. Consequently, $\delta_1 \to +\infty$, thus $\lim_{E \to 0} \mathcal{N}(\delta_1) = 1$. Therefore,

$$\int_0^{\infty} xf(x)dx = S_0 e^{\mu(T-t_0)} \qquad \text{Hence,} \qquad \int_0^E xf(x)dx = S_0 e^{\mu(T-t_0)} - S_0 e^{\mu(T-t_0)} N(\delta_1)$$

$$= S_0 e^{\mu(T-t_0)} \left(1 - \mathcal{N}(\delta_1)\right)$$

$$= S_0 e^{\mu(T-t_0)} \mathcal{N}(-\delta_1)$$

which completes the proof.

Using Lemma 3. *2* and Lemma 3. 3, the expected values of call and put options at expiry $t = T$ are

$$\bar{C}(T) = S_0 e^{\mu(T-t_0)} \mathcal{N}(\delta_1) - E\mathcal{N}(\delta_2) \tag{3.35}$$

$$\bar{P}(T) = E\mathcal{N}(-\delta_2) - S_0 e^{\mu(T-t_0)} \mathcal{N}(-\delta_1). \tag{3.36}$$

The following theorem define the present values at time $t < T$ of the expected expiry values $\bar{C}(T)$ and $\bar{P}(T)$, denoted as $\hat{C}(S,t)$ and $\hat{P}(S,t)$.

Theorem 3. 2

At $t < T$, the present values of call and put options are

$$\hat{C}(S,t) = \left(S\mathcal{N}(\delta_1) - Ee^{-\mu(T-t)}\mathcal{N}(\delta_2)\right) e^{(\mu-r)(T-t)} \tag{3.37}$$

$$\hat{P}(S,t) = \left(Ee^{-\mu(T-t)}\mathcal{N}(-\delta_2) - S\mathcal{N}(-\delta_1)\right) e^{(\mu-r)(T-t)} \tag{3.38}$$

where

$$\delta_1 = \frac{\ln(S/E) + \left(\mu + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{3.39}$$

$$\delta_2 = \frac{\ln(S/E) + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = \delta_1 - \sigma\sqrt{T-t} \tag{3.40}$$

where r represents the continuously compounded risk-free interest rate.

It is easy to observe that the results in Theorem 3. 2 violate the non-arbitrage principle and the put-call parity condition. However, the structure is correct and needs a minor adjustment to produce option prices formulae that are consistent with the no-arbitrage principle. We will consider this issue more closely in the next section.

3.4.3.4 The Black-Scholes partial differential equation

The first satisfactory theory of option pricing was published by Black and Scholes (1973). The theory indicates that option prices must satisfy a given partial differential equation known as the Black-Scholes equation. The main assumptions and notation used for deriving this equation are:

- Options are of European style.
- The underlying asset price $S = S(t)$ follows the geometric random walk and follows log-normally distributed.
- r denotes the continuously compounded risk-free interest rate per unit time, assumed to be the same rate for borrowing and lending and constant over the lifetime of any option.
- The no-arbitrage principle holds.
- All assets may be sold short and no penalties on short selling.
- Continuous trading is allowed with respect to both time and asset amount.
- The underlying asset does not pay a dividend.
- No transaction costs or taxes.

All other symbols (E, r, μ, σ, T) have the meaning previously defined, in particular, denote the theoretically correct call and put option prices as

$$\mathrm{C} = \mathrm{C}(\mathrm{S},\mathrm{t}) = \mathrm{C}(\mathrm{S},\mathrm{t};\mathrm{E},\mathrm{r},\dots,\mathrm{T})$$

$$\mathrm{P} = \mathrm{P}(\mathrm{S},\mathrm{t}) = \mathrm{P}(\mathrm{S},\mathrm{t};\mathrm{E},\mathrm{r},\dots,\mathrm{T}).$$

Use $V = V(S,t)$ to denote both call and put option prices. At time t, open a portfolio consists of $-\Delta$ units of option of the underlying asset. (If $\Delta < 0$ then the asset is held long, the opening transaction of an investor is a buy; if $\Delta > 0$ then the asset is held short, the opening transaction of an investor is a sell). The value of this portfolio Π at the initial time t is

$$\Pi(\mathrm{S},\mathrm{t}) = \mathrm{V}(\mathrm{S},\mathrm{t}) - \Delta\mathrm{S}. \qquad (3.41)$$

The change in the value of this portfolio $d\Pi$ can be derived using Itô's lemma (Lemma 3. 1) as

$$\mathrm{d}\Pi = \frac{\partial\Pi}{\partial\mathrm{S}}\mathrm{dS} + \frac{\partial\Pi}{\partial\mathrm{t}}\mathrm{dt} + \frac{1}{2}\sigma^2\mathrm{S}^2\frac{\partial^2\Pi}{\partial\mathrm{S}^2}\mathrm{dt}$$

$$= \left(\frac{\partial V}{\partial S} - \Delta\right) dS + \frac{\partial V}{\partial t} dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt.$$

Note that the above expression for $d\Pi$ follows the assumption that the underlying asset does not pay a dividend. If dividend is paid then it would need to be taken into account when determining the increase in value of the portfolio (see Section 3.5.1). Observe that in the above expression, for $d\Pi$, all partial derivatives are evaluated at time t and asset price $S = S(t)$. All terms in this expression are known at time t, apart from the differential dS, which is a random quantity. However, this random term to $d\Pi$ can be removed by choosing

$$\Delta = \frac{\partial V}{\partial S}. \tag{3.42}$$

This choice is perfectly possible since $\frac{\partial V}{\partial S}$ is evaluated at time t and hence is known at the time when the portfolio is opened. Therefore, the change in value of the portfolio is $d\Pi = \frac{\partial V}{\partial t} dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt$ which is completely determined at time t and hence, is a risk-free increase over the time interval $[t, t + dt]$.

An alternative risk-free use for the initial amount $\Pi(S, t)$ is to invest at the market-risk free rate r and the investment will increase by $\Pi(S, t)rdt$ over the time interval $[t, t + dt]$. According to the no-arbitrage principle, two alternative risk-free investments produce the same income, hence

$$\frac{\partial V}{\partial t} dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt = \Pi(S, t)rdt = \left(V - \frac{\partial V}{\partial S} S\right) rdt\,.$$

Cancelling out dt, we have the Black-Scholes partial differential equation as

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - rV = 0. \tag{3.43}$$

Therefore, the theoretical prices of the call and put options must be the solutions to the Black-Scholes partial differential equation. Finally, we can define the Black-Scholes differential operator $\mathcal{L} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} + \frac{\partial}{\partial t} - r$.

3.4.3.5 No-Arbitrage Argument and Boundary conditions

No-arbitrage argument for the Black-Scholes equation only holds for European style options (see section below for the American style option). Suppose that $\Pi rdt < d\Pi$, an investor can gain risk-free profit by doing the following:

- At time t, borrow Π from the bank and pay risk-free rate r. Then, use the capital to buy portfolio.
- At time $t + dt$, sell the portfolio and receive $\Pi + d\Pi$; repay back $\Pi + \Pi rdt$ to the bank.

The instantaneous profit is $d\Pi - \Pi r dt > 0$ which contradicts the non-arbitrage principle. Now suppose $\Pi \mathrm{rdt} > d\Pi$, the investor can gain risk-free profit by doing the following:

- At time t, short the portfolio with income Π; invest the fund at the risk-free market at rate r.
- At time $t + dt$, sell the investment and receive $\Pi + \Pi r dt$; buy back the portfolio for $\Pi + d\Pi$.

The instantaneous profit is $\Pi r dt - d\Pi > 0$ which contradicts the non-arbitrage principle. Thus, we proved $\Pi r dt = d\Pi$.

Similarly, as any other regular partial differential equation, the Black-Scholes equation has an infinite number of possible solutions. One shall specify a domain and boundary condition to determine a unique solution.

Consider a call option, where $V = C = C(S,t)$. At $t = T$, $C(S,T) = max(S - E, 0)$; at $S = 0$, $C(0,t) = 0$; at $S = \infty$, using the put-call parity and given that $\lim_{S \to \infty} P(S,t) = 0$, $C(S,T) \to S - Ee^{-r(T-t)}$ as $S \to \infty$. Simplify the results

$$\lim_{S \to \infty} \frac{C(S,t)}{S} = 1$$

Lemma 3. 4

The value of call option price $C(S,t)$ is the solution to the Black-Scholes equation

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - rC = 0 \quad (3.44)$$

which satisfies the boundary conditions

$$C(0,t) = 0, C(S,T) = \max(S - E, 0), \lim_{S \to \infty} \frac{C(S,t)}{S} = 1 \quad (3.45)$$

on the domain $0 < S < \infty, t < T$.

