Minimum-Variance State and Fault Estimation for Multi-Rate Systems with Dynamical Bias

Yuxuan Shen, Zidong Wang and Hongli Dong

***Abstract*—This paper is concerned with the joint state and fault estimation problem for a class of multi-rate systems with dynamical bias. To reflect real practice, the multi-rate sampling is considered which allows the sensor sampling rate and the state update rate to be different. The sensor is subject to the sensor fault that changes according to a dynamic equation. Instead of applying the traditional lifting technique, we introduce a time-varying delay into the measurement equation so as to transform the multi-rate systems into single-rate ones. The aim of this paper is to develop a joint state and fault estimation algorithm with minimized estimation error covariance. The recursion of the estimation error covariance is first derived, and appropriate estimator gains are then characterized that minimizes the estimation error covariance. A simulation example on the DC servo system is given to confirm the usefulness of the developed recursive state and fault estimation algorithm.**

***Index Terms*—Fault estimation, sensor fault, multi-rate sampling, dynamical bias.**

I. Introduction

In the area of signal processing and control engineering, state estimation has been a long-standing research topic that has received considerable research interest [1], [2], [21]. To date, plenty of research results have been obtained on the state estimation problems where the developed algorithms can be generally classified into H_∞, Kalman, set-membership and moving-horizon state estimation approaches [11]–[13], [19]. In industrial systems such as aluminium electrolysis cells and power networks, owing to the diverse physical features of the system components, it is often the case that the state update rate is different from the sensor sampling rate, i.e., the system is a *multi-rate system* (MRS) [15]. For MRSs, the state estimation algorithms developed for single-rate systems (SRSs) are no longer applicable, and this triggers the recent research attention on the state estimation algorithms for MRSs [4], [16], [17], [20]. Up to now, most available results for MRSs have been obtained based on the traditional lifting technique which leads to a high computation burden due to the augmentation.

In engineering practice, the measurement output of the sensors are often subject to abrupt changes due to a variety of reasons such as sensor aging and random sensor failure. Such a phenomenon, customarily known as sensor fault, would largely degrade the estimation performance and it is therefore necessary to acquire the information of the sensor fault with hope to mitigate the impact from the sensor fault. Recently, the fault estimation problems for SRSs with sensor fault have been widely investigated [3], [9]. In [5], the joint estimation problem for the state and the sensor fault has been studied for discrete-time systems and, subsequently, a fault-tolerant controller design scheme has been proposed. Unfortunately, the joint state and fault estimation (JSFE) problem for MRSs has received little attention despite its practical significance, and this gives rise to the main motivation of our current investigation.

In practical systems, it is quite common that the system noises consist of white noises and the strongly correlated noises, where the latter are called *random biases* that could be either constant or dynamic [14]. As early as in 1990, the random bias has been characterized by a dynamical equation in [7] where the joint state and random bias estimation problem has been considered. Thereafter, the state estimation problems for SRSs with random bias have received considerable research attention [8], [10], [18]. For example, in [18], the state estimation problem has been studied for a class of two-dimensional systems with random bias and measurement quantization, and a recursive state estimation algorithm has been designed. Note that the corresponding state estimation problems for MRSs with random bias have not been considered yet, and this constitutes another motivation of this paper.

Motivated by the above discussions, in this paper, we aim to solve the JSFE problem for MRSs subject to dynamical bias. The main contributions of this paper are: *1) the JSFE estimation problem is, for the first time, studied for MRSs with dynamical bias where the considered fault model is quite general; 2) different from the traditional lifting technique that often leads to high computational burden, a novel method is put forward to convert the MRSs into SRSs through the introduction of a time-varying delay into the measurement equation; and 3) the proposed JSFE algorithm is in the recursive form and therefore suitable for online application.*

Notation The notation used here is fairly standard. G^T and G^{-1} represent the transpose and the inverse of the matrix G, respectively. $\mathbb{E}\{\alpha\}$ represents the expectation of the random variable α. The Kronecker delta function $\delta(m,n)$ is a binary function that equals 1 if $m=n$, and equals 0 otherwise.

