H_∞ Observer Design for Networked Hamiltonian Systems with Sensor Saturations and Missing Measurements

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Abstract

In this paper, the H_∞ observer design problem is investigated for discrete-time Hamiltonian systems subject to missing measurement and sensor saturations governed by Bernoulli distributed random variables. Our purpose is to design an observer such that the error dynamics of the state estimation is exponentially mean-square stable with prescribed H_∞ performance. By resorting to the Lyapunov function and the Hamiltonian system property, sufficient conditions are derived to guarantee the existence of the desired observer. Moreover, observer gains are designed in forms of the solutions to certain matrix inequalities. Finally, an illustrative example is utilized to testify the effectiveness of our observer design scheme.

Keywords: H_∞ observer, discrete-time Hamiltonian systems, missing measurement, sensor saturations, mean-square stability.

1. Introduction

For a few decades, Hamiltonian dynamical systems have attracted considerable research interest due to their extensive applications in modeling port-based networks of complex real world systems (e.g., satellites, robot manipulators, electrical networks and electric vehicles), see [24, 32, 35, 43] and the references therein. As a matter of fact, the Hamiltonian function (i.e., the entire energy of energy storing elements) in port-Hamiltonian systems, which is often seen as a satisfactory Lyapunov function candidate, is capable of reflecting essential system properties and facilitating the stability analysis of the underlying system. Motivated by such an appealing advantage of the Hamiltonian dynamical systems, a great deal of research attention has been paid towards the analysis/synthesis of practical control problems under the Hamiltonian system framework, see e.g., [20, 28, 29, 30, 31, 44].

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A noticeable point is that most existing results about port-Hamiltonian systems have restrained themselves to the continuous-time case, despite the fact that the computer-controlled discrete-time case has gained more and more attention in physical scenarios. Recently, the discrete-time Hamiltonian systems have begun to draw some initial research attention, see e.g. [1, 19, 17, 14, 27, 42]. For example, a passive integrator dedicated to input/output Hamiltonian systems approximation has been presented in [1], where the discrete-time model has admitted a Dirac structure representation. The finite-time H_∞ control problem has been investigated in [17] for the nonlinear discrete-time Hamiltonian descriptor system. Very recently, a novel state representation has been exploited in [19] for the port-Hamiltonian dynamics in a discrete-time setting by proposing an energy management strategy.

Owing to their merits of low maintenance costs and high operation reliability, networked systems have found a wide range of applications in many realms including industrial automation, information collection, and smart grid (see e.g., [47, 18, 16, 15, 23, 6, 11]). In networked systems, the signal transmissions between different components are carried out via shared communication networks, and the resultant competition on limited network resource would give rise to network-induced phenomena. In comparison with other network-induced phenomena, the missing measurement is more often encountered in reality which, if not well addressed, are likely to degrade system performances or even provoke instability. On the other hand, it is often the case in engineering practice that a sensor can only produce measurements with limited amplitude, and such an issue is customarily referred to as sensor saturation in existing literature. So far, much effort has been made towards investigating filtering problems subject to missing measurement and/or sensor saturations with representative results reported in [8, 21, 26, 22, 33, 39, 40, 34]. Nevertheless, the corresponding investigation on Hamiltonian systems has received inadequate research attention and this gives rise to the main motivation of this paper.

Summarizing our discussions, we focus on dealing with the H_∞ state observer design problem for discrete-time Hamiltonian systems subject to missing measurement and sensor saturations. A state observer is designed such that the estimation errors satisfy not only the exponentially mean-square stability constraints but also the prescribed H_∞ performance index. Furthermore, observer gains are derived through resorting to the approach of linear matrix inequalities (LMIs). Finally, simulations are provided to testify the effectiveness of our observer scheme.

The main contributions we achieved lie in three aspects: 1) a novel discrete-time Hamiltonian system model is presented to characterize the dynamics of nonlinear systems; 2) we make one of the first attempts towards the observer design problem for discrete-time networked Hamiltonian systems with missing measurement and sensor saturations; and 3) a new H_∞ observer approach is proposed under the Hamiltonian system framework.

Notation: For a matrix M, M^{T} denotes its transpose and the asterisk "$*$" in a matrix is used to denote the term induced by symmetry. The notation $M > 0$ (respectively, $M < 0$) for $M \in \mathbb{R}^{n\times n}$ means that matrix M is real symmetric positive definite (respectively, negative definite). $\lambda_{\max}\{M\}$ (respectively, $\lambda_{\min}\{M\}$) represents the largest (respectively, smallest) eigenvalue of the matrix M. The notation "sign" denotes the signum function. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem Formulation

2.1. Port-Hamiltonian systems and its discretization

To study the discrete-time Hamiltonian systems, we recall the following definitions first.

Definition 1. *([38]) The dynamic system*

$$\dot{z} = f(z), \quad z \in \mathbb{R}^n \tag{1}$$

has a constant Hamiltonian realization if the Hamiltonian function $H(z)$ and the suitable coordinate chart exist such that (1) can be constructed as

$$\dot{z} = J\nabla H(z) \tag{2}$$

where $J \in \mathbb{R}^{n\times n}$ is called the structure matrix and

$$\nabla H(z) := \frac{\partial H(z)}{\partial z}$$

is the gradient of $H(z)$.

From (2), the derivative of $H(z)$ is expressed as

$$\dot{H}(z) = \nabla^{\mathrm{T}} H(z) J \nabla H(z). \tag{3}$$

It is obvious that, if the structure matrix satisfies $J^{\mathrm{T}} + J \leqslant 0$ (i.e., it is expressed by $J = F - R$ where F is skew-symmetric and R is positive semi-definite), then we are able to rewrite (3) as a passivity equation with storage function $H(x)$ with

$$\dot{H}(z) = -\nabla^{\mathrm{T}} H(z) R \nabla H(z) \leqslant 0. \tag{4}$$

Here, the term $\nabla^{\mathrm{T}} H(z) R \nabla H(z)$ is the dissipation rates of systems, and the realization (2) under the condition $J^{\mathrm{T}} + J \leqslant 0$ is called a dissipative Hamiltonian realization. Clearly, the total stored energy $H(z)$ is a good Lyapunov function candidate for investigating the stability of the system's equilibrium points, which is actually a common practice in the analysis of system stability.

Remark 1. *So far, a number of effective approaches have been reported to solve the general (dissipativity) Hamiltonian realization problem for nonlinear systems. It is worth noting that, the Hamiltonian method can be employed to solve control problems for nonlinear systems by transforming such systems into dissipative Hamiltonian systems [37]. Within the Hamiltonian system framework, there has been a wealth of literature concerning the modelling and passivity-based control of practical systems.*

As stated in [27], the discrete-time port-Hamiltonian systems can be described as continuous ones where port variables are frozen within a sample interval T. For convenience of subsequent derivation, we denote $z(s)$ as the discrete counterpart of $z(t)$ in the interval $t \in [sT,\ (s+1)T)$, and then the discrete-time Hamiltonian function is described as $H(z(s))$, where $z(s) = \begin{bmatrix} z_1(s) & z_2(s) & \cdots & z_n(s) \end{bmatrix}^{\mathrm{T}}$. Accordingly, we present the following definitions of the discrete gradients.

Definition 2. *([1, 12]) A discrete gradient $\bar{\nabla} H(\cdot):\ \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is an approximation of the gradient of discrete-time Hamiltonian system $H(\cdot):\ \mathbb{R}^n \mapsto \mathbb{R}$, which satisfies:*

$$(\bar{\nabla} H(s, s'))^{\mathrm{T}}(s' - s) = H(s') - H(s), \tag{5}$$

where $s,\ s' \in \mathbb{R}^n$ and $\bar{\nabla} H(s, s) = \nabla H(s)$.

