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Matrix Wiener-Hopf factorisation

by

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Abstract

A direct method is described for effecting the explicit Wiener-Hopf factorisation of a class of (2 x 2)-matrices. The class is determined such that the factorisation problem can be reduced to a matrix Hilbert problem which involves an upper or lower triangular matrix. Then the matrix Hilbert problem can be further reduced to three scalar Hilbert problems on a half-line, which are solvable in the standard manner. The factorisation technique is applied to the matrices that arise from two problems in diffraction theory, thus permitting these diffraction problems to be solved in closed form (at least in principle).

1. Introduction

In a recent paper by Rawlins and Williams [1] (see also Rawlins [2]), it was shown how a class of (2×2)-matrices could be explicitly factorised. In this paper a different class of matrices is constructively factorised. By using the idea of Rawlins [3] and evaluating the matrix to be factorised on both sides of an assumed branch cut that commonly arises in diffraction problems, the problem of factorisation reduces to a matrix Hilbert problem along the branch cut. In the work. of Rawlins and Williams [1], and Rawlins [2], the form of the original matrix was chosen so that the matrix Hilbert problem was reducible to two uncoupled scalar Hilbert problem. These could be solved without difficulty by the well-known methods given in Muskhelishvili's book on, singular integral equations [4]. The reduction to these two scalar Hilbert problems required that <u>two</u> diagonal elements of the matrix involved in the Hilbert problem were zero. However, it is known, see Gohberg and Krein [5], that upper and lower triangular matrix Hilbert problems can also be solved explicitly. Thus we need only require <u>one</u> off-diagonal element of the matrix Hilbert problem to vanish, in order to effect a Wiener-Hopf factorisation of the original matrix. We shall apply the present theory to the factorisation of matrices which arise in the following physical applications: (a) the reflection and radiation of a guided acoustic wave at the open end of a semi—infinite waveguide bounded by a soft/hard half-planes, (b) the diffraction of an acoustic wave by a thick half-plane. A factorisation of these matrices does not necessarily mean that a closed form solution can be obtained for the stated diffraction problems. It may be that there are insuperable problems depending on the factor matrices associated with completing the Wiener-Hopf solution of the stated diffraction problems. However, a new technique of matrix Wiener-Hopf factorisation may go some way towards an eventual solution of some hitherto intractable diffraction problems. The author hopes to complete, in detail the solution of the above diffraction problems in later separate publication.

We mention that the type of matrix factorised in this paper does not fall into the class considered by Daniele [6], Rawlins [7]. Jones [8] has devised a method for the Wiener-Hopf factorisation of a special type of (2 x 2)-matrix, that ensures that the Wiener-Hopf factors commute. In addition, the factors of various matrices whose Wiener-Hopf factors do not commute were also determined by Jones [8]. The matrices factorised here do not seem to fall straightforwardly into the class considered by Jones [8]. Other work which is related to Wiener-Hopf factorisation of matrices has been done by Williams [9] who has written an interesting and useful paper on methods for recognising when a (2 x 2)-matrix can be explicitly factorised. Jones [10] has extended the class of (2 x 2)-matrices whose factors commute to a class of (n x n)-matrices whose factors commute.

In section 2 of the paper a general matrix will be considered, and its general form is appropriately specified in order that the Wiener-Hopf factorisation problem reduces to a triangular matrix Hilbert problem. In section 3 this class of matrix will be constructively factorised by solving appropriate Hilbert problems. In section 4, to elucidate the method, the factorisation procedure is applied to two matrices which arise from specific problems from diffraction theory, as mentioned above.

Finally, we include in this report appendices A and B . In A we state a conjecture related to the factorisation of (n x n)-matrices. In appendix B we indicate a number of other diffraction problems,whose matrices can be factorised by the present technique.

2. Determination of the class of matrices whose Wiener-Hopf factorisation reduces to a triangular matrix Hilbert problem on a half-line.

Consider the general (2 x 2)-matrix

$$\underset{\sim}{A}(\alpha) = \begin{pmatrix} a_{11}(\alpha) & a_{12}(\alpha) \\ a_{21}(\alpha) & a_{22}(\alpha) \end{pmatrix} ,$$

where the elements $a_{ij}(\alpha)$, $i,j = 1,2$ are functions of the complex variable α. These functions will be assumed to have only branch point singularities, specifically we shall assume that the branch points arise through the function $\gamma = \sqrt{\alpha^2 - k^2}$, where k has positive real and imaginary parts, and the branch cuts C and C' lie along the half-lines C: $\alpha = -k - \delta$, C': $\alpha = k + \delta$, $\delta \geq 0$. The elements $a_{ij}(\alpha)$ are also assumed to be analytic functions in the cut α-plane; and $\det \underset{\sim}{A}(\alpha) \neq 0$ within the strip $-k_i < \mathrm{Im}(\alpha) < k_i$, where k_i denotes the imaginary part of k. The occurrence of a complex k with $\mathrm{Im}(k) > 0$ is traditional in Wiener-Hopf-type problems and is needed to have a common strip of analyticity; in the final solution the complex k is removed by taking the limit as $k \downarrow 0$.

The Wiener-Hopf factorisation problem requires the determination of (2 x 2)-matrices $\underset{\sim}{U}(\alpha)$ and $\underset{\sim}{L}(\alpha)$, whose elements are analytic for $\mathrm{Im}(\alpha) > -k_i$ and $\mathrm{Im}(\alpha) < k_i$ respectively, such that

$$\underset{\sim}{A}(\alpha) = \underset{\sim}{U}(\alpha)\underset{\sim}{L}^{-1}(\alpha). \tag{1}$$

$\underset{\sim}{U}(\alpha)$ and $\underset{\sim}{L}(\alpha)$ are also required to be non-singular in their respective regions of analyticity. Obviously any matrix $\underset{\sim}{A}'(\alpha)$ which can be expressed in the form

$$\underset{\sim}{A}'(\alpha) = \underset{\sim}{C}_U(\alpha)\underset{\sim}{A}(\alpha)\underset{\sim}{B}_L(\alpha) , \tag{2}$$

where $\underset{\sim}{B}_L(\alpha)$ and $\underset{\sim}{C}_U(\alpha)$ are matrices whose elements are analytic functions of α for $\mathrm{Im}(\alpha) < k_i$ and $\mathrm{Im}(\alpha) > -k_i$, respectively, can also be factorised if $\underset{\sim}{A}(\alpha)$ can be factorised.

