Recursive State Estimation for Multi-Rate Time-Varying Systems with Multiplicative Noises: Dealing with Sensor Resolutions

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Abstract

In this paper, the recursive state estimation problem is investigated for a class of multi-rate systems with multiplicative noises where the measurement outputs are collected from sensors with certain resolutions. Due to the existence of the sensor resolution, the actual measurement output of the sensor might deviate from its true value and such a deviation, if not adequately taken into account, would lead to serious degradation of the estimation performance, and we are therefore motivated to develop an effective state estimation algorithm that is insensitive to the sensor-resolution-induced measurement distortions. The aim of the considered estimation problem is to design a state estimator such that an upper bound on the estimation error covariance is first guaranteed and then minimized by properly choosing the estimator gain. Moreover, a simulation example with application background on moving target tracking problem is presented to verify the validity of the developed recursive state estimation algorithm.

Index Terms

Recursive state estimation, sensor resolution, multi-rate systems, multiplicative noises.

I. Introduction

In engineering practice such as petroleum extraction and chemical engineering, it is vitally important to acquire the state information of the underlying plant in order to make sure that the production process

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is operational yet safe. Due to resource limits and physical constraints, it is quite common that the state information cannot be directly obtained, and an alternative way is therefore to estimate the system state by using the available measurements that could be contaminated by stochastic noises [8], [13], [37]–[39], [54]. For decades, the state estimation problems have been attracting an ever-increasing research interest from various communities ranging from signal processing and mathematics to control engineering [1], [9], [15], [21], [24], [32], [34].

So far, many state estimation algorithms have been developed in the literature for a variety of systems including, but are not limited to, linear systems [5], [28], nonlinear systems [7], [18], [36], [42], stochastic systems [3], [55], and uncertain systems [2], [17], [43], [53]. According to the performance indices, the state estimation algorithms can be generally categorized into four types, namely, the H_∞ state estimation method [23], [33], the recursive state estimation scheme [4], [22], [40], the set-membership state estimation technique [6], [30] as well as moving-horizon state estimation mechanism [16], [56]. Among others, the recursive state estimation (RSE) method, whose main aim is to minimize the estimation error covariance at each time instant, has proven to be one of the most popular ones due primarily to its advantages in easy implementation and online computation. Note that the RSE problem has been recently investigated in [49] for sensor networks and in [27] for complex networks.

It is worth mentioning that, up to now, the focus of almost all RSE problems has been paid on the single-rate systems, that is, the update rate of the system state and the sampling rate of the sensor are the same. Unfortunately, due to different physical features of different system components, it is difficult to unify the state update rate and the sensor sampling rate in engineering practice. Also, for systems with slowly changing states, it is unnecessary and uneconomic to sample the measurement at each state update instant [45]. In this sense, the multi-rate sampling mechanism appears to be more reasonable which has been applied in many industrial systems such as structural health monitoring system [11], aluminium electrolysis cells [46], as well as power grids [35]. Until now, despite the significant engineering background of the multi-rate sampling mechanism, the RSE problems for multi-rate systems still need extra research attentions.

The multi-rate sampling, though practically appealing, does bring multiple time sequences to the system and would invalidate the state estimation algorithms designed specifically for single-rate systems. Therefore, it is theoretically important to make dedicated efforts in dealing with the multi-rate sampling issue in the RSE problems, see [12], [14] for some latest results. In particular, the filtering problem has been tackled in [52] for multi-rate systems (MRSs) with asynchronous sensors and the lifting technique has been used to transform the MRS into a single-rate one. In [41], an outlier-resistant recursive filter has been designed for MRSs under the weighted Try-Once-Discard protocol.

Multiplicative noises, also known as state-dependent noises, have recently gained considerable research interest [20], [29]. Note that many practical plants can be modeled by systems with multiplicative noises, and the corresponding RSE problem for such kind of systems demands extra care, see e.g. [19], [31] for some representative results on the single-rate systems with multiplicative noises. When it comes to the MRSs, the relevant RSE results have been really scattered because of the essential difficulties in handling the coupling between the multi-rate sampling and the multiplicative noises. For example, the traditional lifting technique is no longer directly applicable for MRSs because of the existence of the multiplicative noises. Therefore, there is a practical need to develop a novel method to solve the RSE problem for MRSs subject to multiplicative noises.

In engineering practice, no sensors could detect arbitrarily small changes of the measurement. The smallest change that a sensor can detect, known as the *sensor resolution*, is one of the important specifications of the sensor. Due to inherent limit of the sensor resolution, the actual measurement output from the sensor is most likely to deviate from the true measurement output. Obviously, estimating the system state by using the deviated measurement would lead to a poor estimation performance. As such, in the state estimation problem, it is vitally important to take the sensor resolution into serious consideration.

The consideration of the phenomenon of sensor resolution is clearly a non-trivial task with some additional challenges outlined as follows: 1) how to establish a model to accurately characterize the sensor resolution? 2) how to mitigate the performance deterioration of the state estimation caused by the deviation of the actual measurement from the true measurement? and 3) how to design a state estimator with guaranteed estimation accuracy in spite of the sensor-resolution-induced measurement distortions (SRIMDs)? Note that some preliminary results have been obtained in [51] on the state estimation problem for single-rate systems with non-logarithmic sensor resolutions. Unfortunately, for multi-rate systems, the corresponding state estimation results have been really scattered.

To tackle the aforementioned challenges, in this paper, we endeavor to develop an effective RSE scheme for MRSs with multiplicative noises and SRIMDs. The main novelties of this paper are stressed as follows: *1) a novel method is put forward to handle the difficulties resulting from the coupling of the multi-rate sampling and the multiplicative noises; 2) a novel state estimation scheme, which is of acceptable computational complexity, is developed for systems undergoing SRIMDs; and 3) a locally minimized upper bound is guaranteed on the estimation error covariance.*

The rest of this paper is organized as follows. In Section II, the sensor resolution is introduced, the considered MRS is presented and then transformed into a single-rate one. In Section III, the state estimation scheme is developed. A simulation study is conducted on the moving target tracking problem in Section IV. Finally, Section V concludes this paper.

Notation The notation used here is fairly standard. I and 0 denote the identity matrix and the zero matrix with appropriate dimensions, respectively. G^T, G^{-1}, and $\mathrm{tr}(G)$ represent the transpose, the inverse, and the trace of the matrix G, respectively. $\mathrm{col}\{\cdots\}$ stands for a column vector composed of elements. $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are the floor function and the ceiling function, respectively. For a random variable α, $\mathbb{E}\{\alpha\}$ is the expectation of α.

II. Problem formulation

Consider the following class of discrete-time systems with multiplicative noises:

$$x(k+1) = A(k)x(k) + \varepsilon(k)B(k)x(k) + E(k)w(k) \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the system state, $\varepsilon(k) \in \mathbb{R}$ is the zero-mean Gaussian multiplicative noise with unity covariance, and $w(k) \in \mathbb{R}^{n_w}$ is the zero-mean process noise with covariance $W(k) > 0$. $A(k)$, $B(k)$ and $E(k)$ are known time-varying matrices with compatible dimensions. The initial value $x(0)$ of the system state is a random variable with mean $\chi(0)$ and covariance $X(0)$.

In practical engineering, a sensor can only detect the change of the measurement that is larger than a certain value. Such a certain value is known as *sensor resolution* and defined as follows.

Definition 1: [51] Let y_i $(i = 1, 2, \ldots, n_y)$ be the i-th element of the measurement output of a sensor. If y_i takes value in the set $\{jr_i | j = 0, \pm 1, \ldots, \pm z\}$ where z is a given positive integer, then $R \triangleq \mathrm{col}\{r_1, r_2, \cdots, r_{n_y}\}$ is the resolution of the sensor.

In this paper, a sensor with the sampling period

$$b \triangleq s_{k+1} - s_k$$

and the sensor resolution

$$R \triangleq \mathrm{col}\{r_1, r_2, \cdots, r_{n_y}\}$$

is deployed to measure the system. Without considering the sensor resolution, the *ideal* measurement output of the sensor is

$$y^{id}(s_k) = C(s_k)x(s_k) + D(s_k)v(s_k) \tag{2}$$

where $y^{id}(s_k) \in \mathbb{R}^{n_y}$ is the ideal measurement output of the sensor, $v(s_k) \in \mathbb{R}^{n_v}$ is the zero-mean measurement noise with covariance $V(s_k) > 0$, and $C(s_k)$ and $D(s_k)$ are known time-varying matrices with compatible dimensions.