In case of small sampling periods, model (2) becomes

$$z(s+1) - z(s) = J\bar{\nabla} H(z(s)), \tag{6}$$

where $\bar{\nabla} H(z(s))$ is the discrete gradient of $H(z(s))$ given by

$$\bar{\nabla} H(z(s)) := \begin{bmatrix} \frac{H(z_1(s+1), z_2(s), \cdots, z_n(s)) - H(z(s))}{z_1(s+1) - z_1(s)} \\ \vdots \\ \frac{H(z_1(s), \cdots, z_{n-1}(s), z_n(s+1)) - H(z(s))}{z_n(s+1) - z_n(s)} \end{bmatrix}. \tag{7}$$

Here, the derived system (6) is customarily referred to as the so-called discrete-time Hamiltonian system.

Note that, when $J^{\mathrm{T}} + J \leqslant 0$, one has

$$\begin{aligned} & H(z(s+1)) - H(z(s)) \\ & = \bar{\nabla}^{\mathrm{T}} H(z(s))(z(s+1) - z(s)) \\ & = -\bar{\nabla}^{\mathrm{T}} H(z(s)) R \bar{\nabla} H(z(s)) \end{aligned} \tag{8}$$

which shows that system (6) is lossless and energy conservative when $R \equiv 0$.

2.2. Hamiltonian system of incomplete sensor information

In this paper, we consider the following Hamiltonian system:

$$\begin{cases} \triangle z(s) = f(z(s)) + Bw(s) \\ \xi(s) = Mz(s), \end{cases} \tag{9}$$

where $\triangle z(s) := z(s+1) - z(s)$; $\xi(s) \in \mathbb{R}^p$ is the signal that needs to be estimated; $w(s) \in \mathbb{R}^q$ is the external disturbance satisfying $L_2([0, \infty), \mathbb{R}^q)$; A, B and M are constant matrices. The nonlinear function $f(x(s))$ is represented by $A\bar{\nabla} H(z(s))$, where $\bar{\nabla} H(\cdot)$ is the discrete gradient described in (7) and $H(\cdot):\ \mathbb{R}^n \mapsto \mathbb{R}$ is the discrete-time Hamiltonian function satisfying $H(z(s)) > 0$ for $z \neq 0$ and $H(0) = 0$.

The measurement output of the plant (9) is

$$y(s) = \alpha(s)\sigma(Gz(s)) + (I - \alpha(s))\beta(s)Gz(s) + Dv(s), \tag{10}$$

where $y(s) = \begin{bmatrix} y_1(s) & y_2(s) & \cdots & y_m(s) \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^m$; $y_i(s)$ $(i = 1, 2, \cdots, m)$ is the measurement output of the i-th sensor; $v(s) \in \mathbb{R}^m$ denotes the measurement noises satisfying $L_2([0, \infty), \mathbb{R}^m)$; G and D are constant matrices; $\alpha(s) = \mathrm{diag}\{\alpha_1(s), \alpha_2(s), \cdots, \alpha_m(s)\}$ and $\beta(s) = \mathrm{diag}\{\beta_1(s), \beta_2(s), \cdots, \beta_m(s)\}$ are diagonal matrices in which $\alpha_i(s),\ \beta_i(s) \in \mathbb{R}$ $(i = 1, 2, \cdots, m)$ are Bernoulli distributed variables satisfying

$$\begin{cases} \mathrm{Prob}\{\alpha_i(s) = 1\} = \mu_i \\ \mathrm{Prob}\{\alpha_i(s) = 0\} = 1 - \mu_i \end{cases}$$

and

$$\begin{cases} \mathrm{Prob}\{\beta_i(s) = 1\} = \nu_i \\ \mathrm{Prob}\{\beta_i(s) = 0\} = 1 - \nu_i \end{cases}$$

respectively; and $\sigma(\cdot)$ is the saturation function to be introduced later. $\alpha_i(s)$ and $\beta_i(s)$ are assumed to be mutually independent for all $i = 1, 2, \cdots, m$.

Remark 2. *In practical engineering (especially networked control systems), the sensor saturation often occurs in a probabilistic way due to the random abrupt changes. In addition, probabilistically missing measurements is also inevitable in a networked environment due to the limited bandwidth of the channels for signal transmission. The proposed sensor model (10) is capable of accounting for the two phenomena in a unified representation. Specifically, if $\alpha_i(s) = 1$, it can be seen that the sensor i is subject to saturation only; if $\alpha_i(s) = 0$ and $\beta_i(s) = 1$, it means that the sensor i works normally; if $\alpha_i(s) = 0$ and $\beta_i(s) = 0$, the sensor i receives the noise only, implying that the information transmitted from system (9) to sensor i is missing.*

The saturation function is given by

$$\sigma(Gz(s)) = \left[\begin{array}{cccc} \sigma(g_1 z(s)) & \sigma(g_2 z(s)) & \cdots & \sigma(g_m z(s)) \end{array} \right]^{\mathrm{T}},$$

where g_i represents the ith row of the matrix G and the saturation function $\sigma : \mathbb{R} \to \mathbb{R}$ is defined as

$$\sigma(u) = \mathrm{sign}(u)\min\{1, |u|\}. \tag{11}$$

Here, we have slightly abused notations through using σ to stand for both vector- and scalar-valued functions.

In this paper, the state observer of the following form is adopted to for the purpose of state estimation:

$$\begin{cases} \triangle\hat{z}(s) = A_H\hat{z}(s) + K[y(s) - G\hat{z}(s)] \\ \hat{\xi}(s) = M\hat{z}(s), \end{cases} \tag{12}$$

where $\hat{\xi}(s)$ and $\hat{z}(s)$ are estimates of $\xi(s)$ and $z(s)$, and A_H and K are observer parameters to be determined.

Let the estimation error be

$$\tilde{\xi}(s) = \xi(s) - \hat{\xi}(s)$$

and define

$$\zeta(s) = \left[\begin{array}{c} z(s) \\ \hat{z}(s) \end{array} \right], \quad \triangle\zeta(s) = \left[\begin{array}{c} \triangle z(s) \\ \triangle\hat{z}(s) \end{array} \right],$$

$$\eta(s) = \left[\begin{array}{c} \bar{\nabla} H(z(s)) \\ \bar{\nabla} H(\hat{z}(s)) \end{array} \right], \quad \bar{w}(s) = \left[\begin{array}{c} w(s) \\ v(s) \end{array} \right].$$

Then, one has the following augmented system:

$$\begin{cases} \triangle\zeta(s) = (\bar{A} - I)\zeta(s) + C\eta(s) + \bar{K}\sigma(GE\zeta(s)) \\ \qquad\qquad +\bar{B}\bar{w}(s) + \Sigma_{i=1}^{m}\Big((1 - \alpha_i(s))\beta_i(s) \\ \qquad\qquad -(1 - \mu_i)\nu_i\Big)\bar{C}GE\zeta(s) \\ \qquad\qquad +\Sigma_{i=1}^{m}\Big(\alpha_i(s) - \mu_i\Big)\bar{C}\sigma(GE\zeta(s)) \\ \tilde{\xi}(s) = \bar{M}\zeta(s), \end{cases} \tag{13}$$

where $\bar{C} = \begin{bmatrix} 0 & (KE_i)^T \end{bmatrix}^T$, $E = \begin{bmatrix} I & 0 \end{bmatrix}$,

$$\bar{A} = \left[\begin{array}{cc} I & 0 \\ K(I - \Lambda_\alpha)\Lambda_\beta G & I - KG + A_H \end{array} \right],$$

$$\bar{B} = \begin{bmatrix} B & 0 \\ 0 & KD \end{bmatrix}, \ C = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \ \bar{K} = \begin{bmatrix} 0 \\ K\Lambda_\alpha \end{bmatrix},$$
$$E_i = \mathrm{diag}\{\underbrace{0, \cdots, 0}_{i-1}, 1, \underbrace{0, \cdots, 0}_{m-i}\}, \ \bar{M} = \begin{bmatrix} M & -M \end{bmatrix},$$
$$\Lambda_\alpha = \mathrm{diag}\{\mu_1, \mu_2, \cdots, \mu_m\}, \ \Lambda_\beta = \mathrm{diag}\{\nu_1, \nu_2, \cdots, \nu_m\}.$$

