TR/2 June 1971

NUMERICAL SOLUTION OF TWO DIMENSIONAL HARMONIC BOUNDARY PROBLEMS CONTAINING SINGULARITIES BY CONFORMAL TRANSFORMATION METHODS

by

J. R. WHITEMAN and N. PAPAMICHAEL

BRUNEL UNIVERSITY
DEC 1974
LIBRARY

ABSTRACT

Numerical solutions to a class of two dimensional harmonic mixed boundary value problems defined on rectangular domains and containing singularities are obtained using conformal transformation methods. These map the original problems into similar ones containing no singularities, and to which analytic solutions are known. Although the mapping technique produces analytic solutions to the original problems, these involve elliptic functions and integrals which have to be evaluated numerically, so that in practice only approximations can be obtained. Results calculated in this manner for model problems compare favourably with those obtained previously by other methods. On this evidence, and because of the ease with which the method can be adapted to different individual problems, we strongly recommend the transformation technique for solving problems of this class.

We express our thanks to Drs. E.L. Wachspress and W.B.Jordan for their initial suggestion of this approach to elliptic problems containing boundary singularities. Some of the work for this report was done during the period September 1968 - June 1970 whilst Dr.Whiteman was at the University of Texas, and was supported in part by Army Research Office (Durham) Grant DA-ARO(D) - 31 - 124 - G1050, and the National Foundation Grant GP - 8442 awarded to the University of Texas at Austin.

1. INTRODUCTION

The slow convergence with decreasing mesh size of finite-difference and finite-element solutions to the true solutions of harmonic mixed boundary value problems containing boundary singularities has caused such problems to be much studied, and special applications of these standard methods have been proposed, (see Fix [5], and Whiteman and Webb [18]). In addition several numerical techniques based on analytic methods have been given for particular problems, Whiteman [13] and Fox and Sankar [6]. As Laplace's equation is invariant under conformal transformation, another attractive technique is that in which the original harmonic problem is mapped into another containing no singularities. When mapping the region in which the original problem is defined, the user of the conformal transformation has two choices. He can either apply a simple transformation and accept the, usually complicated, resulting region, as for example in [13], or he can devise a transformation which leads to a previously chosen simple region. The latter course is adopted here to solve numerically several harmonic mixed boundary problems defined on rectangular regions. The method involves a new application of a well known technique (see Bowman [2], Chapter VII), and is used to produce accurate solutions to a number of model problems; in particular to one which has been the subject of much interest recently, [8], [10] and [12] - [19].

In all the problems considered here the function u(x,y) satisfies

$$\left.\begin{aligned} \Delta[u(x,y)] &= 0, \quad (x,y)\in R,\\ u(x,y) &= \ell, \quad (x,y)\in S_2,\\ \frac{\partial u(x,y)}{\partial v} &= 0, \quad (x,y)\in S_2,\\ u(x,y) &= m, \quad (x,y)\in S_3,\\ \frac{\partial u(x,y)}{\partial v} &= 0, \quad (x,y)\in S_4. \end{aligned}\right\} \qquad (1)$$

In (1) the ℓ and m are constants, Δ is a Laplacian operator, R is an open domain with a rectangular boundary S, where S $=\cup S_i$, i = 1 ,2 ,3,4 , S_i and S_{i+1} being adjacent sub arcs of S, and $\partial/\partial v$ is the derivative in the direction of the outward normal to the boundary. The application of two Schwarz-Christoffel transformations, with an intermediate bilinear transformation, maps the region G = RUS in the w = x + iy plane into the region G' in the $W' = \xi + i\eta$ plane, where G' = $G' = \{(\xi,\eta): 0\le\xi\le 1, 0\le\eta\le H\}$, and H is a known constant. The original problem is thus transformed into the problem

$$\left.\begin{aligned} \Delta[v(\xi,\eta)] &= 0, \qquad 0<\xi<1, 0<\eta<H,\\ \frac{\partial v(\xi,0)}{\partial v} &= \frac{\partial v(\xi,H)}{\partial v} = 0, 0<\xi<1,\\ v(o,\eta) &= \ell, v(1,\eta) = m, 0\le\eta\le H, \end{aligned}\right\} \qquad (2)$$

which has solution

$$v(\xi,\eta) = (m-1)\left\{\xi + \frac{\ell}{m-\ell}\right\}. \qquad (3)$$

Thus if $P=(x,y)\in G$ is mapped into $P'=(\xi,\eta)\in G'$, it follows that u(P)=v(P'), and so from (3) the solution of (1) at P is known immediately if the real co-ordinate of the point P' is found. In practice the transformations are performed numerically, thus introducing both rounding and truncation errors, and so only an approximation U(P) to u(P) is obtained.

2. TRANSFORMATIONS

2.1 Bounded Rectangular Domains

We consider the general problem of type (1) where G≡BCDE is the rectangle $|x| \le a, 0 \le y \le b,$ as in Figure 1, and where the four distinct end points L, M, N, P of the sub arcs Sj, can be taken anywhere on S. In this problem the boundary conditions on adjacent S_i's are one Diriohlet and the other Neumann, $G \in W$ -plane, and L,M,N,P are respectively
the points w_1, w_2, w_3 and w_4.