Assumption 1: The random variables $x(0)$, $\varepsilon(k)$, $w(k)$, and $v(s_k)$ are mutually uncorrelated.

By taking the sensor resolution into consideration, the *actual* measurement from the sensor with sensor resolution R is

$$y_i^{ac}(s_k) = \begin{cases} \left\lfloor \dfrac{y_i^{id}(s_k)}{r_i} \right\rfloor r_i, & y_i^{id}(s_k) \geq r_i \\ 0, & y_i^{id}(s_k) \in (-r_i, r_i) \\ \left\lceil \dfrac{y_i^{id}(s_k)}{r_i} \right\rceil r_i, & y_i^{id}(s_k) \leq -r_i \end{cases} \tag{3}$$

where $y_i^{ac}(s_k)$ is the i-th element of the actual measurement $y^{ac}(s_k)$ from the sensor, $y_i^{id}(s_k)$ is the i-th element of $y^{id}(s_k)$, and r_i is the i-th element of the resolution R.

Noting that the system under consideration is a MRS, we are going to transform the MRS into a single-rate one. First, we rewrite the system (1) as

$$x(k+1) = F(k)x(k) + E(k)w(k)$$

where $F(k) \triangleq A(k) + \varepsilon(k)B(k)$. Then, setting

$$\bar{x}(s_k) \triangleq \mathrm{col}\{x(s_{k-1}+1), \cdots, x(s_k - 1), x(s_k)\},$$

we have

$$\bar{x}(s_{k+1}) = \bar{A}(s_k)\bar{x}(s_k) + \bar{E}(s_k)\bar{w}(s_k) \tag{4}$$

where

$$\begin{aligned}
\bar{w}(s_k) &\triangleq \mathrm{col}\{w(s_k), w(s_k+1), \cdots, w(s_{k+1}-1)\},\\
\bar{A}(s_k) &\triangleq \begin{bmatrix} \underbrace{0_{bn_x\times n_x} \quad \cdots \quad 0_{bn_x\times n_x}}_{b-1} & \mathscr{A}(s_k) \end{bmatrix},\\
\bar{E}(s_k) &\triangleq \mathscr{B}(s_k)\mathscr{E}(s_k),\ \mathscr{F}_m^n(s_k) \triangleq \prod_{i=m}^{n} F(s_{k+1}-i),\\
\mathscr{A}(s_k) &\triangleq \mathrm{col}\{\mathscr{F}_b^b(s_k), \mathscr{F}_{b-1}^b(s_k), \cdots, \mathscr{F}_1^b(s_k)\},\\
\mathscr{B}(s_k) &\triangleq \begin{bmatrix} I & 0 & \cdots & 0\\ \mathscr{F}_{b-1}^{b-1}(s_k) & I & \cdots & 0\\ \mathscr{F}_{b-2}^{b-1}(s_k) & \mathscr{F}_{b-2}^{b-2}(s_k) & \cdots & 0\\ \vdots & \vdots & \cdots & 0\\ \mathscr{F}_1^{b-1}(s_k) & \mathscr{F}_1^{b-2}(s_k) & \cdots & I \end{bmatrix},\\
\mathscr{E}(s_k) &\triangleq \mathrm{diag}\{E(s_k), E(s_k+1), \cdots, E(s_{k+1}-1)\}.
\end{aligned}$$

Moreover, the ideal measurement model (2) is reformulated as

$$y^{id}(s_k) = \bar{C}(s_k)\bar{x}(s_k) + D(s_k)v(s_k)$$

where $\bar{C}(s_k) \triangleq \begin{bmatrix} 0 & \cdots & 0 & C(s_k) \end{bmatrix}$.

Due to the sensor resolution, the measurement received by the estimator is $y^{ac}(s_k)$ (instead of the ideal measurement $y^{id}(s_k)$). In this paper, the state estimator is of the following form

$$\begin{aligned}\hat{x}(s_{k+1}) = {} & \mathbb{E}\{\bar{A}(s_k)\}\hat{x}(s_k)\\ & + K(s_k)\big(y^{ac}(s_k) - \bar{C}(s_k)\hat{x}(s_k)\big)\end{aligned} \tag{5}$$

where $\hat{x}(s_k)$ is the estimate of $\bar{x}(s_k)$ and $K(s_k)$ is the estimator gain matrix to be determined. The initial condition of the estimator is $\hat{x}(s_0) = \mathbb{E}\{\bar{x}(s_0)\}$.

Denoting the estimation error as

$$e(s_k) \triangleq \bar{x}(s_k) - \hat{x}(s_k)$$

and the difference between the actual measurement and the ideal measurement as

$$\Delta(s_k) \triangleq y^{ac}(s_k) - y^{id}(s_k),$$

we have the following estimation error dynamics:

$$\begin{aligned}e(s_{k+1}) = {} & \breve{A}(s_k)e(s_k) + \tilde{A}(s_k)\bar{x}(s_k) + \bar{E}(s_k)\bar{w}(s_k)\\ & - K(s_k)\Delta(s_k) - K(s_k)D(s_k)v(s_k)\end{aligned} \tag{6}$$

where

$$\tilde{A}(s_k) \triangleq \bar{A}(s_k) - \mathbb{E}\{\bar{A}(s_k)\},$$
$$\check{A}(s_k) \triangleq \mathbb{E}\{\bar{A}(s_k)\} - K(s_k)\bar{C}(s_k).$$

Remark 1: It can be seen from (4) that, due to the existence of the multiplicative noise $\varepsilon(k)$, the parameter matrices $\bar{A}(s_k)$ and $\bar{E}(s_k)$ of the augmented system (4) are essentially random matrices. Therefore, in the estimator (5), the expectation $\mathbb{E}\{\bar{A}(s_k)\}$ of the random matrix $\bar{A}(s_k)$ is used. It can be seen from the estimation error dynamic system (6) that the appearance of the random matrices $\bar{A}(s_k)$ and $\bar{E}(s_k)$ would bring additional difficulties in the derivation of the estimation error covariance (or its upper bound), and therefore a novel method is needed that can tackle the random matrices in an adequate way.

The aim of the considered estimation problem is to develop a state estimator (5) such that the estimation error covariance

$$P(s_k) \triangleq \mathbb{E}\{e(s_k)e^T(s_k)\}$$

has a certain upper bound and, moreover, an appropriate gain matrix $K(s_k)$ is designed such that the derived upper bound is minimized.

III. Main results

In this section, we will first derive an upper bound on the estimation error covariance, and then the estimator gain matrix will be characterized so as to minimize the obtained upper bound.

The following lemma will be useful in our later analysis.

Lemma 1: Let a random matrix

$$M(k) \triangleq \Big[M_{ij}(k)\Big]_{b\times b}, \quad M_{ij}(k) \in \mathbb{R}^{n_x\times n_y}$$

and a vector

$$x(k) \triangleq \mathrm{col}\{x_1(k), x_2(k), \cdots, x_b(k)\}, \quad x_i(k) \in \mathbb{R}^{n_y}$$

be given. The term $\mathbb{E}\{M(k)x(k)x^T(k)M^T(k)\}$ can be obtained by

$$\begin{aligned}&\mathbb{E}\{M(k)x(k)x^T(k)M^T(k)\}_{mn}\\ &\quad= \sum_{j=1}^{b}\sum_{i=1}^{b}\mathbb{E}\{M_{mi}(k)x_i(k)x_j^T(k)M_{nj}^T(k)\}\end{aligned}$$

where $\mathbb{E}\{M(k)x(k)x^T(k)M^T(k)\}_{mn} \in \mathbb{R}^{n_x\times n_x}$ is the (m,n)-th submatrix of $\mathbb{E}\{M(k)x(k)x^T(k)M^T(k)\}$.

Proof: The proof is easily accessible by matrix operations and is therefore omitted here. ■

In the following, the recursion of the estimation error covariance is presented.