Similarly, the value and the boundary conditions of a put option can be found using put-call parity.

Lemma 3. 5

The value of put option price $P(S,t)$ is the solution to the Black-Scholes equation

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} + \frac{\partial P}{\partial t} - rP = 0 \quad (3.46)$$

which satisfies the boundary conditions

$$P(0,t) = Ee^{-r(T-t)}, P(S,T) = \max(E - S, 0), \lim_{S \to \infty} P(S,t) = 0 \quad (3.47)$$

on the domain $0 < S < \infty, t < T$.

3.4.3.6 Black-Scholes Formulae for Option Pricing

We now need to check whether the present value formulae for the option values formula $\hat{C}(S,t)$ and $\hat{P}(S,t)$ given in Theorem 3. 2 are consistent with Black–Scholes. To examine this, we first collect some technical results which we put together in the end.

Lemma 3. 6

With respect to the notation in Theorem 3. 2

$$S\mathcal{N}'(\delta_1) = Ee^{-\mu(T-t_0)}\mathcal{N}'(\delta_2).$$

Proof:

$\delta_1 - \delta_2 = \sigma\sqrt{T-t}$ and $\delta_1 + \delta_2 = 2\left(\frac{\ln(S/E)+\mu(T-t)}{\sigma\sqrt{T-t}}\right)$. Hence

$$\frac{(\delta_1-\delta_2)(\delta_1+\delta_2)}{2} = \ln(S/E) + \mu(T-t).$$

Therefore, $exp\left(\frac{1}{2}\delta_1^2 - \frac{1}{2}\delta_2^2\right) = \frac{S}{E}exp\big(\mu(T-t)\big)$, re-arranging the formula and obtain $Ee^{-\mu(T-t)}e^{-\frac{1}{2}\delta_2^2} = Se^{-\frac{1}{2}\delta_1^2}$. Recall that $\mathcal{N}'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, which proves the lemma.

Lemma 3. 7

With respect to the notation in Theorem 3. 2

$$\frac{\partial\hat{C}}{\partial S} = e^{(\mu-r)(T-t)}\mathcal{N}(\delta_1)$$

$$\frac{\partial^2\hat{C}}{\partial S^2} = \frac{e^{(\mu-r)(T-t)}\mathcal{N}'(\delta_1)}{S\sigma\sqrt{T-t}}$$

$$\frac{\partial\hat{C}}{\partial t} = -Se^{(\mu-r)(T-t)}\left(\frac{\sigma\mathcal{N}'(\delta_1)}{2\sqrt{T-t}} - \mu\mathcal{N}(\delta_1)\right) + r\hat{C}.$$

Proof:

Recall that $\hat{C}(S,t) = \left(S\mathcal{N}(\delta_1) - Ee^{-\mu(T-t_0)}\mathcal{N}(\delta_2)\right)e^{(\mu-r)(T-t)}$, thus

$$\frac{\partial\hat{C}}{\partial S} = e^{(\mu-r)(T-t)}(\mathcal{N}(\delta_1) + S\mathcal{N}'(\delta_1)\frac{\partial\delta_1}{\partial S} - Ee^{-\mu(T-t_0)}\mathcal{N}'(\delta_2)\frac{\partial\delta_2}{\partial S})$$

with Itô's lemma (Lemma 3. 1), it becomes

$$\frac{\partial\hat{C}}{\partial S} = e^{(\mu-r)(T-t)}(\mathcal{N}(\delta_1) + S\mathcal{N}'(\delta_1)\frac{\partial}{\partial S}(\delta_1 - \delta_2)).$$

However, $\delta_1 - \delta_2 = \sigma\sqrt{T-t}$ does not depend on S, therefore, $\frac{\partial}{\partial S}(\delta_1 - \delta_2) = 0$, which proves the first equation in the lemma. Differentiating with respect to S gives

$$\frac{\partial^2 \hat{C}}{\partial S^2} = e^{(\mu-r)(T-t)} \mathcal{N}'(\delta_1) \frac{\partial \delta_1}{\partial S}.$$

Recall that $\delta_1 = \frac{ln(S/E)+\left(\mu+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$, thus $\frac{\partial \delta_1}{\partial S} = \frac{1}{\sigma S\sqrt{T-t}}$, which proves the second formula. Differentiating $\hat{C}$ with respect to t,

$$\frac{\partial \hat{C}}{\partial t} = e^{(\mu-r)(T-t)} \left(S\mathcal{N}'(\delta_1)\frac{\partial \delta_1}{\partial t} - Ee^{-\mu(T-t_0)}\mathcal{N}'(\delta_2)\frac{\partial \delta_2}{\partial t} - \mu Ee^{-\mu(T-t_0)}\mathcal{N}(\delta_2) \right)$$
$$- (\mu - r)\hat{C}$$
$$= e^{(\mu-r)(T-t)} \left(S\mathcal{N}'(\delta_1)\frac{\partial}{\partial t}(\delta_1 - \delta_2) - \mu Ee^{-\mu(T-t_0)}\mathcal{N}(\delta_2) \right) - (\mu - r)\hat{C}.$$

Here, $\frac{\partial}{\partial t}(\delta_1 - \delta_2) = -\frac{\sigma}{2\sqrt{T-t}}$. Multiple μ to both sides of the present value formula $\hat{C}$ then rearrange as $\mu E e^{-\mu(T-t_0)}\mathcal{N}(\delta_2) = \mu S\mathcal{N}(\delta_1)e^{(\mu-r)(T-t)} - \mu\hat{C}$, which proves the final formula.

Theorem 3. 3

$\hat{C}(S,t)$ satisfies the partial differential equation

$$\mathcal{L}\hat{C} = (r-\mu)S\frac{\partial \hat{C}}{\partial S}$$

where $\mathcal{L} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} + \frac{\partial}{\partial t} - r$ is the Black-Scholes differential operator.

Proof:

Applying Lemma 3. 7

$$\mathcal{L}\hat{C} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \hat{C}}{\partial S^2} + rS\frac{\partial \hat{C}}{\partial S} + \frac{\partial \hat{C}}{\partial t} - r\hat{C}$$
$$= \frac{1}{2}\sigma^2 S^2 \frac{e^{(\mu-r)(T-t)}\mathcal{N}'(\delta_1)}{S\sigma\sqrt{T-t}} + rSe^{(\mu-r)(T-t)}\mathcal{N}(\delta_1)$$
$$+ \left(-Se^{(\mu-r)(T-t)} \left(\frac{\sigma\,\mathcal{N}'(\delta_1)}{2\sqrt{T-t}} - \mu\mathcal{N}(\delta_1) \right) + r\hat{C} \right) - r\hat{C}$$
$$= (r-\mu)Se^{(\mu-r)(T-t)}\mathcal{N}(\delta_1)$$
$$= (r-\mu)S\frac{\partial \hat{C}}{\partial S}$$

which proves the theorem.

Hence, by endorsing the assumptions of the Black–Scholes theory we can conclude that the present value formulae $\hat{C}(S,t)$ and $\hat{P}(S,t)$ do not constitute valid option prices, as they do not satisfy the Black–Scholes equation. This is not surprising, since $\hat{C}(S,t)$ and $\hat{P}(S,t)$ violate put-call parity which is a consequence of the no-arbitrage principle. However, substituting $\mu = r$, then obtain the alternative expression of the call option $C(S,t) = S\mathcal{N}(d_1) - Ee^{-r(T-t_0)}\mathcal{N}(d_2)$, where d_1, d_2 correspond to δ_1, δ_2. The new expressions for call and put satisfy Black-Scholes differential equation $\mathcal{L}\hat{C} = 0$ and $\mathcal{L}\hat{P} = 0$.

The alternated expression $C(S,t)$ would result if the drift of the share price's geometric random walk is the same as the continuously compounded risk-free interest rate r, i.e., risk-neutral geometric random walk. This is simply an algebraic fact that $C(S,t) = S\mathcal{N}(d_1) - Ee^{-r(T-t_0)}\mathcal{N}(d_2)$ satisfies the Black-Scholes equation, but not assuming the drift μ is the same as the interest rate r. The Black-Scholes option pricing formula is derived by solving the Black-Scholes partial differential equation, subject to the boundary and terminal conditions.

