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by

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Abstract

In this note we give a real variable approach for calculating the constant term that arises in the application of the Euler-Maclaurin expansion for a special class of series of the form $\sum_{r=1}^{n} f(r)$ as $n \to \infty$.

In particular the method is used to derive the approximate summation of the expression $\sum_{r=1}^{n} r^{\ell} \ell n r$, where ℓ is a non negative integer.

Introduction

A problem of frequent occurrence in analysis and applied mathematics is to find an approximate expression for sums of the form

$$S_n = \sum_{r=1}^{n} f(r), \text{ as } n \to \infty$$

This is particularly important when f(x) is a slowly varying function of x, when the above expression for S_n would be useless for calculating $\lim_{n\to\infty} S_n$. An effective mathematical method for dealing with this type of problem is the Euler-Maclaurin summation formula. One form of this summation formula gives an estimate for the sum S_n, by the integral:

$$\int_1^n f(r)dr,$$

with correction terms, involving a constant, and the values of f(r) and its odd derivatives at t = n. A very good and comprehensive treatment of the Euler-Maclaurin summation formula is given in the book by Olver[l], chapter 8. The evaluation of the constant term requires some ingenuity, especially when f is real with only a finite number of continuous derivatives, (that is, f(x) cannot be analytically continued off the the real x-axis).

We shall describe a method for the evaluation of the constant term which seems more direct than that usually used in text books. The method works when high enough derivatives of f can be expressed in inverse powers of x^2 or x^3. The method uses the periodicity of the Bernoulli polynomials $B_{2s}(x-[x])$ and the fact that

$$\frac{d^2}{dx^2}\ell n\Gamma\Gamma(+1) = \sum_{n=0}^{\infty}\frac{1}{(x+n)^2}.$$

where $\Gamma(x)$ is the Gamma function, O1ver[l]. As an application we consider the sum $\sum_{r=1}^{n} r^{\ell}\ell nr$, ℓ a non-negative integer.

Let f(x) have 2m continuous derivatives $f^{(2m)}(x)$ for $x\geq 1$, and let $f^{(2m-1)}(x) \gtrless 0$, $f^{(2m)}(x) \gtrless 0$, $f^{(2m-1)}(x)\to 0$ as $x\to\infty$, then the Euler-Maclaurin sum formula gives

$$\sum_{r=1}^{n} f(r) = \int_1^n f(x)dx + \frac{1}{2}f(n) + \sum_{s=1}^{m}\frac{B_{2s}}{(2s)!}f^{(2s-1)}(n) + C_{2m} + 0(f^{2m-1}(n)), \qquad (1)$$

$$m = 1,2,...,$$

see Olver [1].

In the above expression on the right hand side of the equality sign, all the terms before the constant C_{2m} increase with n, $C_{2m} = 0(1)$, and the order term is o(l) as $n\to\infty$. The constant term C_{2m} is given by the expression

$$C_{2m} = \frac{1}{2}f(1) - \sum_{s=1}^{m}\frac{B_{2s}}{(2s)!}f^{(2s-1)}(1) - \int_1^{\infty}\frac{B_{2m}((x-[x])}{(2m)!}f^{(2m)}(x)dx \qquad (2)$$

$$= \frac{1}{2}f(1) - \sum_{s=1}^{m}\frac{B_{2s}}{(2s)!}f^{(2s-1)}(1) - \int_0^1 \frac{B_{2m}(x)}{(2m)!}\sum_{r=0}^{\infty} f^{(2m)}(x+r+1)dx, \qquad (3)$$

provided the infinite series in (3) converges uniformly.

Evaluation of the constant term C_{2m}.

For the applications we have in mind we will need to consider two situations.
(i) $f^{(2m)}(x) = a(2m)x^{-2}$, a(2m) independent of x.
Then we can write (3) in the form

$$C_{2m} = \frac{1}{2}f(1) - \sum_{s=1}^{m}\frac{B_{2s}}{(2s)!}f^{(2s-1)}(1) - \frac{a(m)}{(2m)!}\int_0^1 B_{2m}(x)\sum_{r=0}^{\infty}\frac{1}{(x+r+1)^2}dx,$$

$$= \frac{1}{2}f(1) - \sum_{s=1}^{m}\frac{B_{2s}}{(2s)!}f^{(2s-1)}(1) - \frac{a(m)}{(2m)!}\int_0^1 B_{2m}(x)\frac{d^2}{dx^2}\ell n\,\Gamma(x+1)\,dx$$

