Power-law deformation of Wishart-Laguerre ensembles of random matrices

GERNOT AKEMANN and PIERPAOLO VIVO

Department of Mathematical Sciences & BURSt Research Centre
Brunel University West London, Uxbridge UB8 3PH, United Kingdom

Abstract

We introduce a one-parameter deformation of the Wishart-Laguerre or chiral ensembles of positive definite random matrices with Dyson index $\beta = 1, 2$ and 4. Our generalised model has a fat-tailed distribution while preserving the invariance under orthogonal, unitary or symplectic transformations. The spectral properties are derived analytically for finite matrix size $N \times M$ for all three β, in terms of the orthogonal polynomials of the standard Wishart-Laguerre ensembles. For large-N in a certain double scaling limit we obtain a generalised Marčenko-Pastur distribution on the macroscopic scale, and a generalised Bessel-law at the hard edge which is shown to be universal. Both macroscopic and microscopic correlations exhibit power-law tails, where the microscopic limit depends on β and the difference $M - N$. In the limit where our parameter governing the power-law goes to infinity we recover the correlations of the Wishart-Laguerre ensembles. To illustrate these findings the generalised Marčenko-Pastur distribution is shown to be in very good agreement with empirical data from financial covariance matrices.

Dedicated to the 70th birthday of Oriol Bohigas

1 Introduction

Ensembles of matrices with random entries have been introduced in the pioneering works by Wigner and Dyson in the early 1950s, in an attempt to provide a statistical description of energy levels of heavy nuclei based on few and minimal symmetry requirements. Today's applications cover all areas in Physics, and we refer to [1] for a review. However, many years before Wigner and Dyson, John Wishart had already introduced a random matrix ensemble in his studies on samples from a multivariate population. This ensemble of random matrices can be obtained as follows: take a rectangular matrix $\mathbf{X}$ of size $M \times N$ ($M > N$), whose entries are independent Gaussian variables with mean zero and variance one in the real, complex or quaternion domain, labelled by the Dyson index $\beta = 1, 2$ and 4, respectively. Then form the (non-normalised) covariance matrix of size $N \times N$, $\mathbf{W} = \mathbf{X}^\dagger \mathbf{X}$, which is positive definite. By comparing to such matrices one can establish which part of the spectrum of empirical covariance matrices carries genuine information and which should be discarded as contaminated by pure noise.

The ensemble of such matrices $\mathbf{W}$ is known in Random Matrix Theory (RMT) as Wishart, Laguerre or chiral ensemble and will be denoted by WL in the following. It has since appeared in many different contexts, such as multivariate statistical data analysis [2], analysis of the capacity of channels with multiple antennae and receivers [3], low-energy Quantum Chromodynamics and other gauge theories [4, 5], Quantum Gravity [6, 7], knowledge networks [8], finance [9] and also in statistical physics problems, such as a class of $(1+1)$-dimensional directed polymer problems [10]. Very recent analytical results include statistics of large deviations for the maximum eigenvalue [11] and distributions related to entangled random pure states [12].

Because of its invariance, the WL ensembles can be written in terms of eigenvalues and completely solved, both for any finite N and M, as well as in the large-N limit. We only give a brief account on what is known, more details follow in the main part later.

The WL ensemble can be exactly solved through the standard orthogonal polynomial technique [13]. For a Gaussian weight functions all spectral correlation functions can be expressed at finite-N in terms of orthogonal ($\beta = 2$) or skew orthogonal ($\beta = 1, 4$) Laguerre polynomials, see [13] for a review. In the large-N limit one has to distinguish between smooth macroscopic, and oscillatory microscopic correlations on the level of mean spacing between eigenvalues. The most known macroscopic result in the spectral density of Marčenko and Pastur (MP) [14], generalising the semi-circle in the Wigner-Dyson class (WD). On the microscopic domain much effort has been spent on the so-called hard edge at the origin for the positive definite eigenvalues, which is absent in the WD class. It is governed by the Bessel-law [15, 4, 16, 17, 18] labelled by β and $M - N$ being finite. The robustness or universality under polynomial deformations of the Gaussian weight function has also been proven [19, 20]. This perturbation destroys the independence of the uncorrelated degrees of freedom of $\mathbf{X}$, without changing the microscopic correlation functions. Connected macroscopic density correlation functions also remain universal under such deformations [6].

What is the motivation to further generalise WL ensembles? In the finance and risk management domain, the empirical covariance of a set of N assets over a temporal window of size M has been under scrutiny for some time [9], and its eigenvalues were shown to be distributed in reasonably good agreement with the MP law, as if they were originated by a completely uncorrelated data series. However, the same analysis repeated by several groups [21, 22, 23] on different data sets have shown that either the part of the spectrum corresponding to extremely low eigenvalues - the most interesting for portfolio selections - or the fat tails are not reproduced by this crude approach. This has led to the appearance of more sophisticated models [23, 24, 25], e.g. the multivariate student distribution where the variance of each matrix entry becomes a random variable. Another deformation of the WL ensemble has been introduced in [26] where sparse matrices $\mathbf{X}$ were considered. This setting has

many applications in communication theory and in complex networks (namely in the study of spectral properties of adjacency matrices). All these generalisations lead to a deformation of the MP law, and thus lie outside the WL class. However, the lack of invariance in such models generically spoils the complete solvability for all correlations functions.

Conversely, a model with power-law tails which is exactly solvable in principle appeared in [25]. However, the analysis in [25] was restricted to the macroscopic spectral density, whereas many more interesting results, and ultimately a complete solution of the model can be obtained. This goal is achieved by exploiting techniques and results introduced in previous works, where the WD class was generalised using non-standard entropy maximisation [27, 28] and super-statistical approaches [29, 30].

We will follow these lines and provide a complete solution for all three β of a generalised WL model with rotational invariance, with an emphasis on the issues of universality and complete integrability for all spectral correlations, both macroscopic and microscopic. After a proper rescaling and normalisation, our N-independent correlations depend only on a single parameter α, given by the power-law. In the limit of a large parameter $\alpha \gg 1$ we recover all standard WL correlations. For completeness, we also mention another generalisation of WL which exploits a different direction. In [31] the unitary WL were generalised to display critical statistics.

Our paper is organised as follows. In the next section 2 we define our generalised WL ensembles as a one-parameter deformation, including a general non-Gaussian potential V for all three $\beta = 1, 2$ and 4. The general solution for finite-N is given applying the method of (skew) orthogonal polynomials to an integral transform of the standard WL ensembles.

In the next section 3 we take the macroscopic large-N limit for the special case of a Gaussian potential. Here macroscopic refers to the smooth part of the spectrum, considering correlations on a distance large compared to the mean level-spacing. We rederive a generalisation of the semi-circle and MP spectral density, see refs. [27, 28, 29] and [25] respectively. As a new result we compute the average position of an eigenvalue and the position of a pseudo edge. The former leads to the correct scaling with N and an N-independent generalised spectral density displaying a power-law decay.

Section 4 is devoted to the microscopic large-N limit, where correlations on the order of the mean level-spacing are computed. Here we can use the known universal WL results as an input to our model. In the case when the difference of the matrix dimension $M - N$ is kept finite the spectrum has a hard edge at the origin. In two subsections we compute its microscopic density there, generalising the universal Bessel-law as well as the corresponding smallest eigenvalue distribution for all three β.

In section 5 the nearest neighbour spacing distribution in the bulk of the spectrum is computed using a Wigner surmise at $N = 2$ for our generalised model.

Our conclusions and outlook are presented in section 6, including a comparison of the generalised MP density to data from financial correlation matrices. In two appendices technical details are collected.

2 Definition of the model and finite-N solution for general potential

The joint probability density of our generalised Wishart-Laguerre ensembles is defined as follows in terms of matrix elements

$$\mathcal{P}_\gamma[\mathbf{X}] \, d\mathbf{X} \ \propto \ \left(1 + \frac{n\beta}{\gamma} \mathrm{Tr} V(\mathbf{X}^\dagger \mathbf{X}) \right)^{-\gamma} d\mathbf{X} \, , \tag{2.1}$$

where $\mathbf{X}$ is a matrix of size $M \times N$ with real, complex or quaternion real elements for the values $\beta = 1, 2$ or 4, respectively. We define $M = N + \nu$ for later convenience, where $\nu \geq 0$ may be either finite or of order $\mathcal{O}(N)$ in the large-N limit. The integration measure $d\mathbf{X}$ is defined by integrating

over all independent matrix elements of $\mathbf{X}$ with a flat measure. Expectation values of an operator $\mathcal{O}$ (denoted by $\langle \mathcal{O} \rangle_\gamma$) are defined with respect to $\mathcal{P}_\gamma$ in the usual sense.

The real positive parameter γ is to be specified below, and we keep an additional variance-like parameter $n > 0$. The so-called potential V is taken to be a polynomial of finite degree d, although some of the universal results we inherit from the WL ensembles are known to hold for a much larger class of functions.

The measure (2.1) is well-defined and integrable only if the following condition holds

$$\gamma d \; > \; \frac{\beta}{2} N(N + \nu) \;. \tag{2.2}$$

This condition, which is derived in appendix A, can be seen when changing to radial coordinates for the matrix $\mathbf{X}^\dagger \mathbf{X}$, see e.g. [32, 28]. In particular, the large-N behaviour of any spectral property cannot be taken for fixed γ, and a prescription about the way both quantities should approach infinity needs to be given, respecting the inequality eq. (2.2).

A similar model was introduced earlier generalising the (non-chiral) Wigner-Dyson ensembles for a Gaussian potential [27, 29, 28], and a similar interplay between the deformation parameter γ and the matrix size N was observed.

In the limit of an infinite deformation parameter

$$\lim_{\gamma \to \infty} \mathcal{P}_\gamma[\mathbf{X}] \; = \; \exp[-n\beta \mathrm{Tr} V(\mathbf{X}^\dagger \mathbf{X})] \; \equiv \; \mathcal{P}[\mathbf{X}] \tag{2.3}$$

we recover the standard Wishart-Laguerre or chiral ensembles denoted by $\mathcal{P}[\mathbf{X}]$. Typically, in these ensembles one chooses $n \sim N$ when taking the large-N limit to obtain an N-independent macroscopic density. In our case, this limit will be more involved, and we will keep n general for the time being.

The generalised ensembles can be related to the standard WL through the following integral representation:

$$(1 + z)^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^\infty d\xi \; e^{-\xi} \; \xi^{\gamma - 1} \; e^{-\xi z} \;. \tag{2.4}$$

Inserting this into the definition eq. (2.1) we obtain

$$\mathcal{P}_\gamma[\mathbf{X}] = \frac{1}{\Gamma(\gamma)} \int_0^\infty d\xi \; e^{-\xi} \; \xi^{\gamma - 1} \exp\left[-\xi \frac{n\beta}{\gamma} \mathrm{Tr} V(\mathbf{X}^\dagger \mathbf{X})\right] \;. \tag{2.5}$$

This relation is the key starting point to solve our generalised model both for finite- and large-N. The same trick was used for the generalisation of the Gaussian Wigner-Dyson ensembles introduced previously in [27, 28, 29]. In fact, a similar technique was employed much earlier in [32] when solving the fixed and restricted trace ensembles by writing them as integral transforms of the Wigner-Dyson ensembles.

An advantage of our model over some other generalisations of WL [25, 23] is its invariance under orthogonal, unitary or symplectic transformations. In particular, for $\beta = 1$ Burda *et al.* [25] considered a very general family of probability distributions of the form: $\mathcal{P}_f[\mathbf{X}] \sim f(\mathrm{Tr}\mathbf{X}^T \mathbf{C}^{-1} \mathbf{X} \mathbf{A}^{-1})$ where $\mathbf{C}$ and $\mathbf{A}$ represent the true correlation and autocorrelation matrices respectively, and f is a non-negative and normalised weight function. Only in the special case $\mathbf{C} = \mathbf{A} = \mathbf{1}$ invariance is recovered. This approach has been modified in [23] to allow for a time-dependent random volatility.

From eq. (2.5) (or eq. (2.1)) we can immediately go to an eigenvalue basis of the positive definite matrix $\mathbf{X}^\dagger \mathbf{X}$, to obtain the following joint probability distribution function (jpdf)

$$\begin{aligned}
\mathcal{P}_\gamma(\lambda_1, \ldots, \lambda_N) \; &\equiv \; \left(1 + \frac{n\beta}{\gamma} \sum_{i=1}^N V(\lambda_i)\right)^{-\gamma} \prod_{i=1}^N \lambda_i^{\frac{1}{2}\beta(\nu+1)-1} \prod_{j>k}^N |\lambda_j - \lambda_k|^\beta \\
&= \; \frac{1}{\Gamma(\gamma)} \int_0^\infty d\xi \; e^{-\xi} \; \xi^{\gamma - 1} \; \mathcal{P}(\lambda_1, \ldots, \lambda_N; \xi) \;.
\end{aligned} \tag{2.6}$$

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It is expressed through the jpdf of the standard WL

$$\mathcal{P}(\lambda_1,\ldots,\lambda_N;\xi) \ \equiv \ \prod_{i=1}^N \lambda_i^{\frac{1}{2}\beta(\nu+1)-1} \exp\left[-\xi\,\frac{n\beta}{\gamma}V(\lambda_i)\right] \prod_{j>k}^N |\lambda_j-\lambda_k|^\beta \ , \tag{2.7}$$

depending on ξ through its weight $\exp\left[-\xi\,\frac{n\beta}{\gamma}V(\lambda)\right]$. In both jpdf's we have suppressed the constant from the integration over the angular degrees of freedom.