Remark 3. *System (13) is essentially stochastic because of the existence of stochastic parameters $\alpha_i(k)$ ($i = 1, 2, \cdots, m$) and $\beta_i(k)$. As such, we are motivated to assess the system stability along with estimation performance in the mean-square sense.*

In this paper, our objective is to design the observer (12) for the system (9) with

$$f(z(s)) := A\bar{\nabla}H(z(s))$$

and measurement (10). More specifically, we endeavor to find the observer gains A_H and K such that system (13) achieves the following two performance requirements.

i) The estimation error system (13) with $\bar{w}(s) = 0$ is exponentially stable in the mean-square sense, i.e., there exist constants $\rho > 0$ and $0 < \tau < 1$ such that

$$\mathbb{E}\{\|\zeta(s)\|^2\} \leqslant \rho\tau^s\mathbb{E}\{\|\zeta(0)\|^2\} \tag{14}$$

for all initial value $\zeta(0)$;

ii) Given the scalar $\gamma > 0$ and zero initial condition, $\tilde{\xi}(s)$ satisfies

$$\sum_{s=0}^{\infty}\mathbb{E}\{\|\tilde{\xi}(s)\|^2\} \leqslant \gamma^2\sum_{s=0}^{\infty}\|\bar{w}(s)\|^2 \tag{15}$$

for all $\bar{w}(k) > 0$.

Assumption 1. *The Hamiltonian function $H(z(s))$ and its discrete gradient $\bar{\nabla}H(z(s))$ satisfy*

1. $H(z(s)) \leqslant \iota_1\|z(s)\|^2$;
2. $\|\bar{\nabla}H(z(s))\|^2 \leqslant \iota_2\|z(s)\|^2$,

where ι_1 and ι_2 are positive scalars.

Lemma 1. *([39]) For the saturation function $\sigma(u)$ defined in (11), there exists a scalar l such that*

$$[\sigma(u) - lu][\sigma(u) - u] \leqslant 0, \tag{16}$$

where the scalar l can be calculated based on the maximum of u.

It should be noted that the scalar l is very important for the observer design issue discussed in this paper. Lemma 1 provides an effective method to deal with the difficulties induced by nonlinear function $\sigma(u)$ through the well-known "sector-bounded" technique. In practical applications, the system states are always bounded due to their physical constraints. Without loss of generality, we assume that the scalar variable u is bounded with the constraint $|u| \leq \bar{u}$ where $\bar{u}$ is a known positive scalar. Then, it is easy to see that the condition (16) holds with the scalar $l \triangleq \frac{1}{\bar{u}}$.

Lemma 2. *(Young's Inequality [2]) Let constants $a > 0$, $b > 1$ and $c > 1$ be given such that $(b-1)(c-1) = 1$. For any $z, y \in \mathbb{R}^n$, we have*

$$z^{\mathrm{T}} y \leqslant \frac{a^b}{b}\|z\|^b + \frac{1}{ca^c}\|y\|^c. \tag{17}$$

Lemma 3. *(Schur Complement [2]) The linear matrix inequality*

$$\begin{bmatrix} \Upsilon_1 & \Upsilon_2 \\ * & \Upsilon_3 \end{bmatrix} > 0 \tag{18}$$

holds if and only if

$$\begin{cases} \Upsilon_1 - \Upsilon_2 \Upsilon_3^{-1} \Upsilon_2^{\mathrm{T}} > 0 \\ \Upsilon_3 > 0, \end{cases} \tag{19}$$

where Υ_i $(i = 1, 2, 3)$ are constant matrices.

3. Main Results

3.1. Performance analysis

In this subsection, we first establish a stability condition for system (13) satisfying $\bar{w}(s) = 0$. Then, we discuss the H_∞ performance. For convenience of subsequent developments, some notations are defined as $z_s = z(s)$, $\hat{z}_s = \hat{z}(s)$, $\zeta_s = \zeta(s)$, $\eta_s = \eta(s)$, $\alpha_s = \alpha(s)$, $\beta_s = \beta(s)$, $\bar{w}_s = \bar{w}(s)$.

Choose a Lyapunov function as

$$V(s) = V_1(s) + V_2(s) \tag{20}$$

where $V_1(s) = \zeta_s^{\mathrm{T}} P \zeta_s$, $V_2(s) = 2\lambda_1 H(z_s) + 2\lambda_2 H(\hat{z}_s)$, and λ_1 and λ_2 are positive scalars to be determined.

Based on (13) and (7), we have

$$\begin{aligned} \mathbb{E}\{\triangle V_1(s)\} &:= \mathbb{E}\{V_1(s+1) - V_1(s)\} \\ &= \mathbb{E}\{\zeta_{s+1}^{\mathrm{T}} P \zeta_{s+1} - \zeta_s^{\mathrm{T}} P \zeta_s\} \\ &= \mathbb{E}\Big\{\zeta_s^{\mathrm{T}} \bar{A}^{\mathrm{T}} P \bar{A} \zeta_s + \zeta_s^{\mathrm{T}} \sum_{i=1}^{m} \Big((1-\alpha_s^i)\beta_s^i - (1-\mu_i)\nu_i\Big)^2 E^{\mathrm{T}} G^{\mathrm{T}} \bar{C}^{\mathrm{T}} P \bar{C} G E \zeta_s \\ &\quad + 2\zeta_s^{\mathrm{T}} \bar{A}^{\mathrm{T}} P C \eta_s + 2\zeta_k^{\mathrm{T}} \bar{A} P \bar{K} \sigma(GE\zeta_s) + 2\zeta_s^{\mathrm{T}} \sum_{i=1}^{m} \Big((1-\alpha_s^i)\beta_s^i - (1-\mu_i)\nu_i\Big) \\ &\quad \times (\alpha_s^i - \mu_i) E^{\mathrm{T}} G^{\mathrm{T}} \bar{C}^{\mathrm{T}} P \bar{C} \sigma(GE\zeta_s) + \eta_s^{\mathrm{T}} C^{\mathrm{T}} P C \eta_s + 2\eta_s^{\mathrm{T}} C^{\mathrm{T}} P \bar{K} \sigma(GE\zeta_s) \\ &\quad + \sigma^{\mathrm{T}}(GE\zeta_s) \bar{K}^{\mathrm{T}} P \bar{K} \sigma(GE\zeta_s) + \sigma^{\mathrm{T}}(GE\zeta_s) \sum_{i=1}^{m} (\alpha_s^i - \mu_i)^2 \bar{C}^{\mathrm{T}} P \bar{C} \\ &\quad \times \sigma(GE\zeta_s) - \zeta_s^{\mathrm{T}} P \zeta_s\Big\} \end{aligned} \tag{21}$$

and

$$\mathbb{E}\{\triangle V_2(s)\} := \mathbb{E}\{V_2(s+1) - V_2(s)\}$$

$$
\begin{aligned}
&= \mathbb{E}\{2\lambda_1(H(z_{s+1})-H(z_s))+2\lambda_2(H(\hat{z}_{s+1})-H(\hat{z}_s))\} \\
&= \mathbb{E}\{2\eta_s^{\mathrm{T}}\Lambda\triangle\zeta_s\} \\
&= 2\eta_s^{\mathrm{T}}\Lambda(\bar{A}-I)\zeta_s+2\eta_s^{\mathrm{T}}\Lambda C\eta_s+2\eta_s^{\mathrm{T}}\Lambda\bar{K}\sigma(GE\zeta_s),
\end{aligned} \tag{22}
$$

where $\Lambda = \mathrm{diag}\{\lambda_1 I, \lambda_2 I\}$.

It follows from Assumption 1 and Lemma 1 that

$$
\epsilon_1(\eta_s^{\mathrm{T}}\bar{E}\eta_s - \iota_2\zeta_s^{\mathrm{T}}\bar{E}\zeta_s) \leqslant 0, \tag{23}
$$

$$
\epsilon_2[\sigma(GE\zeta_s)-LGE\zeta_s]^{\mathrm{T}}[\sigma(GE\zeta_s)-GE\zeta_s] \leqslant 0, \tag{24}
$$

where $L = \mathrm{diag}\{l_1, l_2, \cdots, l_m\}$, $0 < l_i < 1$ $(i = 1, 2, \cdots, m)$, $\bar{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, ϵ_1 and ϵ_2 are positive scalars.