Lemma 2: The estimation error covariance $P(s_{k+1})$ of the estimator (5) is recursively calculated by

$$\begin{aligned}P(s_{k+1}) = {}& \check{A}(s_k)P(s_k)\check{A}^T(s_k) + \Gamma_1(s_k) + \Gamma_2(s_k)\\ &+ K(s_k)\mathbb{E}\{\Delta(s_k)\Delta^T(s_k)\}K^T(s_k)\end{aligned}$$

$$
\begin{aligned}
&+ K(s_k)D(s_k)V(s_k)D^T(s_k)K^T(s_k) \\
&- \mathcal{A}(s_k) - \mathcal{A}^T(s_k) + \mathcal{B}(s_k) + \mathcal{B}^T(s_k)
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
\Gamma_1(s_k) &\triangleq \mathbb{E}\{\tilde{A}(s_k)\bar{x}(s_k)\bar{x}^T(s_k)\tilde{A}^T(s_k)\}, \\
\Gamma_2(s_k) &\triangleq \mathbb{E}\{\bar{E}(s_k)\bar{w}(s_k)\bar{w}^T(s_k)\bar{E}^T(s_k)\}, \\
\mathcal{A}(s_k) &\triangleq \mathbb{E}\{\check{A}(s_k)e(s_k)\Delta^T(s_k)K^T(s_k)\}, \\
\mathcal{B}(s_k) &\triangleq \mathbb{E}\{K(s_k)\Delta(s_k)v^T(s_k)D^T(s_k)K^T(s_k)\}.
\end{aligned}
$$

Proof: With the help of Assumption 1 and (6), it is obvious that (7) is true. The proof is complete. ■

Note that the estimation error dynamics (6) contains the augmented state $\bar{x}(s_k)$. To facilitate the derivation of $P(s_k)$, in the following, the covariance of the augmented state $\bar{x}(s_k)$ is given.

Lemma 3: The state covariance $X(k) \triangleq \mathbb{E}\{x(k)x^T(k)\}$ is recursively calculated by

$$
\begin{aligned}
X(k+1) = &\ A(k)X(k)A^T(k) + B(k)X(k)B^T(k) \\
&+ E(k)W(k)E^T(k)
\end{aligned} \tag{8}
$$

with initial value $X(0)$. Moreover, the covariance of the augmented state $\bar{X}(s_k) \triangleq \mathbb{E}\{\bar{x}(s_k)\bar{x}^T(s_k)\}$ is derived by

$$
\bar{X}(s_k) = \Big[\bar{X}_{i,j}(s_k)\Big]_{b\times b} \tag{9}
$$

where

$$
\bar{X}_{i,j}(s_k) \triangleq \mathbb{E}\{x(s_{k-1}+i)x^T(s_{k-1}+j)\} \in \mathbb{R}^{n_x \times n_x}
$$

is obtained according to

$$
\bar{X}_{i,j}(s_k) = \begin{cases}
\prod_{l=b-i+1}^{b-j} A(s_k - l)X(s_{k-1}+j), & i > j \\
X(s_{k-1}+i), & i = j \\
X(s_{k-1}+i)\prod_{l=i}^{j-1} A^T(s_{k-1}+l), & i < j.
\end{cases}
$$

Proof: From (1), it is easily obtained that

$$
\begin{aligned}
X(k+1) &= \mathbb{E}\big\{A(k)x(k)x^T(k)A^T(k) \\
&\quad + \varepsilon(k)B(k)x(k)x^T(k)B^T(k)\varepsilon^T(k) \\
&\quad + E(k)w(k)w^T(k)E^T(k)\big\} \\
&= A(k)X(k)A^T(k) + B(k)X(k)B^T(k) \\
&\quad + E(k)W(k)E^T(k).
\end{aligned}
$$

From the definition of $\bar{x}(s_k)$, we know that $\bar{X}_{i,j}(s_k)$ is the (i,j)-th submatrix of $\bar{X}(s_k)$. For $i = j$, one has $\bar{X}_{i,i}(s_k) = X(s_{k-1}+i)$. For $i > j$, it is known from (1) that

$$
\bar{X}_{i,j}(s_k) = A(s_k - b + i - 1)\bar{X}_{i-1,j}(s_k)
$$

$$= \prod_{l=b-i+1}^{b-j} A(s_k - l)\bar{X}_{j,j}(s_k)$$
$$= \prod_{l=b-i+1}^{b-j} A(s_k - l)X(s_{k-1} + j).$$

For $i < j$, we have

$$\begin{aligned}\bar{X}_{i,j}(s_k) &= \bar{X}_{i,j-1}(s_k)A^T(s_{k-1} + j - 1)\\ &= \bar{X}_{i,i}(s_k)\prod_{l=i}^{j-1} A^T(s_{k-1} + l)\\ &= X(s_{k-1} + i)\prod_{l=i}^{j-1} A^T(s_{k-1} + l).\end{aligned}$$

Then, $\bar{X}(s_k)$ is calculated by (9). The proof is complete. ■

For simplification, the following notations are introduced:

$$\begin{aligned}\Theta_i(s_k) &\triangleq \mathbb{E}\left\{\mathscr{F}_i^b(s_k)x(s_k)x^T(s_k)(\mathscr{F}_i^b(s_k))^T\right\},\\ \Omega_i^m(s_k) &\triangleq \mathbb{E}\{\bar{E}_{mi}(s_k)w(s_k + i - 1)\\ &\qquad \times w^T(s_k + i - 1)\bar{E}_{mi}^T(s_k)\},\\ \bar{\mathscr{F}}_m^n(s_k) &\triangleq \prod_{i=m}^{n} A(s_{k+1} - i),\\ \bar{W}(s_k) &\triangleq E(s_k)W(s_k)E^T(s_k)\end{aligned}$$

where $\bar{E}_{mi}(s_k) \in \mathbb{R}^{n_x \times n_w}$ is the (m, i)-th submatrix of $\bar{E}(s_k)$.

In order to calculate the estimation error covariance, one also needs to calculate $\Gamma_1(s_k)$ and $\Gamma_2(s_k)$ with $\tilde{A}(s_k)$ and $\bar{E}(s_k)$ being random matrices. Obviously, the random matrices make the calculations nontrivial. In the following, a novel method is provided to handle such difficulties.

Lemma 4: The term $\Gamma_1(s_k)$ is obtained by

$$\begin{aligned}\Gamma_1(s_k) &= \mathbb{E}\{\bar{A}(s_k)\bar{x}(s_k)\bar{x}^T(s_k)\bar{A}^T(s_k)\}\\ &\quad - \mathbb{E}\{\bar{A}(s_k)\}\bar{X}(s_k)\mathbb{E}\{\bar{A}^T(s_k)\}\end{aligned}$$

where

$$\mathbb{E}\{\bar{A}(s_k)\} = \Big[\underbrace{0_{bn_x \times n_x} \quad \cdots \quad 0_{bn_x \times n_x}}_{b-1} \quad \bar{\mathscr{A}}(s_k)\Big],$$
$$\bar{\mathscr{A}}(s_k) \triangleq \mathrm{col}\{\bar{\mathscr{F}}_b^b(s_k), \bar{\mathscr{F}}_{b-1}^b(s_k), \cdots, \bar{\mathscr{F}}_1^b(s_k)\}$$

and $\mathbb{E}\{\bar{A}(s_k)\bar{x}(s_k)\bar{x}^T(s_k)\bar{A}^T(s_k)\}$ is derived according to

$$\mathbb{E}\{\bar{A}(s_k)\bar{x}(s_k)\bar{x}^T(s_k)\bar{A}^T(s_k)\}_{mn} = \begin{cases} \bar{\mathscr{F}}_{b-m+1}^{b-n}(s_k)\Theta_{b-n+1}(s_k), & m > n\\ \Theta_{b-m+1}(s_k), & m = n\\ \Theta_{b-m+1}(s_k)\left(\bar{\mathscr{F}}_{b-n+1}^{b-m}(s_k)\right)^T, & m < n.\end{cases}$$