For completeness we also define the corresponding partition function

$$\begin{aligned}
\mathcal{Z}_\gamma \ &\equiv \ \int_0^\infty \prod_{i=1}^N d\lambda_i \ \left(1+\frac{n\beta}{\gamma}\sum_{i=1}^N V(\lambda_i)\right)^{-\gamma} \prod_{i=1}^N \lambda_i^{\frac{1}{2}\beta(\nu+1)-1} \prod_{j>k}^N |\lambda_j-\lambda_k|^\beta \tag{2.8} \\
&= \ \frac{1}{\Gamma(\gamma)}\int_0^\infty d\xi \ e^{-\xi}\ \xi^{\gamma-1}\mathcal{Z}(\xi) \ ,
\end{aligned}$$

which is again an integral over the standard, ξ-dependent WL partition function

$$\mathcal{Z}(\xi) \ \equiv \ \int_0^\infty \prod_{i=1}^N d\lambda_i \ \lambda_i^{\frac{\beta}{2}(\nu+1)-1} \exp\left[-\xi\,\frac{n\beta}{\gamma}V(\lambda_i)\right] \prod_{j>k}^N |\lambda_j-\lambda_k|^\beta \ . \tag{2.9}$$

Because of this linear relation between the generalised and standard ensembles we can immediately express all k-point eigenvalue density correlation functions, denoted by R for finite-N, in terms of each other. They are defined in the usual way [13]

$$\begin{aligned}
R_\gamma(\lambda_1,\ldots,\lambda_k) \ &\equiv \ \frac{N!}{(N-k)!}\frac{1}{\mathcal{Z}_\gamma}\int_0^\infty d\lambda_{k+1}\cdots d\lambda_N \mathcal{P}_\gamma(\lambda_1,\ldots,\lambda_N) \tag{2.10} \\
&= \ \int_0^\infty d\xi \ e^{-\xi}\ \xi^{\gamma-1}\frac{\mathcal{Z}(\xi)}{\Gamma(\gamma)\mathcal{Z}_\gamma}R(\lambda_1,\ldots,\lambda_k;\xi) \ , \tag{2.11}
\end{aligned}$$

where the k-point correlation functions of the standard ensembles depend on ξ through the exponent in the measure

$$R(\lambda_1,\ldots,\lambda_k;\xi) \ \equiv \ \frac{(N-k)!}{k!}\frac{1}{\mathcal{Z}(\xi)}\int_0^\infty d\lambda_{k+1}\cdots d\lambda_N \mathcal{P}(\lambda_1,\ldots,\lambda_N;\xi) \ . \tag{2.12}$$

The latter can be solved using the method of (skew) orthogonal polynomials [13], expressing them through the determinant of the kernel of the orthogonal polynomials for $\beta=2$, or the Pfaffian of the matrix kernel of skew orthogonal polynomials for $\beta=1$ and 4. We only detail the simpler $\beta=2$ case here and briefly outline $\beta=1$ and 4, where we refer to [13] for more details.

Let us define monic orthogonal polynomials and their norms for $\beta=2$ as follows

$$\int_0^\infty d\lambda \ \lambda^\nu \exp\left[-\xi\,\frac{2n}{\gamma}V(\lambda)\right] P_k(\lambda)P_l(\lambda) \ = \ h_k\delta_{kl} \ . \tag{2.13}$$

Introducing their kernel

$$K_N(\lambda,\mu) \ \equiv \ (\lambda\mu)^{\frac{\nu}{2}}e^{-\xi\frac{n}{\gamma}(V(\lambda)+V(\mu))} \sum_{k=0}^{N-1} h_k^{-1}P_k(\lambda)P_k(\mu) \ , \tag{2.14}$$

and applying the Christoffel-Darboux identity for $\lambda \neq \mu$

$$\sum_{k=0}^{N-1} h_k^{-1} P_k(\lambda) P_k(\mu) \;=\; h_{N-1}^{-1} \frac{P_N(\lambda) P_{N-1}(\mu) - P_N(\mu) P_{N-1}(\lambda)}{\lambda - \mu} \;, \tag{2.15}$$

we can express all eigenvalue correlations of the standard ensemble through this kernel [13],

$$R(\lambda_1, \ldots, \lambda_k; \xi) = \det_{1 \leq i,j \leq k} [K_N(\lambda_i, \lambda_j)] . \tag{2.16}$$

We thus arrive at

$$R_\gamma(\lambda_1, \ldots, \lambda_k) \;=\; \int_0^\infty d\xi \; e^{-\xi} \; \xi^{\gamma-1} \frac{\mathcal{Z}(\xi)}{\Gamma(\gamma)\mathcal{Z}_\gamma} \det_{1 \leq i,j \leq k} [K_N(\lambda_i, \lambda_j)] \;, \tag{2.17}$$

which is the main result of this section. The simplest example is the spectral density $R_\gamma(\lambda)$ given by the integral over the single kernel $K_N(\lambda, \lambda)$. Notice that it is normalised to $N = \int_0^\infty d\lambda R_\gamma(\lambda)$.

In order to take N large we only need to know the asymptotic of the polynomials P_N, take a finite determinant of size k of the asymptotic kernel and integrate once with respect to ξ. In appendix B the orthogonal polynomials, the corresponding densities and partition functions are worked out in detail for finite-N and a Gaussian potential $V(\lambda) = \lambda$ at $\beta = 2$, given in terms of Laguerre polynomials and their norms.

As pointed out already the same result eq. (2.17) holds for $\beta = 1$ and 4 when replacing the determinant by a Pfaffian, $\mathrm{Pf}[\kappa_N(\lambda_i, \lambda_j)]$, where κ_N is a 2×2 matrix kernel. For a Gaussian weight its skew orthogonal polynomials are explicitly known as well in terms of Laguerre polynomials [17, 18].

For completeness, we also give the partition function occurring inside the integrand in eq. (2.17) in terms of the norms h_i of orthogonal polynomials

$$\mathcal{Z}(\xi) \;=\; N! \prod_{i=0}^{N-1} h_i \;=\; N! \, h_0^N \prod_{i=0}^{N-1} r_i^{N-i} \;, \tag{2.18}$$

or their ratios $r_i \equiv \frac{h_{i+1}}{h_i}$. An identical result holds for $\beta = 1$ and 4 in terms of the skew orthogonal norms [13].

In the Gaussian case the ratio of partition functions $\mathcal{Z}(\xi)/\Gamma(\gamma)\mathcal{Z}_\gamma$ in eq. (2.17) can be obtained most explicitly at finite-N for all 3 values of β by changing to radial coordinates, see appendix A for a derivation.

3 Macroscopic large-N limit for a Gaussian potential

In this section, we will restrict ourselves to the Gaussian potential $V(\lambda) = \lambda$, deriving a generalisation of the MP spectral density of WL for all three β. It exhibits a non-compact support and power-law behaviour for large arguments, and we will distinguish two cases. In the first subsection, we will deal with matrices that become asymptotically quadratic with $M - N = \nu = \mathcal{O}(1)$. In the standard WL ensembles, at large-N the spectral support is an interval on the positive semi-axis, where the origin represents a hard edge. In the second subsection, we take the limit

$$\lim_{M,N \to \infty} \frac{N}{M} \;\equiv\; c \;, \tag{3.1}$$

with $c < 1$, corresponding to the case $M - N = \nu = \mathcal{O}(N)$. In the WL ensembles, the resulting MP macroscopic spectral density takes support on a positive interval. Conversely, in both cases $c = 1$ and $c < 1$ our generalised macroscopic density will have support on the full positive real semi-axis.

3.1 Generalised semi-circle for $c = 1$

The finite-N density for the WL ensembles $R(\lambda; \xi)$ is well known in terms of Laguerre polynomials, see eq. (B.7) for $\beta = 2$ in the appendix B. This results into an explicit integral representation for the spectral density of our generalised ensembles, see eq. (B.10). Despite this result, it is rather difficult to extract information about the macroscopic large-N limit from those analytical formulae, both for the standard and generalised WL ensembles.

Hence, we follow here an alternative route, already exploited in [27, 28, 29]: we directly insert the large-N result into eq. (2.11). The $N \gg 1$ asymptotic of the WL density is known and given by the Marčenko-Pastur law, which is in fact the semi-circle law in squared variables at $c = 1$

$$\lim_{N \gg 1} R(\lambda) = \frac{n}{\pi} \sqrt{\frac{2N}{n\lambda} - 1} \ , \quad \text{with} \ \ \lambda \in (0, 2N/n] \ . \tag{3.2}$$

It is given here for the Gaussian weight $\exp[-n\beta\lambda]$ for all three β, and can be derived easily using the Coulomb gas approach and saddle point method (see also [15]).

In order to obtain an N-independent macroscopic density we have to rescale the argument of the density by the mean eigenvalue position, $\lambda \to \langle\lambda\rangle x$, and divide by N to normalise the density to unity. The mean position of an eigenvalue or first moment, $\langle\lambda\rangle_\gamma$, can be computed in the generalised model for both finite-N and -ν (see appendix A),

$$\langle\lambda\rangle_\gamma \ \equiv \ \frac{\int_0^\infty d\lambda \ \lambda \ R_\gamma(\lambda)}{\int_0^\infty d\lambda \ R_\gamma(\lambda)} \ = \ \frac{1}{N}\langle\mathrm{Tr}(\mathbf{X}^\dagger\mathbf{X})\rangle_\gamma$$

$$= \ \frac{\gamma(N + \nu)}{2n(\gamma - \frac{\beta}{2}N(N + \nu) - 1)} \ . \tag{3.3}$$

We will comment on the existence of this first moment later, and we will also need this equation again when we consider $c < 1$. It correctly reproduces the known result for WL in the limit $\lim_{\gamma\to\infty}\langle\lambda\rangle_\gamma = \langle\lambda\rangle = (N + \nu)/2n$. For the WL case we thus obtain the following known N- and β-independent macroscopic density from eq. (3.2)

$$\rho(x) \ \equiv \ \lim_{N\to\infty} \frac{1}{N}\langle\lambda\rangle R(x\langle\lambda\rangle) \ = \ \frac{1}{2\pi} \sqrt{\frac{4}{x} - 1} \ , \quad \text{with} \ \ x \in (0, 4] \ . \tag{3.4}$$

It is normalised to unity and has mean $\langle x\rangle = 1$.

We can now repeat the same steps for our generalised model, where we will need eq. (3.2) for the weight $\exp[-\xi n\beta\lambda/\gamma]$ (see eq. (2.7)). Because of the rescaling with respect to $\langle\lambda\rangle_\gamma$ we now have to specify the N-dependence of γ. We keep the following combination fixed,

$$\alpha \ \equiv \ \lim_{N, \gamma\to\infty} \left[\gamma - \frac{\beta}{2}N(N + \nu) - 1 \right] \ , \tag{3.5}$$

with $\alpha > 0$ finite. This satisfies the constraint (2.2). We thus obtain for the generalised macroscopic density

$$\rho_\alpha(x) \ \equiv \ \lim_{N, \gamma\to\infty} \frac{1}{N}\langle\lambda\rangle_\gamma R_\gamma(x\langle\lambda\rangle_\gamma)$$

$$= \ \lim_{N, \gamma\to\infty} \frac{1}{N}\langle\lambda\rangle_\gamma \int_{\mathcal{I}} d\xi \ e^{-\xi}\xi^{\gamma-1}\frac{\mathcal{Z}(\xi)}{\Gamma(\gamma)\mathcal{Z}_\gamma}\frac{n\xi}{\gamma\pi} \sqrt{\frac{2N\gamma}{n\xi x\langle\lambda\rangle_\gamma} - 1} \ , \tag{3.6}$$

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where the integration is restricted to the interval $\mathcal{I} = (0, \frac{2N\gamma}{nx\langle\lambda\rangle_\gamma}] = (0, 4\alpha/x]$. In order to compute the integral we still need the following quantity inside the integrand,

$$\frac{\mathcal{Z}(\xi)}{\Gamma(\gamma)\mathcal{Z}_\gamma} = \frac{\xi^{-\frac{\beta}{2}N(N+\nu)}}{\Gamma\left(\gamma - \frac{\beta}{2}N(N+\nu)\right)} \ , \tag{3.7}$$

the ratio of partition functions given here for finite-N. This result is derived in appendix A, see eq. (A.5). Inserting all ingredients into eq. (3.6) and taking limits with the definition (3.5) we arrive at the following result after changing variables,

$$\begin{aligned}
\rho_\alpha(x) &= \frac{1}{2\alpha\pi\Gamma(\alpha+1)} \left(\frac{4\alpha}{x}\right)^{\alpha+2} \int_0^1 dt \ \exp\left[-\frac{4\alpha}{x}t\right] \ t^{\alpha+1}\sqrt{\frac{1}{t}-1} \\
&= \frac{\Gamma(\alpha+\frac{3}{2})}{4\alpha\sqrt{\pi}\Gamma(\alpha+1)\Gamma(\alpha+3)} \left(\frac{4\alpha}{x}\right)^{\alpha+2} {}_1F_1\left(\alpha+\frac{3}{2}, \alpha+3; -\frac{4\alpha}{x}\right) \ . \tag{3.8}
\end{aligned}$$

Here we have introduced the confluent or Kummer hypergeometric function

$$_1F_1(a,b;z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 dt \ e^{zt} \ t^{a-1}(1-t)^{b-a-1} \ . \tag{3.9}$$

Eq. (3.8) is the main result of this subsection, the macroscopic spectral density of our generalised WL. It has an unbounded support $(0, \infty)$, and the density as well as its first moment are normalised to unity

$$\int_0^\infty dx\rho_\alpha(x) = 1 = \int_0^\infty dx \ x \ \rho_\alpha(x) \ . \tag{3.10}$$

Note that due to this normalisation, the parameter n has completely dropped out. We are left with a one-parameter class of densities depicted in fig. 1, which approach the WL density for $\alpha \to \infty$ as discussed below.