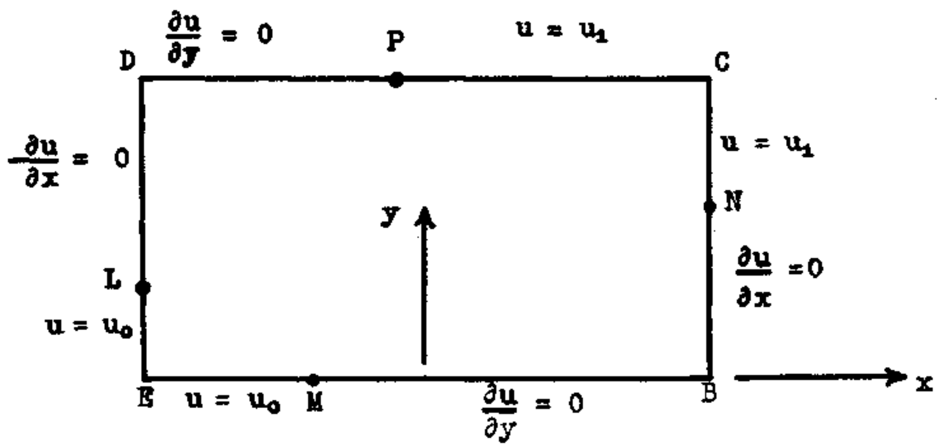


Figure 1.

The transformations used and their effects are

(i) $$\frac{G \in w-plane \to G_1 \in z-plane, (z = \alpha + i\beta),}{z = sn\left(\frac{kw}{a}, k\right),} \qquad (4)$$

where sn denotes the Jacobian elliptic sine, and

$$K = K(k) = \int_0^1 \frac{dt}{\left\{(1-t^2)(1-k^2t^2)\right\}^{\frac{1}{2}}}$$

is the complete elliptic integral of the first kind with modulus k.

This transformation is the inverse of a Schwarz-Christoffel transformation, and, if the modulus k is chosen so that

$$\frac{K\left\{\left(1-k^2\right)^{\frac{1}{2}}\right\}}{K(k)} = \frac{b}{a} \quad , \qquad (5)$$

equation (4) maps G onto $G_1 \equiv$ a the upper half z-plane.
In particular under this transformation,

w-plane		z-plane
(0,0)	→	(0,0),
$(\pm a,0)$	→	$(\pm 1,0)$,
$(\pm a,b)$	→	$(\pm 1/k,0)$,
(0,b)	→	(∞,0) ,

and

w-plane		z-plane
$L \equiv w_1$	→	$(\alpha_1,0) \equiv L_1$,
$M \equiv w_2$	→	$(\alpha_2,0) \equiv M_1$,
$N \equiv w_3$	→	$(\alpha_3,0) \equiv N_1$,
$P \equiv w_4$	→	$(\alpha_4,0) \equiv P_1$,

Where $\alpha_1 = sn\left(\frac{1}{a}kw_i, k\right), \qquad i = 1,2,3,4.$

Further details of transformations of this type can be found in Bowman [2], and Markushevich [9].

(ii) $$\frac{G_1 \in z-plane \rightarrow G_2 \in t-plane, (t = g+ih),}{t = \left(\frac{\alpha_4-\alpha_2}{\alpha_4-\alpha_1}\right)\cdot\left(\frac{z-\alpha_1}{z-\alpha_2}\right).} \qquad (6)$$

This bilinear transformation maps G_1 onto $G_2 \equiv$ the upper half

t - plane , with

z-plane	t-plane
$L_1 \equiv (\alpha_1, 0)$ →	$(0,0) \equiv L_2$,
$M_1 \equiv (\alpha_2, 0)$ →	$(\infty,0) \equiv M_2$,
$N_1 \equiv (\alpha_3, 0)$ →	$(g_1,0) \equiv N_1$,
$P_1 \equiv (\alpha_4, 0)$ →	$(1,0) \equiv P_1$,

where

$$g_1 = \left(\frac{\alpha_4 - \alpha_2}{\alpha_4 - \alpha_1}\right) \cdot \left(\frac{\alpha_3 - \alpha_1}{\alpha_3 - \alpha_2}\right) \quad > \quad 1 \ .$$

(iii) $$\frac{G_2 \in t - plane \rightarrow G' \in w' - plane, \left(w' = \xi + i\eta\right)}{w' = \frac{1}{K(m)} sn^{-1}(t^{\frac{1}{2}}, m) \ ,} \ , \qquad (7)$$

where K(m) is the complete elliptic integral of the first kind with modulus $m = (1/g_1)^{\frac{1}{2}}$. Note that, since $g_1 > 1$, the condition $0<m<1$ for the modulus m is satisfied. The Schwarz-Christoffel transformation (7) maps G_2 onto the rectangle

$$G' = \left\{ (\xi, \eta) : 0 \le \xi \le 1, 0 \le \eta \le \frac{K\left\{ \left(1 - m^2\right)^{\frac{1}{2}} \right\}}{K(m)} \equiv H \right\}$$

in the w'-plane, with

t-plane	w′-plane
$L_2 \equiv (0,0)$ →	$(0,0) \equiv L_3$,
$M_2 \equiv (\infty,0)$ →	$(0,H) \equiv M_3$,
$N_2 \equiv (g_1,0)$ →	$(g_1,0) \equiv N_3$,
$P_2 \equiv (1,0)$ →	$(1,0) \equiv P_3$,

