# A UNIFIED DYNAMIC SIMILITUDE MODEL FOR SOLID CONTINUUM

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## ABSTRACT

*Dynamic similitude has proven to be a valuable tool, which is widely adopted in fluid mechanics. However, even with the ever-growing interest in dynamic similitude in solid mechanics, there is still no unified scaling law applicable to any given solid structure or system, and this has prevented the broad adoption of similitude in the field. Here we develop a unified similitude model for solid mechanics using the momentum and the energy conservation. The model allows for the use of different materials in both elastic and plastic regimes. Never reported dimensionless numbers are derived for the first time in this article, and this set of numbers is sufficient for strictly accurate dynamic similitude of any solid structure. Very different case studies are considered, and the perfect agreement seen in compared results confirms the accuracy of the developed scaling model. The exactness of the dimensionless numbers is also confirmed through analytical solutions. The model allows for the scaling of strain rate and, for the first time, the scaling of the strain state between the full-scale structure and its scaled replica.*

## 1 INTRODUCTION

The need to conduct experimental tests in engineering applications is often unavoidable, where the successful design and analysis of new dynamical engineering systems is broadly dependent on many investigations conducted through theoretical, computational and experimental verifications. The experimental observation of phenomena also aids the development and understanding of scientific laws. However, the experimental testing of dynamical systems can be challenging due to high costs and the lack of experimental resources. Even when ample resources are available, it could still be very challenging or impractical to conduct tests due to uncontrollable factors (for example, space exploration related experiments involving differing gravitational forces), while for large and oversized systems, creating the actual working conditions for testing the prototype can be very time-consuming or impossible. An often-viable alternative, if thoroughly considered, is the experimental observation on a "dynamically similar" model and the subsequent scaling of results to obtain relevant results for the prototype; an alternative that is even more feasible with the advancement of digital non-contact measurement technologies.

In this paper, we present the unified approach for exact dynamic similitude in solid mechanics, first developed by Adetoro and Cardoso [1], along with new applications in space application. The set of non-dimensional coefficients for achieving dynamic similarity in any dynamical system is presented using both the momentum and the energy conservation. The solution from the proposed model is valid over any temporal and spatial domain scale of interest. Some of these coefficients have never been reported in the literature. The coefficients unlock for the first time the same strain state constraints of the dynamically similar solutions for both the "model" and the prototype. The coefficients also allow for the rate of straining of the scaled model to be controlled, which has been a major obstacle for the scaling of strain-rate sensitive materials. The solutions presented in this paper focuses solely on purely mechanical processes, where heat sources and heat fluxes are ignored.

## 2 UNIFIED MODEL

The approach presented here is applicable to solid structures or systems for which the continuum assumption holds and structures for which a homogenised representative volume element can be defined. The momentum equation for such a structure is defined as,

$$\rho \frac{Du_i}{Dt} = \rho f_i + \frac{\partial \sigma_{ji}}{\partial x_j}. \qquad i, j = 1,2,3 \tag{1}$$

For a given continuous domain, equations (1) at any given time can be defined as,

$$\int_V \rho \frac{Du_i}{Dt} \delta x_i \; dV = \int_V \rho f_i \delta x_i \; dV - \int_V S_{ij} \delta\varepsilon_{ij} \; dV + \int_\Gamma T_i \delta x_i \; d\Gamma \tag{2}$$

where, $\delta x_i$ are the generic virtual displacements along the rectangular axes of $x_i$; $u_i$ are the components of the velocity; $f_i$ are the components of the body force per unit mass; $T_i$ are the components of the stress vector at the boundary or surface force; $t$ is time, $\rho$ the density, respectively. $S_{ij}$ and $\varepsilon_{ij}$ are the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensors, respectively; both are defined in a co-rotational coordinate system at the reference configuration.