Theorem 3. 4

The Black-Scholes option pricing formulae are

$$C(S,t) = S\mathcal{N}(d_1) - Ee^{-r(T-t)}\mathcal{N}(d_2) \tag{3.48}$$

$$P(S,t) = Ee^{-r(T-t)}\mathcal{N}(-d_2) - S\mathcal{N}(-d_1) \tag{3.49}$$

where

$$d_1 = \frac{\ln(S/E) + \left(r + \frac{1}{2}\sigma^2\right)(T - t_0)}{\sigma\sqrt{T - t_0}} \tag{3.50}$$

$$d_2 = \frac{\ln(S/E) + \left(r - \frac{1}{2}\sigma^2\right)(T - t_0)}{\sigma\sqrt{T - t_0}} \tag{3.51}$$

with the boundary conditions $C(0,t) = 0, C(S,T) = max(S - E, 0), \lim_{S \to \infty} \frac{C(S,t)}{S} = 1$ on the domain $0 < S < \infty, t < T$.

Theorem 3. 4 states that the correct option pricing formulae are the present values of the expected expiry values that would be obtained if the underlying asset price followed the risk-neutral geometric random walk. Black-Scholes option prices do not depend on the drift μ of the actual random walk followed by the asset price S.

Some useful results can be obtained by substituting $\mu = r$ in the results of Lemma 3. 6 and Lemma 3. 7:

$$S\mathcal{N}'(d_1) = Ee^{-r(T-t_0)}\mathcal{N}'(d_2) \tag{3.52}$$

$$\frac{\partial C}{\partial S} = \mathcal{N}(d_1) \tag{3.53}$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{\mathcal{N}'(d_1)}{S\sigma\sqrt{T - t}} \tag{3.54}$$

$$\frac{\partial C}{\partial t} = -S\left(\frac{\sigma \mathcal{N}'(d_1)}{2\sqrt{T-t}} - r\mathcal{N}(d_1)\right) + rC. \tag{3.55}$$

The fair value of an option is the present value of the expected payoff at expiry under a risk-neutral random walk for the underlying. The fact that it is the risk-neutral random walk that matters, is due to using a perfect hedge, or delta hedging strategy (see Section 3.7).

3.5 Limitations and extensions of option pricing models

The Black-Scholes model derive the value of options that can be exercised only at maturity and underlying assets do not pay dividends. In addition, options are estimated based on the assumption that underlying asset value does not change due to exercising the option. In practice, there are various extensions have been developed including allowing dividend payment and early exercise. This section provides adjustments to the Black-Scholes model and will consider some of the extensions of the options pricing.

3.5.1 Treatment of Dividends

The derivation of the Black-Scholes equation in the previous section assumes that the underlying share does not pay a dividend. However, in practice, the majority of quoted shares pay dividends, usually twice a year. Denote t_d as the time at the close of business immediately prior to the ex-dividend date, i.e., ex-dividend time. Officially registered shareholders at close of business prior to the ex-dividend date are entitled to receive the next dividend payment. Shareholders who purchase these shares after closing time do not receive the dividend. Immediately after the close of business prior to the ex-dividend date, the share price drops by an amount equal to the dividend per share. If this drop did not occur, then there might be an arbitrage opportunity involving buying the share at time $t_d - \epsilon$ (thereby, at least in theory, establishing a right to receive the dividend) and selling the share at time $t_d + \epsilon$. Consequently, call options become less valuable and put options become more valuable with higher expected dividend payments. This section will derive the option value when the European assumption is retained, and dividend paid discrete or continuously.

Denote the closing share price as $S_d = \lim_{t \to t_d - 0} S(t)$ and d_y as the dividend yield (d_y = Dividends/Current value of the asset), so the share price is discontinuous at $t = t_d$ and must drop by the same amount as the dividend payment $d_y S_d$. In other words,

$\lim_{t \to t_d+0} S(t) = (1 - d_y)S_d$. Let $V_d = V_d(S,t)$ denote the price of an option, $C_y = C_y(S,t)$ for call option and $P_y = P_y(S,t)$ for a put option. The owner of an option on a share has no entitlement to receive dividends. The value of the option must not be discontinuous across the dividend payment time, even though the underlying stock price vary due to the dividend payment. Otherwise, there will be an arbitrage opportunity at time t_d. Thus, the price of the option should be unaffected by a dividend and should be continuous at time $t = t_d$, $\lim_{t \to t_d-0} V(S,t) = \lim_{t \to t_d+0} V(S,t)$.

Assume that $t \geq t_d$ and that there is no other ex-dividend time in the lifetime of the option and the option price equals the standard non-dividend Black-Scholes formula $V(S,t)$ as previously derived. As $t \to t_d - 0$, so, $S \to S_d$ and as $t \to t_d + 0$, so $S \to (1 - d_y)S_d$. The continuity requirements require that the value of option while $t < t_d$ as, $V\left((1 - d_y)S, t\right)$. Thus, the value of an option V_d with a discrete dividend underlying share is $V_y(S,t) = \begin{cases} V\left((1 - d_y)S, t\right), t < t_d \\ V(S,t), t \geq t_d \end{cases}$.

Theorem 3. 5

The Black-Scholes option pricing formulae with a discrete dividend yield d_y paid at time t ($t < t_d$) are

$$C_y(S,t) = (1 - d_y)S\mathcal{N}(d_1) - Ee^{-r(T-t)}\mathcal{N}(d_2) \tag{3.56}$$

$$P_y(S,t) = Ee^{-r(T-t)}\mathcal{N}(-d_2) - (1 - d_y)S\mathcal{N}(-d_1) \tag{3.57}$$

with

$$d_1 = \frac{\ln\left((1-d_y)S/E\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t_0)}{\sigma\sqrt{T - t_0}} \tag{3.58}$$

$$d_2 = \frac{\ln\left((1-d_y)S/E\right) + \left(r - \frac{1}{2}\sigma^2\right)(T - t_0)}{\sigma\sqrt{T - t_0}}. \tag{3.59}$$

In the non-dividend case, the law of put-call parity states $S + P(S,t) = C(S,t) + Ee^{-r(T-t)}$.

The above is an algebraic identity which holds for all values of $S > 0$. In particular, if we replace the symbol S by the symbol $(1 - d_y)S$, we obtain

$$(1 - d_y)S + P\left((1 - d_y)S, t\right) = C\left((1 - \lambda)S, t\right) + Ee^{-r(T-t)}.$$

Therefore, the put-call parity law for the discrete dividend case, when $t < t_d$, is

$$(1 - d_y)S + P_y(S,t) = C_y(S,t) + Ee^{-r(T-t)}, \tag{3.60}$$

and when $t \geq t_d$

$$S + P_y(S,t) = C_y(S,t) + Ee^{-r(T-t)}. \tag{3.61}$$

Similarly, multiple dividend payments at times $t_1, t_2, \ldots$ can be derived by subtracting the discounted value of each dividend payment from the stock price. However, for stocks with stable dividend pay-out patterns, continuous payment provides decent approximations to observe option prices.

The main application of the continuous dividend payment is to the pricing of options on indices. For example, the FTSE100 index is constructed from the 100 largest companies by market capitalization in the U.K. Most of these companies pay a dividend twice a year. Assumes that there are about 50 working weeks in the year then, on average, four FTSE100 companies will be paying a dividend in any given week. Continuous payment of dividend is a reasonable assumption to model the indices with frequent dividend pay-out patterns and can be modelled as follows: Suppose that the underlying asset pays dividends at a constant rate D_y, also known as dividend yield. Thus, if S is the price of one unit of the underlying asset, then over the next time instant dt the dividend received are $D_y S dt$.

Assume that the asset price follows the geometric random walk $dS = \mu S dt + \sigma S dX - D_y S dt = (\mu - D_y) S dt + \sigma S dX$. Proceeding in the same fashion as in the derivation of the Black-Scholes partial differential equation to construct a portfolio $\Pi(S,t) = V_c(S,t) - \Delta S$, where $V_c = V_c(S,t)$ is the price of the option ($C_c = C_c(S,t)$ for call option, and $P_c = P_c(S,t)$ for put option) for the underlying asset with continuous dividend yield D_y. Hence, the increase in value of the portfolio over the next time instant is

$$d\Pi = dV_c - \Delta(dS + D_y S dt) = \left(\frac{\partial V_c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_c}{\partial S^2}\right) dt + \frac{\partial V_c}{\partial S} dS - \Delta dS - \Delta D_y S dt.$$

If invest the portfolio Π in a risk-free market, the return over the next time instant equals to $(V_c - \Delta S) r dt$. Pick $\Delta = \frac{\partial V_c}{\partial S}$. Then in order to make the portfolio instantaneously risk-free and on comparing this investment

$$d\Pi = \left(\frac{\partial V_c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_c}{\partial S^2}\right) dt - \Delta D_y S dt = r(V_c - \Delta S) dt$$

and finally leads to the following equation

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_c}{\partial S^2} + (r - D_y) S \frac{\partial V_c}{\partial S} + \frac{\partial V_c}{\partial t} - rV_c = 0.$$

Change of dependent variables V_c to $\widehat{V}$, where $V_c(S,t) = e^{-D_y(T-t)}\widehat{V}(S,t)$.