The combined effect of (4), (6) and (7) is to transform the

original problem in G into a problem of type (2) in G' , as in Figure 2, which has the solution

$$v(\xi,\eta)=(u_1-u_0)\left(\xi+\frac{u_0}{u_1-u_0}\right). \qquad (8)$$

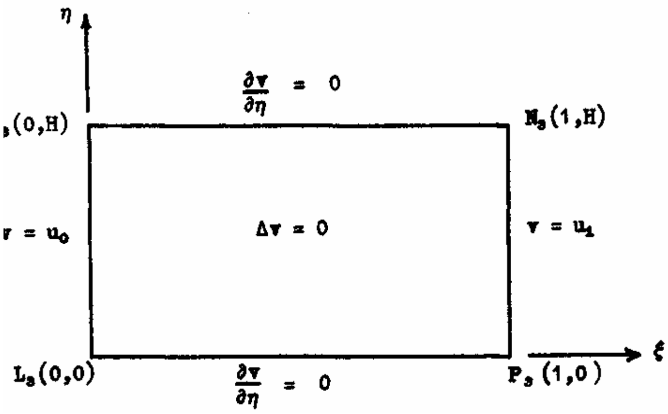


Figure 2.

We note that the bilinear transformation is not unique,and in fact four such suitable transformations exist. For example, (6) can be replaced by

$$t=\left(\frac{\alpha_1-\alpha_3}{\alpha_1-\alpha_4}\right)\cdot\left(\frac{z-\alpha_4}{z-\alpha_3}\right).$$

The half place G_1 is again mapped on to G_2 but now

z – plane	t – plane
$L_1 \equiv (\alpha_1,0)$	$\rightarrow (1,0) \equiv L_2$,
$M_1 \equiv (\alpha_2,0)$	$\rightarrow (g_2,0) \equiv M_2$,
$N_1 \equiv (\alpha_3,0)$	$\rightarrow (\infty,0) \equiv N_2$,
$P_1 \equiv (\alpha_4,0)$	$\rightarrow (0,0) \equiv P_2$,

Where

$$g_2 = \left(\frac{\alpha_1 - \alpha_3}{\alpha_1 - \alpha_4}\right) . \left(\frac{\alpha_2 - \alpha_4}{\alpha_2 - \alpha_5}\right) \rangle \; 1.$$

Transformation (4) followed by this bilinear transformation, and then (7) with m = $(1/g_2)^{\frac{1}{2}}$, transform the original problem into a problem of type (2) with solution

$$v(\xi, \eta) = (u_0 - u_1)\left(\xi + \frac{u_1}{u_0 - u_1}\right) . \qquad (9)$$

The two remaining bilinear transformations are given for reference.
These are

$$t = \left(\frac{\alpha_3 - \alpha_1}{\alpha_3 - \alpha_2}\right) \cdot \left(\frac{z - \alpha_2}{z - \alpha}\right) \qquad (10)$$

and

$$t = \left(\frac{\alpha_2 - \alpha_4}{\alpha_2 - \alpha_5}\right) \cdot \left(\frac{z - \alpha_3}{z - \alpha_4}\right). \qquad (11)$$

Use of (10) in place of (6) with the modulus

$$m = \left\{\left(\frac{\alpha_3 - \alpha_1}{\alpha_3 - \alpha_2}\right) \cdot \left(\frac{\alpha_4 - \alpha_2}{\alpha_4 - \alpha_1}\right)\right\}^{-\frac{1}{2}}$$

in (7), produces a problem in G' with solution (8).
Similarly (11) with

$$m = \left\{\left(\frac{\alpha_2 - \alpha_4}{\alpha_2 - \alpha_3}\right) \cdot \left(\frac{\alpha_1 - \alpha_3}{\alpha_1 - \alpha_4}\right)\right\}^{-\frac{1}{2}}$$

produces a problem with solution (9).

2.2 Unbounded Rectangular Domains

A technique similar to that of Subsection 2.1 is used transform problems of type (1) in which G is a semi-infinite strip into problems of type (2). In this case the only change necessary is that (4) must be replaced by a transformation mapping the strip onto the upper half z-plane. As an illustration we consider the following mixed boundary value problem in which u(x,y) satisfies

$$\left.\begin{aligned} &\Delta u = 0,\ 0 < x < a,\ y > 0, \\ &u = u_0, \frac{a}{3} \le x \le \frac{2a}{3},\ y = 0, \\ &\frac{\partial u(x,0)}{\partial y} = 0,\ 0 < x < \frac{a}{3},\ \frac{2a}{3} < x < a, \\ &u = u_1,\quad x = a,\ y > 0, \\ &u \to u_1 \text{ as } y \to \infty,\ 0 < x < a, \\ &\frac{\partial u(o,y)}{\partial x} = 0, \qquad y > 0, \end{aligned}\right\} \qquad (12)$$

so that $G = \{(x,y) : 0 \le x \le a, y \ge 0\}$. The transformation

$$z = \cos\frac{nw}{a} \qquad (13)$$

maps G onto $G_1 \equiv$ the upper half z-plane with

w − plane	*z − plane*
$L \equiv \left(\frac{a}{3}, 0\right)$	$\to \left(\frac{1}{2}, 0\right) \equiv L_1$,
$M \equiv \left(\frac{2a}{3}, 0\right)$	$\to \left(-\frac{1}{2}, 0\right) \equiv M_1$,
$N \equiv (a, 0)$	$\to (-1, 0) \equiv N_1$,
$P \equiv (0, \infty)$	$\to (\infty, 0) \equiv P_1$.