Taking (21)-(24) into consideration, we have

$$
\begin{aligned}
&\mathbb{E}\{\triangle V(s)\} \\
&\leqslant \zeta_s^{\mathrm{T}}\Big[\bar{A}^{\mathrm{T}}P\bar{A}-P+\sum_{i=1}^{m}\delta_i E^{\mathrm{T}}G^{\mathrm{T}}\bar{C}^{\mathrm{T}}P\bar{C}GE\Big]\zeta_s \\
&\quad + 2\zeta_s^{\mathrm{T}}[\bar{A}^{\mathrm{T}}PC+(\bar{A}^{\mathrm{T}}-I)\Lambda]\eta_s \\
&\quad + 2\zeta_s^{\mathrm{T}}\Big[\bar{A}^{\mathrm{T}}P\bar{s}-\sum_{i=1}^{m}\varepsilon_i E^{\mathrm{T}}G^{\mathrm{T}}\bar{C}^{\mathrm{T}}P\bar{C}\Big]\sigma(GE\zeta_s) \\
&\quad + \eta_s^{\mathrm{T}}(C^{\mathrm{T}}PC+2\Lambda C)\eta_s+2\eta_s^{\mathrm{T}}(C^{\mathrm{T}}P\bar{K}+\Lambda\bar{K}) \\
&\quad \times\sigma(GE\zeta_s)+\sigma^{\mathrm{T}}(GE\zeta_s)\Big[\bar{K}^{\mathrm{T}}P\bar{K}+\sum_{i=1}^{m}\varrho_i\bar{C}^{\mathrm{T}}P\bar{C}\Big] \\
&\quad \times\sigma(GE\zeta_s)-\epsilon_1(\eta_s^{\mathrm{T}}\bar{E}\eta_s-\iota_2\zeta_s^{\mathrm{T}}\bar{E}\zeta_s) \\
&\quad -\epsilon_2[\sigma(GE\zeta_s)-LGE\zeta_s]^{\mathrm{T}}[\sigma(GE\zeta_s)-GE\zeta_s],
\end{aligned} \tag{25}
$$

where $\delta_i = (1-\mu_i)\nu_i - (1-\mu_i)^2\nu_i^2$, $\varepsilon_i = (1-\mu_i)\mu_i\nu_i$, and $\varrho_i = \mu_i(1-\mu_i)$.

By using Lemma 2 and setting $b = c = 2$, one has

$$
\begin{aligned}
&-2\zeta_s^{\mathrm{T}}E^{\mathrm{T}}G^{\mathrm{T}}\bar{C}^{\mathrm{T}}P\bar{C}\sigma(GE\zeta_s) \\
&\leqslant a^2\zeta_s^{\mathrm{T}}E^{\mathrm{T}}G^{\mathrm{T}}\bar{C}^{\mathrm{T}}P\bar{C}GE\zeta_s + a^{-2}\sigma^{\mathrm{T}}(GE\zeta_s)\bar{C}^{\mathrm{T}}P\bar{C}\sigma(GE\zeta_s).
\end{aligned} \tag{26}
$$

Denoting $\xi_s = \begin{bmatrix} \zeta_s^{\mathrm{T}} & \eta_s^{\mathrm{T}} & \sigma^{\mathrm{T}}(GE\zeta_s) \end{bmatrix}^{\mathrm{T}}$, one has

$$
\mathbb{E}\{\triangle V(s)\} \leqslant \xi_s^{\mathrm{T}}\Theta\xi_s, \tag{27}
$$

where

$$
\Theta = \begin{bmatrix} \vartheta_{11} & \vartheta_{12} & \vartheta_{13} \\ * & \vartheta_{22} & \vartheta_{23} \\ * & * & \vartheta_{33} \end{bmatrix}, \tag{28}
$$

$$
\vartheta_{11} = \bar{A}^{\mathrm{T}}P\bar{A}-P+\sum_{i=1}^{m}(\delta_i+a^2\varepsilon_i)E^{\mathrm{T}}G^{\mathrm{T}}\bar{C}^{\mathrm{T}}P\bar{C}GE
$$

$$
\begin{aligned}
&-\epsilon_2 E^{\mathrm{T}} G^{\mathrm{T}} L^{\mathrm{T}} G E + \epsilon_1 \iota_2 \bar{E}, \\
\vartheta_{12} &= \bar{A}^{\mathrm{T}} P C + (\bar{A}^{\mathrm{T}} - I)\Lambda, \\
\vartheta_{13} &= \bar{A}^{\mathrm{T}} P \bar{K} + \frac{1}{2}\epsilon_2 E^{\mathrm{T}} G^{\mathrm{T}} (L + I), \\
\vartheta_{22} &= C^{\mathrm{T}} P C + 2\Lambda C - \epsilon_1 \bar{E}, \\
\vartheta_{23} &= C^{\mathrm{T}} P \bar{K} + \Lambda \bar{K}, \\
\vartheta_{33} &= \bar{K}^{\mathrm{T}} P \bar{K} + \sum_{i=1}^{m} (\varrho_i + a^{-2}\varepsilon_i) \bar{C}^{\mathrm{T}} P \bar{C} - \epsilon_2 I.
\end{aligned}
$$

Based on the above analysis, we are in a position to present a sufficient condition for the exponential mean-square stability of the system (13) with $\bar{w}_s = 0$.

Theorem 1. *Let positive scalars μ_i, ν_i $(i = 1, 2, \cdots, m)$ and observer gains A_H and K be given. If there exist a positive definite matrix P, a diagonal matrix L $(0 < L < I)$ and positive scalars λ_1, λ_2, ϵ_1, ϵ_2, a such that*

$$
\Theta < 0, \tag{29}
$$

then system (13) with $\bar{w}_s = 0$ is exponentially mean-square stable.

Proof. Choose Lyapunov candidate (20) and denote $\hbar = \lambda_{\max}(P) + 2\max\{\lambda_1, \lambda_2\}\iota_1$. Then, based on Assumption 1, we have

$$
\lambda_{\min}(P)\|\zeta_s\|^2 \leqslant V(s) \leqslant \hbar\|\zeta_s\|^2. \tag{30}
$$

For all nonzero ρ_s, it follows from (29) that $\mathbb{E}\{\triangle V(s)\} < 0$. Therefore, one obtains

$$
\mathbb{E}\{\triangle V(s)\} \leqslant -\lambda_{\min}(-\Theta)\|\rho_s\|^2. \tag{31}
$$

Given scalars $\bar{\tau} > 1$ and $N > 1$, one verifies

$$
\begin{aligned}
&\mathbb{E}\{\bar{\tau}^N V(N)\} - \mathbb{E}\{V(0)\} \\
&= \sum_{s=0}^{N-1} \big(\mathbb{E}\{\bar{\tau}^{s+1} V(s+1)\} - \mathbb{E}\{\bar{\tau}^s V(s)\}\big) \\
&= \sum_{s=0}^{N-1} \big(\bar{\tau}^{s+1}\mathbb{E}\{\triangle V(s)\} + \bar{\tau}^s(\bar{\tau} - 1))\mathbb{E}\{V(s)\}\big) \\
&\leqslant -\lambda_{\min}(-\Theta)\bar{\tau} \sum_{s=0}^{N-1} \bar{\tau}^s \|\rho_s\|^2 + (\bar{\tau} - 1)\hbar \sum_{s=0}^{N-1} \bar{\tau}^s \|\zeta_s\|^2 \\
&\leqslant \big(-\lambda_{\min}(-\Theta)\bar{\tau} + (\bar{\tau} - 1)\hbar\big) \sum_{s=0}^{N-1} \bar{\tau}^s \|\zeta_s\|^2.
\end{aligned} \tag{32}
$$