In order to effect the factorisation it will be assumed that $\underset{\sim}{U}(\alpha)$ is analytic except along the branch cut C through α = -k, whilst $\underset{\sim}{L}(\alpha)$ is analytic except along the branch cut C' through α = k. Evaluation of equation (1) on both sides of the cut C(C: ↑ = -k - δ,δ ≥ 0) through α = -k gives, on using the suffices ± to denote values evaluated on the upper and lower sides of C ,

$$\underset{\sim}{A}_+(\alpha) = \underset{\sim}{U}_+(\alpha)\,\underset{\sim}{L}^{-1}(\alpha) \ , \tag{3}$$

$$\underset{\sim}{A}_-(\alpha) = \underset{\sim}{U}_-(\alpha)\,\underset{\sim}{L}^{-1}(\alpha) \ ,, \tag{4}$$

($\underset{\sim}{L}$ (α) is analytic except along the branch cut C' through α = k and therefore takes the same values on both sides of C). Eliminating $\underset{\sim}{L}$ (α) between (3) and (4) gives the matrix Hilbert problem:

$$\underset{\sim}{U}_+(\alpha) = \underset{\sim}{G}(\alpha)\ \underset{\sim}{U}_-(\alpha) \ , \ \alpha \in C \ , \tag{5}$$

where

$$\underset{\sim}{G}(\alpha) = \underset{\sim}{A}_+(\alpha)\ \underset{\sim}{A}_-^{-1}(\alpha) \ .$$

More explicitly

$$\begin{pmatrix} g_{11}(\alpha) & g_{12}(\alpha) \\ g_{12}(\alpha) & g_{22}(\alpha) \end{pmatrix} = \frac{1}{\det \underset{\sim}{A}(\alpha)} \begin{pmatrix} \overset{+}{a}_{21}\overset{-}{a}_{22} - \overset{+}{a}_{12}\overset{-}{a}_{21} & \overset{+}{a}_{22}\overset{-}{a}_{11} - \overset{+}{a}_{11}\overset{-}{a}_{12} \\ \overset{+}{a}_{21}\overset{-}{a}_{22} - \overset{+}{a}_{22}\overset{-}{a}_{21} & \overset{+}{a}_{22}\overset{-}{a}_{11} - \overset{+}{a}_{21}\overset{-}{a}_{12} \end{pmatrix}$$

The problem (5) reduces to an upper or lower triangular matrix Hilbert problem along C , if the condition $g_{12}(\alpha) = 0$ or $g_{21}(\alpha) = 0$ is satisfied. That is

$$\overset{+}{a}_{ij}(\alpha)\,\overset{-}{a}_{ii}(\alpha) = \overset{+}{a}_{ii}(\alpha)\,\overset{-}{a}_{ij}(\alpha), \ \ i=1,\ j=2 \ \text{ or } \ i=2,\ j=1,\ \alpha \in C \ ;$$

or ignoring the trivial case of $\overset{+}{a}_{ij}(\alpha) \equiv 0$, and assuming $a_{ij}(\alpha) \neq 0$ on C ,

$$\left(\frac{a_{ii}(\alpha)}{a_{ij}(\alpha)}\right)^+ - \left(\frac{a_{ii}(\alpha)}{a_{ij}(\alpha)}\right)^- = 0 \ , \qquad \alpha \in C \ . \tag{6}$$

We shall assume that $a_{ij}(\alpha)$ can have isolated zeros in the cut α-plane, (but not on C), and consequently $a_{ii}(\alpha)/a_{ij}(\alpha)$ can have isolated poles in the cut α-plane. Then provided $a_{ii}(\alpha)/a_{ij}(\alpha)$ - $O(|k+\alpha|^{-\mu}$, $0 \le u < 1$ as $\alpha \to -k$, a solution of (6) follows immediately (see

Muskhelishvili [4], section 15) as

$$a_{ii}(\alpha) - a_{ij}(\alpha)F_i(\alpha) \quad . \tag{7}$$

Here the function $F_i(\alpha)$ is analytic except along the branch cut C' through $\alpha = k$, [Footnote: If $A(\alpha) = A(-\alpha)$ for a in the cut plane, then it is not difficult to show (see Rawlins [3]) that $F_i(\alpha)$ must also be analytic along the branch cut C'], and except for pole s at the zeros of $a_{ij}(\alpha)$; the multiplicity of hese poles is not greater than the multiplicity of the corresponding zeros. Thus $G(\alpha)$ will be of lower or upper triangular form if:

(i) $$\underset{\sim}{A}(\alpha) = \begin{pmatrix} a_{12}(\alpha)F_1(\alpha) & a_{12}(\alpha) \\ a_{21}(\alpha) & a_{22}(\alpha) \end{pmatrix} , \tag{8}$$

or

(ii) $$\underset{\sim}{A}(\alpha) = \begin{pmatrix} a_{11}(\alpha)F_1(\alpha) & a_{12}(\alpha) \\ a_{21}(\alpha) & a_{21}(\alpha)F_2(\alpha) \end{pmatrix} ; \tag{9}$$

and $\det \underset{\sim}{A}(\alpha) \neq 0$ in the cut α-plane.