From the expansion for small arguments ${}_1F_1(a,b;z) = 1 + \frac{a}{b}z + \ldots$ we can immediately read off the power law decay of our new density eq. (3.8)

$$\lim_{x\to\infty} \rho_\alpha(x) = x^{-(\alpha+2)}\frac{\Gamma(\alpha+\frac{3}{2})}{2\alpha\sqrt{\pi}\Gamma(\alpha+1)\Gamma(\alpha+3)}4\alpha^{\alpha+2} \left(1 + \mathcal{O}(1/x)\right) \ . \tag{3.11}$$

Because of $\alpha > 0$ the decay is always faster than quadratic. However, if we drop the requirement for the existence of the first moment we can allow for α to take values $-1 < \alpha < 0$ while satisfying the constraint eq. (2.2). Keeping the same formal rescaling in eq. (3.6) we arrive at the same result eq. (3.8) now valid for $-1 < \alpha$ and $\alpha \neq 0$ [1]. Our density can thus describe power laws in between linear and quadratic decay as well, see fig. 1. The same feature could be incorporated in the generalised Gaussian Wigner-Dyson model [27, 29, 28].

As a check we can take the limit $\alpha \to \infty$ on our final result eq. (3.8). This amounts to decoupling the γ- and N-dependence, and thus we expect to recover the MP density at $\gamma = \infty$. By taking a saddle point approximation, we find that this is indeed the case, $\lim_{\alpha\to\infty} \rho_\alpha(x) = \rho(x)$. Hence, the density $\rho_\alpha(x)$ is a well-behaved deformation of the MP density for $c = 1$.

We can also derive the behaviour of the density $\rho_\alpha(x)$ close to the origin. Using the large argument asymptotic for the confluent hypergeometric function at negative argument, $\lim_{z\to\infty} {}_1F_1(a,b;-z) = z^{-a}\Gamma(b)/\Gamma(b-a)(1+\mathcal{O}(1/z))$, we find that

$$\lim_{x\to 0} \rho_\alpha(x) = x^{-\frac{1}{2}}\frac{\Gamma(\alpha+\frac{3}{2})}{\pi\Gamma(\alpha+1)\sqrt{\alpha}} \left(1 + \mathcal{O}(x)\right) \ . \tag{3.12}$$

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[1]We use the obvious notation $\alpha^{\alpha+1} = \exp[(\alpha+1)\ln|\alpha|]$.

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Therefore all our generalised densities have a square root singularity at the origin, just as the MP density.

To illustrate our findings we first plot eq. (3.8) in Fig. 1 (left) for different values of α, and compare it to the semi-circle density eq. (3.4). In order to visualise the square root singularity for all α we map the density from the positive to the full real axis by defining

$$\vartheta_\alpha(y) \equiv |y|\rho_\alpha(y^2) , \tag{3.13}$$

that is to a normalised density on $\mathbb{R}$, $\int_{-\infty}^{\infty} dy\vartheta_\alpha(y) = 1$. In this form it equals the deformed semi-circle law derived from generalising the Gaussian Wigner-Dyson ensembles in [27, 29, 28], where we have eliminated all irrelevant parameters.

The same map eq. (3.13) takes the MP density eq. (3.4) to the semi-circle, as mentioned already several times,

$$\vartheta(y) = \frac{1}{2\pi} \sqrt{4 - y^2} . \tag{3.14}$$

Both densities are shown in Fig. 1 (right). For $\alpha = 14$ it already approximates the semi-circle very well, whereas for $\alpha \to 0$ the generalised density rapidly approximates a δ-function. For negative $-1 < \alpha < 0$ the height of the maximum in fig. 1 left goes down again, for $\alpha = -0.5$ the curve is even below the semi-circle.
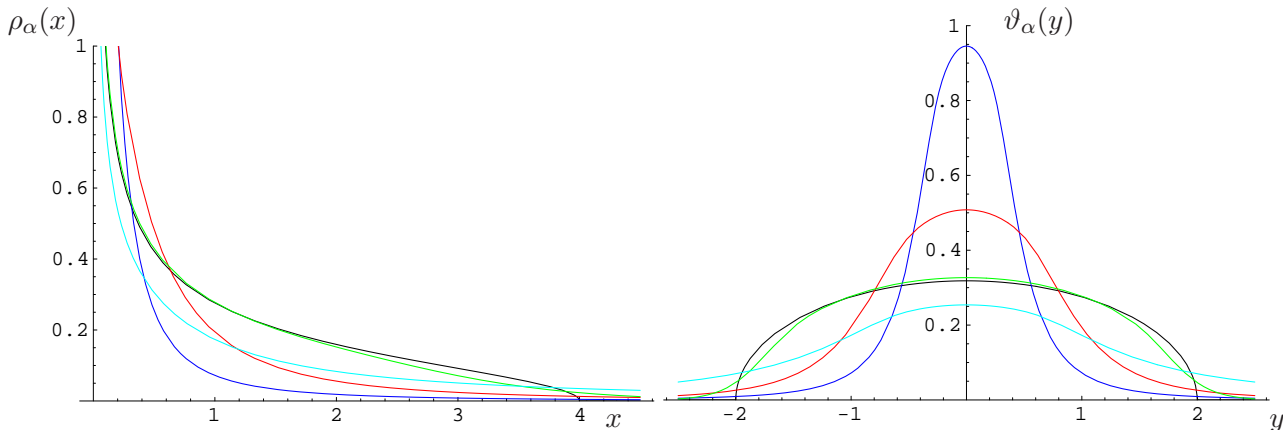

Figure 1: The macroscopic generalised density eq. (3.8) $\rho_\alpha(x)$ shown on the positive real line $\mathbb{R}_+$ for $\alpha = -0.5,\ 0.1,\ 0.5$ and 14 in light blue, blue, red and green, respectively (left), and its map $\vartheta_\alpha(y) = |y|\rho_\alpha(y^2)$ to the full real line $\mathbb{R}$ (right). Note that the MP or semi-circle density given in black for comparison has compact support on $(0, 4]$ and $[-2, 2]$ respectively.

As a final point of this subsection let us discuss the issue of macroscopic universality of our generalised density, eq. (3.8). This is a direct consequence of the (non-)universality of the MP or semi circular density, due to the linear relationship eq. (2.11).

The semi-circle possesses a certain weak universality in the sense that it is the same for all three Gaussian ensembles at $\beta = 1, 2, 4$, as well as for independently distributed random variables, as was shown already by Wigner. Consequently our generalised density is universal in this weak sense, too.

On the other hand the invariant deformation of the Gaussian potential by a polynomial in the definition of our model is clearly *non-universal*, as the semi-circle becomes a polynomial times one or several square root cuts, and we refer to [6] for details at $\beta = 2$. Remarkably, it was found in [6] for $\beta = 2$ that all macroscopic two- or higher k-point *connected* density correlation functions are universal under such perturbations V, depending only on a finite number of parameters for any degree d. Does

this universality persist in our model? The answer is no, simply by looking at the definition of the connected two-point density:

$$R_\gamma^{conn}(\lambda, \mu) \equiv R_\gamma(\lambda, \mu) - R_\gamma(\lambda)R_\gamma(\mu) . \tag{3.15}$$

We use the same definition for the standard WL density. It no longer relates linearly to the corresponding connected WL two-point density as we would subtract an integral of the product of two 1-point densities, instead of the product of two integrated 1-point densities:

$$
\begin{aligned}
R_\gamma^{conn}(\lambda, \mu) &= \int_0^\infty d\xi \ e^{-\xi} \ \xi^{\gamma-1} \frac{\mathcal{Z}(\xi)}{\Gamma(\gamma)\mathcal{Z}_\gamma} R(\lambda, \mu; \xi) \\
&\quad - \int_0^\infty d\xi \ e^{-\xi} \ \xi^{\gamma-1} \frac{\mathcal{Z}(\xi)}{\Gamma(\gamma)\mathcal{Z}_\gamma} R(\lambda; \xi) \int_0^\infty d\xi \ e^{-\xi} \ \xi^{\gamma-1} \frac{\mathcal{Z}(\xi)}{\Gamma(\gamma)\mathcal{Z}_\gamma} R(\mu; \xi) \\
&\neq \int_0^\infty d\xi \ e^{-\xi} \ \xi^{\gamma-1} \frac{\mathcal{Z}(\xi)}{\Gamma(\gamma)\mathcal{Z}_\gamma} \ R^{conn}(\lambda, \mu; \xi) .
\end{aligned}
\tag{3.16}
$$

The universal macroscopic connected two-point function obtained in the large-N limit $\rho^{conn}(x, y; \xi)$ in [6] will thus mix with the non-universal density $\rho(x; \xi)$, and the same feature persists for higher k-point connected correlators. This should not come as a surprise as the same situation was encountered in the fixed or restricted trace ensembles [32], being an integral transformation of the classical Wigner-Dyson ensembles. As observed there [33] our microscopic correlations will remain universal, see section 4.

In the next section we will study the limit $\frac{N}{M} \to c < 1$. The corresponding standard WL ensemble can be mapped to the so-called generalised Penner model [7] with positive definite matrices. The extra determinant from the Jacobian of this change of variables can be written as an extra logarithm in the potential $V \to V + N \ln|\lambda|$. For the same reason as given above the universal findings made in [7] for the unitary ensemble $\beta = 2$ do not translate to the macroscopic limit in the next section either.

3.2 Generalised Marčenko-Pastur law for $c < 1$

In this subsection we deal with the limit in which the matrix $\mathbf{X}$ remains rectangular, that is both M and $N = cM$ become large such that $\lim_{N,M \to \infty} N/M = c < 1$. This limit is particularly relevant for applications to real data series.

We will follow the same steps as in the previous subsection. It is known that in the standard WL ensembles with weight $\exp[-n\beta\lambda]$ the corresponding density is given by the following MP law,

$$\lim_{N \gg 1} R(\lambda) = \frac{n}{\pi\lambda} \sqrt{\left(\lambda - \frac{N}{2n}X_-\right)\left(\frac{N}{2n}X_+ - \lambda\right)} , \quad \text{with} \ \lambda \in \left[\frac{N}{2n}X_-, \frac{N}{2n}X_+\right] . \tag{3.17}$$

For later convenience we have defined the bounds

$$X_\pm \equiv (c^{-\frac{1}{2}} \pm 1)^2 , \quad \text{with} \ 0 < c < 1 . \tag{3.18}$$

In the limit $c \to 1$ we recover from eq. (3.17) the semi-circle eq. (3.2) from the last section.

The N-independent density is again obtained after rescaling by the mean eigenvalue position

$$\langle\lambda\rangle_\gamma = \frac{\gamma N}{2nc(\gamma - \frac{\beta}{2c}N^2 - 1)} , \tag{3.19}$$

where we have used $M = N + \nu = N/c$. For large γ we obtain the quantity $\langle\lambda\rangle = \frac{N}{2nc}$, which is the average position for the standard WL in our limit $c < 1$. We thus obtain for the rescaled density MP density

$$\rho(x) \equiv \lim_{N \to \infty} \frac{1}{N}\langle\lambda\rangle R(x\langle\lambda\rangle) = \frac{1}{2\pi cx} \sqrt{(x - cX_-)(cX_+ - x)} , \quad \text{with} \ x \in [cX_-, cX_+] . \tag{3.20}$$

It is normalised to unity with mean $\langle x \rangle = 1$.