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Y. Shen and H. Dong are with the Artificial Intelligence Energy Research Institute, Northeast Petroleum University, Daqing 163318, China, and are also with the Heilongjiang Provincial Key Laboratory of Networking and Intelligent Control, Northeast Petroleum University, Daqing 163318, China. (Email: shiningdhl@vip.126.com)

Z. Wang is with the Department of Computer Science, Brunel University London, Uxbridge, Middlesex, UB8 3PH, United Kingdom. (Email: Zidong.Wang@brunel.ac.uk)

II. Problem formulation

Consider the following class of discrete-time systems:

$$x(s+1) = A(s)x(s) + B(s)w(s) + E(s)h(s) \tag{1}$$

where $x(s) \in \mathbb{R}^{n_x}$ is the system state and $w(s) \in \mathbb{R}^{n_w}$ is the process noise. The initial value $x(0)$ is a random variable with the mean $\bar{x}(0)$ and the covariance $X(0)$. $h(s) \in \mathbb{R}^{n_h}$ stands for the random bias with the following dynamics:

$$h(s+1) = H(s)h(s) + \lambda(s) \tag{2}$$

where $\lambda(s) \in \mathbb{R}^{n_h}$ is a zero-mean Gaussian sequence with the covariance $\Lambda(s) > 0$. The initial value $h(0)$ of the random bias is a zero-mean Gaussian random variable with the covariance $\Pi(0)$. $H(s)$, $A(s)$, $B(s)$, and $E(s)$ are known time-varying matrices with compatible dimensions.

In this paper, the sampling period of the sensor is $b \triangleq l_{m+1} - l_m$ where $b \geq 1$ is a positive integer. l_m is the sampling instant of the sensor with m being the order number of the sampling instant. The measurement model of the sensor is

$$y(l_m) = C(l_m)x(l_m) + D(l_m)v(l_m) + F(l_m)f(l_m) \tag{3}$$

where $y(l_m) \in \mathbb{R}^{n_y}$ is the measurement output and $v(l_m) \in \mathbb{R}^{n_v}$ is the measurement noise. $C(l_m)$, $D(l_m)$, and $F(l_m)$ are known time-varying matrices with compatible dimensions. $f(l_m)$ is the unknown sensor fault evolving according to [6]

$$f(l_{m+1}) = G(l_m)f(l_m) \tag{4}$$

where $G(l_m)$ is a known time-varying matrix with compatible dimensions.

The process noise $w(s)$ and the measurement noise $v(l_m)$ satisfy

$$\begin{aligned} &\mathbb{E}\{w(s)\} = 0,\ \mathbb{E}\{w(s)w^T(t)\} = W(s)\delta(s,t), \\ &\mathbb{E}\{v(l_m)\} = 0,\ \mathbb{E}\{v(l_m)v^T(l_n)\} = V(l_m)\delta(l_m, l_n) \end{aligned}$$

where $W(s)$ and $V(l_m)$ are known time-varying matrices with compatible dimensions.

Assumption 1: The random variables $x(0)$, $h(0)$, $\lambda(s)$, $w(s)$, and $v(l_m)$ are mutually independent.

In literature, a typical approach to dealing with MRSs is to use the lifting technique to obtain an augmented state equation with an increased state update period, which gives rise to heavy computational load. In this paper, instead of utilizing the lifting technique, we aim to reconstruct a new measurement equation with a decreased sampling period. Here, the zero-order hold strategy is adopted to compensate the measurements at the non-sampling instants of the sensor.

With the zero-order hold strategy, the actual measurement used by the estimator is

$$\bar{y}(s) \triangleq y(\bar{l}_n), \quad \bar{l}_n \leq s < \bar{l}_{n+1}$$

with $\bar{l}_n$ being the largest measurement sampling instant that is not larger than s and $\bar{l}_{n+1} \triangleq \bar{l}_n + b$.