We shall now carry out the explicit Wiener-Hopf factorisation of the matrix given in case (ii) above. The procedure for factorising the matrix given in case (i) will be completely analogous.

3. Wiener-Hopf factorisation of the matrix defined by (9).

We assume the matrix $\underset{\sim}{A}(\alpha)$ has the form (9) and the same general properties as outlined in the first paragraph of section 2. If we carry out the same evaluation on the branch cut C, as described in section 2, the equation (5) reduces to the upper triangular matrix Hilbert problem

$$\underset{\sim}{U}_+(\alpha) = \underset{\sim}{G}(\alpha)\, \underset{\sim}{U}_-(\alpha) , \quad \alpha \in C , \tag{10}$$

where

$$g_{11}(\alpha)=\frac{(a_{11}(\alpha)F_2(\alpha)-a_{12}(\alpha))^+}{(a_{11}(\alpha)F_2(\alpha)-a_{12}(\alpha))^-},\quad g_{12}(\alpha)=\frac{\overset{+}{a}_{12}(\alpha)\overset{-}{a}_{11}(\alpha)-\overset{+}{a}_{11}(\alpha)\overset{-}{a}_{12}(\alpha)}{\overset{-}{a}_{21}(\alpha)(a_{11}(\alpha)F_2(\alpha)-a_{12}(\alpha))^-}\tag{11}$$

$$g_{21}(\alpha) = 0 \qquad , g_{22}(\alpha) = \frac{\overset{+}{a}_{21}(\alpha)}{\overset{-}{a}_{21}(\alpha)} \quad ,$$

$$\underset{\sim}{U}(\alpha) = \begin{pmatrix} u_{11}(\alpha) & u_{12}(\alpha) \\ u_{21}(\alpha) & u_{22}(\alpha) \end{pmatrix} \quad . \tag{12}$$

Evaluating the matrix expression (10) and equating corresponding elements of the matrices on both sides of the equality sign gives the following equations:

$$\left.\begin{aligned} \overset{+}{u}_{ij}(\alpha) &= g_{11}(\alpha)\overset{-}{u}_{1j}(\alpha) + g_{12}(\alpha)\overset{-}{u}_{2j}(\alpha) \\ \overset{+}{u}_{2j}(\alpha) &= g_{22}(\alpha)\overset{-}{u}_{2j}(\alpha) \end{aligned}\right\} \alpha \in C, \quad j = 1,2 \ . \tag{13}$$

The four equations (13) can clearly be solved if the coupled system

$$\left.\begin{aligned} \overset{+}{v}_1(\alpha) &= g_{11}(\alpha)\overset{-}{v}_1(\alpha) + g_{12}(\alpha)\overset{-}{v}_2(\alpha) \\ \overset{+}{v}_2(\alpha) &= g_{22}(\alpha)\overset{-}{v}_2(\alpha) \end{aligned}\right\} \alpha \in C, \tag{14}$$

(15)

can be solved. The equation (15) is a standard Hilbert problem whose fundamental solution is given directly by the methods of Muskhelishvili [4], chapter 10. Similarly we can determine the fundamental solution of the standard auxiliary Hilbert problem

$$v^+(\alpha) \ - \ g_{11}(\alpha)v^-(\alpha) \ , \qquad \alpha \ \varepsilon \ C \tag{16}$$

for $v(\alpha)$. Then the equation (14) can be written as

$$u^+(\alpha) - u^-(\alpha) = g_{12}(\alpha)\overset{-}{v}_2(\alpha)/v^+(\alpha) \ , \quad \alpha \in C \ , \tag{17}$$

where

$$u(\alpha) = v_1(\alpha)/v(\alpha) \ .$$

In the equation (17) the right-hand side is a known quantity and therefore we have a standard Hilbert problem whose fundamental solution is given by using the techniques described in Muskhelishvili [4], chapter 10.

Suppose therefore we have found fundamental solutions $v_2^{(0)}(\alpha)$, $v^{(0)}(\alpha)$ and $u^{(0)}(\alpha)$ of the equations (15), (16) and (17), respectively. To determine the general solution we set $v_2(\alpha) = v_2^{(0)}(\alpha) v_2^*(\alpha)$, $v(\alpha) - v^{(0)}(\alpha)^* v(\alpha)$, and $u(\alpha) = u^{(0)}(\alpha) + u^*(\alpha)$), then we are led to the Hilbert problems

$$[v_2^*(\alpha)]^* = [v_2^*(\alpha)]^-, \quad [v^*(\alpha)]^+ = v^*(\alpha)]^-,$$

and

$$[u^*(\alpha)]^+ - [u^*(\alpha)]^- = g_{12}(\alpha)\,[v_2^{(0)}(\alpha)]^- /[v^{(0)}(\alpha)]^+ \{[v_2^*(\alpha)]^- /[v^*(\alpha)]^+ - 1\},$$

which have a solution, (Muskhelishvili [4], section 15)

$$v_2^*(\alpha) = P_2(\alpha),\ v^*(\alpha) = P_2(\alpha),\ u^*(\alpha) = P_1(\alpha), \tag{18}$$

where $P_1(\alpha), P_2(\alpha)$ are entire functions of α.

Thus a suitably general solution of (14) and (15) is given by

$v_2(\alpha) = P_2(\alpha) v_2^{(0)}(\alpha)$ and $v_1(\alpha) = u(\alpha)v(\alpha) = (u^{(\circ)}(a) + P_1(\alpha))P_2(\alpha)v^{(0)}(\alpha)$.