Now integrating by parts twice gives

$$C_{2m} = \frac{1}{2}f(1) - \sum_{s=1}^{m}\frac{B_{2s}}{(2s)!}f^{(2s-1)}(1) - \frac{a(m)}{(2m)!}B_{2m} - \frac{a(m)}{(2m-2)!}\int_0^1 B_{2m-2}(x)\ell n\,\Gamma(1+x)\,dx. \qquad (4)$$

(ii) $f^{(2m)}(x)=b(m)x^{-3}$, b(m) independent of x.
Then we can write (3) in the form

$$C_{2m} = \frac{1}{2}f(1) - \sum_{s=1}^{m}\frac{B_{2s}}{(2s)!}f^{(2s-1)}(1) - \frac{b(m)}{(2m)!}\int_0^1 B_{2m}(x)\sum_{r=0}^{\infty}\frac{1}{(x+r+1)^3}dx,$$

$$= \frac{1}{2}f(1) = \sum_{s=1}^{m}\frac{B_{2s}}{(2s)!}r^{(2s-1)}(1) + \frac{b(m)}{(2m)!}\int_0^1 B_{2m}(x)\frac{d^3}{dx^3}\ell n\Gamma(x+1)dx.$$

Now integrating by parts thrice gives

$$C_{2m} = \frac{1}{2}f(1) - \sum_{s=1}^{m}\frac{B_{2s}}{(2s)!}f^{(2s-1)}(1) - \frac{b(m)}{2(2m)!}B_{2m} - \frac{b(m)}{2(2m-3)!}\int_0^1 B_{2m-3}(x)\ell n\Gamma(x+1)\,dx. \qquad (m>1) \quad (5)$$

To obtain (4) and (5) we have used the results (see O1ver[l]).

$$\psi^{(m)}(2) - \psi^{(m)}(1) = (-)^m m! \qquad m = 0,1,....., \qquad \text{where } \psi^{(m)}(z) - \frac{d^{m+1}}{dz^{m+1}}\ell n\Gamma(z).$$

$$B_s = B_s(0) = B_s(m), B_s^{'}(x) = sB_{s-1}(x), s = 1,2...,$$

The integrands of the integrals appearing in the expressions (4) and (5) consist of a polynomial multiplied by ℓnΓ(x+l). The smooth behaviour of these integrands over the finite range of integration (0,1) is such that they can be numerically evaluated without difficulty, and hence give a numerical value for the constant C_{2m} to a desired degree of accuracy. Further if integrals of the form

$$\int_0^1 x^{\Gamma} \ell n\Gamma(x+1)dx, \qquad r = 0,1,2......, \tag{6}$$

can be evaluated in closed form, then one can obtain explicit analytic expressions for the constants C_{2m}. The result (6) for r = 0:

$$\int_0^1 \ell n\Gamma(dx+1)dx = \frac{1}{2}\ell n(2\pi) - 1, \tag{7}$$

is well known, sometimes called Raabe's result. However, it does not seem to be known that (6) can be evaluated in closed form for non-negative integers in terms of the Riemann Zeta function ζ(z), see Olver[1]. Specifically:

$$\int_0^1 x^r \ell n\Gamma(x+1)dx = \frac{\ell n(2\pi)}{2(r+1)} - \frac{1}{(r+1)^2}$$
$$+\frac{1}{4\pi}\sum_{k=0}^{r-1}\frac{r!}{(r-k)!}\frac{\sin(\frac{1}{2}k\pi)}{(2\pi)^k}\zeta(k+2)$$
$$-\frac{(\gamma+\ell n(2\pi))}{2\pi^2}\sum_{k=0}^{r-1}\frac{r!}{(r-k)!}\frac{\cos(\frac{1}{2}k\pi)}{(2\pi)^k}\zeta(k+2)$$
$$-\frac{1}{2\pi^2}\sum_{k=0}^{r=1}\frac{r!}{(r-k)!}\frac{\cos(\frac{1}{2}k\pi)}{(2\pi)^k}\zeta'(k+2), \tag{8}$$
$$r = 0,1,2\ldots.$$

The derivation of the result (8) is as follows:

$$\int_0^1 x^r \ell n\Gamma(x+1)dx = \int_0^1 x^r \ell n x dx + \int_0^1 x^r \ell n\Gamma(x)dx,$$

$$= -\frac{1}{(r+1)^2} + \int_0^1 x^r \ell n\Gamma(x)dx.$$

We now use Kummer's Fourier Series representation for $\ell n\Gamma(x)$ given by

$$\ell n\Gamma(x) = \frac{1}{2}\ell n(2\pi(+ \sum_{n=1}^{\infty}\left\{\frac{1}{2n}\cos 2\pi\, nx \frac{1}{n\pi}(\gamma + \ell n(2\pi(2\pi n))\sin 2\pi nx.\right\}$$
$$0 < x < 1.$$