Noting that $\lambda_{\min}(-\Theta) > 0$ and $\hbar > 0$, there exists $\bar{\tau}_0 > 1$ such that $-\lambda_{\min}(-\Theta)\bar{\tau}_0 + (\bar{\tau}_0 - 1)\hbar = 0$. Then, one has

$$
\mathbb{E}\{\bar{\tau}_0^N V(N)\} \leqslant \mathbb{E}\{V(0)\}. \tag{33}
$$

Furthermore, in view of (30), we have

$$
\lambda_{\min}(P)\mathbb{E}\{\|\zeta_s\|^2\} \leqslant \mathbb{E}\{V(N)\} \leqslant \bar{\tau}_0^{-N}\hbar\mathbb{E}\{\|\zeta_0\|^2\}, \tag{34}
$$

which implies that

$$\mathbb{E}\{\|\zeta_s\|^2\} \leqslant \frac{\hbar}{\lambda_{\min}(P)}(\frac{1}{\bar{\tau}_0})^N \mathbb{E}\{\|\zeta_0\|^2\}. \tag{35}$$

Letting

$$\rho = \frac{\hbar}{\lambda_{\min}(P)}, \quad \tau = \frac{1}{\bar{\tau}_0},$$

we arrive at (14). □

Next, we are ready to analyze the H_∞ performance. To this end, a sufficient condition is provided in the following theorem for the exponential mean-square stability and the H_∞ performance of the system (13) with $\bar{w}_k \neq 0$.

Theorem 2. *Let the positive scalars μ_i, ν_i $(i = 1, 2, \cdots, m)$ and observer gains A_H, K be given. If there exist a positive definite matrix P, a diagonal matrix L satisfying $0 < L < I$, and positive scalars λ_1, λ_2, ϵ_1, ϵ_2, a such that*

$$\Psi < 0, \tag{36}$$

where

$$\Psi = \begin{bmatrix} \vartheta_{11} + \bar{M}^{\mathrm{T}}\bar{M} & \vartheta_{12} & \vartheta_{13} & \bar{A}^{\mathrm{T}}P\bar{B} \\ * & \vartheta_{22} & \vartheta_{23} & (C^{\mathrm{T}}P + \Lambda)\bar{B} \\ * & * & \vartheta_{33} & \bar{K}^{\mathrm{T}}P\bar{B} \\ * & * & * & -\gamma^2 I + \bar{B}^{\mathrm{T}}P\bar{B} \end{bmatrix},$$

then the system (13) with $\bar{w}_k = 0$ is exponentially mean-square stable while satisfying the requirement (15).

Proof. Note that inequality (29) can be implied by (36) under zero initial conditions. Then, system (13) with $\bar{w}_k = 0$ is exponentially mean-square stable.

Choosing Lyapunov function as in (20), it follows from the inequalities (23), (24) and (26) that

$$\begin{aligned}
&\mathbb{E}\{\triangle V(s)\} + \mathbb{E}\{\|\tilde{\xi}_s\|^2\} - \gamma^2\|\bar{w}_s\|^2 \\
&\leqslant \zeta_s^{\mathrm{T}}[\bar{A}^{\mathrm{T}}P\bar{A} - P + \sum_{i=1}^{m}(\delta_i + a^2\varepsilon_i)E^{\mathrm{T}}G^{\mathrm{T}}\bar{C}^{\mathrm{T}}P\bar{C}GE \\
&\quad - \epsilon_2 E^{\mathrm{T}}G^{\mathrm{T}}LGE + \epsilon_1\iota_2\bar{E} + \bar{M}^{\mathrm{T}}\bar{M}]\zeta_s + 2\zeta_s^{\mathrm{T}}[\bar{A}^{\mathrm{T}}PC + (\bar{A}^{\mathrm{T}} - I)\Lambda]\eta_s \\
&\quad + 2\zeta_s^{\mathrm{T}}(\bar{A}^{\mathrm{T}}P\bar{K} + \frac{1}{2}\epsilon_2 E^{\mathrm{T}}G^{\mathrm{T}}(L + I))\sigma(GE\zeta_s) \\
&\quad + \eta_s^{\mathrm{T}}(C^{\mathrm{T}}PC + 2\Lambda C - \epsilon_1\bar{E})\eta_s + 2\eta_s^{\mathrm{T}}(C^{\mathrm{T}}P\bar{K} + \Lambda\bar{K}) \\
&\quad \times \sigma(GE\zeta_s) + \sigma^{\mathrm{T}}(GE\zeta_s)[\bar{K}^{\mathrm{T}}P\bar{K} + \sum_{i=1}^{m}(\varrho_i + a^{-2}\varepsilon_i) \\
&\quad \times \bar{C}^{\mathrm{T}}P\bar{C} - \epsilon_2 I]\sigma(GE\zeta_s) + 2\zeta_s^{\mathrm{T}}\bar{A}^{\mathrm{T}}P\bar{B}\bar{w}_s \\
&\quad + 2\eta_s^{\mathrm{T}}(C^{\mathrm{T}}P + \Lambda)\bar{B}\bar{w}_s + 2\sigma^{\mathrm{T}}(GE\zeta_s)\bar{K}^{\mathrm{T}}P\bar{B}\bar{w}_s \\
&\quad - \gamma^2\bar{w}_s^{\mathrm{T}}\bar{w}_s + \bar{w}_s^{\mathrm{T}}\bar{B}^{\mathrm{T}}P\bar{B}\bar{w}_s \\
&\leqslant \mathbb{E}\{\ell_s^{\mathrm{T}}\Psi\ell_s\}
\end{aligned} \tag{37}$$

where $\ell_s = \begin{bmatrix} \zeta_s^{\mathrm{T}} & \eta_s^{\mathrm{T}} & \sigma^{\mathrm{T}}(GE\zeta_s) & \bar{w}_s^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$. Then, it follows from (36) that $\mathbb{E}\{\ell_s^{\mathrm{T}}\Psi\ell_s\} < 0$ holds for all $\ell_s > 0$.

By considering zero initial values, it is easy to verify that (15) holds for any $\bar{w}_s > 0$, $s \in [0, \infty)$. □

Remark 4. *In case of $H(z(s)) = z^{\mathrm{T}}(s)z(s)$ and $A \neq I$, the corresponding discrete gradient is $z(s+1)+z(s)$, and then system (9) is rewritten as*

$$\begin{cases} z(s+1) = \tilde{A}z(s) + \tilde{B}w(s) \\ \xi(s) = Mz(s) \end{cases} \tag{38}$$

where

$$\tilde{A} = \frac{I+A}{I-A}, \quad \tilde{B} = \frac{B}{I-A}.$$

Note that the results presented in Theorems 1-2 are also applicable to the system (38).

Remark 5. *Clearly, based on the given observer gain matrix, Theorems 1 and 2 provide some sufficient conditions for the exponential mean-square stability and the H_∞ performance of the system (13) with $\bar{w}_k \neq 0$. As the design of the gain parameter matrix is the main objective of this work, the corresponding H_∞ observer design method will be given in following subsection.*

3.2. A solution to the H_∞ observer design

In this subsection, we are going to provide an H_∞ observer design method to satisfy the prescribed requirements in Section 2. Specifically, a gain parameter matrix is designed to guarantee the exponentially mean-square stability of system (13) with the given H_∞ performance index.