Here, $\mathbb{E}\{\bar{A}(s_k)\bar{x}(s_k)\bar{x}^T(s_k)\bar{A}^T(s_k)\}_{mn} \in \mathbb{R}^{n_x\times n_x}$ is the (m,n)-th submatrix of $\mathbb{E}\{\bar{A}(s_k)\bar{x}(s_k)\bar{x}^T(s_k)\bar{A}^T(s_k)\}$. $\Theta_i(s_k)$ is calculated by repeating

$$\begin{aligned}\Theta_{i-1}(s_k) = &\, A(s_{k+1}-i+1)\Theta_i(s_k)A^T(s_{k+1}-i+1)\\ &+ B(s_{k+1}-i+1)\Theta_i(s_k)B^T(s_{k+1}-i+1)\end{aligned}$$

with

$$\Theta_b(s_k) = A(s_k)X(s_k)A^T(s_k) + B(s_k)X(s_k)B^T(s_k).$$

Proof: Noting the fact that $\varepsilon(s_{k+1}-i)$ $(i = 1,2,\ldots,b)$ are uncorrelated with $x(s_{k-1}+j)$ $(j = 1,2,\ldots,b)$, it is derived that

$$\begin{aligned}&\mathbb{E}\Big\{\tilde{A}(s_k)\bar{x}(s_k)\bar{x}^T(s_k)\tilde{A}^T(s_k)\Big\}\\ &= \mathbb{E}\{\bar{A}(s_k)\bar{x}(s_k)\bar{x}^T(s_k)\bar{A}^T(s_k)\}\\ &\quad + \mathbb{E}\Big\{\mathbb{E}\{\bar{A}(s_k)\}\bar{x}(s_k)\bar{x}^T(s_k)\mathbb{E}\{\bar{A}^T(s_k)\}\Big\}\\ &\quad - \mathbb{E}\Big\{\mathbb{E}\{\bar{A}(s_k)\}\bar{x}(s_k)\bar{x}^T(s_k)\bar{A}^T(s_k)\Big\}\\ &\quad - \mathbb{E}\Big\{\bar{A}(s_k)\bar{x}(s_k)\bar{x}^T(s_k)\mathbb{E}\{\bar{A}^T(s_k)\}\Big\}\\ &= \mathbb{E}\{\bar{A}(s_k)\bar{x}(s_k)\bar{x}^T(s_k)\bar{A}^T(s_k)\}\\ &\quad - \mathbb{E}\{\bar{A}(s_k)\}\mathbb{E}\{\bar{x}(s_k)\bar{x}^T(s_k)\}\mathbb{E}\{\bar{A}^T(s_k)\}.\end{aligned}$$

From the definition of $\bar{A}(s_k)$, by resorting to Lemma 1, the calculation of $\mathbb{E}\{\bar{A}(s_k)\bar{x}(s_k)\bar{x}^T(s_k)\bar{A}^T(s_k)\}$ reduces to the calculation of

$$\begin{aligned}&\mathbb{E}\{\bar{A}(s_k)\bar{x}(s_k)\bar{x}^T(s_k)\bar{A}^T(s_k)\}_{mn}\\ &\quad = \mathbb{E}\{\mathscr{F}^b_{b-m+1}(s_k)x(s_k)x^T(s_k)(\mathscr{F}^b_{b-n+1}(s_k))^T\}.\end{aligned}$$

For $m = n$, it is obvious that

$$\mathbb{E}\{\bar{A}(s_k)\bar{x}(s_k)\bar{x}^T(s_k)\bar{A}^T(s_k)\}_{mm} = \Theta_{b-m+1}(s_k).$$

For $m > n$, we have

$$\begin{aligned}&\mathbb{E}\{\bar{A}(s_k)\bar{x}(s_k)\bar{x}^T(s_k)\bar{A}^T(s_k)\}_{mn}\\ &\quad = \mathbb{E}\{\prod_{i=b-m+1}^{b-n} F(s_{k+1}-i)\\ &\qquad\quad \times \mathscr{F}^b_{b-n+1}(s_k)x(s_k)x^T(s_k)(\mathscr{F}^b_{b-n+1}(s_k))^T\}\\ &\quad = \prod_{i=b-m+1}^{b-n} \mathbb{E}\{F(s_{k+1}-i)\}\Theta_{b-n+1}(s_k)\\ &\quad = \bar{\mathscr{F}}^{b-n}_{b-m+1}(s_k)\Theta_{b-n+1}(s_k).\end{aligned}$$

Similarly, for $m < n$, we have

$$\mathbb{E}\{\bar{A}(s_k)\bar{x}(s_k)\bar{x}^T(s_k)\bar{A}^T(s_k)\}_{mn}$$

$$
\begin{aligned}
&= \Theta_{b-m+1}(s_k)\left(\prod_{i=b-n+1}^{b-m} \mathbb{E}\{F(s_{k+1}-i)\}\right)^T \\
&= \Theta_{b-m+1}(s_k)\left(\bar{\mathscr{F}}_{b-n+1}^{b-m}(s_k)\right)^T .
\end{aligned}
$$

Since $\varepsilon(k)$ are mutually uncorrelated in time, we can derive the following relationship between $\Theta_{i-1}(s_k)$ and $\Theta_i(s_k)$:

$$
\begin{aligned}
\Theta_{i-1}(s_k) &= \mathbb{E}\{\mathscr{F}_{i-1}^b(s_k)x(s_k)x^T(s_k)(\mathscr{F}_{i-1}^b(s_k))^T\} \\
&= \mathbb{E}\{F(s_{k+1}-i+1)\mathscr{F}_i^b(s_k)x(s_k) \\
&\qquad\qquad \times x^T(s_k)(\mathscr{F}_i^b(s_k))^T F^T(s_{k+1}-i+1)\} \\
&= A(s_{k+1}-i+1)\mathbb{E}\{\mathscr{F}_i^b(s_k)x(s_k) \\
&\qquad\qquad \times x^T(s_k)(\mathscr{F}_i^b(s_k))^T\}A^T(s_{k+1}-i+1) \\
&\quad + B(s_{k+1}-i+1)\mathbb{E}\{\mathscr{F}_i^b(s_k)x(s_k) \\
&\qquad\qquad \times x^T(s_k)(\mathscr{F}_i^b(s_k))^T\}B^T(s_{k+1}-i+1) \\
&= A(s_{k+1}-i+1)\Theta_i(s_k)A^T(s_{k+1}-i+1) \\
&\quad + B(s_{k+1}-i+1)\Theta_i(s_k)B^T(s_{k+1}-i+1).
\end{aligned}
$$

Moreover, it is known that

$$
\begin{aligned}
\Theta_b(s_k) &= \mathbb{E}\{F(s_k)x(s_k)x^T(s_k)F^T(s_k)\} \\
&= A(s_k)\mathbb{E}\{x(s_k)x^T(s_k)\}A^T(s_k) \\
&\quad + B(s_k)\mathbb{E}\{x(s_k)x^T(s_k)\}B^T(s_k) \\
&= A(s_k)X(s_k)A^T(s_k) + B(s_k)X(s_k)B^T(s_k).
\end{aligned}
$$

Then, $\Theta_{b-m+1}(s_k)$ and $\Theta_{b-n+1}(s_k)$ are calculated by repeating the above relationship. The proof is complete. ■

Lemma 5: The term $\Gamma_2(s_k)$ is obtained according to

$$
\{\Gamma_2(s_k)\}_{mn} = \begin{cases}
\bar{\mathscr{F}}_{b-m+1}^{b-n}(s_k)\sum_{i=1}^{n}\Omega_i^n(s_k), & m > n \\
\sum_{i=1}^{m}\Omega_i^m(s_k), & m = n \\
\sum_{i=1}^{m}\Omega_i^m(s_k)\left(\bar{\mathscr{F}}_{b-n+1}^{b-m}(s_k)\right)^T, & m < n.
\end{cases}
$$

Here, $\{\Gamma_2(s_k)\}_{mn} \in \mathbb{R}^{n_x \times n_x}$ is the (m,n)-th submatrix of $\Gamma_2(s_k)$. Moreover, $\Omega_i^m(s_k)$ is derived by repeating

$$
\begin{aligned}
\Omega_i^m(s_k) &= A(s_k+m-1)\Omega_i^{m-1}(s_k)A^T(s_k+m-1) \\
&\quad + B(s_k+m-1)\Omega_i^{m-1}(s_k)B^T(s_k+m-1)
\end{aligned}
$$

with

$$
\Omega_i^i(s_k) = \bar{W}(s_k+i-1).
$$

Proof: The proof of this lemma is similar to that of Lemma 4 and is therefore omitted here. ■

Remark 2: In Lemmas 4 and 5, a novel method is put forward to calculate $\Gamma_1(s_k) = \mathbb{E}\{\tilde{A}(s_k)\bar{x}(s_k)\bar{x}^T(s_k)\tilde{A}^T(s_k)\}$ and $\Gamma_2(s_k) = \mathbb{E}\{\bar{E}(s_k)\bar{w}(s_k)\bar{w}^T(s_k)\bar{E}^T(s_k)\}$. Note that the structures of the matrices $\tilde{A}(s_k)$ and $\bar{E}(s_k)$ complicate the calculations of $\Gamma_1(s_k)$ and $\Gamma_2(s_k)$, while the multiplicative noise $\varepsilon(k)$ makes the calculation even more complicated. Accordingly, it is quite difficult to directly calculate $\Gamma_1(s_k)$ and $\Gamma_2(s_k)$. With the help of Lemma 1, the calculations of $\Gamma_1(s_k)$ and $\Gamma_2(s_k)$ reduce to the calculation of the submatrices of $\Gamma_1(s_k)$ and $\Gamma_2(s_k)$, thereby reducing the computational complexity to a great extent.