Thus

$$\int_0^1 x^r \ell n\Gamma(x+1)dx = \frac{\ell n(2\pi)}{2(r+1)} - \frac{1}{(r+1)^2} + \sum_{m=1}^{\infty}\frac{1}{2m}\int_0^1 x^r \cos 2\pi mx\ dx$$
$$+\sum_{m=1}^{\infty}\left\{\frac{(\gamma+\ell n(2\pi m))}{m\pi}\int_0^1 x^r \sin 2\pi mx\, dx\right\}$$

A simple application of integration by parts gives the results:

$$\int_0^1 x^r \cos 2\pi nx\, dx = \sum_{k=0}^{r-1} \frac{r!}{(r-k)!} \frac{1}{(2\pi n)^{k+1}} .\sin\left(\frac{1}{2}k\pi\right),$$

$$\int_0^1 x^r \sin 2\pi nx\ dx = -\sum_{k=0}^{r-1} \frac{1}{(r-k)!} \frac{1}{(2\pi n)^{k+1}} .\cos\left(\frac{1}{2}k\pi\right), \quad (9)$$

$$r=0,1\ldots$$

Thus the result (8) follows with $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}, z > 1, \zeta'(z) = \sum_{n=2}^{\infty} \ell n n. n^{-z}, z > 1$.

Thus in principle we can evaluate all terms of the expression (1) explicitly, in term of functions which are well tabulated.

Application

We shall now consider an application of the previous results to obtain an approximate expression for the sum $\sum_{r=1}^{n} r^{\ell} \ell nr$ where ℓ is an integer, and n→∞.

Let us consider therefore $f_\ell(x) = x^\ell \ell n\, x$, then

$$\frac{d^m}{dx^m} f_\ell(x) \equiv D^{(m)}(x^\ell \ell nx) = \sum_{r=0}^{m} \binom{m}{r} D^{(r)}(\ell nx) D^{(m-r)}(x^\ell),$$

$$= \ell! x^{\ell-m}\left(\frac{\ell nx}{\Gamma(\ell-m+1)} - \sum_{r=1}^{m} \frac{(-)^r}{r} . \frac{\{m(m-1)....(m-r+1)\}}{\Gamma(\ell-m+r+1)}\right) \quad (10)$$

If $m \geq \ell+1$ then $\frac{d^m}{dx^m} f_\ell(x) = -\ell! x^{\ell-m} \sum_{r=1}^{m} \frac{(-)^r}{r} \frac{\{m(m-1)....(m-r+1)\}}{\Gamma(\ell-m+r+1)}$.

If we choose: $m = 2m = \ell+2$, if ℓ even ,

$m = 2m = \ell+3$, if ℓ odd ,

we get

$$\frac{d^{2m}}{dx^{2m}} f(x) = \tilde{a}(\ell) x^{-2} \quad , \quad \ell \text{ even}$$

$$= \tilde{b}(\ell) x^{-3} \quad , \quad \ell \text{ odd.} \quad (11)$$

where

$$\tilde{a}(\ell) = -\ell! \sum_{r=2}^{\ell+2} \frac{(-)^r}{r} . \frac{\{(\ell+2)(\ell+1)....(\ell-r+3)\}}{(r-2)!} \quad (12)$$

$$\tilde{b}(\ell) = -\ell! \sum_{r=3}^{\ell+3} \frac{(-)^r}{r} \frac{\{(\ell+3)(\ell+2)....(\ell-r+4)\}}{(r-3)!} \quad (13)$$

Hence the expressions (1) and (4) give for ℓ even

$$\sum_{r=1}^{n} r^{\ell}\ell nr = \frac{n^{\ell-1}\ell nn}{\ell+1} - \frac{n^{\ell+1}}{(\ell+1)^2} + \frac{1}{2}n^{\ell}\ell nn$$

$$+\ell!\sum_{s=1}^{(\ell+2)/2}\frac{B_{2s}}{(2s)!}\left[\frac{\ell nn}{\Gamma(\ell-2s+2)} - \sum_{r=1}^{2S-1}\frac{(-)^r}{r}\frac{\{(2s-1)(2s-2)....(2s-r)\}}{\Gamma(\ell-2s+r+2)}\right]n^{\ell-2s+1}$$