A suitably general solution of the equation (13) and consequently of (10) is therefore given by

$$\underset{\sim}{U}(\alpha) = \begin{pmatrix} (u^{(0)}(\alpha) + p_{11}(\alpha))p_{21}(\alpha)v^{(0)}(\alpha) & (u^{(0)}(\alpha) + p_{12}(\alpha))p_{22}(\alpha)v^{(0)}(\alpha) \\ p_{21}(\alpha)v_2^{(0)}(\alpha) & p_{22}(\alpha)v_2^{(0)}(\alpha) \end{pmatrix}, \tag{19}$$

where $\det \underset{\sim}{U}(\alpha) = P_{21}(\alpha)P_{22}(\alpha)v^{(0)}{}'(\alpha)\, v_2^{(0)}(\alpha)\,(P_{11}(\alpha) - P_{12}(\alpha))$, and $P_{ij}.(\alpha)$, $i, j = 1, 2$ are entire functions. The choice of the entire functions $P_{ij}(\alpha)$ are further restricted by the condition that $\underset{\sim}{U}(\alpha)$ is non-singular; and the requirement that the corresponding matrix $L(\alpha) = A(\alpha)U(\alpha)$ is non-singular, and its elements should be analytic except along the branch cut C' through $\alpha = k$. In particular, the elements of $L(\alpha)$ should not have poles at $\alpha = -k$. For the applications we have in mind it is sufficient to let $P_{21} = P_{22} = P_{11} = -P_{12} = 1$, giving

$$\underset{\sim}{U}^{(0)}(\alpha) = \begin{pmatrix} (u^{(0)}(\alpha) + 1)v^{(0)}(\alpha) & (u^{(0)}(\alpha) - 1)v^{(0)}(\alpha) \\ v_2^{(0)}(\alpha) & v_2^{(0)}(\alpha) \end{pmatrix}, \tag{20}$$

$$\det \underset{\sim}{U}^{(0)}(\alpha) = 2v^{(0)}(\alpha)v_2^{(0)}(\alpha).$$

In a completely analogous way it can be shown that a Wiener-Hopf factorisation of the matrix defined by (8) is given by

$$\underset{\sim}{A}(\alpha) = \underset{\sim}{U}(\alpha)\underset{\sim}{L}^{-1}(\alpha).$$

where

$$\underset{\sim}{U}(\alpha) = \begin{pmatrix} P_{11}(\alpha)v_1^{(0)}(\alpha) & P_{12}(\alpha)v_1^{(0)}(\alpha) \\ P_{11}(\alpha)v^{(0)}(\alpha)\,(u^{(0)}(\alpha)+P_{21}(\alpha)) & P_{12}(\alpha)v^{(0)}(\alpha)\,(u^{(0)}(\alpha)+P_{22}(\alpha)) \end{pmatrix}, \qquad (21)$$

and $P_{ij}(\alpha)$, $i,j = 1,2$ are entire functions,

$\det \underset{\sim}{U}(\alpha) = p_{11}(\alpha)\,p_{12}(\alpha)\,v_1^{(0)}(\alpha)\,(p_{22}(\alpha) - p_{21}(\alpha))$; and $v_1^{(0)}(\alpha), v^{(0)}(\alpha)$,

and $u^{(0)}(\alpha)$ are fundamental solutions of the standard Hilbert problems

$$\left.\begin{aligned} v_1^{*}(\alpha) &= g_{11}(\alpha)\, v_1^{-}(\alpha)\ , \\ v^{*}(\alpha) &= g_{22}(\alpha)\, v^{-}(\alpha)\ , \\ u^{+}(\alpha) - u^{-}(\alpha) &= g_{21}(\alpha)\, v_1^{-}(\alpha) / v^{+}(\alpha)\ . \end{aligned}\right\} \quad \alpha \in C\ .$$

$g_{ij}.(\alpha)$ are the elements of the lower triangular matrix $\underset{\sim}{G}(\alpha) = \underset{\sim+}{A}(\alpha)\,\underset{\sim-}{A}^{-1}(\alpha)$

Imposition of the further restriction that $\underset{\sim}{U}(\alpha)$ *and* $\underset{\sim}{L}(\alpha)$ are non-singular, and analytic everywhere except along the branch cuts C and C' respectively, dictates the choice $P_{11} = P_{12} = P_{22} = -P_{21} = 1$, giving

$$\underset{\sim}{U}(\alpha) = \begin{pmatrix} v_1^{(0)}(\alpha) & v_1^{(0)}(\alpha) \\ v^{(0)}(\alpha)\,(u^{(0)}(\alpha)-1) & v^{(0)}(\alpha)\,(u^{(0)}(\alpha)+1) \end{pmatrix} \qquad (22)$$

The elements of $U(\alpha)$ have been constructed by assuming that the matrix $\underset{\sim}{L}(\alpha)$ in equation (2) is continuous across C and therefore $\underset{\sim}{L}(\alpha)$ defined by

$$\underset{\sim}{L}(\alpha) = \underset{\sim}{A}^{-1}(\alpha)\,\underset{\sim}{U}(\alpha)\ ,$$

with the elements defined by one of the equations (19) to (22), should from the method of construction, be continuous across C, Equation (5) and the equat ion above gives

$$\underset{\sim+}{L}(\alpha) = \underset{\sim+}{A}^{-1}(\alpha)\underset{\sim+}{U}(\alpha) = \underset{\sim+}{A}^{-1}(\alpha)\,\underset{\sim+}{A}(\alpha)\,\underset{\sim-}{A}^{-1}(\alpha)\underset{\sim}{U}(\alpha) = \underset{\sim-}{L}(\alpha)$$

thus verifying explicitly that $\underset{\sim}{L}(\alpha)$ is indeed continuous across C .

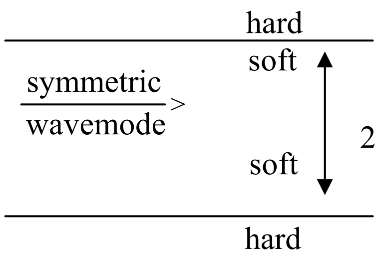
4. <u>Factorisation of matrices that arise in practical applications.</u>

(a) <u>Reflection and radiation at the open end of a waveguide</u>.