Theorem 3. *For Hamiltonian system (9) and measurement model (10), the addressed H_∞ observer design problem is solvable if there exist the observer gain matrices A_H and K, positive definite matrix $P = \mathrm{diag}\{Q_1, Q_2\}$, positive scalars λ_1, λ_2, ϵ_1, ϵ_2, a such that*

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 & \Pi_4 \\ * & Q_2 - 2I & 0 & 0 \\ * & * & \bar{Q}_2 - 2I & 0 \\ * & * & * & \bar{Q}_2 - 2I \end{bmatrix} < 0, \tag{39}$$

where

$$\Pi_1 = \begin{bmatrix} \pi_{11} & -M^{\mathrm{T}}M & Q_1A & \pi_{41} & \epsilon_2 G^{\mathrm{T}}(L+I) & Q_1B & 0 \\ * & -Q_2 + M^{\mathrm{T}}M & 0 & \pi_{42} & 0 & 0 & 0 \\ * & * & \pi_{33} & 0 & 0 & \lambda_1 B + A^{\mathrm{T}}Q_1B & 0 \\ * & * & * & 0 & \lambda_2 K\Lambda_\alpha & 0 & \lambda_2 KD \\ * & * & * & * & -\epsilon_2 I & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I + B^{\mathrm{T}}Q_1B & 0 \\ * & * & * & * & * & * & -\gamma^2 I \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} K(I-\Lambda_\alpha)\Lambda_\beta G & I - KG + A_H & 0 & 0 & K\Lambda_\alpha & 0 & KD \end{bmatrix}^{\mathrm{T}},$$

$$\Pi_3 = \begin{bmatrix} S_{\delta\varepsilon}G & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \ \Pi_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & S_{\varrho\varepsilon} & 0 & 0 \end{bmatrix}^{\mathrm{T}},$$

$$\pi_{11} = -Q_1 + M^{\mathrm{T}}M + \epsilon_1 \iota_2 I - \epsilon_2 G^{\mathrm{T}}LG, \ \pi_{33} = \lambda_1(A + A^{\mathrm{T}}) + A^{\mathrm{T}}Q_1A - \epsilon_1 I,$$

$$\pi_{41} = \lambda_2 G^{\mathrm{T}}\Lambda_\beta^{\mathrm{T}}(I - \Lambda_\alpha^{\mathrm{T}})K^{\mathrm{T}}, \ S_{\delta\varepsilon} = \begin{bmatrix} \sqrt{\delta_1 + a^2\varepsilon_1}E_1K^{\mathrm{T}} & \cdots & \sqrt{\delta_m + a^2\varepsilon_m}E_mK^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

$$\pi_{42} = \lambda_2(A_H - G^{\mathrm{T}}K^{\mathrm{T}}), \ S_{\varrho\varepsilon} = \begin{bmatrix} \sqrt{\varrho_1 + a^{-2}\varepsilon_1}E_1K^{\mathrm{T}} & \cdots & \sqrt{\varrho_m + a^{-2}\varepsilon_m}E_mK^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

$$\bar{Q}_2 = \mathrm{diag}\{\underbrace{Q_2, \cdots, Q_2}_{m}\}.$$

PROOF. Due to the fact that $-Q_2^{-1} < Q_2 - 2I$, it is easy to show that (39) implies the following inequality:

$$\check{\Pi} = \begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 & \Pi_4 \\ * & -Q_2^{-1} & 0 & 0 \\ * & * & -\bar{Q}_2^{-1} & 0 \\ * & * & * & -\bar{Q}_2^{-1} \end{bmatrix} < 0. \tag{40}$$

Recalling that $P = \mathrm{diag}\{Q_1, Q_2\}$, the inequality (40) is equivalent to

$$\hat{\Pi} = \begin{bmatrix} \bar{\Pi}_1 & \bar{\Pi}_2 & \bar{\Pi}_3 & \bar{\Pi}_4 \\ * & -P^{-1} & 0 & 0 \\ * & * & -\bar{P}^{-1} & 0 \\ * & * & * & -\bar{P}^{-1} \end{bmatrix} < 0 \tag{41}$$

with $\bar{P} = \mathrm{diag}\{\underbrace{P, \cdots, P}_{m}\}$,

$$\bar{\Pi}_1 = \begin{bmatrix} \bar{\pi}_{11} & -\Lambda + \bar{A}^{\mathrm{T}}\Lambda & \epsilon_2 E^{\mathrm{T}} G^{\mathrm{T}}(L+I) & 0 \\ * & 2\Lambda C - \epsilon_1 \bar{E} & \Lambda \bar{K} & \Lambda \bar{B} \\ * & * & -\epsilon_2 I & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix}, \quad \bar{\Pi}_2 = \begin{bmatrix} \bar{A}^{\mathrm{T}} \\ C^{\mathrm{T}} \\ \bar{K}^{\mathrm{T}} \\ \bar{B}^{\mathrm{T}} \end{bmatrix},$$

$$\bar{\pi}_{11} = \begin{bmatrix} \pi_{11} & -M^{\mathrm{T}}M \\ * & -Q_2 + M^{\mathrm{T}}M \end{bmatrix}, \quad \bar{\Pi}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ S_{\delta\varepsilon}GE & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \quad \bar{\Pi}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & S_{\varrho\varepsilon} & 0 \end{bmatrix}^{\mathrm{T}}.$$

Note that Ψ in Theorem 2 can be rewritten as follows:

$$\Psi = \bar{\Pi}_1 + \bar{\Pi}_2 P \bar{\Pi}_2^{\mathrm{T}} + \bar{\Pi}_3 \bar{P} \bar{\Pi}_3^{\mathrm{T}} + \bar{\Pi}_4 \bar{P} \bar{\Pi}_4^{\mathrm{T}}. \tag{42}$$

It follows from Lemma 3 that $\hat{\Pi} < 0$ holds if and only if $\Psi < 0$ is true. The rest of the proof is similar to that of Theorem 2. □

Remark 6. *So far, the design of the H_∞ state observer has been accomplished for Hamiltonian systems subject to missing measurement and sensor saturations. A state observer has been constructed such that estimation errors are exponentially mean-square stable with the prescribed H_∞ performance. Furthermore, the desired state observer gains have been calculated by exploiting the LMI approach. Within the established framework, our results can be directly generalized to more general systems with more complicated dynamics with more complex network-induced phenomena.*

Remark 7. *In comparison to the rich body of existing literature on observer design problems [13, 9, 5, 10, 48, 41, 45], our results have two distinguishing features: 1) the solved Hamiltonian system model is new, which can capture the characteristics of the dynamics of Lipschitz-like nonlinear systems; and 2) the proposed observer design problem is new in the sense that both missing measurement and sensor saturation issues are addressed under networked Hamiltonian frameworks.*

4. Numerical Example

Consider a normalized undamped pendulum with small external disturbance, where the dynamics of the system is characterized by the following Hamiltonian system:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -0.6 & 0.3 \\ 0.2 & -0.7 \end{bmatrix} \begin{bmatrix} \sin z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0.05 \\ 0.05 \end{bmatrix} w. \tag{43}$$

Here, z_1 is the vertical angle (i.e., the generalized position) and z_2 is the generalized momentum.

Denoting $z = [z_1,\ z_2]^{\mathrm{T}} \in \mathbb{R}^2$, the Hamiltonian function is expressed as

$$H(z) = \frac{1}{2}z_2^2 + 1 - \cos z_1.$$

By approximating the derivatives with respect to state variables in (43) and replacing gradient terms with $\bar{\nabla}H(z(s))$ as defined as (7), the discrete-time representation of (43) can be written as (9) where $M = \mathrm{diag}\{1, 1\}$. Then, we easily verify that Assumption 1 is satisfied by the Hamiltonian function.

The measurement model (10) is specified with the following parameters:

$$G = \begin{bmatrix} 0.1 & 0.2 \\ -0.2 & 0.1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix}.$$

The occurring probabilities of the incomplete measurements are chosen as $\mu_1 = \nu_1 = 0.7$, $\mu_2 = 0.6$, and $\nu_2 = 0.75$. We set the H_∞ performance index as $\gamma = 0.8$.

Referring to Theorem 3, LMI (39) can be solved with the following H_∞ observer parameters:

$$\begin{aligned} K &= \begin{bmatrix} -0.0021 & -0.0130 \\ 0.0467 & 0.1111 \end{bmatrix}, \\ A_H &= \begin{bmatrix} -0.3469 & -0.0046 \\ 0.0124 & -0.3462 \end{bmatrix} \end{aligned}$$

with $P = \mathrm{diag}\{Q_1, Q_2\} > 0$ being given by

$$P = \begin{bmatrix} 6.8109 & 2.5724 & 0 & 0 \\ 2.5724 & 5.7960 & 0 & 0 \\ 0 & 0 & 2.2196 & -0.0176 \\ 0 & 0 & -0.0176 & 2.2333 \end{bmatrix}. \tag{44}$$

In addition, we set

$$\begin{aligned} w(s) &= \sin(s)\exp(-0.1s), \\ v(s) &= \begin{bmatrix} \frac{\sin(10s+1)}{3s+1} & \frac{\cos(10s+1)}{3s+1} \end{bmatrix}^{\mathrm{T}}, \end{aligned}$$

$z(0) = [0.5,\ 0.8]^{\mathrm{T}}$ and $\hat{z}(0) = [0.5,\ 1]^{\mathrm{T}}$.