Due to the consideration of the sensor resolution, it is extremely hard (if not impossible) to obtain the exact estimation error covariance. As such, an alternative way is to obtain an upper bound on the estimation error covariance and then the gain matrix is characterized so as to minimize the obtained upper bound.

Lemma 6: Denote

$$\Delta_i(s_k) \triangleq y_i^{ac}(s_k) - y_i^{id}(s_k)$$

as the i-th element of $\Delta(s_k)$. Then, one has

$$|\Delta_i(s_k)| < r_i.$$

Proof: For a constant $a \geq 1$, from the definition of the floor function $\lfloor\cdot\rfloor$, we know that $-1 < \lfloor a\rfloor - a \leq 0$ and therefore

$$-1 < \left\lfloor \frac{y_i^{id}(s_k)}{r_i} \right\rfloor - \frac{y_i^{id}(s_k)}{r_i} \leq 0, \quad y_i^{id}(s_k) \geq r_i$$

which is equivalent to $-r_i < y_i^{ac}(s_k) - y_i^{id}(s_k) \leq 0$ for $y_i^{id}(s_k) \geq r_i$.

Similarly, from the definition of the ceiling function $\lceil\cdot\rceil$, we know that $0 \leq \lceil b\rceil - b < 1$ with $b \leq -1$ being a constant. Therefore, we have

$$0 \leq y_i^{ac}(s_k) - y_i^{id}(s_k) < r_i, \quad y_i^{id}(s_k) \leq -r_i.$$

Moreover, it is easily known that $-r_i < y_i^{ac}(s_k) - y_i^{id}(s_k) < r_i$ for $-r_i < y_i^{id}(s_k) < r_i$.

Summarizing the above discussions, we have $|\Delta_i(s_k)| = \left|y_i^{ac}(s_k) - y_i^{id}(s_k)\right| < r_i$, and the proof is complete. ■

Theorem 1: Let the positive scalars $\gamma_1(s_k)$ and $\gamma_2(s_k)$ be given. Denote

$$\begin{aligned}
\bar{\gamma}_1(s_k) &\triangleq 1 + \gamma_1(s_k), \\
\bar{\gamma}_2(s_k) &\triangleq 1 + \gamma_1^{-1}(s_k) + \gamma_2(s_k), \\
\bar{\gamma}_3(s_k) &\triangleq 1 + \gamma_2^{-1}(s_k).
\end{aligned}$$

The solution $\bar{P}(s_{k+1})$ for the following recursion

$$\begin{aligned}
\bar{P}(s_{k+1}) = {} & \bar{\gamma}_1(s_k)\check{A}(s_k)\bar{P}(s_k)\check{A}^T(s_k) + \Gamma_1(s_k) + \Gamma_2(s_k) \\
& + \bar{\gamma}_2(s_k)\sum_{i=1}^{n_y} r_i^2 K(s_k)K^T(s_k) \\
& + \bar{\gamma}_3(s_k)K(s_k)D(s_k)V(s_k)D^T(s_k)K^T(s_k)
\end{aligned} \tag{10}$$

with initial value $\bar{P}(s_0) = P(s_0)$ is an upper bound on the estimation error covariance $P(s_{k+1})$. Here, $\Gamma_1(s_k)$ and $\Gamma_2(s_k)$ can be obtained according to Lemma 4 and Lemma 5, respectively. Moreover, the upper bound $\bar{P}(s_{k+1})$ is minimized with the estimator gain $K(s_k)$ chosen as

$$K(s_k) = \Upsilon(s_k)\Sigma^{-1}(s_k) \tag{11}$$

and the minimized upper bound is

$$\begin{aligned}\bar{P}(s_{k+1}) =& -\Upsilon(s_k)\Sigma^{-1}(s_k)\Upsilon^T(s_k) + \Gamma_1(s_k) + \Gamma_2(s_k)\\ &+ \bar{\gamma}_1(s_k)\mathbb{E}\{\bar{A}(s_k)\}\bar{P}(s_k)\mathbb{E}^T\{\bar{A}(s_k)\}\end{aligned} \tag{12}$$

where

$$\begin{aligned}\Sigma(s_k) \triangleq\ & \bar{\gamma}_1(s_k)\bar{C}(s_k)\bar{P}(s_k)\bar{C}^T(s_k)\\ &+ \bar{\gamma}_2(s_k)\sum_{i=1}^{n_y} r_i^2 I\\ &+ \bar{\gamma}_3(s_k)D(s_k)V(s_k)D^T(s_k),\\ \Upsilon(s_k) \triangleq\ & \bar{\gamma}_1(s_k)\mathbb{E}\{\bar{A}(s_k)\}\bar{P}(s_k)\bar{C}^T(s_k).\end{aligned}$$

Proof: The proof is completed by using the mathematical induction method. First, it is obvious that $P(s_0) \leq \bar{P}(s_0)$ holds. Assuming that $P(s_k) \leq \bar{P}(s_k)$ holds, we need to prove $P(s_{k+1}) \leq \bar{P}(s_{k+1})$.

With the help of the elementary inequality $ab^T + ba^T \leq \delta aa^T + \delta^{-1}bb^T$ where a, b are vectors of appropriate dimensions and δ is a known scalar, it follows from Lemmas 2-4 that

$$\begin{aligned}P(s_{k+1}) \leq\ & \bar{\gamma}_1(s_k)\check{A}(s_k)P(s_k)\check{A}^T(s_k)\\ &+ \Gamma_1(s_k) + \Gamma_2(s_k)\\ &+ \bar{\gamma}_2(s_k)K(s_k)\mathbb{E}\{\Delta(s_k)\Delta^T(s_k)\}K^T(s_k)\\ &+ \bar{\gamma}_3(s_k)K(s_k)D(s_k)V(s_k)D^T(s_k)K^T(s_k).\end{aligned}$$

Moreover, from Lemma 6, we know that

$$\begin{aligned}\mathbb{E}\{\Delta(s_k)\Delta^T(s_k)\} &\leq \mathbb{E}\big\{\mathrm{tr}\{\Delta(s_k)\Delta^T(s_k)\}\big\}I\\ &= \sum_{i=1}^{n_y}\mathbb{E}\{\Delta_i^2(s_k)\}I \leq \sum_{i=1}^{n_y} r_i^2 I.\end{aligned}$$

Accordingly, one has

$$\begin{aligned}P(s_{k+1}) \leq\ & \bar{\gamma}_1(s_k)\check{A}(s_k)P(s_k)\check{A}^T(s_k)\\ &+ \Gamma_1(s_k) + \Gamma_2(s_k)\\ &+ \bar{\gamma}_2(s_k)\sum_{i=1}^{n_y} r_i^2 K(s_k)K^T(s_k)\\ &+ \bar{\gamma}_3(s_k)K(s_k)D(s_k)V(s_k)D^T(s_k)K^T(s_k).\end{aligned}$$

Therefore, $P(s_{k+1}) \leq \bar{P}(s_{k+1})$ holds.

Now, we are going to characterize the estimator gain that minimizes $\bar{P}(s_{k+1})$. From the definition of $\breve{A}(s_k)$, (10) is rewritten as

$$\begin{aligned}\bar{P}(s_{k+1}) = &\ \bar{\gamma}_1(s_k)\mathbb{E}\{\bar{A}(s_k)\}\bar{P}(s_k)\mathbb{E}^T\{\bar{A}(s_k)\}\\ &+ \bar{\gamma}_1(s_k)K(s_k)\bar{C}(s_k)\bar{P}(s_k)\bar{C}^T(s_k)K^T(s_k)\\ &- \bar{\gamma}_1(s_k)\mathbb{E}\{\bar{A}(s_k)\}\bar{P}(s_k)\bar{C}^T(s_k)K^T(s_k)\\ &- \bar{\gamma}_1(s_k)K(s_k)\bar{C}(s_k)\bar{P}(s_k)\mathbb{E}^T\{\bar{A}(s_k)\}\\ &+ \Gamma_1(s_k) + \Gamma_2(s_k)\\ &+ \bar{\gamma}_2(s_k)\sum_{i=1}^{n_y} r_i^2 K(s_k)K^T(s_k)\\ &+ \bar{\gamma}_3(s_k)K(s_k)D(s_k)V(s_k)D^T(s_k)K^T(s_k).\end{aligned}$$