Consider the matrix

$$\underset{\sim}{A}'(\alpha) = \begin{pmatrix} 1 & 1/\gamma \\ \gamma\tanh\gamma & -1 \end{pmatrix}, \quad \gamma = \sqrt{a^2 - k^2}\ , \tag{23}$$

where the branch of the multivalued function γ is chosen such that γ = -ik when a = 0, and the branch cuts are taken along the half lines α = k + δ, α = -k - δ, $\delta \geq 0$. The quantity k is here assumed to be real and positive. A Wiener-Hopf factorisation of this matrix is required for the solution, in closed form, of the problem of a symmetric acoustic wave mode propagating down a two-dimensional semi-infinite waveguide, whose guide walls, a distance two units apart, are internally soft and externally rigid, see figure 1. The "wavenumber" k is related to the wave propagating down the guide by k = 2π/wavelength.



Geometry of the diffraction problem (a).

fig 1.

Although det $\underset{\sim}{A}'(\alpha)$ (a) = -(1+tanhy) H in the cut ct-plane, the matrix (23) does not fall into the class of matrices considered in section 3 since the element $a'_{21}(\alpha)$ has poles at the roots of coshy = 0. To overcome this defect we can rewrite (23) in the form

$$\underset{\sim}{A}'(\alpha) = \frac{1}{\cosh y}\begin{pmatrix} \cosh y & 0 \\ 0 & 1 \end{pmatrix}\underset{\sim}{A}(\alpha)\ , \tag{24}$$

where

$$\underset{\sim}{A}(\alpha) = \begin{pmatrix} 1 & 1/\gamma \\ \gamma\sinh\gamma & -\cosh y \end{pmatrix}\ . \tag{25}$$

The expression (24) can now be written in the from (1) where

$$\underset{\sim}{C}_U(\alpha) = \frac{1}{g_U(\alpha)}\begin{pmatrix}\cosh\gamma & 0\\ 0 & 1\end{pmatrix},\quad \underset{\sim}{B}_L(\alpha) = \frac{1}{g_L(\alpha)}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},$$

$$\cosh\gamma = g_U(\alpha)g_L(\alpha) = \prod_{n=1}^{\infty}\left\{(1-k^2/\alpha_n^2)+\alpha^2/\alpha_n^2\right\},\quad \alpha_n = (n-\tfrac{1}{2})\pi$$

$$g_U(\alpha) = g_L(-\alpha) = \prod_{n=1}^{\infty}\{1-k^2/\alpha_n^2)^{\frac{1}{2}} - i\,\alpha/\alpha_n\}\ \exp[i\alpha/\alpha_n]\ .$$

The matrix (23) can therefore be factorised if we can factorise the matrix (25). Clearly the matrix (25) falls within the class considered in section 3; since in (9) we can let $a_{11}(\alpha) = 1$, $a_{12}(\alpha) = 1/\gamma$, $a_{21}(\alpha) = y\sinh y$, $F_2(\alpha) = -\coth y/y$. The simple poles of $F_2(\alpha) = -\coth y/\gamma$ correspond to the first-order zeros of $a_{21}(\alpha) = y\sinh y$. We also note that $\det \underset{\sim}{A}(\alpha) = -e^{\gamma} \neq 0$ in the cut α-plane.

Thus the factorisation of $\underset{\sim}{A}(\alpha)$ proceeds as follows:

$$\underset{\sim+}{A}(\alpha) = \begin{pmatrix}1 & 1/\gamma(\alpha)\\ \gamma(\alpha)\sinh\gamma(\alpha) & -\cosh\gamma(\alpha)\end{pmatrix},\ \underset{\sim}{A}^{-1}(\alpha) = \exp[\gamma(\alpha)]\begin{pmatrix}\cosh\gamma(\alpha) & 1-/\gamma(\alpha)\\ \gamma(\alpha)\sinh\gamma(\alpha) & -1\end{pmatrix},\ \alpha \in C\ ,$$

where we have used the fact that $\gamma^{\pm} = \pm|\gamma|$, and where $\gamma(\alpha) = |\alpha^2 - k^2|^{\frac{1}{2}}$.

Thus

$$\underset{\sim}{G}(\alpha) = \underset{\sim+}{A}(\alpha)\,\underset{\sim+}{A}^{-1}(\alpha) = \begin{pmatrix}\exp[2\gamma(\alpha)] & -2\exp[\gamma(\alpha)]/\gamma(\alpha)\\ 0 & 1\end{pmatrix},\quad \alpha \in C. \tag{26}$$

Thus the equations (13) are in this particular problem:

$$\left.\begin{aligned} u_{ij}^{+}(\alpha) &= \exp[2\gamma(\alpha)]u_{1j}^{-}(\alpha) - 2\exp[\gamma(\alpha)]/\gamma(\alpha)\cdot u_{2j}^{-}(\alpha)\,,\\ u_{2j}^{+}(\alpha) &= u_{2j}^{-}(\alpha)\ , \end{aligned}\right\}\ \alpha \in C,\quad j = 1,2. \tag{27}$$

and a solution of these equations is given by a fundamental solution of the equations

$$\left.\begin{aligned} v_1^{*}(\alpha) &= \exp[2\gamma(\alpha)]\, v_1^{-}(\alpha) - 2\exp[\gamma(\alpha)]/\gamma(\alpha)\, v_2^{-}(\alpha) \quad (28) \\ v_2^{+}(\alpha) &= v_2^{-}(\alpha) \quad (29) \end{aligned}\right\} \quad \alpha \in C$$

A solution of (29) is obviously

$$v_2^{(0)}(\alpha) = 1 \ . \tag{30}$$

A solution of the auxiliary problem

$$v^{+}(\alpha) = \exp[2\gamma(\alpha)]v^{-}(\alpha) \ ,$$

is given by (Muskhelishvili [4], chapter 10)

$$v^{(0)}(\alpha) = \exp\left[2\gamma^{2}_{\cdot}\ \frac{1}{2\pi i}\int_{-\infty}^{-k} \frac{dt}{(t-)\gamma(t)}\right\} = \exp\left[\frac{i\gamma}{\pi}\ell n\,[(\alpha+\gamma)/k]\right] \ , \tag{31}$$

where the branch of the logarithm is specified by $\ell n[(\alpha+\Upsilon)/k = -i\pi/2$ when $\alpha = 0$.