By defining the displacement, strain and stresses in an incremental fashion, we have for an isotropic material,

$$\begin{aligned}\int_V \left( \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} \right) \delta x_i \; dV = \int_V \rho f_i \delta x_i \; dV - \int_V \lambda C_{ijmn_1} \Delta\varepsilon^l_{mn} \delta\varepsilon^l_{ij} \; dV - \int_V 2\mu C_{ijmn_2} \Delta\varepsilon^l_{mn} \delta\varepsilon^l_{ij} \; dV - \\ \int_V {}^{t+\Delta t} S_{ij} \delta\varepsilon^{nl}_{ij} \; dV - \int_V {}^{t+\Delta t} S_{ij} \delta\varepsilon^l_{ij} \; dV + \int_\Gamma T_i \delta x_i \; d\Gamma.\end{aligned} \tag{3}$$

where, subscripts $m, n = 1,2,3$, $\delta\varepsilon_{ij}^l$, $\Delta\varepsilon_{ij}^l$ and $\delta\varepsilon_{ij}^{nl}$, $\Delta\varepsilon_{ij}^{nl}$ are the infinitesimal and incremental linear and nonlinear parts of the strain tensor; $\lambda$ and $\mu$ are the first and second Lame constants; $C_{ijmn_1}$ and $C_{ijmn_2}$ are the material stiffness matrices, respectively. Equations (3) can also be written for orthotropic materials.

For fluids continuum, Stokes [2] argued the unnecessary complexity in the complete determination of the time integrals of his equations similar to (3), since it would be necessary to put $t = 0$ in the equations and equate the results to the initial velocities. It is often impossible and unnecessary to describe the motion of a fluid continuum with respect to initial conditions or a reference configuration since the behaviour of a Newtonian fluid is generally independent of its history. On the other hand, for solid continuum, the stresses generally depend on the history of deformation and the strain is defined in relation to the initial conditions because the behaviour of solids is history-dependent.

Therefore, obtaining a complete similitude solution for the phenomena in question (the prototype) calls for the integration of equations (3) over time. If we assume that the physical situation exists from a state of rest where initial velocities are zero, we can, therefore, integrate (3) with respect to time and we have,

$$\begin{aligned}\int_V \rho u_i \delta x_i \, dV - \left(\rho u_i V \delta x_i\right)\Big|_t = & -\int\int_V \rho u_j \frac{\partial u_i}{\partial x_j} \delta x_i \, dV \, dt + \int\int_V \rho f_i \delta x_i \, dV \, dt - \int\int_V \lambda C_{ijmn_1} \Delta\varepsilon^l_{mn} \delta\varepsilon^l_{ij} \, dV \, dt - \\ & \int\int_V 2\mu C_{ijmn_2} \Delta\varepsilon^l_{mn} \delta\varepsilon^l_{ij} \, dV \, dt - \int\int_V^{t+\Delta t} S_{ij} \delta\varepsilon^{nl}_{ij} \, dV \, dt - \int\int_V^{t+\Delta t} S_{ij} \delta\varepsilon^l_{ij} \, dV \, dt + \\ & \int F_i^{ext} \delta x_i \, dt \end{aligned} \tag{4}$$

where, $F_i^{ext}$ are the components of the external force. By non-dimensionalising (4), we obtain,

$$\begin{aligned}\int_V \overline{\rho}\overline{u}_i \delta\overline{x}_i \, d\overline{V} - \left(\overline{\rho}\overline{u}_i V \delta\overline{x}_i\right)\Big|_{\overline{t}} = & -\int\int_{\overline{V}} \overline{\rho}\overline{u}_j \frac{\partial \overline{u}_i}{\partial \overline{x}_j} \delta\overline{x}_i \, d\overline{V} \, d\overline{t} + \frac{f_{ref} L_{ref}}{u^2_{ref}} \int\int_V \overline{\rho}\overline{f}_i \delta\overline{x}_i \, d\overline{V} \, d\overline{t} - \\ & \frac{E_{ref}}{\rho_{ref} u^2_{ref}} \int\int_V \overline{\lambda}\overline{C}_{ijmn_1} \Delta\overline{\varepsilon}^l_{mn} \delta\overline{\varepsilon}^l_{ij} \, d\overline{V} \, d\overline{t} - \frac{E_{ref}}{\rho_{ref} u^2_{ref}} \int\int_V 2\overline{\mu}\overline{C}_{ijmn_2} \Delta\overline{\varepsilon}^l_{mn} \delta\overline{\varepsilon}^l_{ij} \, d\overline{V} \, d\overline{t} - \\ & \frac{E_{ref}}{\rho_{ref} u^2_{ref}} \int\int_V^{t+\Delta t} \overline{S}_{ij} \delta\overline{\varepsilon}^{nl}_{ij} \, d\overline{V} \, d\overline{t} - \frac{E_{ref}}{\rho_{ref} u^2_{ref}} \int\int_V^{t+\Delta t} \overline{S}_{ij} \delta\overline{\varepsilon}^l_{ij} \, d\overline{V} \, d\overline{t} + \\ & \frac{J_{ref}}{\rho_{ref} u_{ref} L^3_{ref}} \int \overline{F}_i^{ext} \delta\overline{x}_i \, d\overline{t} \end{aligned} \tag{5}$$