Thus, $\alpha_1 = 1/2, \alpha_2 = -1/2, \alpha_3 = -1, \alpha_4 = \infty$, and (6) becomes

$$t = \frac{2z-1}{2z+1} \quad , \tag{14}$$

where $g_1 = 3$. Hence (13), (14), and (7) with m = 1/$\sqrt{3}$ map (12) into a problem of type (2), as in Figure 2, with solution (8).

2.3 Numerical Algorithms.

Details of the methods for implementing the transformations are now given. For the bounded domain problems the first transformation, the elliptic sine (4), is found from a series expansion, Copson [3] p.412, so that

$$\alpha + i\beta = sn\left(\frac{Kw, k}{a}\right) = \frac{1}{\sqrt{k}} \frac{\sum_{n-0}^{\infty}\left[(-1)^n e^{\frac{-\pi b}{a}\left(n+\frac{1}{2}\right)^2} \sin\left\{\left(n+\frac{1}{2}\right)\frac{\pi w}{a}\right\}\right]}{\frac{1}{2} + \sum_{n=1}^{\infty}\left[(-1)^n e^{\frac{-\pi b m^2}{a}} \cos\left(\frac{n\pi w}{a}\right)\right]}, \tag{15}$$

whereas for problems in unbounded domains equation (13) is used directly. In both cases the bilinear transformation which follows gives immediately the image (g,h) $\in G_2$ of $(\alpha, \beta) \in G_1$.
In order that the solutions (8) and (9) may be used, the real co-ordinaie ξ of the image $w' \equiv (\xi, \eta) \in G'$ of (g,h) $\in G_2$ must be found. For this the following analysis is necessary. Equation (7) gives

$$sn^2(k(m)w', m) = t = g + ih \;,$$

so that

$$cn^2(K(m)w', m) = 1 - sn^2(K(m)w', m)$$
$$= (1-g) - ih.$$

When A and B are defined by

$$A = \left|cn^2(K(m)w', m)\right| = \left\{(1-g)^2 + h^2\right\}^{\frac{1}{2}}, \tag{16}$$

and

$$B = \left|sn^2(k(m)w', m)\right| = \left\{g^2 + h^2\right\}^{\frac{1}{2}} \quad , \tag{17}$$

and, when K(m)w' = p+ iq, the expressions for the absolute values of the elliptic functions, (Bowman [2], p.41, equations 37 and 38), give

$$A = \frac{dn(2p,m) + cn(2p,m)dn(2p,m')}{dn(2q,m') + dn(2p,m)cn(2q,m')} \quad , \tag{18}$$

and

$$B = \frac{1 - cn(2p,m)on(2q,m')}{dn(2q,m') + dn(2p,m)cn(2q,m')} \tag{19}$$

The $m' \equiv (1-m^2)^{\frac{1}{2}}$ in (18) and (19) is the complementary modulus, and dn is the Jacobian elliptic function defined by

$$dn(2p,m) = \{1 - m^2 sn^2(2p,m)\}^{\frac{1}{2}}$$

$$= \{1 - m^2 + m^2 cn^2(2p,m)\}^{\frac{1}{2}} \quad . \tag{20}$$

Equations (18) and (19) give

A - B dn(2p,m) = cn(2p,m) ,

so that

$$cn^2(sp,m) - 2A cn(2p,m) + A^2 - B^2 dn^2(2p,m) = 0.$$

This together with (20) gives

$$(1 - B^2 m^2) cn^2(2p,m) - 2A\, cn(2p,m) + A^2 - B^2 = 0. \tag{21}$$

Equation (21) is a quadratic in on(2p,m) with solutions

$$cn(2p,m) = \frac{A \pm B\{A^2 m^2 + (1 - B^2 m^2)(1 - m^2)\}^{\frac{1}{2}}}{(1 - B^2 m^2)} , \tag{22}$$

one of which is extraneous. To determine this we note that under transformation (7) the image of (1,0) $\in G_2$ is $(1,0) \in G'$.

Thus, when g = 1 and h = 0 , ξ = 1 and η = 0 so that w' = 1 and cn(2p,m) = cn(2K(m),m) = -1. Again when g = 1 and h = 0, it follows from (17) and (18) that A = 0 and B= 1, and therefore for these values of A and B the right hand side of (22) must have value -1. Thus

$$\mathrm{cn}(2p,m) = \mathrm{cn}(2k(m)\xi,m)$$

$$= \frac{A - B\{A^2m^2 + (1 - B^2m^2)(1 - m^2)\}^{\frac{1}{2}}}{\{1 - B^2m^2\}} . \qquad (23)$$

With g and h known, equations (17) and (18) give A and B, and hence (23) gives cn(2K(m) ξ ,m). When $\phi = \cos^{-1}(cn(2K(n)\xi, m)$, it follows from Abramowitz and Stegun [1], Section 17, p. 589, that

$$\xi = F(\phi \backslash \alpha)/2K(m), \qquad (24)$$

where $F(\phi \backslash \alpha)$ is the incomplete elliptic integral of the first kind of amplitude ϕ and modular angle a =$\sin^{-1}$m

In (15) the modulus k is determined from (5) to 10 decimal places either directly from tables [1], Table 17.3, or, for values of b/a outside the interval [0.3,3], by means of the iterative method, [1], Ex 6,p.602. The series in (15) are truncated after N terms. These series converge very rapidly and, for the model problems solved, it is found that the calculated approximations to the elliptic sine remain constant for N≥3. Thus, in all calculations N is taken as 3. Two algorithms, due to Hofsommer and Van de Riet [7] are used to calculate respectively the elliptic integrals $F(\phi \backslash \alpha)$ and K(m) in (24). The authors of these algorithms claim twelve accuracy, and this is confirmed by experiments carried out on an I.C.L. 1903A.