For the generalised model we have to insert eq. (3.17), now with weight $\exp[-\xi\frac{n\beta}{\gamma}\lambda]$, into eq. (2.11) and rescale with respect to eq. (3.19). As previously we keep fixed

$$\alpha \equiv \lim_{N,\gamma\to\infty} \left[\gamma - \frac{\beta}{2c}N^2 - 1 \right] , \quad \text{with } \alpha > 0 , \tag{3.21}$$

as in eq. (3.5). The rescaled generalised density is thus given by

$$\rho_\alpha(x) \equiv \lim_{N,\gamma\to\infty} \frac{1}{N} \langle\lambda\rangle_\gamma R_\gamma(x\langle\lambda\rangle_\gamma) \tag{3.22}$$

$$= \lim_{N,\gamma\to\infty} \frac{1}{N} \langle\lambda\rangle_\gamma \int_{\mathcal{I}} d\xi \; e^{-\xi}\xi^{\gamma-1} \frac{\mathcal{Z}(\xi)}{\Gamma(\gamma)\mathcal{Z}_\gamma} \frac{n\xi}{\gamma\pi x\langle\lambda\rangle_\gamma} \sqrt{\left(x\langle\lambda\rangle_\gamma - \frac{N\gamma}{2n\xi}X_- \right)\left(\frac{N\gamma}{2n\xi}X_+ - x\langle\lambda\rangle_\gamma \right)} ,$$

where $\mathcal{I} \equiv \left[\frac{c\alpha}{x}X_-, \frac{c\alpha}{x}X_+ \right]$. Filling in all definitions and changing variables we finally arrive at the following main result of this subsection:

$$\rho_\alpha(x) = \frac{1}{2\pi c\alpha\Gamma(\alpha+1)} \left(\frac{c\alpha}{x} \right)^{\alpha+2} \int_{X_-}^{X_+} dt \exp\left[-\frac{c\alpha}{x}t \right] t^\alpha \sqrt{(t - X_-)(X_+ - t)} . \tag{3.23}$$

Our density in normalised to unity and has first moment $\langle x \rangle = 1$. This result was derived previously (modulo different notations) for $\beta = 1$ in [25], using different methods. The integral in eq. (3.23) can be computed in principle in terms of a confluent hypergeometric series in two variables (see [34], formulae 3.385 and 9.261(1)), but the integral form is more convenient for numerical evaluations and an asymptotic analysis. As a first check we recover eq. (3.8) in the limit $c \to 1$. Furthermore, one can show that the following limit holds, $\lim_{\alpha\to\infty} \rho_\alpha(x) = \rho(x)$, recovering the MP density eq. (3.20). Thus the large-N limit and the large-γ limit are again well behaved. We have also checked in appendix B that the convergence with N towards the density eq. (3.8) is very fast, see fig. 9, in fact faster than for WL. Our remark from the previous subsection allowing $-1 < \alpha < 0$ applies here too.

It is easy to see how the density decays for large $x \gg 1$

$$\rho_\alpha(x) \sim x^{-(\alpha+2)} \frac{(c\alpha)^{\alpha+2}}{2\pi c\alpha\Gamma(\alpha+1)} \mathcal{C}(1 + \mathcal{O}(1/x)) , \tag{3.24}$$

with the same power law as for $c = 1$. The constant $\mathcal{C}$ is given by

$$\mathcal{C} \equiv \frac{1}{8}(X_+ - X_-)^2 X_-^\alpha \pi \; {}_2F_1\left(\frac{3}{2}, -\alpha, 3, -\frac{X_+ - X_-}{X_-} \right) . \tag{3.25}$$

The asymptotic for small values of x is less obvious to obtain, and we find

$$\rho_\alpha(x) \sim x^{-\alpha-1/2} \exp\left[-\frac{c\alpha}{x}X_- \right] \mathcal{D} , \tag{3.26}$$

where the constant $\mathcal{D} = X_-^\alpha(X_+ - X_-)^{1/2}(c\alpha)^{\alpha-1/2}/16\Gamma(\alpha+1)$.

While the MP density of the standard WL ensemble eq. (3.20) has compact support between $[cX_-, cX_+]$, our generalised density is non-vanishing on the entire real positive axis, even for $c < 1$. To the left of the edge in MP, $x < cX_-$, our density decreases but remains non-zero. Below a certain point that we will call pseudo edge, $\mathcal{X}_-$, our density becomes exponentially suppressed. From the asymptotic (3.26) it is possible to give an estimate for $\mathcal{X}_-$, below which the density becomes negligible. The reasoning goes as follows: writing the asymptotic (3.26) as

$$\rho_\alpha(x) \approx \exp\left[-\left((\alpha + 1/2)\log x + \frac{c\alpha}{x}X_- \right) \right] \tag{3.27}$$
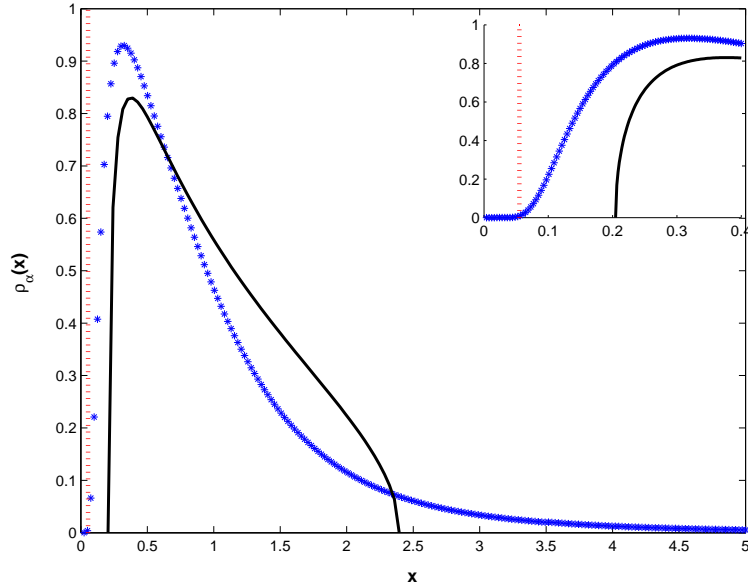
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Figure 2: The macroscopic generalised density $\rho_\alpha(x)$ eq. (3.23) for $\alpha = 3$ and $c = 0.3$ (blue), compared to the MP distribution $\rho(x)$ (3.20) (black). In the inset, the behaviour close to the origin is shown. The red dashed line corresponds to the pseudo edge $\mathcal{X}_-$ of our generalised MP density (see main text for details).

the exponential damping conventionally begins at the point $\mathcal{X}_-$ where

$$(\alpha + 1/2) \log \mathcal{X}_- + \frac{c\alpha}{\mathcal{X}_-} X_- \approx 1 \tag{3.28}$$

For the case depicted in fig. 2 ($c = 0.3$ and $\alpha = 3$), the above estimate reads $\mathcal{X}_- \approx 0.055...$, in reasonable agreement with the inset.

4 Universal microscopic large-N limit for a general potential V

In this section we consider a different large-N limit, the microscopic limit, which takes us to the scale of the mean level spacing and thus to the distribution of individual eigenvalues. Our findings will be universal for a general non-Gaussian polynomial potential $V(\lambda)$ for all three $\beta = 1, 2, 4$, inheriting the corresponding universality from the standard ensembles.

We will only consider the case $c = 1$ and the so-called hard edge here, deriving a generalised Bessel-law for the microscopic densities and their first eigenvalue distributions. For $c < 1$ the local distribution at the inner (and outer) soft edge of the standard WL ensembles follows the Tracy–Widom-law. Although it would be very interesting to derive the corresponding generalisation we have not managed so far, and leave this task for future investigation.

In the first subsection the microscopic densities are derived while the second subsection is devoted to the first eigenvalue distributions. The matching of the two is illustrated in many pictures throughout this section, being an important consistency check.

4.1 Generalised universal Bessel-law

Let us first recall the definition of the microscopic limit in the standard WL ensembles, resulting into the universal Bessel-law. We will discuss in detail the case $\beta = 2$ and then only quote the results for $\beta = 1, 4$.

For simplicity consider the Gaussian case first. Starting from the weight $\exp[-n\beta\lambda]$ we first scale out the mean eigenvalue $\lambda \to x\langle\lambda\rangle$, just as in eq. (3.4) for the macroscopic limit. On top of that we make a further rescaling by the mean level spacing $4N^2x = y$, keeping y fixed. We therefore define the microscopic limit as

$$\rho_\nu^{(\beta)}(y) \equiv \lim_{N\to\infty} \frac{1}{2N^2}\langle\lambda\rangle R\left(\frac{y}{4N^2}\langle\lambda\rangle\right) , \tag{4.1}$$

where $R(\lambda)$ is the spectral density for finite-N in one of the three WL ensembles. The result now depends on β and ν as indicated through the indices, in contrast to the semi-circle law. We can then apply the following asymptotic

$$\lim_{k\to\infty} k^{-\nu}L_k^\nu\left(\frac{z^2}{4k}\right) = \left(\frac{z}{2}\right)^{-\nu} J_\nu(z) \tag{4.2}$$

to the orthogonal Laguerre polynomials in the finite-N density, see eq. (B.7) for $\beta = 2$. We obtain

$$\rho_\nu^{(2)}(y) = \lim_{N\to\infty} \frac{2n}{2N^2}\langle\lambda\rangle\left(\frac{y\langle\lambda\rangle 2n}{4N^2}\right)^\nu e^{-\frac{2ny}{4N^2}\langle\lambda\rangle} \sum_{k=0}^{N-1} \frac{k!}{(k+\nu)!}L_k^\nu\left(\frac{2ny\,k}{4N^2 k}\langle\lambda\rangle\right)^2$$

$$= \frac{1}{2}\int_0^1 dt\, J_\nu(\sqrt{ty})^2 = \frac{1}{2}(J_\nu(\sqrt{y})^2 - J_{\nu-1}(\sqrt{y})J_{\nu+1}(\sqrt{y})) , \tag{4.3}$$

after replacing the sum by an integral[2] with variable $t = k/N$. The Bessel density is plotted for different values of ν in fig. 4 together with the first eigenvalue from the next subsection, after changing to the conventional squared variables $y \to y^2$ (see eq. (3.13)),

$$\vartheta_\nu^{(2)}(y) = \frac{|y|}{2}(J_\nu(y)^2 - J_{\nu-1}(y)J_{\nu+1}(y)) . \tag{4.4}$$

This result is universal [19] being valid for any potential V with spectral support including the origin. We only have to rescale in eq. (4.1) by the macroscopic density in terms of the squared variables $\pi\vartheta(0)$ for a general potential V, instead of the Gaussian macroscopic density eq. (3.14) where $\pi\vartheta(0) = 1$. In other words all orthogonal polynomials eq. (2.13) tend to Bessel-J functions in the microscopic limit.

We can now repeat the above analysis for our generalised microscopic density. The scaling of γ with N keeping α fixed is kept throughout this entire section. In the previous section we found that the density diverges exactly like the standard density as an inverse square root, see eq. (3.12). For that reason the microscopic rescaling is the same, without changing powers of N. However, the constant in front of the *macroscopic* density at the origin is not the standard Gaussian result, $1/\pi = \vartheta(0)$, but given by eq. (3.12), $\vartheta_\alpha(0) = b/\pi$ with

$$b \equiv \frac{\Gamma(\alpha + \frac{3}{2})}{\Gamma(\alpha + 1)\sqrt{\alpha}} . \tag{4.5}$$

We therefore define the microscopic limit as

$$\rho_{\alpha,\nu}^{(\beta)}(y) \equiv \lim_{N,\gamma\to\infty} \frac{1}{2N^2 b^2}\langle\lambda\rangle_\gamma R\left(\frac{y}{4N^2 b^2}\langle\lambda\rangle_\gamma\right) . \tag{4.6}$$

---

[2] Instead of eq. (B.7) we could have applied the Christoffel-Darboux identity eq. (B.9), leading to the same result.