Therefore, we have the following first three coefficients from (5), and the fourth coefficient is obtained by non-dimensionalising (3),

$$\frac{f_{ref} L_{ref}}{u^2_{ref}}; \frac{E_{ref}}{\rho_{ref} u^2_{ref}}; \frac{J_{ref}}{\rho_{ref} u_{ref} L^3_{ref}}; \frac{T_{ref}}{\rho_{ref} u^2_{ref}} \tag{6}$$

where $E_{ref}$ is the reference modulus of elasticity, $J_{ref}$ is the characteristic impulse imposed at the boundary of the domain,

$$\overline{x}_i = \frac{x_i}{L_{ref}}; \overline{u}_i = \frac{u_i}{u_{ref}}; \overline{\rho} = \frac{\rho}{\rho_{ref}}; \overline{t} = \frac{t}{t_{ref}}; \overline{V} = \frac{V}{L^3_{ref}}; \overline{f}_i = \frac{f_i}{f_{ref}}; \overline{\lambda} = \frac{\lambda}{E_{ref}}; \overline{\mu} = \frac{\mu}{E_{ref}}; \overline{S}_{ij} = \frac{S_{ij}}{E_{ref}}; \overline{T}_i = \frac{T_i}{T_{ref}} \tag{7}$$

The first coefficient in (6) is Froude's number, the second and fourth coefficients are analogous to Cauchy number and Johnson's damage number. The second coefficient is the ratio between the elastic and the inertial force. The fourth coefficient is the ratio of applied surface force to the inertial force, and the third is a ratio of the externally applied impulse at the boundary to the characteristic momentum. To the best of the authors' knowledge, the third number is a ratio that has never been defined in the literature, and neither is it analogous to any existing number. The same coefficient is obtained for the linear and nonlinear terms on the right-hand side; hence dynamic similarity is guaranteed for large deformations.

The coefficients in (6) are sufficient for achieving dynamic similitude for any given solid domain with varying material and any given spatial and temporal scale, provided the continuum assumption remains valid. The coefficients in (6) remain strictly accurate in the elastic regime and also in the plastic regime if the same material or if different materials are used. The dynamically similar solutions obtained at any given time will always exist at the same strain state right through the evolution of deformation because the strain tensor in the domain is dimensionless and so cannot be scaled. This constraint is seen across different studies, including the recent finite similitude theory proposed by Davey et al. [3].

It is possible, however, to still have dynamic similarity whilst enforcing the ratio of deformation to the characteristic length (i.e. strain) to be variable, which to date has not been possible.

For this, the energy equation can be used, and following the proposed formulation above, the following non-dimensional coefficients are obtained,

$$\frac{f_{ref} L_{ref}}{w_{ref}^{\mathrm{int}}}; \frac{u_{ref}^2}{w_{ref}^{\mathrm{int}}}; \frac{u_{ref} J_{ref}}{\rho_{ref} L_{ref}^3 w_{ref}^{\mathrm{int}}} \tag{8}$$

where,

$$\overline{w}^{\mathrm{int}} = \frac{w^{\mathrm{int}}}{w_{ref}^{\mathrm{int}}}$$

The first coefficient in (8) is analogous to Froude's number, and the second is the ratio of the kinetic energy to internal energy. The third coefficient is the ratio of the externally applied energy at the boundary to the internal energy in the domain. To the best of the authors' knowledge, the second and third coefficients have never been reported in the literature. When considering two dynamically similar domains (model and prototype) with the same spatial or geometrical scale, a third coefficient can be defined as,

$$\left(\frac{\varepsilon_{ref}^{eq} F_{ref}^{ext}}{U_{ref}^{\mathrm{int}}}\right)_{\mathrm{model}} = \left(\frac{\varepsilon_{ref}^{eq} F_{ref}^{ext}}{U_{ref}^{\mathrm{int}}}\right)_{\mathrm{prototype}}$$