3. APPLICATIONS TO MODEL PROBLEMS AND NUMERICAL RESULTS

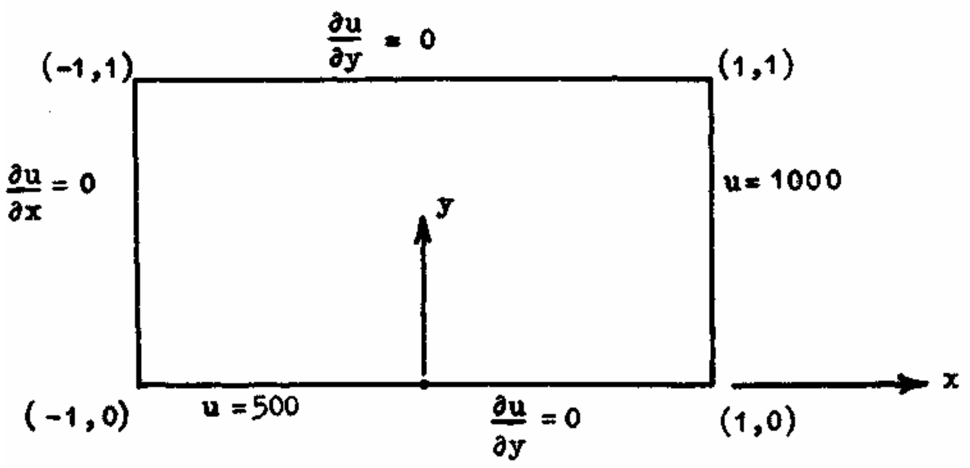
3.1 Bounded Rectangular Domains

Problem 1

The function u(x,y) satisfies $\Delta u = 0$ in a square region $-1 < x,\ y < 1$ with the slit y=0, $0 < x < 1$ and boundary conditions

$$\left.\begin{aligned} &\frac{\partial u(x,\pm 1)}{\partial y} = 0, \qquad -1<x<1 \\ &u(1,y) = \begin{Bmatrix} 1000, & 0 \le y \le 1, \\ 0\ \ , & -1 \le y \le 0, \end{Bmatrix} \\ &\frac{\partial u(-1,y)}{\partial x} = 0, \qquad -1<y<1\,, \\ &\frac{\partial u(x,0+)}{\partial y} = \frac{\partial u(x,0-)}{\partial y} = 0, \quad 0<x<1, \end{aligned}\right\}$$

where 0+ and 0- represent respectively the upper and lower arms of the slit. Because of the antisymmetry it is sufficient to consider only the problem in the upper half region G, where $G \equiv \{(x,y); -1 \le x \le 1,\ 0 \le y \le 1\}$, and to add the boundary condition $u(x,0) = 500$, $-1 \le x \le 0$, as in Figure 3. This problem is a special case of (1) and has a singularity at the origin.



The details relevant to Subseotion 2.1 are:

Value of k in (4) : $1/\sqrt{2}$,

Bilinear transformation : $t = \frac{2z}{1+z}$,

Value of m in (7) : $(1/2 + 1/2\sqrt{2})^{\frac{1}{2}}$,

Solution : $u(x,y) = v(\xi, \eta) = 500\ (\xi+1)$.

The numerical results for the conformal transformation method (CTM) show everywhere four figure agreement with those obtained by whiteman [16]. In the neighbourhood of the singularity the results show at nearly all points five figure agreement with those obtained by Whiteman [15]. The CTM results are given on a mesh of length 2/7 throughout R in Table 1, and on a mesh of length 1/28 in the neighbourhood of 0, $|x| < 1/7$, $0 < y < 1/7$, in Table 2. Results of [15], [16] and of Wait and Mitchell [12] are included for comparison. Execution time on the ICL 1903A for the CTM is 15 seconds for the solution of Table 1.

591.33	590 608.87	607 645.46	644 702.11	701 776.27	775 862.00	861 953.48	953
591.34	608.89	645.49	702.14	776.29	862.02	953.46	
574.10	572 589.79	588 624.74	623 683.89	682 764.82	763 856.66	855 951.96	951
574.10	589.80	624.76	683.92	764.84	856.68	951.98	
541.77	541 551.97	551 578.55	577 641.55	641 743.80	742 848.63	847 949.92	950
574.76	551.97	578.56	641.56	743.81	848.64	949.93	
(-1,0 500	500	500	500	728.47	727 844.36	844 948.95	949 (1,0)
500.00	500.00	500.00	500.00	728.47	844.37	948.93	

TABLE-1 Mesh Length=2/7.