Because of the universality we just stated we can restrict ourselves to do the computation for the orthogonal polynomials with Gaussian weight $\exp[-2n\xi\lambda/\gamma]$ in eq. (2.13). Taking the microscopic limit eq. (4.6) and inserting eq. (4.3) we obtain

$$\vartheta^{(2)}_{\alpha,\nu}(y) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty d\xi \; e^{-\xi} \; \xi^\alpha \frac{\xi|y|}{2\alpha b^2} \left(J_\nu\left(\frac{y}{b}\sqrt{\xi/\alpha}\right)^2 - J_{\nu-1}\left(\frac{y}{b}\sqrt{\xi/\alpha}\right)J_{\nu+1}\left(\frac{y}{b}\sqrt{\xi/\alpha}\right) \right). \quad (4.7)$$

This is the first main result of this subsection given here in terms of squared values. The integral could be expressed in terms of generalised hypergeometric functions, but for plots this representation is preferable. Note that in our the calculation we have inserted the ratio of partition functions eq. (3.7) for the Gaussian models. This quantity is again universal as in the large-N limit not only the polynomials themselves, but also their norms become universal. Starting from eq. (2.18) the standard WL partition functions is given as follows

$$\lim_{N\to\infty} (\log[\mathcal{Z}(\xi)] - \log[N!\,h_0^N]) = \lim_{N\to\infty} N \sum_{i=0}^{N-1} \left(1 - \frac{i}{N}\right) \log[r_i] = \int_0^1 dt(1-t)\log[r(t)] \quad (4.8)$$

The ξ-dependent ratios of the norms determined by the so-called string or recursion equation at finite-N have a universal limit $r(t)$ [19]. Re-exponentiating and inserting this universal result eq. (4.8) into eq. (2.8), the generalised partition function $\mathcal{Z}_\gamma$ also becomes universal in the large-N limit, and thus the ratio eq. (3.7) as well. The constant factor that we have subtracted on the left hand side of eq. (4.8) cancels out when taking the ratio.
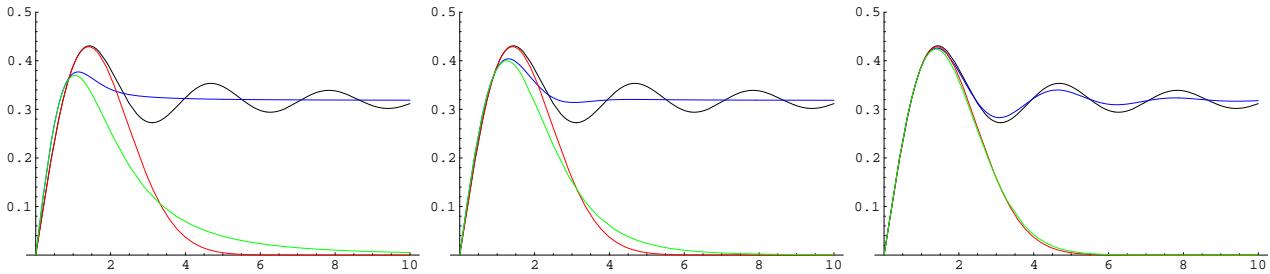

Figure 3: Varying α at $\beta = 2$: the generalised microscopic density $\vartheta^{(2)}_{\alpha,\nu}(y)$ eq. (4.7) (blue) and its first eigenvalue (green) at $\alpha = 0.1$ (left), $\alpha = 2$ (middle), and $\alpha = 20$ (right), vs the corresponding WL Bessel density $\vartheta^{(2)}_\nu(y)$ eq. (4.4) (black) and its first eigenvalue (red).
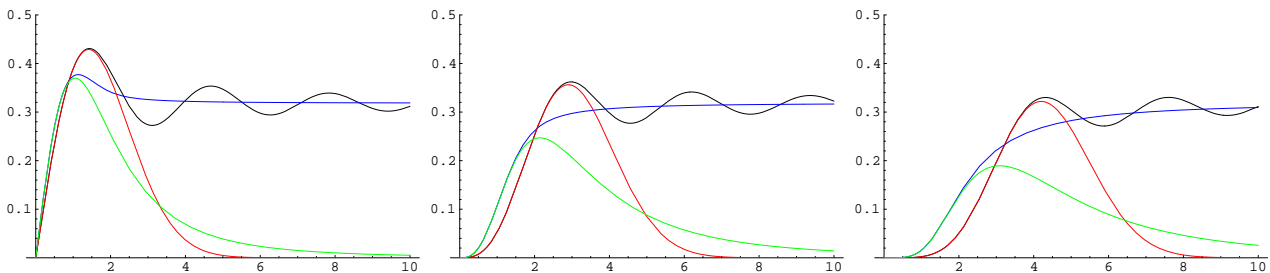

Figure 4: Varying ν at $\beta = 2$: the generalised microscopic density $\vartheta^{(2)}_{\alpha,\nu}(y)$ eq. (4.7) (blue) and its first eigenvalue (green) at $\alpha = 0.1$ vs the corresponding WL Bessel density $\vartheta^{(2)}_\nu(y)$ eq. (4.4) (black) and its first eigenvalue (red): $\nu = 0$ (left), $\nu = 1$ (middle) and $\nu = 2$ (right). It is clearly visible that even the first eigenvalue of the generalised model has fat tails.

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The new microscopic density eq. (4.7) generalising the Bessel-law eq. (4.4) is shown in figs. 3 and 4 for various values of α and ν. The plots include the corresponding first eigenvalues to be derived later. As a check we can take the limit $\alpha \to \infty$ to analytically reobtain eq. (4.4) from eq. (4.7). This is illustrated in fig. 3 where we observe that the convergence is rather slow. We have checked that for $\alpha \sim \mathcal{O}(150)$ the first three maxima become indistinguishable.

The procedure for $\beta = 1$ and 4 is the same as above and so we can be more concise. The corresponding microscopic densities of the WL ensembles are again universal [20]. Computed initially in [17, 18] the obtained expressions can be simplified. They can be expressed through the $\beta = 2$ density eq. (4.4) plus extra terms as shown for $\beta = 1$ [35] and $\beta = 4$ [36],

$$\vartheta_\nu^{(1)}(y) \;=\; \vartheta_\nu^{(2)}(y) + \frac{1}{2}J_\nu(y)\left(1 - \int_0^y dt\, J_\nu(t)\right), \tag{4.9}$$

$$\vartheta_\nu^{(4)}(y) \;=\; \vartheta_{2\nu}^{(2)}(2y) - \frac{1}{2}J_{2\nu}(2y)\int_0^{2y} dt\, J_{2\nu}(t) \;. \tag{4.10}$$

The generalised densities immediately follow. Because of the linear relationship they are also expressed through the generalised $\beta = 2$ density eq. (4.7):

$$\vartheta_{\alpha,\nu}^{(1)}(y) \;=\; \vartheta_{\alpha,\nu}^{(2)}(y) + \frac{1}{\Gamma(\alpha+1)}\int_0^\infty d\xi\, e^{-\xi}\,\xi^\alpha\sqrt{\frac{\xi}{\alpha b^2}}\frac{1}{2}J_\nu\left(\frac{y}{b}\sqrt{\xi/\alpha}\right)\left(1 - \int_0^{y\sqrt{\xi/\alpha b^2}} dt\, J_\nu(t)\right), \tag{4.11}$$

$$\vartheta_{\alpha,\nu}^{(4)}(y) \;=\; \vartheta_{\alpha,2\nu}^{(2)}(2y) - \frac{1}{\Gamma(\alpha+1)}\int_0^\infty d\xi\, e^{-\xi}\,\xi^\alpha\sqrt{\frac{\xi}{\alpha b^2}}\frac{1}{2}J_{2\nu}\left(\frac{2y}{b}\sqrt{\xi/\alpha}\right)\int_0^{2y\sqrt{\xi/\alpha b^2}} dt\, J_{2\nu}(t). \tag{4.12}$$

For a given ν the inner integral over the single Bessel-J function can be performed analytically. It is given terms of Bessel-J functions for odd values of ν, e.g. $\int_0^v dt\, J_1(t) = J_0(v)$, and additional Struve functions for even ν.

The generalised microscopic densities $\vartheta_{\alpha,\nu}^{(\beta)}(y)$ are compared to the standard ones below in fig. 5 for $\beta = 1$, and in fig. 6 for $\beta = 4$.

Higher order correlation functions can be computed along the same lines by inserting the asymptotic Bessel kernels into eq. (2.17), and we only quote the simplest final result for $\beta = 2$:

$$\vartheta_{\alpha,\nu}^{(2)}(y_1,\ldots,y_k) \;=\; \frac{\prod_{i=1}^k \frac{1}{2}|y_i|}{\Gamma(\alpha+1)}\int_0^\infty d\xi\, e^{-\xi}\,\xi^\alpha\prod_{j=1}^k\sqrt{\frac{\xi}{\alpha}}\frac{y_j}{b}$$

$$\times \det_{1\le i,j\le k}\left[\frac{J_\nu\left(\frac{y_i}{b}\sqrt{\xi/\alpha}\right)J_{\nu+1}\left(\frac{y_j}{b}\sqrt{\xi/\alpha}\right) - (i\leftrightarrow j)}{y_i - y_j}\right]. \tag{4.13}$$

The corresponding results for $\beta = 1, 4$ are given in terms of a Pfaffian of a matrix kernel [18], and for a discussion of a relation between the three universal kernels we refer to [20].

A feature we observe for all three β is that for $\alpha \le \mathcal{O}(1)$ the oscillations of the Bessel density are completely smoothed out, apart from the first peak. A similar feature was observed in a generalisation of the unitary WL ensemble for critical statistics [31]. However, no power law tails seem to be present in such a model where only the generalised microscopic density and number variance were computed.

It is known that for standard WL the maxima of the Bessel density correspond to the location of individual eigenvalues [37], as we will see in the next subsection. On the other hand the microscopic density of the WL ensembles in the bulk is completely flat, equalling $\frac{1}{\pi}$ in our normalisation. We may thus suspect that in the generalised model the bulk is approached much faster than in the standard WL, where localised maxima persist to $y \gg 10$. We therefore focus mainly on the first eigenvalue distribution in the generalised model which is the subject of the next subsection.

4.2 Generalised universal first eigenvalue distribution at the hard edge

The probability that the interval $(0, s]$ is empty of eigenvalues is defined as follows,

$$E_\gamma(s) \equiv \frac{1}{\mathcal{Z}_\gamma} \int_s^\infty d\lambda_1 \cdots d\lambda_N \mathcal{P}_\gamma(\lambda_1, \ldots, \lambda_N) \tag{4.14}$$

$$= \int_0^\infty d\xi \ e^{-\xi} \ \xi^{\gamma-1} \frac{\mathcal{Z}(\xi)}{\Gamma(\gamma)\mathcal{Z}_\gamma} E(s; \xi) , \tag{4.15}$$

where the gap probability of the WL ensembles is defined as

$$E(s; \xi) \equiv \frac{1}{\mathcal{Z}(\xi)} \int_s^\infty d\lambda_1 \cdots d\lambda_N \mathcal{P}(\lambda_1, \ldots, \lambda_N; \xi) . \tag{4.16}$$

Both quantities are normalised to unity at $s = 0$ and vanish at $s = \infty$. The distribution of the first eigenvalue $p(s)$ simply follows by differentiation.

$$p_\gamma(s) \equiv -\frac{\partial}{\partial s} E_\gamma(s) , \tag{4.17}$$

and likewise for WL. In WL the gap probability $E(s)$ and the first eigenvalue distribution $p(s)$ are explicitly known and universal in the microscopic large-N limit for all ν at $\beta = 2$, for odd values of ν and 0 at $\beta = 1$, and for $\nu = 0$ at $\beta = 4$. This has been shown by various authors independently [16, 38, 39, 37]. In some cases only finite-N results are know in terms of a hypergeometric function of a matrix valued argument [40, 41], from which limits are difficult to extract.

Although $p(s)$ follows from $E(s)$, the most compact universal formulas are known directly for $p(s)$ for all three β [37]. There, also the second and higher eigenvalue distributions are given, which we will not consider here.

For pedagogical reasons we start once more with an explicit calculation for $\beta = 2$ and the Gaussian ensemble. At $\nu = 0$ the pre-exponential factor is absent and we have for WL with weight $\exp[-2n\lambda]$

$$E(s) = \frac{1}{\mathcal{Z}} \int_s^\infty d\lambda_1 \cdots d\lambda_N \exp\left[-2n \sum_{i=1}^N \lambda_i\right] \prod_{j>k}^N |\lambda_j - \lambda_k|^2 = \exp[-2nNs] , \tag{4.18}$$

shifting all integration variables by s and using the invariance of the Vandermonde determinant. This result is exact for any N and identical to the properly rescaled large-N result when keeping Ns fixed.

The generalised ensemble follows easily, by inserting this result into eq. (4.15)

$$E_\gamma(s) = \int_0^\infty d\xi \ e^{-\xi} \ \xi^{\gamma-1} \frac{\mathcal{Z}(\xi)}{\Gamma(\gamma)\mathcal{Z}_\gamma} \exp\left[-\frac{2n\xi Ns}{\gamma}\right] = \left(1 + \frac{2nNs}{\gamma}\right)^{-(\gamma-N^2)} , \tag{4.19}$$

which is also exact for finite and infinite N. This very fact implies that we have full control of the large-N limit. Because of the fat tails in our distributions it was not a priori clear if the scales at the hard edge and in the bulk would mix. This implicit assumption of a separation of scales in the previous subsection is thus fully justified.

The microscopic limit can be taken following eq. (4.1) for WL:

$$\mathcal{E}_\nu^{(\beta)}(y) \equiv \lim_{N\to\infty} E\left(\frac{y}{4N^2} \langle \lambda \rangle\right) , \tag{4.20}$$

where we explicitly indicate the dependence on β and ν, as in the previous subsection. As a result we obtain for $\beta = 2$ and $\nu = 0$

$$\mathcal{E}_{\nu=0}^{(2)}(y) = \exp\left[-\frac{1}{4}y\right] , \tag{4.21}$$

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and for the corresponding generalised gap probability

$$\mathcal{E}^{(2)}_{\alpha,\,\nu=0}(y) \equiv \lim_{N,\gamma\to\infty} E_\gamma\left(\frac{y}{4N^2b^2}\langle\lambda\rangle_\gamma\right) = \left(1+\frac{y}{4\alpha b^2}\right)^{-(\alpha+1)} . \tag{4.22}$$

The first eigenvalue distribution can be compared to the microscopic densities $\vartheta_\alpha(y)$ in squared variables:

$$\wp^{(2)}_{\nu=0}(y) \equiv -\frac{\partial}{\partial y}\mathcal{E}^{(2)}_0(y^2) = \frac{1}{2}|y|\exp\left[-\frac{1}{4}y^2\right] , \tag{4.23}$$

for WL, and for the generalised ensemble

$$\wp^{(2)}_{\alpha,\,\nu=0}(y) \equiv -\frac{\partial}{\partial y}\mathcal{E}^{(2)}_{\alpha,0}(y^2) = |y|\frac{(\alpha+1)}{2\alpha b^2}\left(1+\frac{y^2}{4\alpha b^2}\right)^{-(\alpha+2)} . \tag{4.24}$$

These distributions are all normalised to unity,

$$\int_0^\infty dy \; \wp^{(\beta)}_{\alpha,\nu}(y) = 1 . \tag{4.25}$$

The restriction to a Gaussian potential in the discussion above can be lifted, as the first eigenvalue distributions for all ν are universal, including the ratio in partition functions that we have inserted again. Eqs. (4.23) and (4.24) are compared to the corresponding densities eqs. (4.4) and (4.7) in fig. 3 for various values of α.