Define $\rho_s \triangleq s - \bar{l}_n$ ($\bar{l}_n \leq s < \bar{l}_{n+1}$). Then, the measurement $\bar{y}(s)$ is rewritten as

$$\begin{aligned} \bar{y}(s) = &C(s-\rho_s)x(s-\rho_s) + D(s-\rho_s)v(s-\rho_s) \\ &+ F(s-\rho_s)f(s-\rho_s). \end{aligned} \tag{5}$$

In the following, we introduce a new notation

$$\bar{f}(s) \triangleq f(\bar{l}_n), \quad \bar{l}_n \leq s < \bar{l}_{n+1}. \tag{6}$$

Similarly, one has $\bar{f}(s) = f(s-\rho_s)$ for $\bar{l}_n \leq s < \bar{l}_{n+1}$. From the definition of $\bar{f}(s)$, it is obvious that

$$\bar{f}(s+1) = \begin{cases} \bar{f}(s), & \text{if } \bar{l}_n < s+1 < \bar{l}_{n+1} \\ G(s-b+1)\bar{f}(s), & \text{if } s+1 = \bar{l}_{n+1}. \end{cases}$$

Then, $\bar{f}(s+1)$ is further rewritten as

$$\bar{f}(s+1) = \bar{G}(s)\bar{f}(s)$$

where

$$\bar{G}(s) \triangleq (1-\delta(s+1, \bar{l}_{n+1}))I + \delta(s+1, \bar{l}_{n+1})G(s-b+1)$$

and $\delta(\cdot,\cdot)$ is the Kronecker delta function. Furthermore, with (6), $\bar{y}(s)$ is rewritten as

$$\begin{aligned} \bar{y}(s) = &C(s-\rho_s)x(s-\rho_s) + D(s-\rho_s)v(s-\rho_s) \\ &+ F(s-\rho_s)\bar{f}(s). \end{aligned} \tag{7}$$

Remark 1: In this paper, to estimate the fault, the dynamics of the fault are required to be known [6]. Nevertheless, noting that the update period of the fault is b, it is impossible to obtain the relationship between $f(s+1)$ and $f(s)$. To solve such a problem, in this paper, we introduce a new notation $\bar{f}(s)$ that satisfies (6), and the relationship between $\bar{f}(s+1)$ and $\bar{f}(s)$ is known. Then, the dynamics of $\bar{f}(s)$ is obtained and an estimator can be designed to estimate $\bar{f}(s)$. Note that, with the help of (6), the estimate of the fault $f(l_m)$ can be easily obtained from the estimate of $\bar{f}(s)$.

From (1) and (7), it is obvious that the MRS is now transformed into a SRS with the time-varying delay ρ_s. To tackle the addressed JSFE problem, we design the estimator of the following form:

$$\begin{cases} \hat{x}(s+1) = A(s)\hat{x}(s) + E(s)\hat{h}(s) + K_1(s)\Big(\bar{y}(s) - \\ \qquad C(s-\rho_s)\hat{x}(s-\rho_s) - F(s-\rho_s)\hat{f}(s)\Big) \\ \hat{h}(s+1) = H(s)\hat{h}(s) + K_2(s)\Big(\bar{y}(s) - \\ \qquad C(s-\rho_s)\hat{x}(s-\rho_s) - F(s-\rho_s)\hat{f}(s)\Big) \\ \hat{f}(s+1) = \bar{G}(s)\hat{f}(s) + K_3(s)\Big(\bar{y}(s) - \\ \qquad C(s-\rho_s)\hat{x}(s-\rho_s) - F(s-\rho_s)\hat{f}(s)\Big) \end{cases} \tag{8}$$

where $\hat{x}(s)$ is the estimate of the state $x(s)$, $\hat{h}(s)$ is the estimate of the bias $h(s)$, $\hat{f}(s)$ is the estimate of $\bar{f}(s)$, and $K_1(s)$, $K_2(s)$, $K_3(s)$ are the estimator gain matrices to be designed. Moreover, we set $\hat{x}(0) = \bar{x}(0)$ and $\hat{h}(0) = \hat{f}(0) = 0$.