Simulation results are given in Figs. 1-5 where true and estimated trajectories of the Hamiltonian system states are depicted in Figs. 1-2. Figs. 3 and 4 display the trajectories of $\xi(s)$ and its estimates. Moreover, the trajectory of the error estimate $\tilde{\xi}(s)$ is depicted in Fig. 5. The above results confirm the good performance of the designed H_∞ observer.

5. Conclusion

In this paper, an H_∞ observer design problem has been investigated for generalized discrete-time Hamiltonian systems subject to missing measurement and sensor saturations. The underlying system under consideration is nonlinear with external disturbance along with sensor noises. An H_∞ observer has been constructed whose existence has been guaranteed by sufficient conditions derived through the Lyapunov function method. Finally, examples have been provided to testify the observer effectiveness. Future work would focus on the filtering and/or control problems for Hamiltonian systems with multiple network-induced complex phenomena caused by limited communication resources.

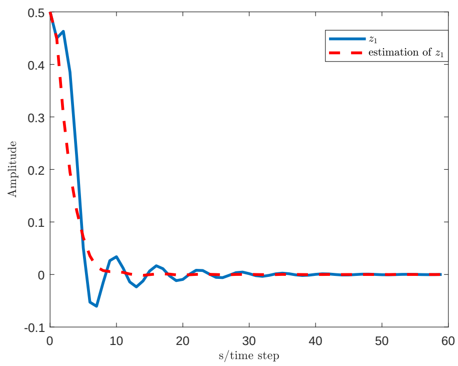

Figure 1: State trajectory of z_1 and its estimate.

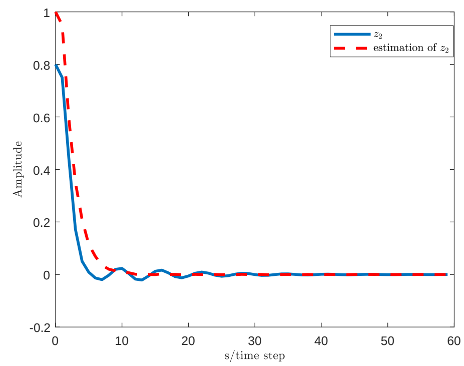

Figure 2: State trajectory of z_2 and its estimate.

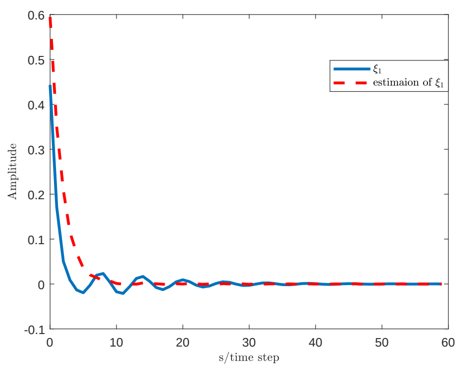

Figure 3: Output ξ_1 and its estimate.

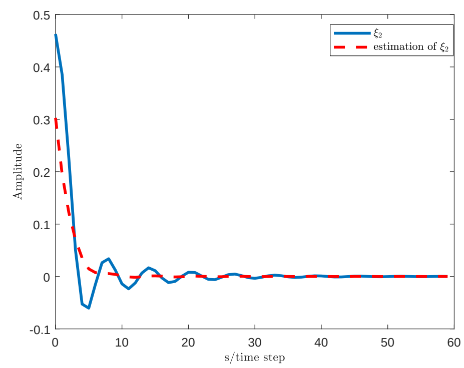

Figure 4: Output ξ_2 and its estimate.

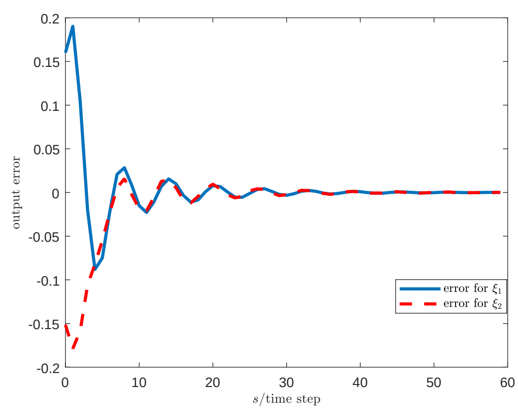

Figure 5: Errors for output ξ_1 and ξ_2.

Acknowledgements

The Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia has funded this project under grant no. (FP-094-43). This work was also supported in part by the National Natural Science Foundation of China under Grants 62073189, 61873148 and 61933007, the AHPU Youth Top-notch Talent Support Program of China under Grant 2018BJRC009, the Natural Science Foundation of Anhui Province of China under Grant 2108085MA07, the Royal Society of the UK, and the Alexander von Humboldt Foundation of Germany.

References

[1] S. Aoues, M. Di Loreto, D. Eberard and W. Marquis-Favre, Hamiltonian systems discrete-time approximation: Losslessness, passivity and composability, *Systems & Control Letters*, vol. 110, pp. 9–14, 2017.

[2] E. Beckenbach and R. Bellman, *Inequalities*. New York, Springer-Verlag, 1965.

[3] W. Chen, Z. Wang, D. Ding and H. Dong, Consensusability of discrete-time multi-agent systems under binary encoding with bit errors, *Automatica*, vol. 133, art. no. 109867, 2021.

[4] Y. Chen, Z. Wang, F. E. Alsaadi and H. Liu, Dynamic output-feedback H_∞ control for discrete time-delayed systems with actuator saturations under Round-Robin communication protocol, *International Journal of Robust and Nonlinear Control*, in press, DOI:10.1002/rnc.5906.

[5] C. Fan, J. Lam and X. Xie, Fault detection observer design for periodic piecewise linear systems, *International Journal of Systems Science*, vol. 51, no. 9, pp. 1622–1636, 2020.

[6] H. Fu, H. Dong, F. Han, Y. Shen and N. Hou, Outlier-resistant H_∞ filtering for a class of networked systems under Round-Robin protocol, *Neurocomputing*, vol. 403, pp. 133–142, Aug. 2020.

[7] C. Gao, Z. Wang, X. He and H. Dong, Encryption-decryption-based consensus control for multi-agent systems: Handling actuator faults, *Automatica*, vol. 134, art. no. 109908, 2021.

[8] C. Gao, Z. Wang, X. He and D. Yue, Sampled-data-based fault-tolerant consensus control for multi-agent systems: A data privacy preserving scheme, *Automatica*, vol. 133, art. no. 109847, 2021.

[9] H. Geng, H. Liu, L. Ma and X. Yi, Multi-sensor filtering fusion meets censored measurements under a constrained network environment: advances, challenges and prospects, *International Journal of Systems Science*, vol. 52, no. 16, pp. 3410–3436, 2021.

[10] J. Hu, C. Jia, H. Liu, X. Yi and Y. Liu, A survey on state estimation of complex dynamical networks, *International Journal of Systems Science*, vol. 52, no. 16, pp. 3351–3367, 2021.

[11] J. Hu, H. Zhang, H. Liu and X. Yu, A survey on sliding mode control for networked control systems, *International Journal of Systems Science*, vol. 52, no. 6, pp. 1129–1147, 2021.

[12] T. Itoh and K. Abe, Hamiltonian-conserving discrete canonical equations based on variational difference quotients, *Journal of Computational Physics*, vol. 77, pp. 85–102, 1988.

[13] X.-C. Jia, Resource-efficient and secure distributed state estimation over wireless sensor networks: a survey, *International Journal of Systems Science*, vol. 52, no. 16, pp. 3368–3389, 2021.