Next we give the first eigenvalue distribution for general ν. Here we directly use the most compact universal expression [37] for $\wp^{(2)}_\nu(y)$ in WL, without making the detour over $\mathcal{E}^{(2)}(y)$ [39],

$$\wp^{(2)}_\nu(y) = \frac{1}{2}|y|\exp\left[-\frac{1}{4}y^2\right]\det_{1\leq i,j\leq\nu}[I_{i-j+2}(y)] . \tag{4.26}$$

In addition to the exponential in eq. (4.23) it contains a determinant of finite size $\nu\times\nu$ over the modified Bessel-I function, which is absent at $\nu=0$. Knowing that the properly rescaled microscopic gap probability is a function of the form $\mathcal{E}^{(2)}_\nu(y;\xi) = \mathcal{E}^{(2)}_\nu(\sqrt{\xi y/\alpha b^2})$, see eq. (4.7), we obtain for the generalised first eigenvalues distribution in terms of squared eigenvalues

$$\begin{aligned}
\wp^{(2)}_{\alpha,\nu}(y) &= \frac{1}{\Gamma(\alpha+1)}\int_0^\infty d\xi \; e^{-\xi} \; \xi^\alpha(-)\frac{\partial}{\partial y}\mathcal{E}^{(2)}(y\sqrt{\xi/\alpha b^2}) \\
&= \frac{1}{\Gamma(\alpha+1)}\int_0^\infty d\xi \; e^{-\xi} \; \xi^\alpha\frac{\xi|y|}{2\alpha b^2}\exp\left[-\frac{\xi y^2}{4\alpha b^2}\right]\det_{1\leq i,j\leq\nu}\left[I_{i-j+2}\left(\frac{y}{b}\sqrt{\xi/\alpha}\right)\right] . \tag{4.27}
\end{aligned}$$

For $\nu=1$ containing only one Bessel-I the integral can be performed and is given in terms of a hypergeometric function

$$\begin{aligned}
\wp^{(2)}_{\alpha,1}(y) &= \frac{|y|}{2\alpha b^2\Gamma(\alpha+1)}\int_0^\infty d\xi \; e^{-\xi} \; \xi^{\alpha+1}\exp\left[-\frac{\xi y^2}{4\alpha b^2}\right]I_2\left(\frac{y}{b}\sqrt{\xi/\alpha}\right) \\
&= \frac{\Gamma(\alpha+3)}{|y|\Gamma(\alpha+1)}\left(1+\frac{y^2}{4\alpha}\right)^{-(\alpha+3)}{}_1F_1\left(\alpha+3,3;\left(1+\frac{4\alpha}{y^2}\right)^{-1}\right) . \tag{4.28}
\end{aligned}$$

It is shown in fig. 4 (middle), together with $\nu=2$ (right). For increasing ν however, the integral representation eq. (4.27) is more convenient.
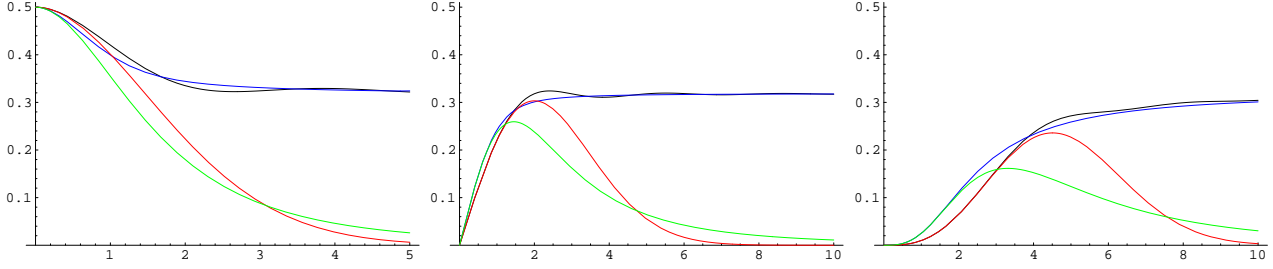
Figure 5: Varying ν for $\beta = 1$: the generalised microscopic density $\vartheta_{\alpha,\nu}^{(1)}(y)$ eq. (4.11) (blue) with its first eigenvalue (green) at $\alpha = 0.1$ vs the corresponding WL Bessel density $\vartheta_{\nu}^{(1)}(y)$ eq. (4.9) (black) and its first eigenvalue (red) at $\nu = 0$ (left), $\nu = 1$ (middle), and $\nu = 3$ (right).

We now turn to $\beta = 1$. Here the first eigenvalue distribution of the WL ensembles is only known explicitly for odd values of ν in the large-N limit [37], given by a Pfaffian with indices running over half integers:

$$\wp_{\nu}^{(1)}(y) = const. \, |y|^{(3-\nu)/2} \exp\left[-\frac{1}{8}y^2\right] \mathrm{Pf}_{-\frac{\nu}{2}+1 \leq i,j \leq \frac{\nu}{2}-1}\left[(i-j)I_{i+j+3}(|y|)\right] . \tag{4.29}$$

The constant in front is determined by the normalisation to unity and can be computed case by case. An exception is $\nu = 0$ where the distribution was calculated in [16]

$$\wp_0^{(1)}(y) = \frac{1}{4}(2 + |y|) \exp\left[-\frac{|y|}{2} - \frac{y^2}{8}\right] . \tag{4.30}$$

For $\nu = 1$ and 3 we have from eq. (4.29)

$$\wp_1^{(1)}(y) = \frac{1}{4}|y| \exp\left[-\frac{1}{8}y^2\right] , \tag{4.31}$$

$$\wp_3^{(1)}(y) = \frac{1}{2} \exp\left[-\frac{1}{8}y^2\right] I_3(|y|) . \tag{4.32}$$

The corresponding generalised formula for general odd ν thus reads

$$\wp_{\alpha,\nu}^{(1)}(y) = const. \, \frac{1}{\Gamma(\alpha+1)} \int_0^\infty d\xi \, e^{-\xi} \, \xi^\alpha \sqrt{\frac{\xi}{\alpha b^2}} \left(\frac{|y|}{b}\sqrt{\frac{\xi}{\alpha}}\right)^{(3-\nu)/2} \exp\left[-\frac{1}{8\alpha b^2}\xi y^2\right]$$

$$\times \mathrm{Pf}_{-\frac{\nu}{2}+1 \leq i,j \leq \frac{\nu}{2}-1}\left[(i-j)I_{i+j+3}\left(\frac{|y|}{b}\sqrt{\xi/\alpha}\right)\right] , \tag{4.33}$$

up to the normalisation constant. For the simplest examples $\nu = 0, 1$ and 3 displayed in fig. 5 we have

$$\wp_{\alpha,0}^{(1)}(y) = \frac{1}{4\sqrt{\alpha}\,\Gamma(\alpha+1)b} \int_0^\infty d\xi \, e^{-\xi}\xi^{\alpha+\frac{1}{2}}\left(2 + \frac{|y|}{b}\sqrt{\frac{\xi}{\alpha}}\right) \exp\left[-\frac{|y|}{2b}\sqrt{\frac{\xi}{\alpha}} - \frac{\xi y^2}{8\alpha b^2}\right] , \tag{4.34}$$

$$\wp_{\alpha,1}^{(1)}(y) = \frac{(\alpha+1)}{4\alpha b^2}|y|\left(1 + \frac{y^2}{8\alpha b^2}\right)^{-(\alpha+2)} , \tag{4.35}$$

$$\wp_{\alpha,3}^{(1)}(y) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty d\xi \, e^{-\xi} \, \xi^\alpha \sqrt{\frac{\xi}{\alpha b^2}} \frac{1}{2} \exp\left[-\frac{\xi y^2}{8\alpha b^2}\right] I_3\left(\frac{|y|}{b}\sqrt{\xi/\alpha}\right) . \tag{4.36}$$

18

While the density is modified only rather mildly compared to WL, the first eigenvalue changes considerably. We have checked that the curves converges to WL for large α, where the convergence to the density is much faster than for $\beta = 2$. The case $\nu = 0$ in fig. 5 left is the only example where the microscopic densities do not vanish at $x = 0$. The fact that they both have the same limit $\frac{1}{2}$ can also be seen analytically, exploiting that $J_\nu(0) = \delta_{\nu,0}$ (see eqs. (4.9) and (4.11)). As mentioned above for $\nu = 2k$ with $k \in \mathbb{N}_+$ the first eigenvalue is not available to date.

Finally we turn to $\beta = 4$. In principle the result is known in the WL ensemble,

$$\wp_0^{(4)}(y) \;=\; const. \; |y|^{\nu+\frac{3}{2}} \exp\left[-\frac{1}{2}y^2\right] Z_{3/2}(\{|y|\}_{2\nu+1}) \;. \tag{4.37}$$

Here $Z_{3/2}(\{y\}_{2\nu+1})$ is the large-N matrix model partition function at topological charge $3/2$ with $2\nu+1$ degenerate masses at value y, and we refer to [37] for a more detailed discussion of these objects. This partition function is generally known explicitly only for an even number of masses, except at $\nu = 0$. There we have

$$\wp_0^{(4)}(y) \;=\; \frac{1}{2}\sqrt{2\pi} \; |y|^{\frac{3}{2}} \exp\left[-\frac{1}{2}y^2\right] I_{3/2}(y) \;=\; |y| \left(\cosh(y) - \frac{1}{y}\sinh(y) \right) \exp\left[-\frac{1}{2}y^2\right] \;. \tag{4.38}$$

Thus the generalised distribution depicted in fig. 6 is given by

$$\wp_0^{(4)}(y) \;=\; \frac{1}{\alpha\Gamma(\alpha+1)b^2} \int_0^\infty d\xi \, e^{-\xi} \, \xi^{\alpha+1} \left(|y| \cosh\left(\frac{y}{b}\sqrt{\xi/\alpha}\right) - b\sqrt{\frac{\alpha}{\xi}} \sinh\left(\frac{|y|}{b}\sqrt{\xi/\alpha}\right) \right) \exp\left[-\frac{\xi y^2}{2\alpha b^2}\right] \;. \tag{4.39}$$
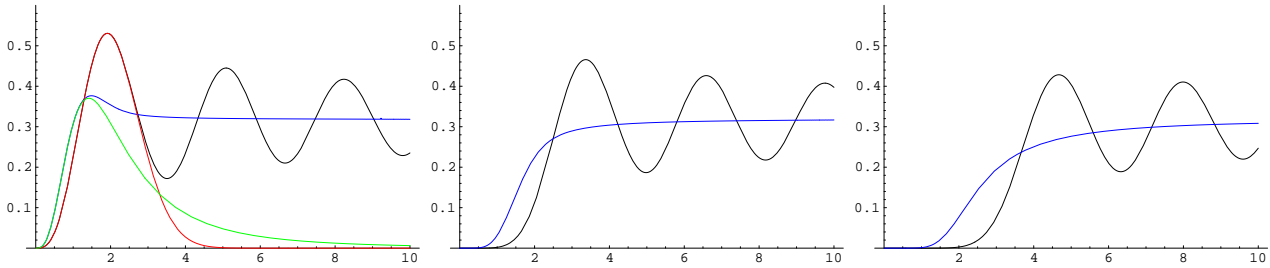

Figure 6: Varying ν for $\beta = 4$: the generalised microscopic density $\vartheta_{\alpha,\nu}^{(4)}(y)$ eq. (4.12) (blue) at $\alpha = 0.1$ vs the corresponding WL Bessel density $\vartheta_\nu^{(4)}(y)$ eq. (4.10) (black) at $\nu = 0$ (left), $\nu = 1$ (middle), and $\nu = 2$ (right). For $\nu = 0$ we also display the respective first eigenvalue for the generalised (green) and WL ensemble (red).

We note that the oscillations get smoothed out as already observed at $\beta = 2$. The convergence in α to the WL quantity is again slow, as for $\beta = 2$.

5   The Wigner surmise in the bulk

In this section we study the spacing distribution between eigenvalues in the bulk in our generalised model. In contrast to the previous sections we do not take N to be large, but use the $N = 2$ results, following the original idea of Wigner.