Denoting the state estimation error as $e_x(s) \triangleq x(s) - \hat{x}(s)$, the bias estimation error as $e_h(s) \triangleq h(s) - \hat{h}(s)$, and the fault estimation error as $e_f(s) \triangleq \bar{f}(s) - \hat{f}(s)$, we have

$$\begin{aligned} e_x(s+1) = &A(s)e_x(s) + B(s)w(s) + E(s)e_h(s) \\ &- K_1(s)\Big(C(s-\rho_s)e_x(s-\rho_s) \\ &+ D(s-\rho_s)v(s-\rho_s) + F(s-\rho_s)e_f(s)\Big) \\ e_h(s+1) = &H(s)e_h(s) + \lambda(s) \end{aligned}$$

$$
\begin{aligned}
&- K_2(s)\Big(C(s-\rho_s)e_x(s-\rho_s)\\
&+ D(s-\rho_s)v(s-\rho_s) + F(s-\rho_s)e_f(s)\Big)\\
e_f(s+1) =& \bar{G}(s)e_f(s) - K_3(s)\Big(C(s-\rho_s)e_x(s-\rho_s)\\
&+ D(s-\rho_s)v(s-\rho_s) + F(s-\rho_s)e_f(s)\Big).
\end{aligned}
$$

Denoting $e(s) \triangleq \begin{bmatrix} e_x^T(s) & e_h^T(s) & e_f^T(s) \end{bmatrix}^T$, we have the following augmented system

$$
\begin{aligned}
e(s+1) = &\tilde{A}(s)e(s) - \bar{K}(s)\bar{C}(s-\rho_s)e(s-\rho_s) + \bar{I}\lambda(s)\\
&- \bar{K}(s)D(s-\rho_s)v(s-\rho_s) + \bar{B}(s)w(s) \qquad (9)
\end{aligned}
$$

where

$$
\begin{aligned}
&\tilde{A}(s) \triangleq \bar{A}(s) - \bar{K}(s)\bar{F}(s-\rho_s),\\
&\bar{A}(s) \triangleq \begin{bmatrix} A(s) & E(s) & 0\\ 0 & H(s) & 0\\ 0 & 0 & \bar{G}(s)\end{bmatrix},\\
&\bar{K}(s) \triangleq \begin{bmatrix} K_1^T(s) & K_2^T(s) & K_3^T(s)\end{bmatrix}^T,\\
&\bar{F}(s-\rho_s) \triangleq \begin{bmatrix} 0 & 0 & F(s-\rho_s)\end{bmatrix},\ \bar{I} \triangleq \begin{bmatrix} 0 & I & 0\end{bmatrix}^T\\
&\bar{B}(s) \triangleq \begin{bmatrix} B^T(s) & 0 & 0\end{bmatrix}^T, \bar{C}(s-\rho_s) \triangleq \begin{bmatrix} C(s-\rho_s) & 0 & 0\end{bmatrix}.
\end{aligned}
$$

The aim of this paper is to design the estimator (8) such that the estimation error covariance (EEC) $P(s) \triangleq \mathbb{E}\{e(s)e^T(s)\}$ is minimized.

III. MAIN RESULTS

Lemma 1: The EEC $P(s+1)$ is calculated by the following recursion:

$$
\begin{aligned}
P(s+1) =& \tilde{A}(s)P(s)\tilde{A}^T(s) + \bar{B}(s)W(s)\bar{B}^T(s) + \bar{I}\Lambda(s)\bar{I}^T\\
&+ \bar{K}(s)\bar{C}(s-\rho_s)P(s-\rho_s)\bar{C}^T(s-\rho_s)\bar{K}^T(s)\\
&+ \bar{K}(s)D(s-\rho_s)V(s-\rho_s)D^T(s-\rho_s)\bar{K}^T(s)\\
&- \mathscr{P}_1(s) - \mathscr{P}_1^T(s) - \mathscr{P}_2(s) - \mathscr{P}_2^T(s) \qquad (10)
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathscr{P}_1(s) \triangleq \tilde{A}(s)\mathbb{E}\{e(s)e^T(s-\rho_s)\}\bar{C}^T(s-\rho_s)\bar{K}^T(s),\\
&\mathscr{P}_2(s) \triangleq \tilde{A}(s)\mathbb{E}\{e(s)v^T(s-\rho_s)\}D^T(s-\rho_s)\bar{K}^T(s).
\end{aligned}
$$

Proof: It is easily known from Assumption 1 and (9) that (10) is true. Therefore, the proof is omitted here. ■

From Lemma 1, we know that the calculation of the EEC needs the calculations of $\mathscr{P}_1(s)$ and $\mathscr{P}_2(s)$, for which some preliminary results are presented as follows.