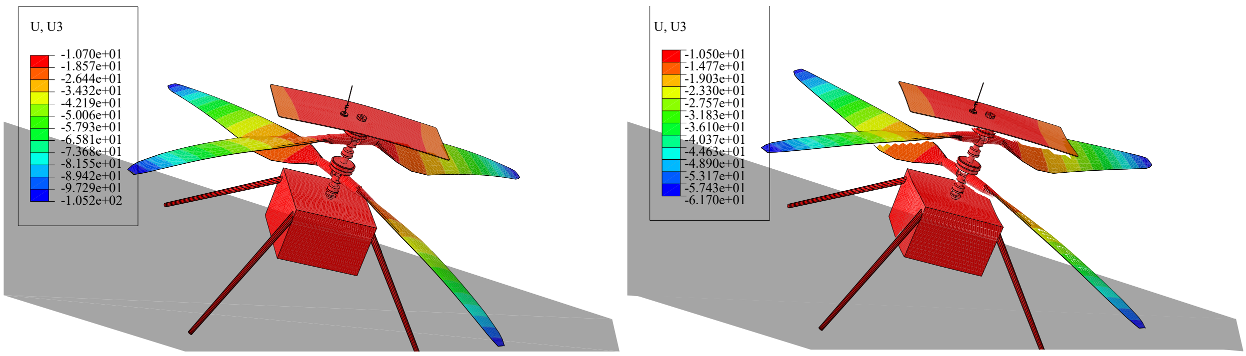
where, $\varepsilon_{ref}^{eq}$ is the equivalent reference strain and $U_{ref}^{int}$ is the internal reference strain energy; either $F_{ref}^{eq}$ or $\varepsilon_{ref}^{eq}$ can be defined for scaling the domain. The coefficients in (8) are also sufficient for achieving dynamic similitude for any given solid domain with varying material that are not strain rate sensitive (the case of strain rate sensitive material was addressed by Adetoro and Cardoso [1]) and any given spatial and temporal scale, provided the continuum assumption remains valid. They also allow for the scaling of the kinematics of the two domains for both similar or different materials.

## 3 CASE STUDY

In this case study, we will consider the impact landing case of the Ingenuity Helicopter on Mars (Figure 1) and define a scaled experimental model for testing on Earth, which shares exact dynamic similitude. The landing phase is one of the crucial points of the space exploration mission, and due to different gravity, care must be taken when conducting experiments on Earth. Several approaches have been adopted for such tests in the past, including a geometrically scaled-down model, the pulley balance approach, balloon buoyancy approach, electromagnetic resistance approach, air cavity simulation approach, and so on. The unified model presented in this paper allows, in a natural fashion, the scaling of body forces to produce dynamically similar models. With the strain state constraint no longer enforced, the model on Earth can exist in a different, yet dynamically similar kinematic state whilst the same materials can be adopted. A simplified model of Ingenuity Helicopter is used in this study, and a constant material is used through the domain. Therefore, an exact replica model can be used in the test. A scaled replica can also be used, in which case the presented unified model allows for the use of different materials. It should be noted that this study does not involve plasticity. Cases studies involving plasticity, contact and so on has been considered by Adetoro and Cardoso [1]. The conditions for dynamic similitude for the Earth model obtained from the Mars model are given in Table 1. A contour plot of the vertical displacement of the helicopter is shown in Figure 1.

| | Mars Model | Earth Model |
|---|---|---|
| Material | Aluminium 2024-T351 | |
| Gravity, $g$ (m.s$^{-2}$) | 3.721 | 9.807 |
| Impact Velocity, $v_0$ (m.s$^{-1}$) | 6.667 | 10.823 |

Table 1. Example of a table.




(a) (b)

Figure 1 - A plot of the displacement of (a) the Earth and (b) the Mars models both at time - $9.8\times10^{-3}$ s.

## 4 CONCLUDING REMARKS

In this article, we have presented a unified approach for scaling solid continuum using non-dimensional coefficients obtained from the momentum and energy conservation. Whilst some of the numbers are analogous to existing numbers, others have never been reported in the literature. A space exploration case study has also been presented which allows for experimental tests to be conducted especially in the case of differing gravitational body forces.

## REFERENCES

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