Thus, since $v^{(0)}(\alpha) \neq 0$ on C , the equation (28) can be put into the form of the standard Hilbert problem:

$$u^{+}(\alpha) - u^{-}(\alpha) = -2\exp\left[\frac{-i\gamma(\alpha)}{\pi}\,\ell n\left|\frac{+\gamma(\alpha)}{k}\right|\right] \Big/ \gamma(\alpha) \ ,$$

where $u(\alpha) = v_1(\alpha)/v(\alpha)$. This problem has the fundamental solution (Muskhelishvili [47], chapter 10)

$$u^{(0)}(\alpha) = -\int_{-\infty}^{-k} \frac{\exp\left[-\frac{i\gamma(t)}{\pi}\ell n\left|\frac{t+\gamma(t)}{k}\right|\right]dt}{\gamma(t)(t-\alpha)} \ . \tag{32}$$

Thus having found particular solutions $v_2^{(0)}(\alpha),\ v^{(0)}(\alpha),\ u^{(0)}(\alpha)$ a Wiener-Hopf factorisation is given by substituting these expressions (30), (31) and (32) into the expression (20) giving

$$\underset{\sim}{A}(\alpha) = \underset{\sim}{U}^{(0)}(\alpha)\ [\underset{\sim}{L}^{(0)}(\alpha)]^{-1}$$

where

$$\underset{\sim}{U}^{(0)}(\alpha) = \begin{pmatrix} 1-I(\alpha))x(\alpha) & -(1+I(\alpha))x(\alpha) \\ 1 & 1 \end{pmatrix} , \tag{33}$$

where

$$\chi(\alpha) = \exp\left[\frac{i\gamma}{\pi}\ell n[(\alpha+\gamma)/k]\right],$$

$$I(\alpha) = \frac{1}{\pi i}\int_{-\infty}^{-k} \frac{\exp\left[-\frac{i\gamma(t)}{\pi}\ell n\left|\frac{t+\gamma(t)}{k}\right|\right] dt}{\gamma(t)\,(t-\alpha)} \quad .$$

We note that x(α) and I(α) are analytic functions everywhere in the a-plane except along the branch cut C, and therefore all the elements of $\underset{\sim}{U}^{(0)}(\alpha)$ are analytic in the α-plane cut along C . Also det $\underset{\sim}{U}^{(0)}(\alpha)$ = 2x(α) ≠ 0 in the α-plane. We also have

$$\underset{\sim}{L}^{(0)}(\alpha) = \underset{\sim}{A}^{-1}(\alpha)\underset{\sim}{U}^{(0)}(\alpha)$$

$$\underset{\sim}{L}^{(0)}(\alpha) = \exp[\gamma]\begin{pmatrix} \cosh\gamma.\ (1-I(\alpha)x(\alpha)+1/\gamma & -\cosh\gamma.\ (1+I(\alpha))x(\alpha)+1/\gamma \\ \gamma\sinh\gamma.\ (1-I(\alpha)x(\alpha)\ -1 & -\gamma\sinh\gamma.\ (1+I(\alpha)x(\alpha)\ -\ 1 \end{pmatrix}. \qquad (34)$$

It can be shown that the elements of L (a) have no poles at a = -k , by analysing the behaviour of L (a) as α → -k. To analyse the behaviour of L(0)(α) at α = -k we need the results:

$$x(\alpha) = 1 + 0(\gamma),\ \ I(\alpha) = y^{-1} + 0(1),\ \text{as } \alpha \to -k\ ,$$

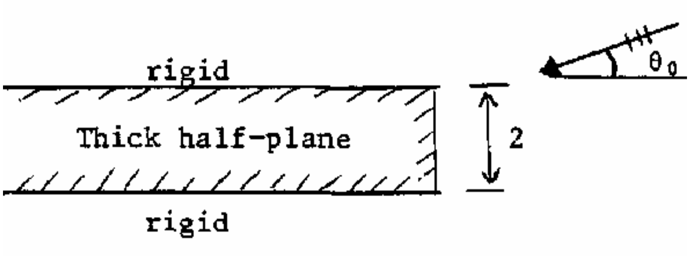
which readily follow by means of Muskhelishvili [4, section 29].

On inserting these results into (34), it is found that the elements of $L^{(0)}(\alpha)$ are bounded near α = -k, hence there are no poles at α = -k. Thus the choice o£ P_{ij} giving (20) is satisfactory.

Finally we remark that the reflection/radiation problem with an anti-symmetric mode leads to a matrix which can also be factorised by the present method. Also the reflection/radiation problem for symmetric or anti-symmetric mode propagation down the same wave guide, but with the hard and soft boundary conditions interchanged, leads to a matrix which can be factorised by the present method. It is hoped to present these problems fully solved in later publications.

(b) Diffraction by a thick half-plane.

As a second application we shall consider the Wiener-Hopf matrix factorisation that arises in the problem of the acoustic diffraction by a rigid thick half-plane, see figure 2.



Geometry of the diffraction problem (b).