At each mesh point the numbers represent:

Linear Programing [16]	Finite-element [12]
CTM	

561.1 561.95 561.95	568.6 569.47 569.47	577.9 578.77 578.77	589.3 590.17 590.18	602.9 603.77 603.77	617.7 619.19 619.19	633.9 635.72 635.72	650.7 652.67 652.67	667.0 669.52 669.54
547.5 548.42 548.42	554.2 555.13 555.13	563.0 563.96 563.96	574.5 575.65 575.65	589.1 590.63 590.63	606.4 608.30 608.30	625.1 627.24 627.24	644.1 646.20 646.20	662.7 664.59 664.59
532.6 533.42 533.42	537.8 538.70 538.70	545.1 546.24 546.24	556.3 557.64 557.64	572.6 574.61 574.61	593.5 596.23 596.23	616.2 618.25 618.85	638.2 640.36 640.37	658.5 660.40 660.40
516.6 517.12 517.12	519.5 520.11 520.11	523.81 524.81 524.81	531.7 533.59 533.59	550.5 553.18 553.19	579.2 583.67 583.67	609.0 611.86 611.86	633.9 636.07 636.07	655.8 657.52 657.52
500.00 500.00	500.00 500.00	500.00 500.00	500.00 500.00	500.00 500.00	572.8 576.41 576.41	606.2 608.91 608.91	632.4 634.45 634.45	655.1 656.48 656.48

TABLE 2. Mesh Length =1/28

At each point the numbere represent : Finite-element [12]
Extend dual series [15]
CTM

In the problems 2-5 that follow G is the rectangle
G ≡ {(x,y) : |x| 1 ,0 y 1} with interior R.

Problem 2.

$$\left.\begin{array}{l} \Delta u = 0 \, in \, R, \\ u(x,0) = 0, -1 \le x \le 0, \\ \dfrac{\partial u(x,0)}{\partial x} = \dfrac{\partial u(x,1)}{\partial y} = 0, \ \ 0 < x < 1, \\ \dfrac{\partial u(\pm 1, y)}{\partial x} = 0, \ \ 0 < y < 1, \\ u(x,1) = 1, \ \ -1 \le x \le 0 \end{array}\right\}$$

The details relevant to Subsection 2.1 are:

Value of k in (4)	:	$1/\sqrt{2}$,
Bilinear transformation	:	$t = \dfrac{z}{1+z}$
Value of m in (7)	:	$\left(1-1/\sqrt{2}\right)^{\frac{1}{2}}$,
Solution	:	u(x,y) = v(ξ, n) = ξ.

CTM results for a mesh of length 1/5 are given in Table 3 together with results obtained using finite elements. On the 1903A the execution times are 17 seconds for the CTM, and, with a square mesh of length 1/20, 2 minutes 45 seconds for the finite element method.

1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.6801	0.5939	0.5224	0.5331	0.5275
						0.692	0.599	0.555	0.535	0.528
0.7997	0.7995	0.7982	0.7938	0.7774	0.7222	0.6436	0.5745	0.5422	0.5628	0.5222
0.800	0.800	0.799	0.795	0.781	0.730	0.644	0.579	0.544	0.528	0.528
0.5998	0.5997	0.5989	0.5964	0.5887	0.5711	0.5472	0.5277	0.5160	0.5102	0.5085
0.599	0.599	0.599	0.597	0.590	0.573	0.550	0.530	0.519	0.511	0.509
0.4002	0.4003	0.4001	0.4036	0.4113	0.4829	0.4528	0.4725	0.4840	0.4898	0.4915
0.400	0.400	0.400	0.403	0.410	0.427	0.450	0.470	0.483	0.489	0.491
0.2003	0.2005	0.2018	0.2062	0.2226	0.2778	0.3654	0.4225	0.4578	0.4732	0.4778
0.200	0.200	0.201	0.205	0.219	0.270	0.356	0.420	0.455	0.472	0.476
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.3199	0.4061	0.4476	.4669	0.4725
						0.308	0.400	0.444	0.465	0.471
(-1,0)										(1,0)

<u>TABLE 3</u> Mesh Length =1/5

At each point the numbers represent :

CTM
Finite element solution Due to M.Lavender.

Problem 3

$$\left.\begin{array}{l} \Delta u = 0 \; in \; R, \\ u(x,0) = 1, -1 \le x \le 0, \\ \dfrac{\partial u(x,0)}{\partial x} = 0, \;\; 0 < x < 1, \\ \dfrac{\partial u(\pm 1, y)}{\partial x} = 0, \;\; 0 < y < 1, \\ u(x,1) = 0, \;\; -1 \le x \le 1. \end{array}\right]$$

The details relevant to Subsection 2.1 are:

Value of k in (4)	:	$1/\sqrt{2}$,
Bilinear transformation	:	$t = (1+1/\sqrt{2})\left(\dfrac{z}{1+z}\right).$
Value of m in (7)	:	$\left(3-2/\sqrt{2}\right)^{\frac{1}{2}}$,
Solution	:	$u(x,y) = v(\xi, \eta) = 1-\xi.$

CTM results obtained on a mesh of length 1/5 are given in Table 4.

