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# ASYMPTOTIC BEHAVIOUR OF THE SOLUTION OF A FUNCTIONAL-DIFFERENTIAL EQUATION.

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Department of Mathematics and Statistics, Brunei University.

Abstract

The asymptotic behaviour as $t\to\infty$ of the solution of the functional-differential equation $y'(t) = -y(t/k)$, with $y(0) = 1$ and $k > 1$, is derived from an integral representation by the method of steepest descents. It is shown that the solution oscillates (that is, has arbitrarily large zeros), that the amplitude of the oscillations growsfasterthan any polynomialbut slower thananyexponential, and that the ratios of successive zeros of the solution decrease to the limiting value k.

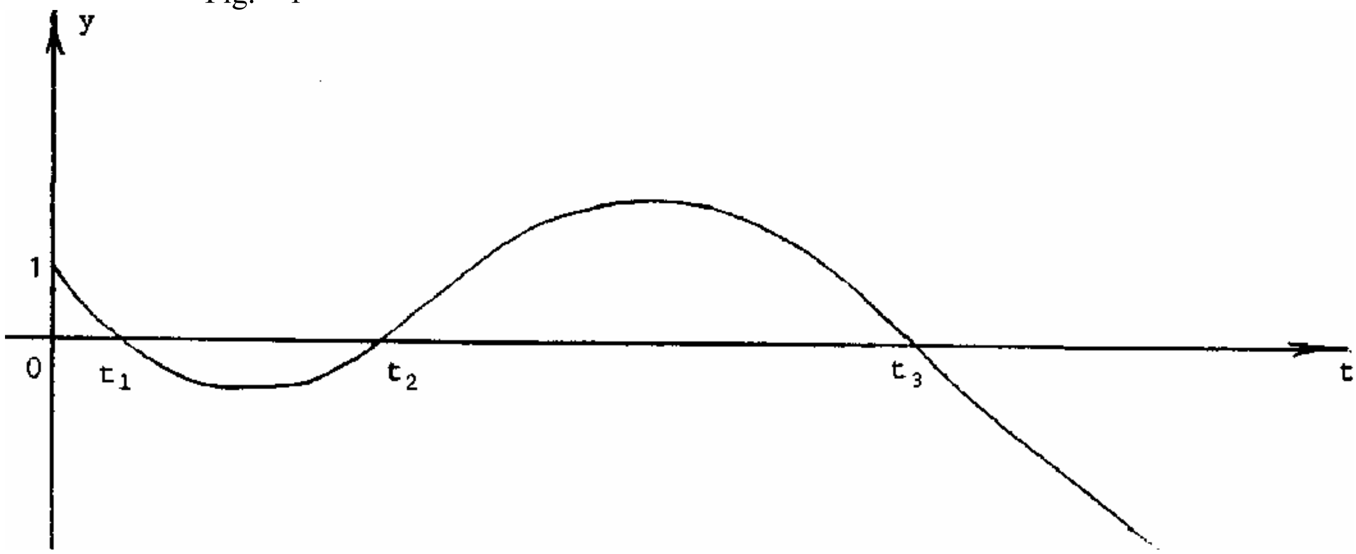
## Introduction

Feldstein and Grafton[1] investigated the solution of the functional-differential equation

$$y'(t) = -y(t/k) \quad , \quad y(0) = 1 \tag{1}$$

for $k > 1$ numerically for various values of $k$. The solution appeared to oscillate with increasing amplitude about the zero value as $t$ increases, and they calculated the first few zeros. From these values they conjectured that the ratios of successive zeros tends to the value $k$ , Later, in 1971, Fox et al[2] obtained a formula giving the rate of growth of solutions of equation (1) by a trialand error method, Feldstein and Grafton'sconjecture will be proved and the exact asymptotic formula for the rate of growth of solutions will be obtained. A sketch of what the solution of equation (1) looks like is shown in figure1.

Fig. 1




Solution of $y'(t) = -y(t/k), y(0) = 1, \ k > 1$

## An integral representation of the solution

The functional-differential equation (1) has the series solution

$$y(t) = \sum_{n=0}^{\infty} (-t)^n / n : k^{\frac{1}{2}n(n-1)} \tag{2}$$

absolutely convergent for all values of t and for k ≥ 1). This may be easily verified by substituting the series into the equation. From the theory of Fourier transforms we have the result

$$\exp(-\alpha s^2) = \frac{1}{2\sqrt{(\pi\alpha)}} \int_{-\infty}^{\infty} (\exp(-ius - u^2/4\alpha)du \tag{3}$$