This diffraction problem was considered by, Jones [11], and Crighton and Leppington [12] for arbitrary angle θ_0 of the incident wave. The solutions obtained by these authors were approximate. The rigorous methods used by these authors enable them to give accurate error bounds; on their approximate solutions. The situation for arbitrary angle of incidence can be treated by superposition of a symmetric and anti-symmetric incidence problem, (see [13] p.181). In the symmetric and anti-symmetric case matrices arise which can be treated by the method of this paper. To simplify matters we shall only consider here the symmetric situation $\theta_0 = 0$, when the incident wave is normally incident on the flat end of the thick half-plane. It is hoped to complete the solution for arbitrary incidence in detail in a later publication. Meanwhile the matrix which must be factorised for an exact closed-form solution to the symmetric incidence situation $\theta_0 - 0$ is given by * [Footnote: This matrix is easily derived, using the terminology of Noble [13], p. 181, by replacing α by $-\alpha$ in the expression (5.8) of [13] and combining the resulting expression with (5.8).]

$$\underset{\sim}{A}(\alpha) = \begin{pmatrix} -\exp[-\gamma]\cosh\gamma & -\exp[-\gamma]\sinh\gamma \\ -\exp[-\gamma](\sinh\gamma + 2\cosh\gamma) & \exp[-\gamma]\cosh\gamma \end{pmatrix} . (35)$$

The matrix (35) falls within the class of matrices considered in Section 3; since in (8) we can let a12(^{a}) = -exp[-γ]γsinh γ, F_1(a) = coth γ / γ , a_{21}(ct) = -exp [-γ] (sinh γ + 2cosh γ)/γ , a_{22}(ci) = exp[-γ]coshy . The simple poles of F_1(ct) = -coth *yly* correspond to the first-order zeros of a_{12}(cx) = γ sinhy; and det A(a) = -1 ≠ 0 in the cut a-plane. Taking k to be real and positive then γ ~ - |γ| , AND denoting γ (ot) - $|a^2-k^2|^{\frac{1}{2}}$ we get

$$\underset{\sim+}{A}(\alpha) = \begin{pmatrix} \exp[-\gamma(\alpha)]\cosh\gamma(\alpha) & -\exp[-\gamma(\alpha)]\,\gamma(\alpha)\sinh\gamma(\alpha) \\ \exp[-\gamma(\alpha)](\sinh\gamma(\alpha) + 2\cosh\gamma(\alpha))/\gamma(\alpha) & \exp[-\gamma(\alpha)]\cosh\gamma(\alpha) \end{pmatrix}, \quad (36)$$

$$\underset{\sim+}{A}(\alpha) = \begin{pmatrix} \exp[\gamma(\alpha)]\cosh\gamma(\alpha) & -\exp[-\gamma(\alpha)]\,\gamma(\alpha)\sinh\gamma(\alpha) \\ \exp[\gamma(\alpha)](\sinh\gamma(\alpha) - 2\cosh\gamma(\alpha))/\gamma(\alpha) & \exp[\gamma(\alpha)]\cosh\gamma(\alpha) \end{pmatrix}, \quad (37)$$

$$\underset{\sim+}{G}(\alpha) = \underset{\sim+}{A}(\alpha)\,\underset{\sim-}{A}^{-1}(\alpha) = \begin{pmatrix} \exp[-2\gamma(\alpha)] & 0 \\ 4\cosh^2\gamma(\alpha)/\gamma(\alpha) & \exp[2\gamma(\alpha)] \end{pmatrix} . \quad (38)$$

Thus the atrix Hilbert problem that needs to be solved is

$$\underset{\sim+}{U}(\alpha) = \underset{\sim}{G}(\alpha)\underset{\sim-}{U}(\alpha), \text{ where } \underset{\sim}{U}(\alpha) = \begin{pmatrix} u_{11}(\alpha) & u_{12}(\alpha) \\ u_{21}(\alpha) & u_{22}(\alpha) \end{pmatrix} , \quad (39)$$

or more explicitly

$$\left.\begin{aligned} \overset{+}{u}_{1j}(\alpha) &= \exp[-2\gamma(\alpha)]\overset{-}{u}_{1j}(\alpha) \\ \overset{+}{u}_{2j}(\alpha) &= \exp[2\gamma(\alpha)]\overset{-}{u}_{1j}(\alpha) + 4\cosh^2\gamma(\alpha)/\gamma(\alpha)\overset{-}{u}_{1j}(\alpha) \end{aligned}\right\} \alpha \in C, \ j = 1,2. \quad (40)$$

A solution of the system of equations (40) is given by a fundamental solution of the equations:

$$\left.\begin{aligned} v_1^+(\alpha) &= \exp[-2\gamma(\alpha)v_1^-(\alpha), & (41) \\ v^+(\alpha) &= \exp[-2\gamma(\alpha)v^-(\alpha), & (42) \\ u^+(\alpha) - u^-(\alpha) &= 4\cosh^2\gamma(\alpha)/\gamma(\alpha).v_1^-(\alpha)/v^+(\alpha) & (43) \end{aligned}\right\} \alpha \in C$$

A fundamental solution $v_1^{(0)}(\alpha)$ and $v^{(0)}(\alpha)$ of the equations (41) and (42) is given by (Muskhelishvili [4], chapter 10)

$$v_1^{(0)}(\alpha) = \exp[-i(\gamma/\pi)\ell n[(\alpha+\gamma)/k]], \tag{44}$$

$$v^{(0)}(\alpha) = \exp[-i(\gamma/\pi)\ell n[(\alpha+\gamma)/k]]. \tag{45}$$

Thus equation (43) can now be written as

$$u^{+}(\alpha) - u^{-}(\alpha) = 4\cosh^2\gamma(\alpha)\exp[-i(2\gamma(\alpha)/\pi/\ell n|\frac{\alpha+\gamma(\alpha)}{k}|]/\gamma(\alpha) \tag{46}$$