Noting that $\Gamma_1(s_k)$ and $\Gamma_2(s_k)$ do not contain the estimator gain $K(s_k)$, one has

$$\begin{aligned}\bar{P}(s_{k+1}) = &\ K(s_k)\Sigma(s_k)K^T(s_k) - \Upsilon(s_k)K^T(s_k)\\ &- K(s_k)\Upsilon^T(s_k) + \Gamma_1(s_k) + \Gamma_2(s_k)\\ &+ \bar{\gamma}_1(s_k)\mathbb{E}\{\bar{A}(s_k)\}\bar{P}(s_k)\mathbb{E}^T\{\bar{A}(s_k)\}\\ = &\ (K(s_k) - \Upsilon(s_k)\Sigma^{-1}(s_k))\Sigma(s_k)\\ &\quad\times (K(s_k) - \Upsilon(s_k)\Sigma^{-1}(s_k))^T\\ &- \Upsilon(s_k)\Sigma^{-1}(s_k)\Upsilon^T(s_k)\\ &+ \bar{\gamma}_1(s_k)\mathbb{E}\{\bar{A}(s_k)\}\bar{P}(s_k)\mathbb{E}^T\{\bar{A}(s_k)\}\\ &+ \Gamma_1(s_k) + \Gamma_2(s_k).\end{aligned}$$

Since $\Sigma(s_k) > 0$, the minimum of $\bar{P}(s_{k+1})$ is (12) with the estimator gain being (11). The proof is complete. ■

In the following, the proposed recursive estimation algorithm is summarized in Algorithm 1.

Algorithm 1 Recursive state estimation with the sensor resolution and multiplicative noises

Step 1. Give positive scalars $\gamma_1(s_k)$, $\gamma_2(s_k)$ and set the initial values $\hat{x}(s_0)$, $\bar{P}(s_0)$;

Step 2. At time instant s_k, calculate $\mathbb{E}\{\bar{A}(s_k)\}$, $\Gamma_1(s_k)$, and $\Gamma_2(s_k)$ according to Lemmas 4-5;

Step 3. Calculate $K(s_k)$ and $\bar{P}(s_{k+1})$ by using (11) and (12), respectively;

Step 4. Compute the estimate $\hat{x}(s_{k+1})$ with the estimator (5). Set $k = k + 1$;

Step 5. If $k < M$, then go to Step 2, else go to Step 6;

Step 6. Stop.

Remark 3: In this paper, the RSE problem is studied for MRSs with multiplicative noises and sensor resolution. In Lemmas 4-5, a novel method is used to handle the difficulties resulting from the coupling of the multi-rate sampling and the multiplicative noises. An upper bound is derived in Lemma 6 as a result of tackling the uncertainties caused by the sensor resolution. Based on the result given in Lemma

6, the estimator gain is characterized in Theorem 1 which minimizes the upper bound on the estimation error covariance and the corresponding minimized upper bound is presented. It is worth mentioning that the obtained minimal upper bound reflects all the system information including the multi-rate sampling, the multiplicative noises, and the sensor resolution.

Remark 4: In this paper, the RSE problem has been solved for MRSs with multiplicative noises and sensor resolution. The main contributions of our results are that 1) a novel recursive state estimation algorithm is developed for systems with sensor resolution; 2) a novel method is proposed which largely reduces the computational complexity for the expectation of matrix multiplication that involves random matrices; and 3) the proposed estimation algorithm is in the recursive form and is suitable for online computation.

IV. Simulation example

In this section, the usefulness of the proposed estimation algorithm is verified on the moving target tracking problem.

The dynamics of the moving target (modified from [44]) is formulated as

$$x(k+1) = A(k)x(k) + \varepsilon(k)B(k)x(k) + E(k)w(k)$$

where

$$x(k) \triangleq \begin{bmatrix} p_x^T(k) & \nu_x^T(k) & p_y^T(k) & \nu_y^T(k) \end{bmatrix}^T$$

with $(p_x(k), p_y(k))$ being the position of the target and $(\nu_x(k), \nu_y(k))$ being the velocity of the target. $\varepsilon(k)$ is the multiplicative noise with zero mean and unity covariance. $w(k)$ is the zero-mean process noise with covariance matrix

$$W(k) = \Lambda \begin{bmatrix} T^3/3 & T^2/2 & 0 & 0 \\ T^2/2 & T & 0 & 0 \\ 0 & 0 & T^3/3 & T^2/2 \\ 0 & 0 & T^2/2 & T \end{bmatrix}$$

where T is the sampling period and Λ is the acceleration variance. Moreover, the parameter matrices are

$$A(k) = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ E(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ B(k) = \begin{bmatrix} 0 & 0.01 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.01 \\ 0 & 0.01 & 0 & 0 \end{bmatrix}.$$

In this example, a sensor with the sampling period $b = 2T$ and the resolution $R = \mathrm{col}\{0.1, 0.01\}$ is deployed to collect the information. The ideal measurement model is given as

$$y^{id}(s_k) = C(s_k)x(s_k) + D(s_k)v(s_k)$$

where $v(s_k)$ is the measurement noise with zero mean and covariance 0.05. The parameter matrices are

$$C(s_k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ D(s_k) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

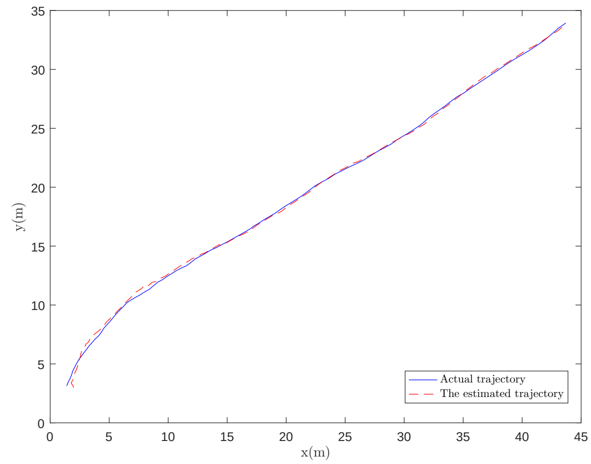

Fig. 1: The trajectory of the moving target and its estimate

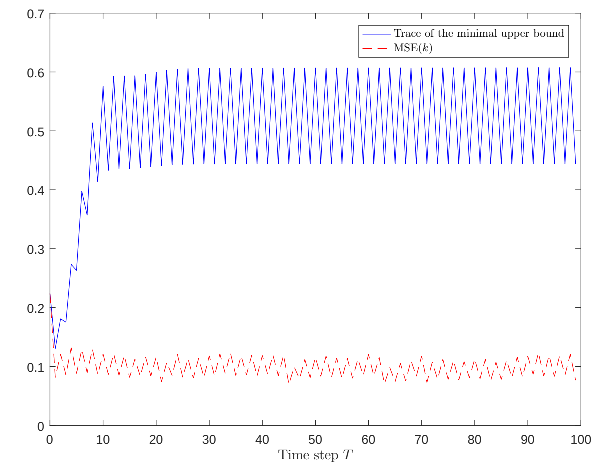

Fig. 2: Trace of the minimal upper bound and the MSE

In the simulation, we set $T = 1\mathrm{s}$ and $\Lambda = 0.04$. The initial condition of the state is $x(0) = \begin{bmatrix} 2\mathrm{m} & 0.1\mathrm{m/s} & 3\mathrm{m} & 0.2\mathrm{m/s} \end{bmatrix}^T$.

The simulation results are given in Figs. 1-2. Fig. 1 shows the actual trajectory of the moving target and the estimate of the trajectory. Fig. 2 gives the trace of the minimal upper bound and the mean square error of the proposed estimation algorithm. The mean square error, denoted as $\mathrm{MSE}(k)$, is defined as

$$\mathrm{MSE}(k) \triangleq \frac{1}{N}\big(x(k) - \vec{x}(k)\big)^T\big(x(k) - \vec{x}(k)\big)$$

with $N = 500$ and $\vec{x}(k)$ being the estimate of $x(k)$. The estimate $\vec{x}(k)$ is obtained by applying matrix operations to $\hat{x}(k)$. The simulation results verify that the developed estimation algorithm is effective in the moving target tracking problem and the derived minimal upper bound is indeed an upper bound of the mean square error.