$\hat{V}$ satisfies the standard Black-Scholes equation with r replaced by $r - D_y$. V can then be determined by the reduction to the heat equation technique for finding the value of the option, and conclude $V_c(S,t) = V\left(e^{-D_y(T-t)}S, t\right)$.

Theorem 3. 6

The Black-Scholes option pricing formulae with a continuous dividend yield D_y are

$$C_c(S,t) = e^{-D_y(T-t)} S\mathcal{N}(d_1) - Ee^{-r(T-t)}\mathcal{N}(d_2) \tag{3.62}$$

$$P_c(S,t) = Ee^{-r(T-t)}\mathcal{N}(-d_2) - e^{-D_y(T-t)} S\mathcal{N}(-d_1) \tag{3.63}$$

with

$$d_1 = \frac{\ln(S/E) + \left(r - D_y + \frac{1}{2}\sigma^2\right)(T - t_0)}{\sigma\sqrt{T - t_0}} \tag{3.64}$$

$$d_2 = \frac{\ln(S/E) + \left(r - D_y - \frac{1}{2}\sigma^2\right)(T - t_0)}{\sigma\sqrt{T - t_0}}. \tag{3.65}$$

The relationship of non-dividend law of put-call parity $S + P(S,t) = C(S,t) + Ee^{-r(T-t)}$ and with S replaced by $e^{-D_y(T-t)}S$, can obtain $e^{-D_y(T-t)}S + P\left(e^{-D_y(T-t)}S, t\right) = C\left(e^{-D_y(T-t)}S, t\right) + Ee^{-r(T-t)}$. Hence, the option prices P_c and C_c satisfy the modified put-call parity law

$$e^{-D_y(T-t)}S + P_c(S,t) = C_c(S,t) + Ee^{-r(T-t)}. \tag{3.66}$$

3.5.2 Early exercise

The option pricing model discussed in the previous sections are designed to value options that can be exercised only at expiration. The American style option is more flexible than the European style option, which can be exercised at any time up to and including to the expiry date. Most options that we encounter in practice, especially option contracts, trade on futures exchanges and are mainly American style.

Unlike Black-Scholes formulae provide closed form solution to European options, the American option consider the specific path that the stock price follows. This makes it more difficult to value the American style option. However, European style option can be treated as a special case of the American-style options. For this reason, if there are both American and European style options available on the same underlying asset, investors would expect to pay more for the American style options for the possibility of early exercise, $C_a \geq C_e$ and $P_a \geq P_e$, where the subscript e represents the European style option and a represents the American style option. From no-arbitrage opportunity principle, $C_a(S,t) \geq max(S - E, 0)$. Otherwise, an investor may buy one call option costing C_a and immediately exercise the options at exercise price E. Afterwards, the investor may sell the share in the market at price S, and earn profit $S - E - C_a \geq 0$. Similarly, $P_a(S,t) \geq max(E - S, 0)$.

Owners of an American style call option can close their position by sell the option with income $C_a(S,t)$, or exercise the option with income $S - E$. Therefore, the investor can earn more from trading, than from the exercising the call options on this occasion. Nevertheless, the only circumstances which generate the same income are when $S \geq E$ and $C_a(S,t) = S - E$.

However, if $S \geq E$ when $t < T$

$$C_a(S,t) \geq C_e(S,t)$$

$$\geq \max(S - Ee^{-r(T-t)}, 0), \text{from put-call parity}$$

$$= S - Ee^{-r(T-t)} > S - E\,,$$

which means $C_a(S,t) = S - E$ does not hold for $t < T$. Therefore, early exercise of an American style call options is never desirable. The apparent flexibility of being able to exercise the American style call option at any time before expiry date is an illusion. The only rational action is to exercise the option on expiration. Thus, American style call option is basically a European style call option, that is

$$C_a = C_e \tag{3.67}$$

An investor of an American style put option can close their position by sell the option with income $P_a(S,t)$, or exercise the option with income $E - S$. Therefore, an investor can earn more from trading than from exercising put options. The only circumstances which generate the same income are when $S \leq E$ and $P_a(S,t) = E - S$

$$P_a(S,t) \geq P_e(S,t)$$

$$\geq \max(Ee^{-r(T-t)} - S, 0), \text{from put-call parity.}$$

It is possible for $P_a(S,t) = E - S$ to occur. Thus, early exercise of the American style put option may be possible in practice, and

$$P_a \geq P_e \tag{3.68}$$

Hence, early exercise will only be done by an investor holding a portfolio with a stock and put combination. The investor needs to also consider the transaction costs associated with making the decision of early exercise: selling the put will incur transaction cost while exercise option only involves delivery. Corresponding to the non-dividend payment put-call parity is

$$S + P_a(S,t) \geq S + P_e(S,t) \tag{3.69}$$

$$=e^{r(T-t)} + C_e(S,t)$$
$$= Ee^{-r(T-t)} + C_a(S,t).$$

However, American option produces path dependence in the option price, make it difficult to derive a closed-form solution for the valuation problem and need to involve techniques from numerical analysis such as Monte Carlo simulation.

3.5.3 Impact of exercise on underlying asset value

The assumption that the underlying asset value is uncorrelated with the exercising option price may not be true. If the exercise of call options issued by a specific firm but not the exchanges, known as warrants, a firm is obligated to issue new stock which bring new cash flows into the firm. Exercise warrants cause dilution of the stocks and affects the stock price. The Black-Scholes model can be modified to price the value of warrants with adjustment for dilution to the stock price

$$\text{Dilution-adjusted } S = \frac{Sn_S + Wn_W}{n_S + n_W}.$$

In the above equation, S represents the current value of the stock while n_S is the number of outstanding shares, and W represents the value of warrants. n_W is the number of outstanding warrants. The sum of $Sn_S + Wn_W$ reflects the market value of equity, when additional warrants are exercised, the number of outstanding shares increases. The reduction in stock value causes the call option value to be reduced. One can assume an initial value for the warrants and re-estimate the warrants' value until there is convergence.

3.5.4 Extensions of option pricing

Standard call or put options are often referred to as vanilla options. There is a huge range of more specialized options beyond these standard options, collectively known as exotic options. Moreover, the options encounter in financial markets take more complicated forms and are often on real assets rather than financial assets.

One of the more popular types of exotic option is capped or barrier option. Consider a simple call option with strike price K_1. In theory, the underlying asset prices can go up infinitely. Thus, there is no upper limit on the profit of a call option. In a capped call option, the investors are entitled to profits up to a specified amount but not above it. Assume that asset price capped at K_2 and the payoff of call option is $[0, K_2 - K_1]$. Note that if the underlying asset price reaches K_2 at any time during the option's life, the option will be exercised immediately. The asset price afterwards will not matter. The value of a capped call is always lower than the value of the same call without the cap and can be estimated as the difference of the values of call options with strike price K_1 and K_2.

Capped calls are part of a family of options called barrier options. There are eight different types of barrier option exist, namely:

$$\begin{cases} \text{Up} \\ \text{Down} \end{cases} \text{and} \begin{cases} \text{In} \\ \text{Out} \end{cases} \text{barrier} \begin{cases} \text{Call} \\ \text{Put} \end{cases}$$

In addition, investors can open a long contract (buy) or a short contract (sell) of any type of barrier options. Consider a down-and-in barrier put option where an investor prepares to bear the risks associated with normal levels of market volatility, but also need some protection against larger falls in the asset price:

1. The option has no rights unless the underlying asset price falls to a prescribed amount which is lower than the current asset price. This amount is known as the *down barrier* price.
2. If the asset price falls to or crossed the down barrier price, then the option acquires the same rights as a standard European style put option. That is the option rights are *knocked in* at the barrier. A knock-in option has no value until the underlying reaches a certain price.

A barrier option has a significantly cheaper price than the standard put option but can still give the same protection as a standard put option against large drop in asset price. A barrier option is often used by portfolio managers to hedge against losses on a long position. It is important to note that if the underlying asset drop below the barrier at any time during the option's life, the option is knocked in, and will remain there until expiration. Similarly, an up-and-out barrier put option has the following terms.