0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1928	0.1915	0.1871	0.1718	0.1656	0.1477	0.1270	0.1073	0.0917	0.0819	0.0786
0.3882	0.3858	0.3781	0.3626	0.3364	0.2982	0.2531	0.2104	0.1774	0.1572	0.1505
0.5879	0.5854	0.5766	0.5575	0.5203	0.4568	0.3761	0.3028	0.2499	0.2190	0.2089
0.7924	0.7907	0.7846	0.7700	0.7335	0.6370	0.4880	0.3730	0.3002	0.2602	0.2474
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.5464	0.4008	0.3185	0.2747	0.2609
(1,0)					**(0,0)**				(-1,0)	

Table 4. Mesh Lenth = 1/5.

Problem 4

$$\left.\begin{aligned}
\Delta u &= 0 \, in \, R, \\
u(x,0) &= 1, \quad -1 \le x \le -0.4, \\
\frac{\partial u(x,0)}{\partial x} &= 0, \quad -0.4 < x < 1, \\
\frac{\partial u(\pm 1, y)}{\partial x} &= 0, \quad 0 < y < 1, \\
u(x,1) &= 0, \quad -1 \le x \le 1.
\end{aligned}\right]$$

The details relevant to Subsection 2.1 are:

Value of k in (4) : $1/\sqrt{2}$,

Bilinear transformation : $t = \left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \cdot \left(\frac{\sqrt{2}+z}{\sqrt{2}-z}\right)$.

Value of m in (7) : $\left\{\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) \cdot \left(\frac{\sqrt{2}-\alpha}{\sqrt{2}+\alpha}\right)\right\}^{\frac{4}{2}}$,

Where α = sn(-0.4K, $1/\sqrt{2}$),

Solution : $u(x,y) = v(\xi, y) = \xi$.

CTM results are given in Table 5.

0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1746	0.1708	0.1595	0.1415	0.1194	0.0969	0.0772	0.0618	0.5511	0.0449	0.0429
0.3567	0.3492	0.3260	0.2873	0.2389	0.1907	0.1498	0.1189	0.0978	0.0857	0.0818
0.5536	0.5434	0.5090	0.4438	0.3577	0.2762	0.2120	0.1660	0.1356	0.1184	0.1128
0.7694	0.7608	0.7257	0.6256	0.4682	0.3423	0.2577	0.1976	0.1604	0.1396	0.1329
1.0000	1.0000	1.0000	1.0000	0.5628	0.3689	0.2718	0.2088	0.1690	0.1469	0.1398
(-1,0)				(0,0)					(1,0)	

Table 5. Mesh Length = 1/5

Problem 5

$$\left.\begin{aligned}
&\Delta u = 0 \text{ in } R,\\
&u(x,0) = 1,\ \ -1 \le x \le -0.4,\\
&u(-1,y) = 1,\ \ 0 \le y \le 0.2,\\
&\frac{\partial u(x,0)}{\partial y} = 0,\ \ -0.4 < x < 1,\\
&\frac{\partial u(1,y)}{\partial x} = 0,\ \ 0 < y < 1,\\
&\frac{\partial u(-1,y)}{\partial x} = 0,\ \ 0.2 < y < 1,\\
&u(x,1) = 0,\ \ -1 \le x \le 1.
\end{aligned}\right]$$

The details relevant to Subsection 2.1 are

Value of k in (4) : $1/\sqrt{2}$,

Bilinear transformation : $t = \left(\dfrac{\alpha-\sqrt{2}}{\alpha+\sqrt{2}}\right)\cdot\left(\dfrac{z+\sqrt{2}}{z-\sqrt{2}}\right)$,

Where $\alpha = \mathrm{sn}\{K(-1+0.21),\ 1/\sqrt{2}\}$,

Value of m in (7) : $\left\{\left(\dfrac{\alpha+\sqrt{2}}{\alpha-\sqrt{2}}\right)\cdot\left(\dfrac{\alpha'-\sqrt{2}}{\alpha'+\sqrt{2}}\right)\right\}^{\frac{4}{2}}$,

Where $\alpha' = \mathrm{sn}(-0.4K,\ 1/\sqrt{2})$,

Solution : $u(x,y) = v(\xi,\ \eta) = \xi$.

CTM results on a mesh of length 1/5 are given in Table 6.

0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1872	0.1821	0.1679	0.1471	0.1230	0.0993	0.0788	0.0631	0.0521	0.0458	0.0437
0.3851	0.3734	0.3424	0.2976	0.2454	0.1950	0.1528	0.1212	0.0996	0.0873	0.0832
0.6090	0.5834	0.5310	0.4562	0.3655	0.2816	0.2160	0.1691	0.1381	0.1205	0.1149
1.0000	0.8077	0.7441	0.6354	0.4754	0.3479	0.2602	0.2011	0.1632	0.1421	0.1353
1.0000	1.0000	1.0000	1.0000	0.5330	0.3745	0.2763	0.2125	0.1720	0.1496	0.1424
(-1,0)					(0,0)					(1,0)

TABLE 6. **Mesh Lenth =1/5.**

3.2. Unbounded Rectangular Domains

Problem 6

The CTM is applied to the problem of Subsection 2.2 with $a = 1$, $u_0 = 1$ and $u_1 = 2$. The numerical results so obtained are given on a mesh of length 1/6 in the x-direction and 1/12 in the y-direction in Table 7. These results show at all points four figure agreement with those obtained by Tranter and Whiteman [11] using a triple cosine series method. Execution time for the CTM on the 1903A is 14 seconds.