To show the monotonicity of the minimal upper bound with respect to the resolution of the sensor, in the following, simulation results with different sensor resolutions are presented. Figs. 3-4 give the minimal

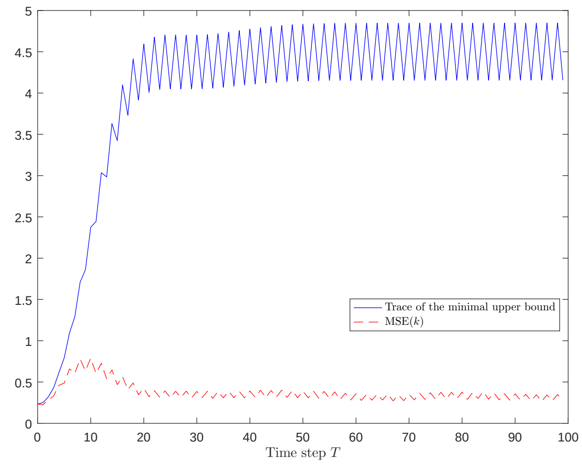

Fig. 3: Estimation performance with resolution $R_1 = \mathrm{col}\{0.5, 0.01\}$

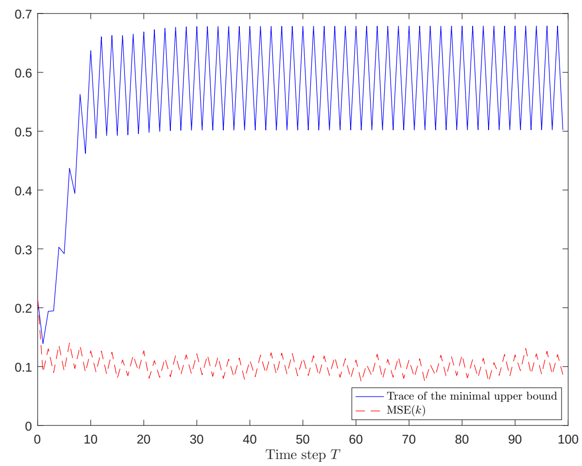

Fig. 4: Estimation performance with resolution $R_2 = \mathrm{col}\{0.1, 0.05\}$

upper bounds and the MSEs with the sensor resolutions $R_1 = \mathrm{col}\{0.5, 0.01\}$ and $R_2 = \mathrm{col}\{0.1, 0.05\}$, respectively. From the simulation results, we know that the minimal upper bound and the MSE are increasing when the sensor resolution increases, which is in agreement with the engineering practice.

V. Conclusions

In this paper, the RSE problem has been investigated for MRSs with multiplicative noises and sensor resolution. By applying the lifting technique, the MRS with multiplicative noises has been converted into a single-rate system with random parameter matrices. A novel method has been put forward to handle the difficulties from the random parameter matrices. A novel state estimation scheme with low mathematical complexity has also been developed which is robust to the uncertainty caused by the sensor resolution. With the developed state estimation scheme, an upper bound has been obtained on the estimation error covariance and the estimator gain has been characterized with which the obtained upper bound is minimized. Finally, the effectiveness of the proposed estimation algorithm has been verified on the moving target tracking problem. The future research topics will be the investigations of the state estimation problems for complex networks [26], [47], [50] with sensor resolution and the control problems for multi-agent systems [10], [25], [48] with sensor resolution.

VI. Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

References

[1] X. Bai, Z. Wang, L. Zou and Z. Zhang, Target tracking for wireless localization systems using set-membership filtering: A component-based event-triggered mechanism, *Automatica*, vol. 132, art. no. 109795, 2021.

[2] M. Basin, J. Rodriguez-Gonzalez and L. Fridman, Integral sliding mode design for robust filtering and control of linear stochastic time-delay systems, *International Journal of Robust and Nonlinear Control*, vol. 15, no. 9, pp. 407–421, 2005.

[3] M. Basin, P. Shi and D. Calderon-Alvarez, Joint state filtering and parameter estimation for linear stochastic time-delay systems, *Signal Processing*, vol. 91, no. 4, pp. 782–792, 2011.

[4] R. Caballero-Águila, A. Hermoso-Carazo and J. Linares-Pérez, Optimal state estimation for networked systems with random parameter matrices, correlated noises and delayed measurements, *International Journal of General Systems*, vol. 44, no. 2, pp. 142–154, 2015.

[5] B. Cai, L. Zhang and Y. Shi, Observed-mode-dependent state estimation of hidden semi-Markov jump linear systems, *IEEE Transactions on Automatic Control*, vol. 65, no. 1, pp. 442–449, 2020.

[6] S. Chen, L. Ma and Y. Ma, Distributed set-membership filtering for nonlinear systems subject to round-robin protocol and stochastic communication protocol over sensor networks, *Neurocomputing*, vol. 385, pp. 13–21, 2020.

[7] W. Chen, J. Hu, Z. Wu, X. Yu and D. Chen, Finite-time memory fault detection filter design for nonlinear discrete systems with deception attacks, *International Journal of Systems Science*, vol. 51, no. 8, pp. 1464–1481, 2020.

[8] Y. Chen, J. Ren, X. Zhao and A. Xue, State estimation of Markov jump neural networks with random delays by redundant channels, *Neurocomputing*, vol. 453, pp. 493–501, 2021.

[9] Y. Cui, Y. Liu, W. Zhang and F. E. Alsaadi, Sampled-based consensus for nonlinear multiagent systems with deception attacks: The decoupled method, *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 1, pp. 561–573, 2021.

[10] L. Ding, Q.-L. Han, X. Ge and X.-M. Zhang, An overview of recent advances in event-triggered consensus of multiagent systems, *IEEE Transactions on Cybernetics*, vol. 48, no. 4, pp. 1110–1123, 2018.

[11] Q. Dong, N. Yao, H. Chang and X. Qiu, Application and analysis of multi-rate sampled in structural health monitoring system, *Journal of Computational Methods in Sciences and Engineering*, vol. 16, no. 2, pp. 317–328, 2016.

[12] A. Fatehi and B. Huang, Kalman filtering approach to multi-rate information fusion in the presence of irregular sampling rate and variable measurement delay, *Journal of Process Control*, vol. 53, pp. 15–25, 2017.

[13] X. Ge, Q.-L. Han, X.-M. Zhang and L. Ding, Distributed event-triggered estimation over sensor networks: A survey, *IEEE Transactions on Cybernetics*, vol. 50, no. 3, pp. 1306–1320, 2020.

[14] H. Geng, Y. Liang, F. Yang, L. Xu and Q. Pan, The joint optimal filtering and fault detection for multi-rate sensor fusion under unknown inputs, *Information Fusion*, vol. 29, pp. 57–67, 2016.

[15] H. Geng, H. Liu, L. Ma and X. Yi, Multi-sensor filtering fusion meets censored measurements under a constrained network environment: advances, challenges and prospects, *International Journal of Systems Science*, vol. 52, no. 16, pp. 3410–3436, 2021.

[16] Y. Gu, Y. Chou, J. Liu and Y. Ji, Moving horizon estimation for multirate systems with time-varying time-delays, *Journal of the Franklin Institute*, vol. 356, no. 4, pp. 2325–2345, 2019.

[17] N. Hou, Z. Wang, D. W. C. Ho and H. Dong, Robust partial-nodes-based state estimation for complex networks under deception attacks, *IEEE Transactions on Cybernetics*, vol. 50, no. 6, pp. 2793–2802, 2020.

[18] J. Hu, D. Chen and J. Du, State estimation for a class of discrete nonlinear systems with randomly occurring uncertainties and distributed sensor delays, *International Journal of General Systems*, vol. 43, no. 3-4, pp. 387–401, 2014.

[19] J. Hu, Z. Wang, B. Shen and H. Gao, Quantised recursive filtering for a class of nonlinear systems with multiplicative noises and missing measurements, *International Journal of Control*, vol. 86, no. 4, pp. 650–663, 2013.

[20] J. Hu, H. Zhang, H. Liu and X. Yu, A survey on sliding mode control for networked control systems, *International Journal of Systems Science*, vol. 52, no. 6, pp. 1129–1147, 2021.

[21] J. Hu, C. Jia, H. Liu, X. Yi and Y. Liu, A survey on state estimation of complex dynamical networks, *International Journal of Systems Science*, vol. 52, no. 16, pp. 3351–3367, 2021.

[22] X.-C. Jia, Resource-efficient and secure distributed state estimation over wireless sensor networks: a survey, *International Journal of Systems Science*, vol. 52, no. 16, pp. 3368–3389, 2021.