1. The option has the same rights as a vanilla European style put option unless the underlying asset price rises to a prescribed amount which is higher than the current asset price. This prescribed price amount is known as the *up-barrier* price.
2. If the asset price rises to or crosses the barrier, then the option rights are cancelled. In this case, we say that the option rights are *knocked out* at the barrier.

Knock-out options expire worthless if the underlying asset reaches a certain price at any time during the option’s life. This limits the profits for the holders and losses for the writer. In the case of a call (put) option, the knockout price is usually set below (above) the exercise price, and this option is called a down-and-out call (an up-and-out put) option. This type of option would provide the immediate protection of a put option with a cheaper price than a vanilla option, while there is a chance of being knocked out of the option which would make it worthless.

For a given barrier price, a given choice of call (put) option type and a given selection of up/down type there will be one option type that knocks in at the barrier and one that knocks out at the barrier. In such cases,

(In barrier option price) + (Out barrier option price) = Vanilla option price. This condition is usually known as in-out parity. The sketch of the proof may consider an investor who owns both the in-barrier option, and the out-barrier option. If the barrier is crossed, then the in option is cancelled and the out option has the same rights as vanilla option. The two-barrier option together equals to a vanilla option. On the other hand, if the barrier has been crossed, then the in option has acquired the same rights as vanilla option and the out option has no value; again, the two-barrier option together equals to a vanilla option. Hence, owning one in and one out-barrier options, gives an investor exactly the same rights as owning a vanilla option in both circumstances. Therefore, the sum of the out-barrier option prices must always equal to the price of the corresponding vanilla option.

Another popular type of options are compound options for which values derived from other options are not from underlying assets. Compound options are options that give an investor the right—not obligation—to buy another option at a specific price on or by a specific date. Those options can take any of four types: a call on a call (CoC), a call on a put (CoP or caput option), a put on a put (PoP), and a put on a call (PoC).

The holder of a compound call (CoC or CoP) option needs to pay the seller of the underlying option a premium, known as back fee, if they wish to exercise the option, called the overlying option. The compound option gives the investor some exposure to the put (or call) option now, but without the cost of paying for a long-term option right now. On the other hand, the premium is more expensive than a simple put (or call) option if they exercise the initial call option and receive the put (or call). PoP or PoC provide the right to sell a put or call as the underlying. These types of options are commonly used in foreign exchange and fixed-income markets, where investors can benefit from large leverage and cheaper initial investment.

In a simple vanilla option, the uncertainty is from the price of the underlying asset. Rainbow option is an option exposed to two or more sources of uncertainty. More generally, rainbow options are multi-asset options which take various other forms, and payoff depends on the assets sorted by their performance at maturity. This process is called best-of (worst-of) which only pays the best (respectively worst) performing asset of the basket. Rainbow options are often used to value natural resource deposits. For example, an undeveloped oil reserve is exposed to two sources of uncertainty – price of oil and quantity of oil in the reserve.

A Bermudan option is an option where the buyer has the right to exercise at a set (always discretely spaced) number of times. This is intermediate between a European option—which allows exercise at a single time, namely expiry and an American option, which allows exercise at any time (the name is jocular: Bermuda, a British overseas territory, is somewhat American and somewhat European—in terms of both option style and physical location—but is nearer to American in terms of both). For example, a typical Bermudian swaption might confer the opportunity to enter into an interest rate swap. The option holder might decide to enter into the swap at the first exercise date (and so enter into, say, a ten-year swap) or defer and have the opportunity to enter in six months (and so enter a nine-year and six-month swap). Most exotic interest rate options are of Bermudan style.

3.6 Volatility

Theoretically, Black-Scholes option prices depend on the primary random variables S, t the parameters E, r, σ, T. If dividends are involved, then the additional parameters t_d, d_y *or* D_y are necessary. At any given time t, the values of all variables and parameters are known and can be obtained directly from the financial news media, or other sources. The volatility parameter σ is not directly observed. There are two approaches to estimating the variance (volatility) σ^2, of the continuously compounded rate of return on the stock $\frac{dS}{S}$. One approach uses the historical time series data for the asset price S to compute σ which gives the so-called historic volatility σ_H, in terms of the standard deviation. The second approach takes the current market price of an option and determines what value σ should take in order for the theoretical price to match the market price. This parameter is called the implied standard deviation or implied volatility $\overline{\sigma}$.

3.6.1 Historic volatility

A fundamental assumption underlying the Black-Scholes theory is that the underlying asset price S follows a geometric random walk with log-normally distributed. In particular,

$$\ln\left(\frac{S(t)}{S(t_0)}\right) = \left(\mu - D_y - \frac{1}{2}\sigma^2\right)(t - t_0) + \sigma\sqrt{t - t_0}\mathcal{N}(0,1).$$

Denote $\eta = \mu - D_y - \frac{1}{2}\sigma^2$ and set $t_0 = t - 1$

$\ln\left(\frac{S(t)}{S(t-1)}\right) = \eta + \sigma\mathcal{N}(0,1).$

Thus, for a sequence of closing prices of the asset $S_1, S_2, \dots, S_n$ on n successive days, the unit of time is one day. Then daily return y_i is $y_i = \ln\left(\frac{S_{i+1}}{S_i}\right) = \eta + \sigma z_i,\ i = 1,2,\dots n-1$ where $z_1, z_2, \dots, z_{n-1}$ is an arbitrary sequence of independent random numbers from the standard normal population $\mathcal{N}(0,1)$.

According to the ordinary least square theory, the best estimate for η is the mean of the daily return $\bar{y}$,

$$\eta \approx \bar{y} = \frac{\sum_{i=1}^{n-1} y_i}{n-1}$$
$$= \frac{1}{n-1}\left(\ln\left(\frac{S_2}{S_1}\right) + \ln\left(\frac{S_3}{S_2}\right) + \cdots + \ln\left(\frac{S_n}{S_{n-1}}\right)\right) = \frac{1}{n-1}\ln\left(\frac{S_2 S_3 \dots S_n}{S_1 S_2 \dots S_{n-1}}\right)$$
$$= \frac{1}{n-1}\ln\left(\frac{S_n}{S_1}\right). \qquad (3.70)$$

Similarly, the best estimate for σ is the standard deviation of the continuously compounded daily return, which is the daily historic volatility σ_H

$$\sigma^2 \approx \sigma_H = \frac{\sum_{i=1}^{n-1}(y_i - \bar{y})^2}{n-2}$$
$$= \frac{n-1}{n-2}\left(\frac{1}{n-1}\sum_{i=1}^{n-1} y_i^2 - \bar{y}^2\right). \qquad (3.71)$$

Here σ_H is measuring in $\text{days}^{-1/2}$. Assuming 252 trading days, the corresponding annualised mean return is $252\bar{y}$, while the historic volatility is $\sqrt{252}\sigma_H$ in $\text{years}^{-1/2}$.

3.6.2 Implied Volatility

Using the historic volatility to produce theoretical Black-Scholes option prices can be inconsistent with observed option prices. With given market option price, Black-Scholes equation can be inverted and produce estimate of volatility. Suppose that the market price of a certain call option is C_0. The implied volatility $\bar{\sigma}$ for this call option is the solution to the non-linear equation $C(S, t; E, r, \bar{\sigma}, T) = C_0$, where S, t, E, r, T are all assumed to be known and the underlying asset does not pay a dividend. If dividends are involved, then the appropriate function C_d or C_c must be used to replace C. Similarly, one can compute an implied volatility from the market price of a put option. The implied volatility is the particular value of the volatility that forces the theoretical option price to exactly match its market price, and both C and P increase monotonically with respect to $\bar{\sigma}$. However, the equation cannot be solved analytically with closed form solutions and must be solved by numerical methods or approximation techniques.

Because the only unobserved variable in the traded option price is volatility, Black-Scholes is more appropriately treated as an equation to determine the estimation of the underlying spot price volatility. In the foreign exchange market, the dealers' metric for the exercise price are the delta and implied volatility when trading an option. The volatility smile is implied the volatility patterns of the option with same underlying asset, and with same maturity but different deltas.

Implied volatility can be used to predict future volatility. Market makers can use econometric volatility predictor such as in GARCH models (see Bollerslev, 1986) to capture volatility or to calculate value at risk (VaR). Using GARCH to estimate expected volatility is a flexible approach as the estimation can be performed under different distributional assumptions. Several studies propose using (weighted) averages of past implied volatilities to capture implied volatility. Moreover, in practice, implied volatilities derived from in, at and out-of-the-money options differ; implied volatilities from at-the-money options are better predictors than deep out-of-the-money and deep in-of-the-money options.