1.5778	1.5830	1.6041	1.6541	1.7415	1.8620	2.0000
1.5307	1.5328	1.5478	1.5967	1.6952	1.8370	2.0000
1.4830	1.4797	1.4827	1.5263	1.6375	1.8068	2.0000
1.4376	1.4258	1.4074	1.4380	1.5643	1.7713	2.0000
1.3988	1.3749	1.3191	1.3246	1.4688	1.7322	2.0000
1.3720	1.3350	1.2120	1.1778	1.3351	1.6968	2.0000
1.3623	1.3189	1.0000	1.0000	1.0000	1.6811	2.0000

TABLE 7.

The mesh lengths are 1/6 in the x - direction, 1/12 in the y - direction

4. DISCUSSION

As a technique for solving harmonic boundary problems containing singularities, the CTM must be compared with two other classes of methods. These are the "analytic" methods such as the dual or triple series techniques of [11], [14] and [15], and the "disorete" methods such as finite-differences and finite elements. Two obvious advantages of the CTM, or any other "analytic" method, over a "discrete" method are firstly that the original problem itself rather than some approximating problem is solved, and secondly that the same technique produces the solution at all points right up to the singularities. The solution can also be obtained at any desired point in the domain without the need to interpolate between the values at mesh points. The CTM is much more versatile than the dual and triple series methods, and, for the class of problems of this report, produces as accurate or more accurate solutions in equal computation times. Further, it is well known that finite-difference tchniques are inadequate near singularities, and, although the CTM has generally less wide application than these "disorete" methods, it is more suited to the problems containing singularities considered here, being computationally several orders of magnitude faster in producing solutions of higher accuracy.

It is important to consider possible extensions of the CTM. Clearly, the technique of Subsection 2.1 may be used to solve a problem of type (1) defined on a non-rectangular domain provided that (4) is replaced by a transformation that maps the domain onto the upper half z-plane. In some cases this transformation is quite simple, and can be determined analytically; as for example in the case of a circular region. However, for a general polygonal region equation (4) must be replaced by the inverse of a Schwarz-Christoffel transformation which has to be determined numerically. Work is proceeding on this, and here a method proposed by Cox [4] may be of use. Other possible extensions include the application of the method to problems with more general boundary conditions, in particular those with Neumann conditions of the form $\frac{\partial u}{\partial v}$ = constant $\neq$ 0, and to eigenvalue problems.

REFERENCES

1. M.Abramowitz and I.A.Stegun, Handbook of Mathematical Functions, Dover, New York, (1965).
2. F.Bowman, Introduction to Elliptic Functions, English Universities Press, London, (1953).
3. E.T. Copson, An Introduction to the Theory of Functions of a Complex Variable, Oxford University Press, London, (1935).
4. M.G. Cox, The numerical Schwartz-Christoffel transformation with applications. A.E.I.Ltd., Internal Report.
5. G. Fix, Higher-order Rayleigh Ritz approximations, Journal of Math.And Mech., 18, 645-657, (1969).
6. L. Fox and R. Sankar, Boundary singularities in linear elliptic differential equations, J.Inst.Maths.Applics,5, 340-350 (1969).
7. D.J. Hofsommer and R.P.van de Riet, On the numerical calculations of elliptic integrals of the first and second kind and the elliptic functions of Jacobi, Numerische Mathematik,5, 291-302, (1963).
8. R.B. Kelman, Harmonic mixed boundary value problems in composite rectangular regions, Q.J.Mech.Appl.Math.XXIII, 549-566, (1970).
9. A.I. Markushevich, Theory of Functions of a Complex Variable, Vol.II, Prentice Hall, New Jersey, (1965).
10. H. Motz, The treatment of singularities of partial differential equations by relaxation methods, Quart.Appl.Math.,4,371-377 (1946).
11. C.J. Tranter and J.R. Whiteman, Triple cosine series and a harmonic mixed boundary value problem, University of Texas at Austin, Computation Center, TNN 100, January 1970.
12. R. Wait and A.R. Mitchell, Corner singularities in elliptic problems by finite element methods. To appear.
13. J.R. Whiteman, Singularities due to re-entrant corners in harmonic boundary value problems, Mathematics Research Center, Technical Summary Report No.829, University of Wisconsin, Madison, (1967).
14. J.R. Whiteman, Treatment of singularities in a harmonic mixed boundary value problem by dual series methods, Q.J.Mech.Appl.Math.,XXI, 41-50, (1968).
15. J.R. Whiteman, Numerical solution of a harmonic mixed boundary value problem by the extension of a dual series method. Q.J.Mech. Appl.Math., XXIII, 449-455 (1970).
16. J.R. Whiteman, Numerical solution of a harmonic mixed boundary value problem by linear programming, Mathematics Research Center, Technical Summary Report No.857, University of Wisconsin, Madison, (1968).
17. J.R. Whiteman and Jerry C. Webb, Convergence of finite-difference techniques for a harmonic mixed boundary value problem, B.I.T.,10, 366-374, (1970).
18. J.R. Whiteman and Jerry C. Webb, Finite-difference solutions to a harmonic mixed boundary value problem, Technical Report, Center for Numerical Analysis, University of Texas. To appear.
19. L.C. Woods, The relaxation treatment of singular points in Poisson's equation, Q.J.Mech.Appl.Math., 6, 163-185. (1953).

XB 2359533 7