[23] Y. Ju, Y. Liu, X. He and B. Zhang, Finite-horizon H_∞ filtering and fault isolation for a class of time-varying systems with sensor saturation, *International Journal of Systems Science*, vol. 52, no. 2, pp. 321–333, 2021.

[24] Y. Ju, X. Tian, H. Liu and L. Ma, Fault detection of networked dynamical systems: a survey of trends and techniques, *International Journal of Systems Science*, vol. 52, no. 16, pp. 3390–3409, 2021.

[25] H. Li, X. Liao, T. Huang and W. Zhu, Event-triggering sampling based leader-following consensus in second-order multi-agent systems, *IEEE Transactions on Automatic Control*, vol. 60, no. 7, pp. 1998–2003, 2015.

[26] N. Li, Q. Li and J. Suo, Dynamic event-triggered H_∞ state estimation for delayed complex networks with randomly occurring nonlinearities, *Neurocomputing*, vol. 421, pp. 97–104, 2021.

[27] W. Li, C. Meng, Y. Jia and J. Du, Recursive filtering for complex networks using non-linearly coupled UKF, *IET Control Theory & Applications*, vol. 12, no. 4, pp. 549–555, 2018.

[28] X. Li, F. Han, N. Hou, H. Dong and H. Liu, Set-membership filtering for piecewise linear systems with censored measurements under Round-Robin protocol, *International Journal of Systems Science*, vol. 51, no. 9, pp. 1578–1588, 2020.

[29] Y. Li, H. R. Karimi, M. Zhong, S. X. Ding and S. Liu, Fault detection for linear discrete time-varying systems with multiplicative noise: The finite-horizon case, *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 65, no. 10, pp. 3492–3505, 2018.

[30] L. Liu, L. Ma, J. Zhang and Y. Bo, Distributed non-fragile set-membership filtering for nonlinear systems under fading channels and bias injection attacks, *International Journal of Systems Science*, vol. 52, no. 6, pp. 1192–1205, 2021.

[31] Q. Liu, Z. Wang, Q.-L. Han and C. Jiang, Quadratic estimation for discrete time-varying non-Gaussian systems with multiplicative noises and quantization effects, *Automatica*, vol. 113, art. no. 108714, 2020.

[32] Y. Liu, Z. Wang and D. Zhou, Resilient actuator fault estimation for discrete-time complex networks: A distributed approach, *IEEE Transactions on Automatic Control*, vol. 66, no. 9, pp. 4214–4221, Sept. 2021.

[33] C.-E. Park, N. K. Kwon and P. Park, Optimal H_∞ filtering for singular Markovian jump systems, *Systems & Control Letters*, vol. 118, pp. 22–28, 2018.

[34] B. Qu, Z. Wang, B. Shen and H. Dong, Distributed state estimation for renewable energy microgrids with sensor saturations, *Automatica*, vol. 131, art. no. 109730, 2021.

[35] S. Roshany-Yamchi, M. Cychowski, R. R. Negenborn, B. D. Schutter, K. Delaney and J. Connell, Kalman filter-based distributed predictive control of large-scale multi-rate systems: Application to power networks, *IEEE Transactions on Control Systems Technology*, vol. 21, no. 1, pp. 27–39, 2013.

[36] X. Tian, G. Wei, L. Wang and J. Zhou, Wireless-sensor-network-based target localization: A semidefinite relaxation approach with adaptive threshold correction, *Neurocomputing*, vol. 405, pp. 229–238, 2020.

[37] Z.-H. Pang, L.-Z. Fan, K. Liu and G.-P. Liu, Detection of stealthy false data injection attacks against networked control systems via active data modification, *Information Sciences*, vol. 546, pp. 192-205, 2021.

[38] Z.-H. Pang, L.-Z. Fan, Z. Dong, Q.-L. Han and G.-P. Liu, False data injection attacks against partial sensor measurements of networked control systems, *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 69, no. 1, pp. 149-153, 2022.

[39] W. Qian, Y. Li, Y. Zhao, and Y. Chen, New optimal method for L_2-L_∞ state estimation of delayed neural networks, *Neurocomputing*, vol. 415, pp. 258–265, 2020.

[40] B. Qu, N. Li, Y. Liu and F. E. Alsaadi, Estimation for power quality disturbances with multiplicative noises and correlated noises: A recursive estimation approach, *International Journal of Systems Science*, vol. 51, no. 7, pp. 1200–1217, 2020.

[41] Y. Shen, Z. Wang, B. Shen and H. Dong, Outlier-resistant recursive filtering for multisensor multirate networked systems under weighted Try-Once-Discard Protocol, *IEEE Transactions on Cybernetics*, vol. 51, no. 10, pp. 4897–4908, 2021.

[42] J. Song, D. Ding, H. Liu and X. Wang, Non-fragile distributed state estimation over sensor networks subject to DoS attacks: The almost sure stability, *International Journal of Systems Science*, vol. 51, no. 6, pp. 1119–1132, 2020.

[43] Q. Song, Y. Chen, Z. Zhao, Y. Liu and F. E. Alsaadi, Robust stability of fractional-order quaternion-valued neural networks with neutral delays and parameter uncertainties, *Neurocomputing*, vol. 420, pp. 70–81, 2021.

[44] W. Song, Z. Wang, J. Wang, F. E. Alsaadi and J. Shan, Distributed auxiliary particle filtering with diffusion strategy for target tracking: A dynamic event-triggered approach, *IEEE Transactions on Signal Processing*, vol. 69, pp. 328–340, 2021.

[45] H. Tan, B. Shen, Y. Liu, A. Alsaedi and B. Ahmad, Event-triggered multi-rate fusion estimation for uncertain system with stochastic nonlinearities and colored measurement noises, *Information Fusion*, vol. 36, pp. 313–320, 2017.

[46] H. Viumdal, S. Mylvaganam and D. Di Ruscio, System identification of a non-uniformly sampled multi-rate system in aluminium electrolysis cells, *Modeling, Identification and Control*, vol. 35, no. 3, pp. 127–146, 2014.

[47] J.-L. Wang, Z. Qin, H.-N. Wu, T. Huang and P.-C. Wei, Analysis and pinning control for output synchronization and H_∞ output synchronization of multiweighted complex networks, *IEEE Transactions on Cybernetics*, vol. 49, no. 4, pp. 1314–1326, 2019.

[48] W. Xu and D. W. C. Ho, Clustered event-triggered consensus analysis: An impulsive framework, *IEEE Transactions on Industrial Electronics*, vol. 63, no. 11, pp. 7133–7143, 2016.

[49] H. Yang, H. Li, Y. Xia and L. Li, Distributed Kalman filtering over sensor networks with transmission delays, *IEEE Transactions on Cybernetics*, vol. 51, no. 11, pp. 5511–5521, 2021.

[50] H. Yang, Z. Wang, Y. Shen, F. E. Alsaadi and F. E. Alsaadi, Event-triggered state estimation for Markovian jumping neural networks: On mode-dependent delays and uncertain transition probabilities, *Neurocomputing*, vol. 424, pp. 226–235, 2021.

[51] J. Zhang, X. He and D. Zhou, Filtering for stochastic uncertain systems with non-logarithmic sensor resolution, *Automatica*, vol. 89, pp. 194–200, 2018.

[52] W.-A. Zhang, G. Feng and L. Yu, Multi-rate distributed fusion estimation for sensor networks with packet losses, *Automatica*, vol. 48, no. 9, pp. 2016–2028, 2012.

[53] D. Zhao, S. X. Ding, H. R. Karimi and Y. Li, Robust H_∞ filtering for two-dimensional uncertain linear discrete time-varying systems: A krein space-based method, *IEEE Transactions on Automatic Control*, vol. 64, no. 12, pp. 5124–5131, 2019.

[54] M. Zhu, Y. Chen, Y. Kong, C. Chen and J. Bai, Distributed filtering for Markov jump systems with randomly occurring one-sided lipschitz nonlinearities under Round-Robin scheduling, *Neurocomputing*, vol. 417, pp. 396–405, 2020.

[55] L. Zou, Z. Wang, J. Hu, Y. Liu and X. Liu, Communication-protocol-based analysis and synthesis of networked systems: Progress, prospects and challenges, *International Journal of Systems Science*, vol. 52, no. 14, pp. 3013–3034, 2021.

[56] L. Zou, Z. Wang, J. Hu and Q.-L. Han, Moving horizon estimation meets multi-sensor information fusion: Development, opportunities and challenges, *Information Fusion*, vol. 60, pp. 1–10, 2020.