In theory, implied volatilities calculated from call or put options should be the same, using the Black-Scholes model. If there is a difference, then there is an indication of mispricing, either that the market has placed too high a price on the put or too low a price on the call, or perhaps a combination of both. The Black-Scholes model also has a significant weakness, by assuming that the return distribution is normal. In reality, returns are not normally distributed particularly at high frequency. That is, the Black-Scholes model fails to capture the volatility clustering and fat-tailedness which are stylized features of returns. Heston (1993) develops a closed-form solution for stochastic volatility with application to Black-Scholes. Specifically, Heston's (1993) model contains a volatility parameter which increases with the degree of kurtosis in returns, thereby providing a better fit to the data compared to the Black-Scholes model. Using the analytical approach of Drăgulescu and Yakovenko (2002) for computing the probability density function for the Heston model, Daniel et al. (2005) show that the Heston model outperforms the Black-Scholes model, particularly at higher data frequency. Even so, the Heston model does not provide the best fit to the data.

3.7 Hedging strategy

3.7.1 Hedging

Derivatives, including options, may be used to protect an exposure against adverse price movements (or volatilities) of the underlying asset. Hedging is a risk management strategy to limit risk and protecting the value of the underlying financial assets or liabilities. Consider the simplest approach to protect an existing shareholding using a long put option. This strategy is known as the protective put strategy. The combined value of these investment $V = V(S,t)$ is $\mathrm{V(S,t) = S + P(S,t)}$.

If S decreases then $P(S,t)$ will increase reversely, and the investor hoping the combined value $V(S,t)$ will not fluctuate very much. However, only at the time of expiry, the investor definitely knows investment value. Denote $S = S(T)$ the share price at the expiry time then, $\mathrm{V(S,T) = S + P(S,T) = S + \max(E - S, 0)} \ = \begin{cases} \mathrm{E}, & \mathrm{S \leq E} \\ \mathrm{S}, & \mathrm{E < S} \end{cases}$.
Thus, no matter how small S may become, the total value at expiry will not fall below the exercise price E, which offsets the risks.

3.7.2 Delta-Neutral hedging

Assume a European style option with the use of the Black-Scholes formulae. Shares in an individual firm can be hedged either by buying a number of underlying equity put options (hedging with long puts) or by writing call options (hedging with short calls). A combination of long puts and short calls is also possible. If a portfolio includes shares in a variety of different companies then the value of this investment may be highly correlated with an index number, and thus this portfolio of shares may be hedged using index options.

Suppose an investor wishes to hedge N shares by buying a certain number, n, of put options. Denote the value of this portfolio at any time t as $\Pi(\mathrm{S,t}) = \mathrm{nP(S,t) + NS}$. Similar, assume the stock S follows a geometric Brownian motion. The portfolio value change is $\mathrm{d}\Pi(\mathrm{S,t}) = \frac{\partial \Pi}{\partial \mathrm{S}}\mathrm{dS} + \frac{\partial \Pi}{\partial \mathrm{t}}\mathrm{dt} + \frac{1}{2}\sigma^2 \mathrm{S}^2 \frac{\partial^2 \Pi}{\partial \mathrm{S}^2}\mathrm{dt}$.

The random component in $d\Pi$ can be eliminated if we set $\frac{\partial \Pi}{\partial \mathrm{S}} = 0$. The quantity of $\frac{\partial \Pi}{\partial \mathrm{S}}$ is called the delta of the portfolio, while the portfolio satisfies called a delta neutral portfolio. A delta neutral portfolio is instantaneously risk-free and its value Π

is insensitive to small changes in the underlying share price S. Therefore, a good hedging strategy is to construct a portfolio that is delta-neutral $\frac{\partial \Pi}{\partial S} = n\frac{\partial P}{\partial S} + N = 0$. Thus, the delta neutral portfolio requires the number of options n should satisfy

$$n = -\frac{N}{\frac{\partial P}{\partial S}} . \tag{3.72}$$

Here, $\frac{\partial P}{\partial S}$ is put option deltas. If dividends are involved, then P shall be substituted as P_d or P_c for the option price.

Suppose an investor wishes to hedge N shares of stock by writing a certain number, m, of call options. Denote the value of this portfolio at any time t as $\Pi(S,t) = -mC(S,t) + NS$. Clearly $\frac{\partial \Pi}{\partial S} = -m\frac{\partial C}{\partial S} + N = 0$, so that a delta neutral portfolio requires the number of options m should satisfy

$$m = \frac{N}{\frac{\partial C}{\partial S}}. \tag{3.73}$$

Here $\frac{\partial C}{\partial S}$ is the call option delta. If dividends are involved, then C shall be substituted as C_d or C_c for the option price.

Suppose an investor wish to hedge N shares of stock by buying n put options and writing m call options. Denote the value of this portfolio at any time t as $\Pi(S,t) = nP(S,t) - mC(S,t) + NS$. Clearly, $\frac{\partial \Pi}{\partial S} = n\frac{\partial P}{\partial S} - m\frac{\partial C}{\partial S} + N$ so that a delta neutral portfolio requires the number of n and m should satisfy

$$n\frac{\partial P}{\partial S} - m\frac{\partial C}{\partial S} + N = 0 \tag{3.74}$$

There is no unique solution for n and m. Investors may wish to arrange for the income initially generated by writing the call options to approximately cover the cost of buying the put option. If dividends are involved than P shall be substituted as P_d or P_c for the put option price, and C shall be substituted as C_d or C_c for the call option price. As is derived in Section 3.4.3.6, the option deltas $\frac{\partial C}{\partial S} = \mathcal{N}(d_1)$ and $\frac{\partial P}{\partial S} = -\mathcal{N}(-d_1) = \mathcal{N}(d_1) - 1$.

As a result of the put-call parity, $1 + \frac{\partial P}{\partial S} = \frac{\partial C}{\partial S}$. The put option delta is always negative, so the number of options n is positive.

If dividends are paid discretely then the option deltas will differ from results above only if $t < t_d$. In this case, recall that $V_d(S,t) = V\left(\left(1-d_y\right)S,t\right),\ \ t < t_d$. Hence the chain rule gives $\frac{\partial V_d}{\partial S}(S,t) = \left(1-d_y\right)\frac{\partial V}{\partial S}\left(\left(1-d_y\right)S,t\right)$.

Therefore, the option deltas with discrete dividend are $\frac{\partial C_d}{\partial S} = \left(1-d_y\right)\mathcal{N}(d_1)$ and $\frac{\partial P_d}{\partial S} = -\left(1-d_y\right)\mathcal{N}(-d_1) = \left(1-d_y\right)(\mathcal{N}(d_1)-1)$, where d_1 need to be evaluated by substitute S with $\left(1-d_y\right)S$. Therefore, the put-call parity gives $1-d_y+\frac{\partial P_d}{\partial S} = \frac{\partial C_d}{\partial S}$.

On the other hand, recall the continuous dividend $V_c(S,t) = V\left(e^{-D_y(T-t)}S,t\right)$. Appling the chain rule and have $\frac{\partial V_c}{\partial S}(S,t) = e^{-D_y(T-t)}\frac{\partial V}{\partial S}\left(e^{-D_y(T-t)}S,t\right)$. Therefore, the option deltas with continuous dividend are $\frac{\partial C_d}{\partial S} = e^{-D_y(T-t)}\mathcal{N}(d_1)$ and $\frac{\partial P_d}{\partial S} = -e^{-D_y(T-t)}\mathcal{N}(-d_1) = e^{-D_y(T-t)}(\mathcal{N}(d_1)-1)$.

Therefore, the put-call parity gives

$$e^{-D_y(T-t)}+\frac{\partial P_c}{\partial S} = \frac{\partial C_c}{\partial S}. \tag{3.75}$$

3.7.3 Dynamic hedging

Delta-hedging is not a perfect hedge if you do not hedge continuously. While a linear approximation to the option value can be obtained, convexity implies that second-order derivatives matter, such that the delta hedge is more effective for smaller price changes. The option delta is itself a function of t. Hence, the ideal value of n (or m) is determined as a function of t. Thus, in order to maintain a delta neutral portfolio, n (m) needs to be recalculated over the time which implies buying or selling option contracts. This is known as rebalancing the portfolio. Delta neutral hedging with regular rebalancing is known as dynamic hedging.

For delta hedging with long puts, if the underlying asset price rises, then additional put contracts will need to be purchased. If the asset price falls, then put contracts already owned may be sold. Because dealers routinely dynamically hedge their option positions, this means that they do not bet on the direction of the underlying. Instead, they are betting on the direction of volatility.