[14] P. Kotyczka and L. Lefevre, Discrete-time port-Hamiltonian systems: A definition based on symplectic integration, *Systems & Control Letters*, vol. 133, art.no. 104530, pp. 1–9, 2019.

[15] Y. Li, H. Li and W. Sun, Event-triggered control for robust set stabilization of logical control networks, *Automatica*, vol. 95, pp. 556–560, 2018.

[16] L. Liu, L. Ma, J. Zhang and Y. Bo, Distributed non-fragile set-membership filtering for nonlinear systems under fading channels and bias injection attacks, *International Journal of Systems Science*, vol. 52, no. 6, pp. 1192–1205, 2021.

[17] X. Lu, X. Zhang and L. Sun, Finite-time H_∞ control for nonlinear discrete Hamiltonian descriptor systems, *Journal of the Franklin Institute*, vol. 354, no. 14, pp. 6138–6151, 2017.

[18] J. Mao, Y. Sun, X. Yi, H. Liu and D. Ding, Recursive filtering of networked nonlinear systems: A survey, *International Journal of Systems Science*, vol. 52, no. 6, pp. 1110–1128, 2021.

[19] A. Moreschini, M. Mattioni, S. Monaco, and D. Normand-Cyrot, Stabilization of discrete port-Hamiltonian dynamics via interconnection and damping assignment, *IEEE Control Systems Letters*, vol. 5, no. 1, pp. 103–108, 2021.

[20] R. Ortega, A. J. van der Schaft and A. Astolfi, Control by interconnection and standard passivity-based control of port-Hamiltonian systems, *IEEE Transactions on Automatica Control*, vol. 11, no. 53, pp. 2527–2542, 2008.

[21] W. Qian, Y. Li, Y. Chen, and W. Liu, L_2-L_∞ filtering for stochastic delayed systems with randomly occurring nonlinearities and sensor saturation, *International Journal of Systems Science*, vol. 51, no. 13, pp. 2360–2377, 2020.

[22] W. Qian, Y. Li, Y. Zhao, and Y. Chen, New optimal method for L_2-L_∞ state estimation of delayed neural networks, *Neurocomputing*, vol. 415, pp. 258–265, 2020.

[23] B. Qu, B. Shen, Y. Shen and Q. Li, Dynamic state estimation for islanded microgrids with multiple fading measurements, *Neurocomputing*, vol. 406, pp. 196–203, Sept. 2020.

[24] J. Schiffer, E. Fridman, R. Ortega and J. Raisch, Stability of a class of delayed port-Hamiltonian systems with application to microgrids with distributed rotational and electronic generation, *Automatica*, vol. 74, no. 12, pp. 71–79, 2016.

[25] Y. Shen, Z. Wang, H. Dong, F. E. Alsaadi and H. Liu, Dynamic event-based recursive filtering for multirate systems with integral measurements over sensor networks, *International Journal of Robust and Nonlinear Control*, in press, DOI: 10.1002/rnc.5884.

[26] H. Song, D. Ding, H. Dong, G. Wei and Q.-L. Han, Distributed entropy filtering subject to DoS attacks in non-Gauss environments, *International Journal of Robust and Nonlinear Control*, vol. 30, no. 3, pp. 1240–1257, Feb. 2020.

[27] S. Stramigioli, C. Secchi, A. J. van der Schaft and C. Fantuzzi, Sampled data systems passivity and discrete port-Hamiltonian systems, *IEEE Transactions on Robotics*, vol. 21, no. 4, pp. 574–587, 2005.

[28] L. Sun, G. Feng and Y. Wang, Finite-time stabilization and H_∞ control for a class of nonlinear Hamiltonian descriptor systems with application to affine nonlinear descriptor systems, *Automatica*, vol. 50, no. 8, pp. 2090–2097, 2014.

[29] W. Sun and B. Fu, Adaptive control of time-varying uncertain nonlinear systems with input delay: A Hamiltonian approach, *IET Control Theory & Applications*, vol. 15, no. 10, pp. 1844–1858, 2016.

[30] W. Sun and L. Peng, Observer-based robust adaptive control for uncertain stochastic Hamiltonian systems with state and input delays, *Nonlinear Analysis: Modelling and Control*, vol. 4, no. 19, pp. 626–645, 2014.

[31] W. Sun, X. Lv and M. Qu, Distributed estimation for stochastic Hamiltonian systems with fading wireless channels, *IEEE Transactions on Cybernetics*, early access, doi: 10.1109/TCYB.2020.3023547.

[32] W. Sun, Y. Wu and L. Wang, Trajectory tracking of constrained robotic systems via a hybrid control strategy, *Neurocomputing*, vol. 330, pp. 188–195, 2019.

[33] Y. Sun, D. Ding, H. Dong and H. Liu, Event-based resilient filtering for stochastic nonlinear systems via innovation constraints, *Information Sciences*, vol. 546, pp. 512–525, Feb. 2021.

[34] H. Tan, B. Shen, K. Peng and H. Liu, Robust recursive filtering for uncertain stochastic systems with amplify-and-forward relays, *International Journal of Systems Science*, vol. 51, no. 7, pp. 1188–1199, 2020.

[35] A. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*, London: Springer, 2000.

[36] E. Vos, J. M. A. Scherpen and A. van der Schaft, Equal distribution of satellite constellations on circular target orbits, *Automatica*, vol. 50, no. 10, pp. 2641–2647, 2014.

[37] Y. Wang, *The Generalized Hamiltonian Control Systems Theory–Implementation, Control and Application (In Chinese)*, Beijing, China, Science Press, 2007.

[38] Y. Wang and D. Cheng, Generalized Hamiltonian realization and its application in construction of energy-based Lyapunov candidates (in Chinese), *Control Theory and Applications*, vol. 19, no. 4, pp. 511–515, 2012.

[39] Z. Wang, B. Shen and X. Liu, H_∞ filtering with randomly occuring sensor saturations and missing measurements, *Automatica*, vol. 48, no. 3, pp. 556–562, 2012.

[40] H. Yan, F. Qian, F. Yang and H. Shi, H_∞ filtering for nonlinear networked systems with randomly occurring distributed delays, missing measurements and sensor saturation, *Information Sciences*, vol. 370–371, pp. 772–782, 2016.

[41] X.-G. Yan, S. K. Spurgeon and C. Edwards, Sliding mode control for time-varying delayed systems based on a reduced-order observer, *Automatica*, vol. 46, no. 8, pp. 1354–1362, 2010.

[42] Y. Yaprak, L. Gören-Sümer and A. Astolfi, Some results on disturbance attenuation for Hamiltonian systems via direct discrete design, *International Journal of Robust and Nonlinear Control*, vol. 13, no. 25, pp. 1927–1940, 2015.

[43] J. Yu, W. Pei and C. Zhang, A loss-minimization port-controlled Hamilton scheme of induction motor for electric vehicles, *IEEE/ASME Transactions on Mechatronics*, vol. 20, no. 6, pp. 2645–2653, 2015.

[44] M. Zhang, P. Borja, R. Ortega, Z. Liu and H. Su, PID passivity-based control of port-Hamiltonian systems, *IEEE Transactions on Automatic Control*, vol. 63, no. 4, pp. 1032–1044, 2018.

[45] X.-M. Zhang and Q.-L. Han, Event-based H_∞ filtering for sampled-data systems, *Automatica*, vol. 51, pp. 55–69, 2015.

[46] Z. Zhang and S. Xu, Observer design for uncertain nonlinear systems with unmodeled dynamics, *Automatica*, vol. 1, no. 51, pp. 80–84, 2015.

[47] K. Zhu, J. Hu, Y. Liu, N. D. Alotaibi and F. E. Alsaadi, On ℓ_2-ℓ_∞ output-feedback control scheduled by stochastic communication protocol for two-dimensional switched systems, *International Journal of Systems Science*, vol. 52, no. 14, pp. 2961–2976, 2021.

[48] L. Zou, Z. Wang, J. Hu, Y. Liu and X. Liu, Communication-protocol-based analysis and synthesis of networked systems: Progress, prospects and challenges, *International Journal of Systems Science*, vol. 52, no. 14, pp. 3013–3034, 2021.