$$+\widetilde{C}_{\ell+2} + 0(n^{-1}) \qquad (14)$$

Where

$$\widetilde{C}_{\ell+2} = \ell!\sum_{S=1}^{(\ell+2)/2}\frac{B_{2S}}{(2s)!}\left[\sum_{r=1}^{2S-1}\frac{(-)^r}{r}\frac{\{(2s-1)(2s-2)....(2s-r)\}}{\Gamma(\ell-2s+r+2)}\right] + \frac{1}{(\ell+1)^2}$$

$$+\frac{B_{\ell+2}}{(\ell+2)(\ell+1)}\sum_{r=2}^{\ell+2}\frac{(-)^r}{r}\frac{\{(\ell-2)(\ell-1)....(\ell-r+3)\}}{(r-2)!}$$

$$+\sum_{r=2}^{\ell+2}\frac{\{(\ell+2)(\ell+1)....(\ell-r+3)\}(-)^r}{r(r-2)!}\int_0^1 B_{\ell}(x)\ell n\Gamma(x+1)dx. \qquad (15)$$

The expression (1) and (5) give for ℓ odd

$$\sum_{r=1}^{n} r^{\ell}\ell nr = \frac{n^{\ell+1}}{(\ell+1)}\ell n\, n - \frac{n^{\ell+1}}{(\ell+1)^2} + \frac{1}{2}n^{\ell}\ell n\, n$$

$$+\ell!\sum_{s=1}^{(\ell+2)/2}\frac{B_{2S}}{(2s)!}\left[\frac{\ell n\, n}{\Gamma(\ell-2s+2)} - \sum_{r=1}^{2S-1}\frac{(-)^r}{r}\frac{\{(2s-1)(2s-2).....(2s-r)\}}{\Gamma(\ell-2s+r+2)}\right]n^{\ell-2S-1}$$

$$+\widetilde{C}_{\ell+3} + 0(n^{-2}) \qquad (16)$$

Where

$$\widetilde{C}_{\ell+3} = \frac{1}{(\ell+1)^2} + \ell!\sum_{S=1}^{(\ell+3)/2}\frac{B_{2S}}{(2s!)}\left[\sum_{r=1}^{2S-1}\frac{(-)^r}{r}\frac{\{(2s-1)(2s-2)....(2s-r)\}}{\Gamma(\ell-2s+2)}\right]$$

$$+\frac{\ell!B_{\ell+3}}{2(\ell+3)!}\sum_{r=3}^{\ell+3}\frac{(-)^r}{r}\frac{\{(\ell+3)(\ell+2)....(\ell-r+4)\}}{(r-3)!}$$

$$+\frac{1}{2}\sum_{r=3}^{\ell+3}\frac{(-)^r}{r}\frac{\{(\ell+3)(\ell+2)....(\ell-r+4)\}}{(r-3)!}\int_0^1 B_{\ell}(x)\ell n\,\Gamma(x+1)\,dx \qquad (17)$$

As specific examples we apply (14) and (15) for

(i) $\ell = 0$, m = 1, giving

$$\sum_{r=1}^{n} \ell n\, r = n\ell n\, n - n + \frac{1}{2}\ell n\, n + C + 0(n^{-1}),$$

$$C = 1 + \int_0^1 \ell n\Gamma(x+1)dx = 1 + \frac{1}{2}\ell n(2\pi) - 1,$$

$$= \frac{1}{2}\ell n(2\pi),$$

where we have used the result (8) with r = 0. This result agrees with Olver[l].

(ii) We also apply (16) and (17) for ℓ=l, m=2 giving

$$\sum_{r=1}^{n} r\ell n\, r = \frac{n^2}{2}\ell n\, n - \frac{n^2}{4} + \frac{1}{2} n\ell n\, n + \frac{1}{12}\ell n\, n + C + 0(n^{-2})$$

$$C + \frac{1}{4} - \int_0^1 (x - \frac{1}{2}) - \ell n\Gamma(x+1)dx.$$

$$= \frac{1}{4} + \frac{1}{4}\ell n(2\pi) - \frac{1}{2} - \int_0^1 x\ell n\Gamma\,(x+1)dx,$$

$$= \frac{(\gamma + \ell n(2\pi))}{12} - \frac{1}{2\pi^2}\zeta'(2),$$

where we have used the result (8) with r=0 and r=l and the fact that $\zeta(2) = \pi^{2}/6$. This result for C agrees with Olver[l] who obtained it by a different method.

References

F.W J. Olver, Asymptotics and Special functions. Academic Press 1974.