Lemma 2: $\Omega_1(s) \triangleq \mathbb{E}\{e(s)v^T(s-\rho_s)\}$ is calculated by

$$
\Omega_1(s) = \begin{cases} 0, & \text{for } \rho_s = 0\\ -\Upsilon_1(s), & \text{for } \rho_s > 0 \end{cases} \qquad (11)
$$

where

$$
\begin{aligned}
\Upsilon_1(s) \triangleq& (1-\delta(1,\rho_s))\sum_{i=2}^{\rho_s}\prod_{j=1}^{i-1}\tilde{A}(s-j)\\
&\times \bar{K}(s-i)D(s-\rho_s)V(s-\rho_s)\\
&+ \bar{K}(s-1)D(s-\rho_s)V(s-\rho_s).
\end{aligned}
$$

Proof: From the definition of ρ_s, we know that ρ_s takes values in the set $\{0,1,2,\ldots,b-1\}$ and

$$
\rho_s = \begin{cases} 0, & \text{for } s = \bar{l}_n\\ \rho_{s-1}+1, & \text{for } \bar{l}_n < s < \bar{l}_{n+1}. \end{cases}
$$

The proof of this lemma is divided in the following two cases.

Case 1: $\underline{\rho_s = 0}$. For $\rho_s = 0$, it is obvious that

$$
\mathbb{E}\{e(s)v^T(s-\rho_s)\} = \mathbb{E}\{e(s)v^T(s)\} = 0.
$$

Case 2: $\underline{\rho_s > 0}$. By introducing

$$
\begin{aligned}
\Delta(t) \triangleq& \mathbb{E}\{\bar{K}(t)\bar{C}(t-\rho_t)e(t-\rho_t)v^T(s-\rho_s)\\
&+ \bar{K}(t)D(t-\rho_t)v(t-\rho_t)v^T(s-\rho_s)\},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\mathbb{E}\{e(s)v^T(s-\rho_s)\}\\
=&\tilde{A}(s-1)\mathbb{E}\{e(s-1)v^T(s-\rho_s)\} - \Delta(s-1)\\
=&\prod_{i=1}^{\rho_s}\tilde{A}(s-i)\mathbb{E}\{e(s-\rho_s)v^T(s-\rho_s)\}\\
&- (1-\delta(1,\rho_s))\sum_{i=2}^{\rho_s}\prod_{j=1}^{i-1}\tilde{A}(s-j)\Delta(s-i) - \Delta(s-1).
\end{aligned}
$$

Accordingly, what we need to do is to calculate $\Delta(s-i)$ $(1 \le i \le \rho_s)$. From the definition of ρ_s, we know that $s-\rho_s = \bar{l}_n$. For $1 \le i < \rho_s$, one has

$$
\bar{l}_n = s-\rho_s < s-i \le s-1 = \bar{l}_n + \rho_s - 1 < \bar{l}_{n+1},
$$

and therefore $s-i-\rho_{s-i} = s-\rho_s$. For $i = \rho_s$, it is obvious that $s-i-\rho_{s-i} = s-\rho_s$. Accordingly, $\Delta(s-i)$ $(1 \le i \le \rho_s)$ is rewritten as

$$
\begin{aligned}
\Delta(s-i) =& \mathbb{E}\{\bar{K}(s-i)\bar{C}(s-\rho_s)e(s-\rho_s)v^T(s-\rho_s)\\
&+ \bar{K}(s-i)D(s-\rho_s)v(s-\rho_s)v^T(s-\rho_s)\}\\
=& \bar{K}(s-i)D(s-\rho_s)V(s-\rho_s).
\end{aligned}
$$