In the Wigner-Dyson ensembles with a Gaussian potential, the spacing distribution has the simple form

$$\mathcal{P}_{WD}^{(\beta)}(s) \;=\; As^\beta e^{-Bs^2} \;, \tag{5.40}$$

known as Wigner's surmise. The known constants A, B follow from normalisation and can be found e.g. in [1]. For the WL case the corresponding expression can be computed from the jpdf:

$$\mathcal{P}^{(\beta)}(s) = Cs^{\beta+\nu+1/2}K_{-1/2-\nu}(n\beta s) \tag{5.41}$$

where $K_\mu(x)$ is a modified Bessel function and the constant C is given by:

$$C = \left(2^{-1/2+\beta+\nu}(n\beta)^{-3/2-\beta-\nu}\Gamma\left(\frac{1+\beta}{2}\right)\Gamma\left(1+\nu+\frac{\beta}{2}\right)\right)^{-1} \tag{5.42}$$

Only for $\nu = 0$, one recovers the Wigner's surmise, using $K_{-1/2}(x) = \sqrt{\pi/2x}\,e^{-x}$ (apart from squared variables in the exponent there). For general ν one easily gets an exponential decay for large spacing $\sim s^{\beta+\nu}e^{-n\beta s}$, from the asymptotic expression of the Bessel function (which in fact equals $K_{-1/2}(x) = \lim_{x\to\infty} K_\mu(x)$). However, we expect that only the $\nu = 0$ expression will lead to a good approximation of the infinite-N case.

Conversely, for the generalised model the spacing distribution has a different expression:

$$\mathcal{P}_\gamma^{(\beta)}(s) = C_\gamma s^{\beta+\nu}\left(\frac{\gamma}{2n\beta}+\frac{s}{2}\right)^{\nu+1-\gamma}\,{}_2F_1\left(-\nu,\nu+1;-\nu+\gamma;\frac{1}{2}-\frac{\gamma}{2n\beta s}\right) \tag{5.43}$$

where ${}_2F_1(a,b;c;z)$ is a hypergeometric function. The constant C_γ can be computed as

$$C_\gamma = \frac{2^{1-\gamma}B(\nu+1,-2\nu-1+\gamma)\Gamma(\gamma)\Gamma(1+\beta/2)}{\Gamma(\nu+1)\Gamma(1+\beta)\Gamma(\nu+1+\beta/2)\Gamma(\gamma-2-\beta-2\nu)}\left(\frac{n\beta}{\gamma}\right)^{2+2\nu+\beta-\gamma} \tag{5.44}$$

and $B(\alpha,\beta)$ is a Beta function. The large-s decay is now a pure power law $\sim s^{-(\varpi+1)}$ where we have defined

$$\varpi \equiv \gamma - \beta - 2\nu - 2 > 0 , \tag{5.45}$$

required to be positive for convergence[3].

For $\nu = 0$, the spacing distribution takes a much simpler form:

$$\mathcal{P}_\gamma^{(\beta)}(s) = \frac{n(n\beta/\gamma)^\beta\Gamma(\gamma-1)}{\gamma\Gamma(\beta)\Gamma(\gamma-\beta-2)}\,s^\beta\left(1+\frac{sn\beta}{\gamma}\right)^{1-\gamma} . \tag{5.46}$$

It agrees with the corresponding quantity in the generalised WD ensembles found in [29]. In this particular case, we can compute the mean level spacing explicitly:

$$\langle\langle s\rangle\rangle_\gamma = \int_0^\infty s\,\mathcal{P}_\gamma^{(\beta)}(s)ds = \left(\frac{\gamma}{n\beta}\right)\frac{1+\beta}{\gamma-\beta-3} , \tag{5.47}$$

which converges to the WL Wigner surmise value for $\gamma \to \infty$:

$$\langle\langle s\rangle\rangle = \int_0^\infty s\,\mathcal{P}^{(\beta)}(s)ds = \frac{1+\beta}{n\beta} . \tag{5.48}$$

After defining the rescaled quantities having mean spacing 1,

$$\hat{\mathcal{P}}_\gamma^{(\beta)}(x) = \langle\langle s\rangle\rangle_\gamma\mathcal{P}_\gamma^{(\beta)}\left(\langle\langle s\rangle\rangle_\gamma x\right) \tag{5.49}$$

$$\hat{\mathcal{P}}^{(\beta)}(x) = \langle\langle s\rangle\rangle\mathcal{P}^{(\beta)}\left(\langle\langle s\rangle\rangle x\right) \tag{5.50}$$

---

[3]For $\beta \neq 2$ this condition is different from eq. (2.2) derived for the partition function.

we can compare the curves for all three β at $\nu = 0$ (and $n = 1$), in fig. 7. The power-law tail modification compared to the standard WL spacing distribution is evident in the plots. Because in the large-N limit γ scales with N, we keep the combination ϖ in eq. (5.45) fixed to be able to compare to spacing distributions at large-N.

Both in this and the previous section the power-law tail of the macroscopic density is seen to persist on the microscopic level of the mean level-spacing. It would be very interesting to confirm this on real data sets.
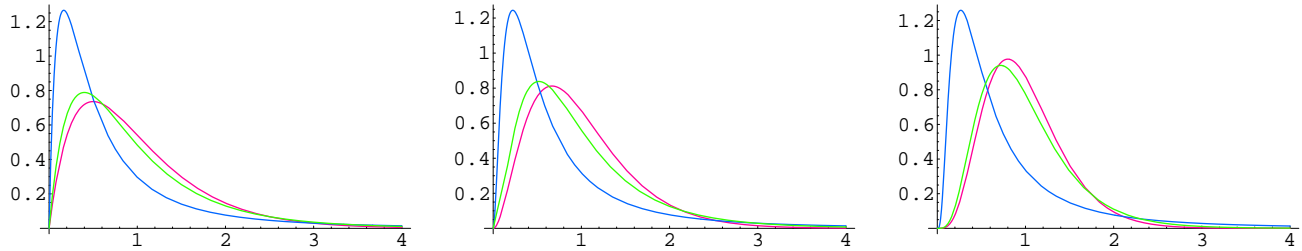

Figure 7: Comparison between $\hat{\mathcal{P}}_\gamma^{(\beta)}(x)$ (blue, green) and $\hat{\mathcal{P}}^{(\beta)}(x)$ (red), for $\beta = 1, 2, 4$ (from left to right). The γ value for the blue curves is chosen is chosen in such a way that the combination ϖ is kept constant to the value 2. The green curves have value $\gamma = 12, 12, 25$ from left to right, and correctly approach the limiting WL curve.

6 Conclusions and outlook

We have introduced a generalisation of all three ensembles of random matrices called Wishart-Laguerre or chiral ensembles, replacing the exponential of a non-Gaussian potential by a fat-tailed distribution with parameter γ. In the limit $\gamma \to \infty$ we can recover the exponential weight and thus the standard ensembles. This modification lead to the appearance correlations with a power-law, governed by a single parameter. Such a behaviour is found in many systems in nature, e.g. in the wide area of complex networks. WL ensembles are often used in comparing to eigenvalues from covariance matrices of real data sets. To illustrate the potential of our generalisation we show a comparison to financial data in fig. 8. The eigenvalues of the covariance matrix from time series of stock data clearly show a power-law behaviour. These are well described by the generalised Marčenko-Pastur density, refining previous comparisons to the standard Marčenko-Pastur law.

The solution of our generalised models relied heavily on the possibility of writing them as an integral transform of the standard WL ensembles. This generalisation is thus in the spirit of super-statistics where other models have been constructed already. The virtue of our model is its invariance for all three symmetry classes, allowing to go to an eigenvalues basis and to study universality. We could show that the generalised macroscopic density which was known in the Gaussian case is only weakly universal.

In contrast all microscopic densities are universal under any invariant deformations by polynomial potentials. This macroscopic/microscopic dichotomy should not come as a surprise, being observed previously for the restricted trace ensembles.

We exploited the linear relation to standard WL to solve our model exactly at finite-N for any polynomial potential, using the formalism of orthogonal polynomials therein. In the subsequent large-N double scaling limit, where γ is scaled with N, we derived all density correlations in the macroscopic limit for quadratic and rectangular matrices, and in the microscopic limit at the hard edge for all

three values of β. Here we have mainly focused on the spectral density itself and the first eigenvalue distribution. The general formalism for higher density or higher individual eigenvalue correlation functions was provided and is straightforward to use if such quantities will be needed.

While the hard edge was solved exhaustively, persisting in our model for asymptotically quadratic matrices, we only provided a Wigner surmise in the bulk. Here, more detailed properties of correlation functions could be investigated, including a possible generalisation of the Tracy-Widom distribution at the soft edges. This is left for future investigations.
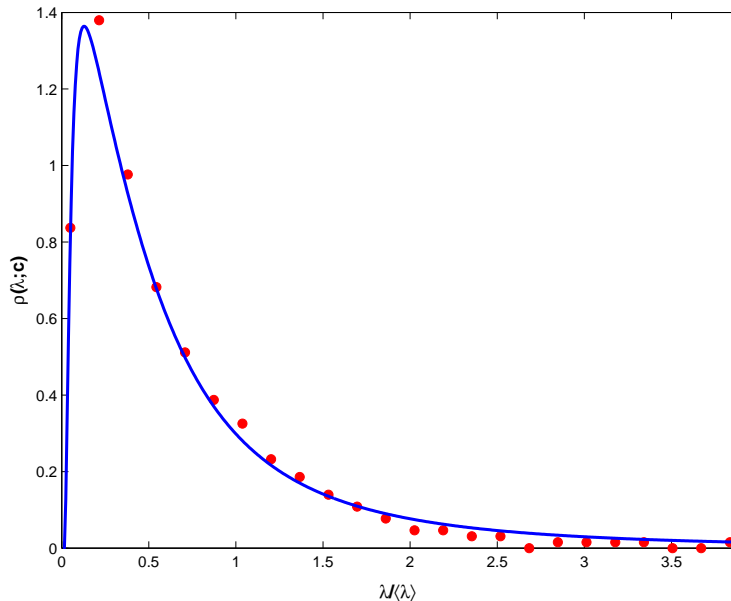

Figure 8: Comparison between the rescaled eigenvalue distribution from financial data [42] and the macroscopic density $\rho_\alpha(x)$ for the generalised model eq. (3.23), in red dots and solid blue respectively. The best fit gives a value of $\alpha \approx 0.95$, which corresponds to a power-law decay as $\rho_\alpha(x) \sim x^{-2.95}$.

A   Partition function and first moment of the Gaussian models

The purpose of this appendix is threefold. First, we derive the condition under which our generalised model defined in eq. (2.1) is convergent. Second, we compute both the generalised and standard Gaussian partition functions in order to determine their ξ-dependent ratio needed in the computation of all eigenvalue correlation functions. Third, we compute the first moment as a function of N, ν, β and γ which is needed for the rescaling of the eigenvalues in the large-N limit.

All three steps will be performed by changing variables from independent matrix elements or its eigenvalues to radial coordinates, following [32].

The generalised partition function reads in terms of eigenvalues

$$
\mathcal{Z}_\gamma \;=\; \int_0^\infty \prod_{i=1}^N d\lambda_i \; \prod_{i=1}^N \lambda_i^{\frac{1}{2}\beta(\nu+1)-1} \prod_{j>k}^N |\lambda_j - \lambda_k|^\beta \frac{1}{\left(1 + \frac{n\beta}{\gamma}\sum_{i=1}^N V(\lambda_i)\right)^\gamma} \tag{A.1}
$$

$$
=\; \int_{\Omega(N)} da_N \int_0^\infty dr\, r^{N-1} r^{N(\frac{1}{2}\beta(\nu+1)-1)} \prod_{i=1}^N \left(\frac{\lambda_i}{r}\right)^{\frac{1}{2}\beta(\nu+1)-1} r^{\frac{N(N-1)}{2}\beta} \frac{\prod_{j>k}^N \left|\frac{\lambda_j}{r} - \frac{\lambda_k}{r}\right|^\beta}{\left(1 + \frac{n\beta}{\gamma}\sum_{i=1}^N V(r\frac{\lambda_i}{r})\right)^\gamma} \;.
$$

Here we have changed to radial coordinates of the N-component vector of the eigenvalues $(\lambda_1, \ldots, \lambda_N)$, and da_N denotes the angular integration over the N-dimensional unit sphere $\Omega(N)$. The eigenvectors $e_i = \lambda_i/r$ of norm unity span $\Omega(N)$ and no longer depend on the radius. Collecting all powers of r in the numerator and comparing to the leading power of the denominator $d\gamma$ at large $r \gg 1$, the integral only converges if the following inequality holds:

$$
\frac{\beta}{2}N(N+\nu) - 1 - \gamma d \;<\; -1 \;, \tag{A.2}
$$

which is exactly eq. (2.2).

Next we compute the partition function $\mathcal{Z}(\xi)$, where for the rest of this appendix we restrict ourselves to the Gaussian potential $V(\lambda) = \lambda$. The same steps can be taken for a purely monic potential $V(\lambda) = \lambda^d$ as well.