This Hilbert problem has a fundamental solution $u^{(0)}(\alpha)$ given, by

$$u^{(0)}(\alpha) = 4\cosh^2\frac{1}{2\pi i}\int_{-\infty}^{-k}\frac{\exp dt[-i(2\gamma(t)/\pi)\ell n|\frac{t+\gamma(t)}{}|]\ dt}{\gamma(t)\,(t-\alpha)}.$$

Thus using the expressing (22), and the fundamental solutions (44), (45) and (47) a Wiener-Hopf factorization of the matrix (35) is given by

$$\underset{\sim}{A}(\alpha) = \underset{\sim}{U}^{(0)}(\alpha)\,[\underset{\sim}{L}^{(0)}(\alpha)]^{-1},$$

$$\underset{\sim}{U}^{(0)}(\alpha) = \begin{pmatrix} 1/x(\alpha) & 1/x(\alpha) \\ x(\alpha)\,(4\cosh^2\gamma J(\alpha)-1) & x(\alpha)\,(4\cosh^2\gamma J(\alpha)+1 \end{pmatrix}$$

where

$$x(\alpha) = \exp[\frac{i\gamma}{\pi}\ \ell n\ (\frac{\alpha+\gamma)}{k})],$$

$$J(\alpha) = \frac{1}{2\pi i}\int_{-\infty}^{-k}\frac{\exp[-i(^{2\gamma(t)}/\pi)\ell n|\frac{t+\gamma(t)}{k}]dt}{\gamma(t)\,(t-\alpha)},$$

and

$$\underset{\sim}{L}^{(0)}(\alpha) = \begin{pmatrix} -\exp[-\gamma]\cosh\gamma os(\alpha) - \exp[-\gamma]\,x(\alpha)\,(4\cosh^2\lambda J(\alpha)-1)\gamma\ \sinh\gamma & -\exp[-\gamma]\cosh\gamma/x(\alpha) - \exp[-\gamma]x(\alpha)\,(4\cosh^2\gamma J(\alpha)-1)\gamma\sinh\gamma \\ -\exp[-\gamma]\,(\sinh\gamma+2\cosh\gamma c/\{x(\alpha)\gamma\} + \exp[-\gamma]x(\alpha)\,(4\cosh^2\gamma J(\alpha)+1)\cosh\gamma & -\exp[-\gamma]\,(\sinh\gamma+2\cosh\gamma c/\{x(\alpha)\gamma\} + \exp[-\gamma]x(\alpha)\,(4\cosh^2\gamma J(\alpha)+1)\cosh\gamma \end{pmatrix}$$

We note that since

$$x(\alpha) = 1 + 0(\gamma), \qquad J(\alpha) = \frac{1}{2}\gamma^{-1} + 0(1) \quad \text{as} \quad \alpha \to -k;$$

the elements of $L^{(0)}(\alpha)$ have no poles at $\alpha = -k$; and this justifies the choice of $P_{ij}(\alpha)$ giving (22).

<u>Conclusions</u>.

We have presented a method for factorising matrices which arise in diffraction problems. This could offer scope for deriving closed-form solution to hitherto unsolved diffraction problems. The applicability of the present method to a given matrix A'(α) (whose elements, besides having the branch point singularities at α = ±k, also have poles; and whose determinant vanishes or becomes infinite in the cut α-plane) can be easily determined. If $\underset{\sim +}{A}'(\alpha)\,[\underset{\sim +}{A}'(\alpha)]^{-1}$ is triangular then the present method can be used to factorise the matrix $\underset{\sim}{A}'(\alpha)$. One merely has to determine the $\underset{\sim U}{C}(\alpha)$ and $\underset{\sim L}{B}(\alpha)$ of the expression (1) which ensures that the elements of $\underset{\sim}{A}(\alpha)$ have no poles and that det A(α) ≠ 0. This can be effected without too much ifficulty by inspection. On the strength of the above remarks there are a number of other diffraction problems (besides those mentioned in this paper), whose matrices can be factorised, see Rawlins [14] .

Finally we mention that the (n x n) triangular matrix Hilbert problem can also be solved explicitly. Thus provided we can find the class of (n x n)-matrices that reduce to the (n x n) triangular matrix Hilbert problem on analytic evaluation about the branch cut C, we will have effected the Wiener-Hopf factorisation of this class of (n x n)-matrices.

Appendix A.

Jones [10] has extended the class of 2×2 matrices whose factors commute to a class of $n \times n$ matrices whose factors commute. It is conjectured that a broader class of matrices, which includes the class considered by Jones [10], can also be explicitly factorised into commutative factors.

Conjecture:

The class of $n \times n$ matrices $\underset{\sim}{R}(\alpha)$ such that

$$\underset{\sim}{R}(\alpha) = \sum_{m=1}^{M} a_m(\alpha) \underset{\sim m}{A}(\alpha) \qquad M \leq n^2 ,$$

where $a_m(\alpha)$ are arbitrary functions of the complex variable α, and $\underset{\sim m}{A}(\alpha)$ are $n \times n$ matrices whose elements are entire functions of α, and these entire matrices are such that

$$\underset{\sim i}{A}(\alpha)\underset{\sim i}{A}(\beta) = \underset{\sim i}{A}(\beta)\underset{\sim i}{A}(\alpha) , \quad i = 1,2 ,...M.$$

can be factorised in the form

$$\underset{\sim}{R}(\alpha) = \underset{\sim +}{A}(\alpha)\underset{\sim -}{A}(\alpha)\underset{\sim +}{A}(\alpha)$$

For the case of $n = 2$ it can be shown that

$$\underset{\sim}{R}(\alpha) = a_1(\alpha)\underset{\sim}{I} + a_2(\alpha)\underset{\sim}{\beta}(\alpha)$$

where $\underset{\sim}{L}$ is the unit matrix and $\underset{\sim}{B}(\alpha)$ is an entire matrix; this is the type of matrix considered by Jones in [8].

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