Therefore, one has

$$
\begin{aligned}
\mathbb{E}\{e(s)v^T(s-\rho_s)\} =& -(1-\delta(1,\rho_s))\sum_{i=2}^{\rho_s}\prod_{j=1}^{i-1}\tilde{A}(s-j)\\
&\times \bar{K}(s-i)D(s-\rho_s)V(s-\rho_s)\\
&- \bar{K}(s-1)D(s-\rho_s)V(s-\rho_s).
\end{aligned}
$$

The proof is complete. ■

Remark 2: In Lemma 2, instead of simply applying the elementary equality to avoid the calculation of $\mathbb{E}\{e(s)v^T(s-\rho_s)\}$, we have derived the exact form of $\mathbb{E}\{e(s)v^T(s-\rho_s)\}$. It is worth mentioning that the calculation of $\mathbb{E}\{e(s)v^T(s-\rho_s)\}$ is nontrivial due to the existence of the time-varying delay ρ_s.

Lemma 3: The term $\Omega_2(s) \triangleq \mathbb{E}\{e(s)e^T(s-\rho_s)\}$ is recursively calculated by

$$
\Omega_2(s) = \begin{cases} P(s), & \text{for } \rho_s = 0\\ \Upsilon_2(s), & \text{for } \rho_s > 0 \end{cases} \qquad (12)
$$

where

$$
\Upsilon_2(s) \triangleq \tilde{A}(s-1)\Omega_2(s-1)
$$

$$- \bar{K}(s-1)\bar{C}(s-\rho_{s-1}-1)P(s-\rho_{s-1}-1).$$

Proof: The proof of this lemma is similar to that of Lemma 2 and is therefore omitted here. ■

Theorem 1: The estimator gains that minimize the EEC $P(s)$ are given as follows:

$$K_1(s) = \begin{bmatrix} I & 0 & 0 \end{bmatrix} \bar{K}(s), \tag{13}$$
$$K_2(s) = \begin{bmatrix} 0 & I & 0 \end{bmatrix} \bar{K}(s), \tag{14}$$
$$K_3(s) = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \bar{K}(s) \tag{15}$$

where

$$\begin{aligned}
\bar{K}(s) &\triangleq \Psi(s)\Theta^{-1}(s), \\
\Theta(s) &\triangleq \bar{F}(s-\rho_s)P(s)\bar{F}^T(s-\rho_s) \\
&\quad + \bar{C}(s-\rho_s)P(s-\rho_s)\bar{C}^T(s-\rho_s) \\
&\quad + D(s-\rho_s)V(s-\rho_s)D^T(s-\rho_s) \\
&\quad + \bar{F}(s-\rho_s)\Omega_2(s)\bar{C}^T(s-\rho_s) \\
&\quad + \bar{C}(s-\rho_s)\Omega_2^T(s)\bar{F}^T(s-\rho_s) \\
&\quad + \bar{F}(s-\rho_s)\Omega_1(s)D^T(s-\rho_s) \\
&\quad + D(s-\rho_s)\Omega_1^T(s)\bar{F}^T(s-\rho_s), \\
\Psi(s) &\triangleq \bar{A}(s)P(s)\bar{F}^T(s-\rho_s) + \bar{A}(s)\Omega_2(s)\bar{C}^T(s-\rho_s) \\
&\quad + \bar{A}(s)\Omega_1(s)D^T(s-\rho_s).
\end{aligned}$$

Moreover, the minimal EEC is given by

$$\begin{aligned}
P(s+1) = &-\Psi(s)\Theta^{-1}(s)\Psi^T(s) + \bar{A}(s)P(s)\bar{A}^T(s) \\
&+ \bar{B}(s)W(s)\bar{B}^T(s) + \bar{I}\Lambda(s)\bar{I}^T.
\end{aligned}$$