3.8 Conclusion

The chapter provided a detailed account of financial options and their applications for speculative and hedging strategies under different economic conditions. We show how various analytical solutions can be derived using the Black-

Scholes approach. We also provide extensions on the standard option pricing model. Financial options have become common place both for speculative and hedging purposes. We emphasise differences between the American and European style options and the conditions under when early exercise of an American style option is likely to be profitable. We show how many of the components of financial options are derived and how they relate to pricing under different economic conditions.

BIBLIOGRAPHY

Abraham, S. A. (2019). *The History of Options Contracts* https://www.investopedia.com/articles/optioninvestor/10/history-options-futures.asp (Accessed, February 15, 2021).

Bachelier, L. (1900). Théorie de la Spéculation. Paris: Gauthier-Villars. Translated by A. J. Boness. In: *The Random Character of Stock Market Prices*. (ed.) P. Cootner. Cambridge: MIT Press.

Black, F., and Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81, 637-654.

Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, 31, 307-327.

Daniel, G., Joseph, N.L., and Brée, D. S. (2005). Stochastic Volatility and the Goodness-of-fit of the Heston model. *Quantitative Finance*, 5, 199-211.

Drăgulescu, A., and Yakovenko, V. (2002). Probability Distribution of Returns in the Heston Model with Stochastic Volatility. *Quantitative Finance*, 2, 443–453.

Einstein, A. (1905). Über die von der Molekularkinetischen Theorie der Wärme geforderte Bewegung von in Ruhenden Flüssigkeiten Suspendierten Teilchen. Annalen der. *Physik*, 4.

Fischer, S. (1975). The Demand for Index Bonds. *Journal of Political Economy*, 83, 509-534.

Géczy, C. C., Minton, B. A., and Schrand, C. M. (2007). Taking a view: Corporate Speculation, Governance, and Compensation. *Journal of Finance*, 62, 2405–2443.

Hakkarainen, A., Joseph, N., Kasanen, E., and Puttonen, V. (1998). The Foreign Exchange Exposure Management Practices of Finnish Industrial Firms. *Journal of International Financial Management and Accounting*, 9, 34-57.

Heston, S. (1993). A Closed Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *Review of Financial Studies*, 6, 327–343.

Itô, K. (1944). Stochastic Integral. *Proceedings of the Imperial Academy*, 20, 519-524.

Itô, K. (1951). On Stochastic Differential Equations. *American Mathematical Society*, 4, 1–51.

Jilling, M. (1978). *Foreign Exchange Risk Management in U.S. Multinational Corporations (Research for business decisions).* UMI Research Press: U.S.

Joseph, N. L. (2000). The Choice of Hedging Techniques and the Characteristics of UK Industrial Firms. *Journal of Multinational Financial Management*, 10, 161–184.

Mackay, C. (1841). *Extraordinary Popular Delusions and the Madness of Crowds*. London, Ch. 3.

Merton, R. C. (1973). Theory of Rational Option Pricing. *Bell Journal of Economics and Management Science*, 4, 141-183.

Merton, R. C. (1975). Optimum Consumption and Portfolio Rules in a Continuous-time Model. In: *Stochastic Optimization Models in Finance* (pp. 621-661). Academic Press.

Milanfar, P. (1996). A Persian Folk Method of Figuring Interest. *Mathematics Magazine*, 69, 376-376.

Pearson, K. (1905). The Problem of the Random Walk. *Nature*, 72(1867), 342-342.

Pegolotti, F. B. (1936). *La Pratica Della Mercatura*. (ed.) A. Evans: Massachusetts, Cambridge, pp. 25, 40-1.

Reichert, S. (2019). e is everywhere. *Nature Physics,* 15**,** 982.

Samuelsson (2021). Biggest futures exchange in the world-2020 (List). *The Robust Trader*. Available at: https://therobusttrader.com/largest-futures-exchanges-in-the-world/ (Accessed January 13, 2021).

Scott, G. (2019). *London International Financial Futures and Options Exchange (LIFFE)*. https://www.investopedia.com/terms/l/liffe.asp (Accessed February 15, 2021).

Statista. (2021). Largest Stock Exchange Operators Worldwide as of March 2020, by Market Capitalization of Listed Companies. Available at:https://www.statista.com/statistics/270126/largest-stock-exchange-operators-by-market-capitalization-of-listed-companies/ (Accessed, February 24, 2021)

The Economic Times. (2021). NSE remains world's largest derivatives exchange for 2nd consecutive year Available at: https://economictimes.indiatimes.com/markets/stocks/news/nse-remains-worlds-largest-derivatives-exchange-for-2nd-consecutive-year/articleshow/80391771.cms (Accessed February 24, 2021)

Thompson, E.A. (2007). The Tulipmania: Fact or Artifact? Public Choice, 130, 1-2, 99-114.