In principle we could repeat the same calculation in terms of eigenvalues as above, but it will be more instructive to start directly from the matrix elements:

$$
\mathcal{Z}(\xi) \;=\; \int d\mathbf{X} \exp\left[-\xi \frac{n\beta}{\gamma}\mathrm{Tr}\mathbf{X}^\dagger\mathbf{X}\right]
$$

$$
=\; \int_{\Omega(\beta N(N+\nu))} da_{\beta N(N+\nu)} \int_0^\infty dr\, r^{\beta N(N+\nu)-1} \exp\left[-\xi\frac{n\beta}{\gamma}r^2\right]
$$

$$
=\; \frac{1}{2}\left(\frac{\gamma}{\xi n\beta}\right)^{\frac{\beta}{2}N(N+\nu)} \Gamma\left(\frac{\beta}{2}N(N+\nu)\right) \int_{\Omega(\beta N(N+\nu))} da_{\beta N(N+\nu)} \;. \tag{A.3}
$$

Here we have used radial coordinates for the $\beta N(N+\nu)$ component vector of all independent matrix elements $\mathbf{X}_{ij}$, with squared norm $r^2 = \mathrm{Tr}\mathbf{X}^\dagger\mathbf{X}$. We don't need to compute the angular integral explicitly as it cancels out below.

If we insert the result eq. (A.3) into the relation (2.8) we immediately obtain

$$
\mathcal{Z}_\gamma \;=\; \frac{1}{\Gamma(\gamma)}\int_0^\infty d\xi\, e^{-\xi}\, \xi^{\gamma-1} \mathcal{Z}(\xi) \tag{A.4}
$$

$$
=\; \frac{1}{2\Gamma(\gamma)}\left(\frac{\gamma}{\beta n}\right)^{\frac{\beta}{2}N(N+\nu)} \Gamma\left(\gamma - \frac{\beta}{2}N(N+\nu)\right)\Gamma\left(\frac{\beta}{2}N(N+\nu)\right) \int_{\Omega(\beta N(N+\nu))} da_{\beta N(N+\nu)} \;.
$$

Combining the last two equations we arrive at

$$
\frac{\mathcal{Z}(\xi)}{\Gamma(\gamma)\mathcal{Z}_\gamma} \;=\; \frac{\xi^{-\frac{\beta}{2}N(N+\nu)}}{\Gamma\left(\gamma - \frac{\beta}{2}N(N+\nu)\right)} \;, \tag{A.5}
$$

the ratio of the two Gaussian partition functions at finite values of N, ν and γ valid for all three β.

In the last step of this appendix we compute the mean eigenvalue position in the Gaussian model. It can be either defined through the spectral density, see eq. (3.3), or in terms of the first moment, where we start with the standard WL ensembles:

$$
\begin{aligned}
\langle \lambda(\xi) \rangle &= \frac{1}{\mathcal{Z}(\xi)} \int d\mathbf{X} \frac{1}{N} \mathrm{Tr} \left(\mathbf{X}^\dagger \mathbf{X} \right) \exp \left[-\xi \frac{n\beta}{\gamma} \mathrm{Tr} \mathbf{X}^\dagger \mathbf{X} \right] \\
&= \frac{1}{N \mathcal{Z}(\xi)} \int_{\Omega(\beta N(N+\nu))} da_{\beta N(N+\nu)} \int_0^\infty dr \, r^{\beta N(N+\nu)-1} r^2 \exp \left[-\xi \frac{n\beta}{\gamma} r^2 \right] \\
&= \frac{\gamma}{2n\xi} (N + \nu) \, .
\end{aligned}
\tag{A.6}
$$

Note that the β-dependence has cancelled out. We can immediately use this result to compute the same quantity for the generalised Gaussian model,

$$
\begin{aligned}
\langle \lambda \rangle_\gamma &= \frac{1}{\mathcal{Z}_\gamma} \int d\mathbf{X} \frac{\frac{1}{N} \mathrm{Tr} \left(\mathbf{X}^\dagger \mathbf{X} \right)}{\left(1 + \frac{n\beta}{\gamma} \mathrm{Tr} \mathbf{X}^\dagger \mathbf{X} \right)^\gamma} \\
&= \frac{1}{N \mathcal{Z}_\gamma \Gamma(\gamma)} \int_0^\infty d\xi \, e^{-\xi} \, \xi^{\gamma-1} \mathcal{Z}(\xi) \langle \lambda(\xi) \rangle \\
&= \frac{\gamma(N + \nu)}{2n \left(\gamma - \frac{\beta}{2} N(N + \nu) - 1 \right)} \, .
\end{aligned}
\tag{A.7}
$$

This result is used in the rescaling of both large-N limits. As a check it reduces to eq. (A.6) in the limit $\gamma \to \infty$, with a weight $\exp[-n\beta r^2]$.

B Explicit $\beta = 2$-solution for all k-point densities at finite N and γ

In this appendix we present all details for the solution of the generalized WL ensemble with unitary invariance $\beta = 2$ and Gaussian potential $V(\lambda) = \lambda$. In this case the orthogonal polynomials of the WL ensemble are know to be Laguerre, allowing for an explicit solution at finite N and finite γ.

The orthogonal polynomials in eq. (2.13) read for the weight function $\exp[-\xi \frac{2n}{\gamma} \lambda]$

$$
P_k(\lambda) = (-)^k k! \left(\frac{\gamma}{2n\xi} \right)^k L_k^\nu \left(\frac{2n\xi}{\gamma} \lambda \right) \, ,
\tag{B.1}
$$

with norms

$$
h_k = k! \, (k + \nu)! \left(\frac{\gamma}{2n\xi} \right)^{2k+\nu+1} \, .
\tag{B.2}
$$

Here the Laguerre polynomials are defined as usual

$$
L_k^\nu(z) = \sum_{j=0}^k (-1)^j \binom{k + \nu}{k - j} \frac{z^j}{j!} \, , \quad \text{with} \quad L_k^\nu(z)' = -L_{k-1}^{\nu+1}(z) \, .
\tag{B.3}
$$

Using eq. (2.18) we can immediately read off the WL partition function from the norms,

$$
\mathcal{Z}(\xi) = \left(\frac{\gamma}{2n\xi} \right)^{N\nu+N^2} \prod_{k=0}^{N-1} (k + 1)! \, (k + \nu)! \, .
\tag{B.4}
$$

As the next step we can compute the partition function eq. (2.8) given by

$$
\begin{aligned}
\mathcal{Z}_\gamma &= \frac{1}{\Gamma(\gamma)} \int_0^\infty d\xi \; e^{-\xi} \; \xi^{\gamma-1} \left(\frac{\gamma}{2n\xi}\right)^{N\nu+N^2} \prod_{k=0}^{N-1} (k+1)! \, (k+\nu)! \\
&= \left(\frac{\gamma}{2n}\right)^{N\nu+N^2} \frac{\Gamma(\gamma - N(N+\nu))}{\Gamma(\gamma)} \prod_{k=0}^{N-1} (k+1)! \, (k+\nu)! \; .
\end{aligned}
\tag{B.5}
$$

This leads to the following ratio needed for example inside the relation (2.11)

$$
\frac{\mathcal{Z}(\xi)}{\Gamma(\gamma)\mathcal{Z}_\gamma} = \xi^{-N(N+\nu)} \frac{1}{\Gamma(\gamma - N(N+\nu))} \; .
\tag{B.6}
$$

It confirms independently part of the result from the previous appendix, eq. (A.5) for $\beta = 2$.

The spectral density for finite N follows by inserting this ratio as well as the standard Laguerre density at finite-N,

$$
R(\lambda;\xi) = \lambda^\nu \; e^{-\xi \frac{2n}{\gamma}\lambda} \sum_{k=0}^{N-1} \frac{k!}{(k+\nu)!} \left(\frac{2n\xi}{\gamma}\right)^{\nu+1} L_k^\nu \left(\frac{2n\xi}{\gamma}\lambda\right)^2 \; ,
\tag{B.7}
$$

into (2.11):

$$
R_\gamma(\lambda) = \frac{1}{\Gamma(\gamma - N(N+\nu))} \int_0^\infty d\xi \; e^{-\xi} \; \xi^{\gamma-1-N(N+\nu)} R(\lambda;\xi) \; .
\tag{B.8}
$$

With the help of eq. (B.3) we can derive and simplify the Christoffel-Darboux identity for Laguerre polynomials of equal arguments

$$
\sum_{k=0}^{N-1} \frac{k!}{\Gamma(k+1+\nu)} L_k^\nu(y)^2 = \frac{N!}{\Gamma(N+\nu)} \left[L_{N-1}^\nu(y) L_{N-1}^{\nu+1}(y) - L_N^\nu(y) L_{N-2}^{\nu+1}(y) \right] \; .
\tag{B.9}
$$

We thus arrive at our final result for the generalised density at finite N and γ:

$$
\begin{aligned}
R_\gamma(\lambda) &= \frac{N!}{\Gamma(\gamma - N^2 - N\nu)\Gamma(N+\nu)} \left(\frac{2n}{\gamma}\right)^{\nu+1} \lambda^\nu \int_0^\infty d\xi \; e^{-\xi\left(1+\frac{2n}{\gamma}\lambda\right)} \xi^{\gamma-N^2-N\nu+\nu} \\
&\quad \times \left[L_{N-1}^\nu \left(\frac{2n\xi}{\gamma}\lambda\right) L_{N-1}^{\nu+1} \left(\frac{2n\xi}{\gamma}\lambda\right) - L_N^\nu \left(\frac{2n\xi}{\gamma}\lambda\right) L_{N-2}^{\nu+1} \left(\frac{2n\xi}{\gamma}\lambda\right) \right] \; .
\end{aligned}
\tag{B.10}
$$

This single integral over an exponential times polynomials can be performed explicitly, at the expense of a double sum. Since this equivalent result is not very illuminating or useful for the asymptotic we do not display it here.

Proceeding along the same lines as above we can write down the general result for the k-point density correlations functions as they follow from eq. (2.17)

$$
\begin{aligned}
R_\gamma(\lambda_1,\ldots,\lambda_k) &= \frac{N!^k \left(\frac{n}{\gamma}\right)^{k\nu}}{\Gamma(\gamma - N^2 - N\nu)\Gamma(N+\nu)^k} \prod_{j=1}^k \lambda_j^\nu \int_0^\infty d\xi \; e^{-\xi\left(1+\frac{2n}{\gamma}\sum_{j=1}^k \lambda_j\right)} \xi^{\gamma-1-N^2-(N-k)\nu} \\
&\quad \times \det_{1\le i,j\le k} \left[\frac{\left(L_N^\nu \left(\frac{2n\xi}{\gamma}\lambda_i\right) L_{N-1}^\nu \left(\frac{2n\xi}{\gamma}\lambda_j\right) - L_N^\nu \left(\frac{2n\xi}{\gamma}\lambda_j\right) L_{N-1}^\nu \left(\frac{2n\xi}{\gamma}\lambda_i\right) \right)}{\lambda_j - \lambda_i} \right] \; .
\end{aligned}
\tag{B.11}
$$

In order to compare the finite-N result (B.10) with the macroscopic γ- and N-independent density $\vartheta_\alpha(x)$ eq. (3.13), we adopt the following procedure:

1. We rescale $R_\gamma(\lambda)$ with mean value $\langle\lambda\rangle_\gamma$ eq. (3.3) and normalise to 1: $\hat{\rho}_\gamma(x) \equiv N^{-1}\langle\lambda\rangle_\gamma R_\gamma\left(\langle\lambda\rangle_\gamma x\right)$.

2. Next, we express γ as a function of α and N, $\gamma = \alpha + N(N + \nu) + 1$, and pass to squared variables: $\hat{\vartheta}_\alpha(x) \equiv |x|\hat{\rho}_\gamma(x^2)$.

3. Then, we compare $\hat{\vartheta}_\alpha(x)$ and $\vartheta_\alpha(x)$ for $\nu = 0$ in Fig. 9. The agreement is already very good for $N = 4$, apart from the region close to the origin.
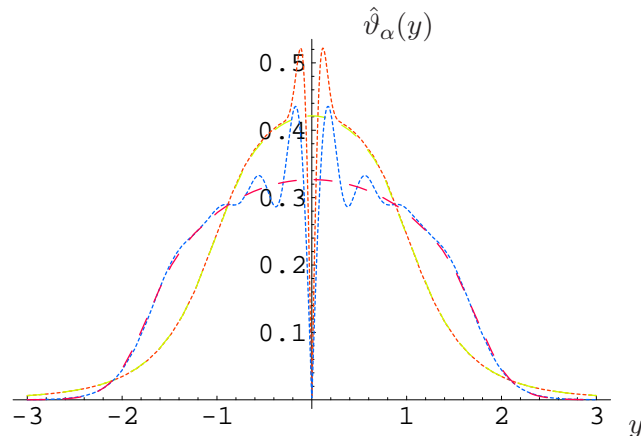

Figure 9: The macroscopic generalised semi-circle density $\vartheta_\alpha(x)$ eq. (3.13) for $\alpha = 1.02$ and 14 (green and dashed red), compared with the finite $N = 4$ result $\hat{\vartheta}_\alpha(x)$ (blue and dash-dotted orange).

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