Proof: With the help of Lemmas 1-3, one has

$$\begin{aligned}
P(s+1) =& \bar{K}(s)\Theta(s)\bar{K}^T(s) + \bar{A}(s)P(s)\bar{A}^T(s) \\
&- \bar{K}(s)\bar{F}(s-\rho_s)P(s)\bar{A}^T(s) \\
&- \bar{A}(s)P(s)\bar{F}^T(s-\rho_s)\bar{K}^T(s) \\
&+ \bar{B}(s)W(s)\bar{B}^T(s) + \bar{I}\Lambda(s)\bar{I}^T \\
&- \bar{A}(s)\Omega_2(s)\bar{C}^T(s-\rho_s)\bar{K}^T(s) \\
&- \bar{K}(s)\bar{C}(s-\rho_s)\Omega_2^T(s)\bar{A}^T(s) \\
&- \bar{A}(s)\Omega_1(s)D^T(s-\rho_s)\bar{K}^T(s) \\
&- \bar{K}(s)D(s-\rho_s)\Omega_1^T(s)\bar{A}^T(s).
\end{aligned}$$

We are now ready to derive the estimator gains that minimize the EEC. The EEC $P(s+1)$ is rewritten as

$$\begin{aligned}
P(s+1) =& \bar{K}(s)\Theta(s)\bar{K}^T(s) - \Psi(s)\bar{K}^T(s) \\
&- \bar{K}(s)\Psi^T(s) + \tilde{A}(s)P(s)\tilde{A}^T(s) \\
&+ \bar{B}(s)W(s)\bar{B}^T(s) + \bar{I}\Lambda(s)\bar{I}^T \\
=& \big(\bar{K}(s) - \Psi(s)\Theta^{-1}(s)\big)\Theta(s) \\
&\times \big(\bar{K}(s) - \Psi(s)\Theta^{-1}(s)\big)^T \\
&- \Psi(s)\Theta^{-1}(s)\Psi^T(s) + \bar{A}(s)P(s)\bar{A}^T(s) \\
&+ \bar{B}(s)W(s)\bar{B}^T(s) + \bar{I}\Lambda(s)\bar{I}^T.
\end{aligned}$$

It is easily known that the EEC $P(s+1)$ is minimized by choosing $\bar{K}(s)$ as $\Psi(s)\Theta^{-1}(s)$. Noting the definition of $\bar{K}(s)$, the estimator gains that minimize $P(s+1)$ are derived by (13)-(15). The proof is complete. ■

Remark 3: In this paper, the fault estimation problem is concerned for a class of MRSs with dynamical bias. A novel method is put forward to convert the MRSs into SRSs and the proposed method has less computation complexity as compared to the lifting technique. First, in Lemma 1, the recursion of the EEC is derived. Then, with the help of Lemmas 2-3, the estimator gains that minimize the EEC as well as the minimal EEC are given in Theorem 1. It is worth noting that both the state and the fault are well estimated with the proposed estimation algorithm.

IV. An illustrative example

In the simulation example, we consider the JSFE problem for a DC servo system [22] subject to random bias where the system parameters in (1)-(2) are given as follows:

$$A(s) = \begin{bmatrix} 1.12 + 0.3\sin(s) & 0.213 & -0.333 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ b = 2,$$

$$E(s) = \begin{bmatrix} 0.45 & 0.26 & 0.12 + 0.2\sin(s) \\ 0.43 & 0.33 + 0.2\cos(s) & 0.28 \\ 0.33 & 0.34 & 0.25 \end{bmatrix},$$

$$H(s) = \begin{bmatrix} 0.31 & 0.12 & 0.26 \\ 0.37 & 0.21 & 0.34 \\ 0.52 & 0.15 & 0.25 \end{bmatrix}, B(s) = \begin{bmatrix} 0.8 \\ 0 \\ 0 \end{bmatrix}, F(l_m) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$C(l_m) = \begin{bmatrix} 1 & 2 + \sin(l_m) & 1 \\ 2 & 1 & 2 \end{bmatrix}, D(l_m) = \begin{bmatrix} 0.43 \\ 0.51 \end{bmatrix}, \ \Lambda(s) = 0.15I.$$

The covariances of the process noise $w(s)$ and the measurement noise $v(l_m)$ are 0.1 and 0.15, respectively. The initial conditions are given as $\hat{x}(0) = \begin{bmatrix} 0.52 & -0.56 & 0.55 \end{bmatrix}^T$.