Figure 3. 1: The four options opening positions

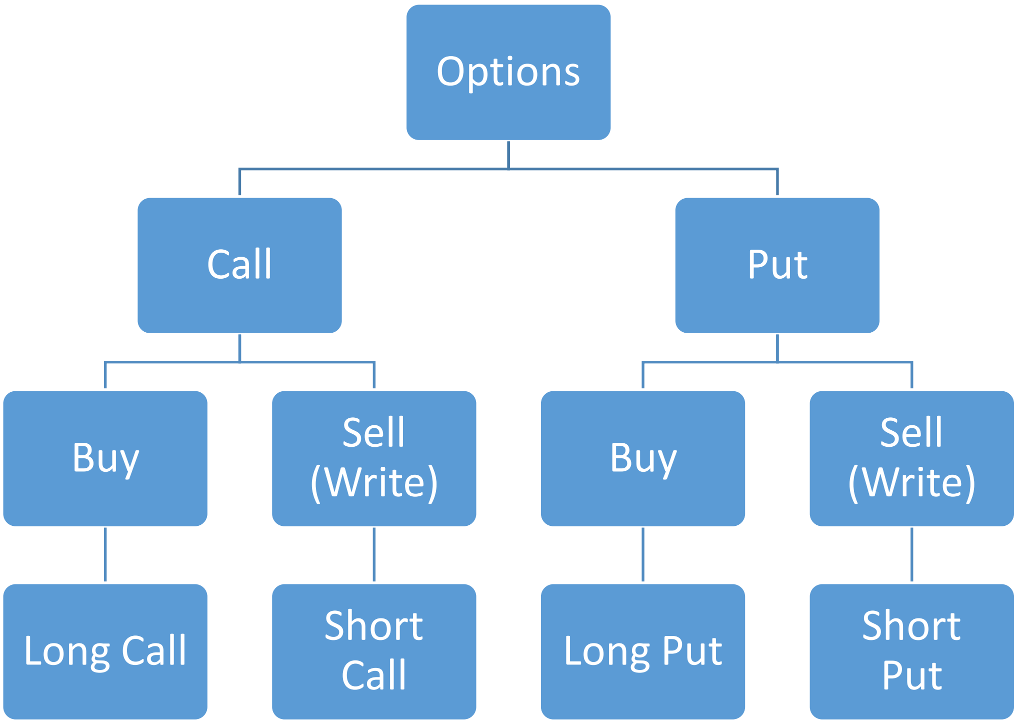

Figure 3. 2: Payoff from long a call option

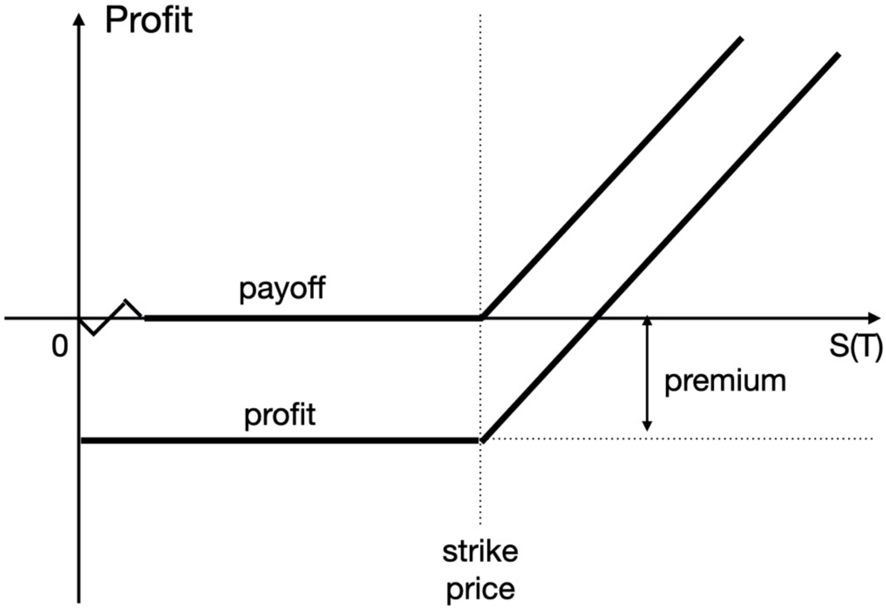

Figure 3. 3: Payoff from short a call option

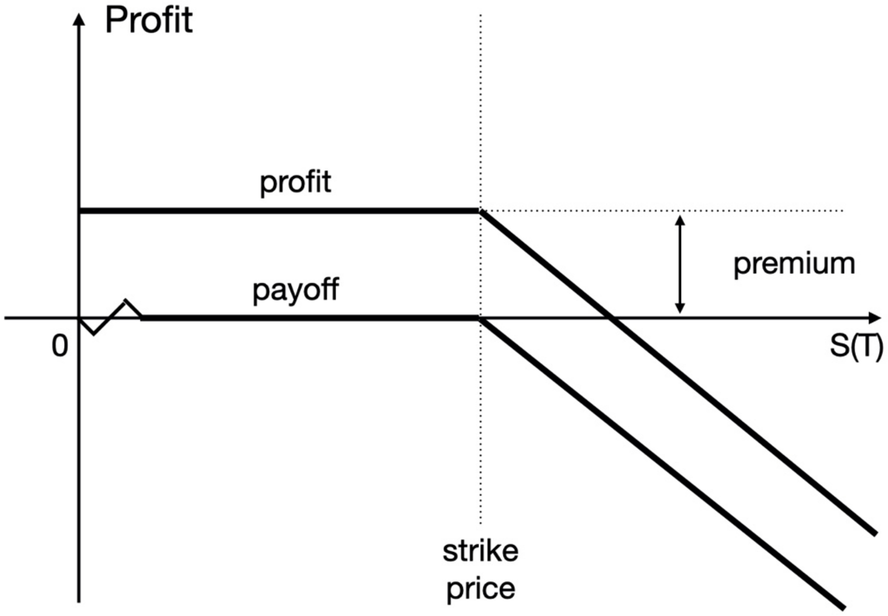

Figure 3. 4: Payoff from long a put option

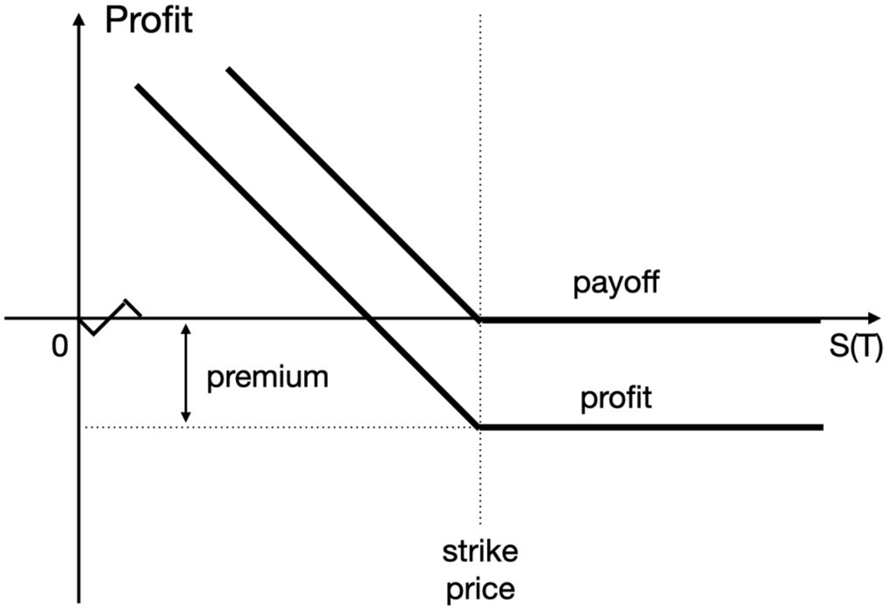

Figure 3. 5: Payoff from short a put option

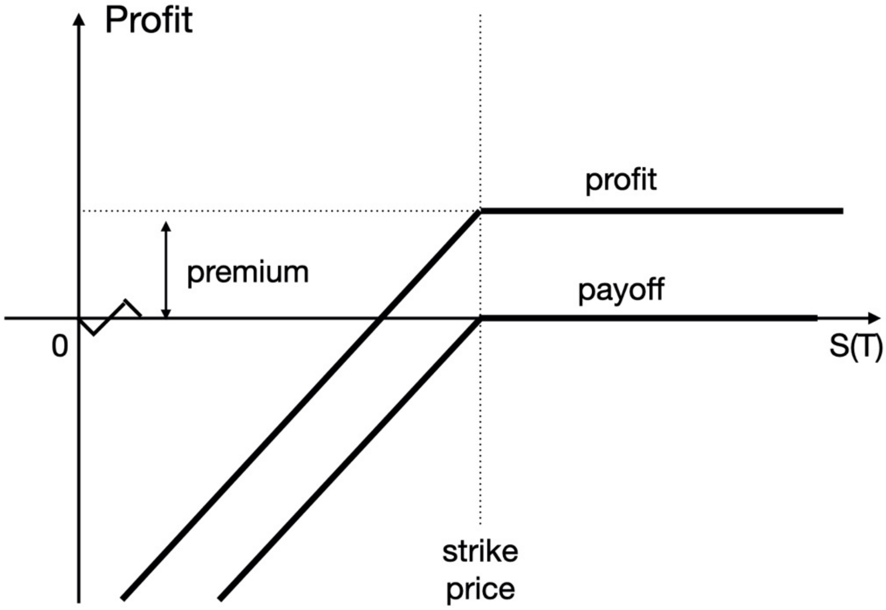

Figure 3. 6: FTSE100 index in 2020

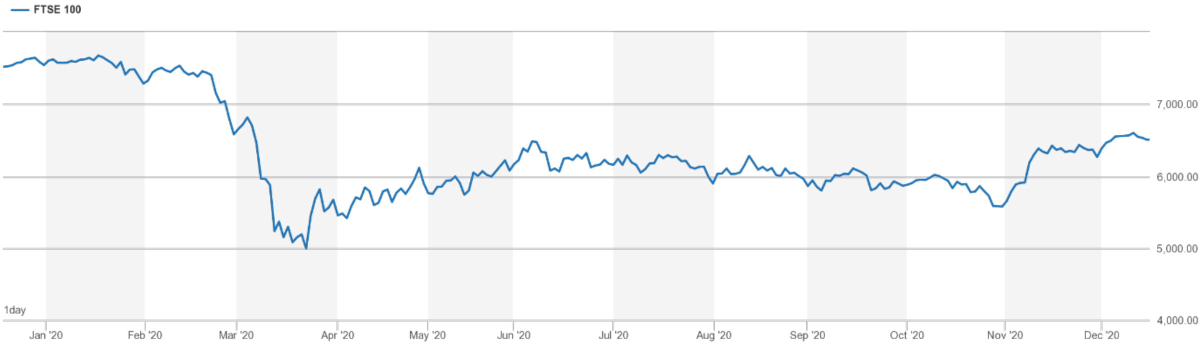

Figure 3. 7: General formulation for Binomial price path

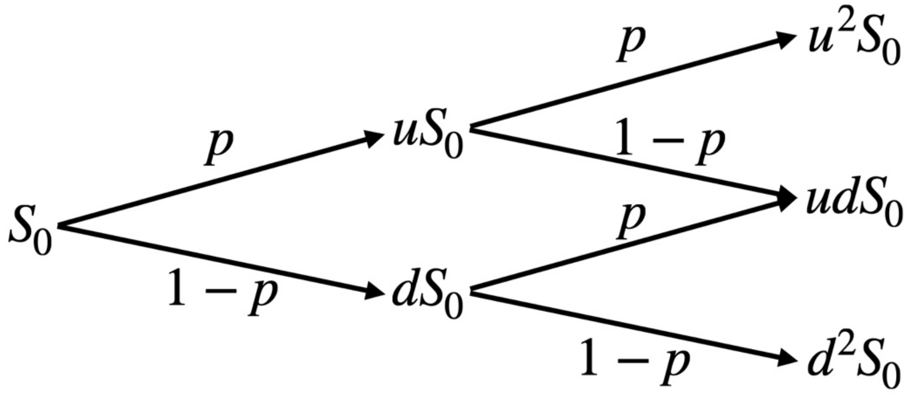

Figure 3. 8: Binomial Call option example