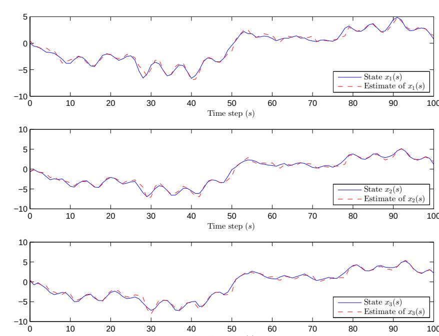

Fig. 1: State $x(s)$ and the estimate

In the simulation, the following sensor fault is considered:

$$f(l_{m+1}) = G(l_m)f(l_m)$$

with $G(l_m) = 1.8\sin(l_m)$. With the given parameters, the estimator gains $K_i(s)$ $(i = 1, 2, 3)$ and the EEC $P(s)$ are derived according to the proposed estimation algorithm. The simulation results are shown in Figs. 1-3. Fig. 1 shows $x_i(s)$ $(i = 1, 2, 3)$ and the corresponding estimates where $x_i(s)$ denotes the ith element of the state $x(s)$. Fig. 2(a) depicts the sensor fault $f(l_m)$ and its estimate. It is known from Fig. 1 and Fig. 2(a) that the proposed estimation scheme can estimate the system state and the sensor fault with a

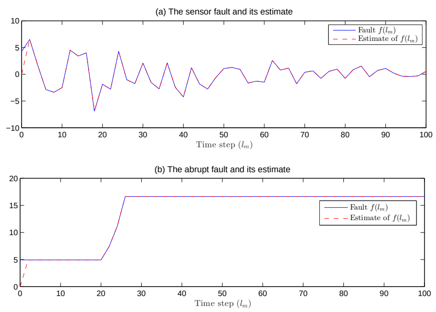

Fig. 2: The fault estimation performance

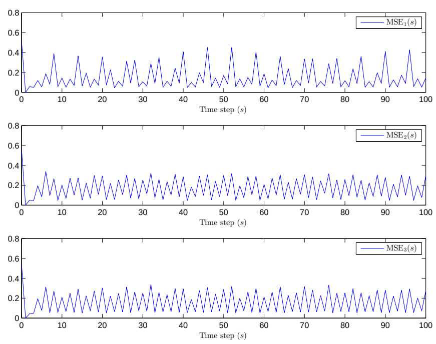

Fig. 3: The mean-square error of the estimation

satisfactory accuracy. Let $\text{MSE}_i(s)$ $(i = 1, 2, 3)$ denote the mean-square error of the estimation of $x_i(s)$, i.e., $\text{MSE}_i(s) = \frac{1}{N}\sum_{j=1}^{N}\left(x_i(s) - \hat{x}_i(s)\right)^2$. The $\text{MSE}_i(s)$ $(i = 1, 2, 3)$ are plotted in Fig. 3 which further verify the estimation accuracy of the developed fault estimation algorithm. The simulation results verify that the proposed estimation scheme is indeed effective.

To further verify the fault estimation performance, let us consider the abrupt fault described by (4) with

$$G(l_m) = \begin{cases} 1, & l_m \leq 20; \\ 1.5, & 20 < l_m \leq 26; \\ 1, & 27 < l_m. \end{cases}$$

The fault and its estimate are shown in Fig. 2(b), from which we can verify the effectiveness of the sensor fault estimation.

V. Conclusion

In this paper, the fault estimation problem has been investigated for a class of MRSs with dynamical bias. To avoid the computational complexity from the lifting technique, a novel method has been developed to transform the MRSs into SRSs. Based on the transformed SRSs, the estimator has been designed that estimates the fault and the system state simultaneously. First, the recursion of the EEC has been derived. Then, the estimator gains have been characterized that minimize the EEC. Finally, a practical simulation has verified the usefulness of the proposed estimation scheme. In the future, we will extend the results of this paper to other systems with multi-rate sampling such as multi-agent systems and sensor networks.

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