valid for $\alpha > 0$ and all real values of s . Putting $\alpha = \frac{1}{2}\ell nk$ , $s = n$ and $u = iz$, we find

$$\frac{1}{k^{\frac{1}{2}n^2}} = \frac{1}{i\sqrt{(2\pi\ell nk)}} \int_{-i\infty}^{i\infty} \exp(nz + z^2 / \ell n k^2)\, dz \tag{4}$$

valid for k > 1 and all real values of n . Using this representation of $1/k^{\frac{1}{2}n^2}$ in the series solution (2) we have

$$y = \frac{1}{i\sqrt{(2\pi\ell n k)}} \int_{-i\infty}^{i\infty} \exp(z^2 / \ell n k^2) \sum_{n=0}^{\infty} \frac{(-tk^{\frac{1}{2}}e^z)^n}{n!} dz \tag{5}$$

where interchange of the order of summation and integration is justified since the resulting sum is absolutely convergent for all values of z . The series sums to $\exp(-t\sqrt{ke^z})$, so

$$y = \frac{1}{i\sqrt{2\pi\ell n\, k)}} \int_{-i\infty}^{i\infty} \exp(\,-t\sqrt{k}e^z \;+\; z^2 \;\; \ell n\, k^2\,)dz \tag{6}$$

We may obtain the asymptotic behaviour of y(t) as $t \to \infty$ from this integral representation, using the method of steepest descents.

## The method of steepest descents

We must examine the behaviour of the modulus of the integrand in (6) in the complex z-plane. This is equivalent to examining the real part of the exponent, so to this end we set

$$H(z) = -t\sqrt{k}e^z + z^2 / \ell n k^2 \tag{7}$$

We aim at deforming the contour $-i\infty \rightarrow i\infty$ in the complex z-plane until it follows paths of steepest descent on the ReH(z) surface from points where it reaches a local maximum. Since H(z) is an analytic function of z, ReH(z) is a harmonic function over the complex z-plane and so cannot have local maxima (or minima). So the places where ReH(z) reaches a local maximum on steepest paths must be saddle-points on the ReH(z) surface. As t increases, these local maxima on the steepest descent paths will become sharper and sharper, so that for large t the behaviour of the integral will be dominated by contributions to the integral from increasingly small neighbourhoods of the highest of these saddle-points.

So first we need to locate these saddle-points. Next we must determine how the path of integration can be deformed so as to pass through some or all of these saddle-points along steepest descent paths from them. This involves examining the global properties of the ReH(z) surface. Lastly, having established the relevant saddle-points, we evaluate the contribution from each of them to the integral. This involves the local behaviour of the integrand at the saddle-points.

Saddle-points on the ReH(z) surface

Saddle-points occur where $H'(z) = 0$, that is for values of z such that

$$-t\, ke^{z} + z/\ell n k = 0 \ .$$

We write this equation, for convenience, in the form

$$ze^{-z} = e^{\lambda} \ , \tag{8}$$

where

$$\lambda = \ell n(t\sqrt{k}\,\ell n k) \ . \tag{9}$$

Equation (8) has an infinity of complex solutions for z, which we estimate asymptotically for $\lambda \rightarrow \infty$ ($t \rightarrow \infty$) using the iterative procedure

$$z^{(k+1)} = -\lambda + \ell n z^{(k)} \ , \quad \text{with } z^{(0)} = -\lambda \ . \tag{10}$$

Neglecting order $\ell n^3\lambda/\lambda^3$ terms, we find for one root

$$z^{(1)} \sim -\lambda + \ell n\lambda + i\pi \ ,$$

$$z^{(2)} \sim -\lambda + \ell n\lambda + i\pi - \frac{\ell n\lambda}{\lambda} - \frac{i\pi}{\lambda} - \frac{\ell n^2\lambda}{2\lambda^2} - \frac{i\pi\ell n\lambda}{\lambda^2} + \frac{\pi^2}{2\lambda^2} \ ,$$

$$z^{(3)} \sim -\lambda + \ell n\lambda + i\pi - \frac{\ell n\lambda}{\lambda} - \frac{i\pi}{\lambda} - \frac{\ell n^2\lambda}{2\lambda^2} - \frac{i\pi\ell n\lambda}{\lambda^2} + \frac{\ell n\lambda}{\lambda^2} + \frac{i\pi}{\lambda^2} + \frac{\pi^2}{2\lambda^2} \ .$$

The terms of the iterate $z^{(4)}$ are the same as those of $z^{(3)}$ up to this order, so we have for one root (which we label $z_0^+$), as $\lambda \to \infty$:

$$z_0^+ = -\lambda + \ell n\lambda - \frac{\ell n\lambda}{\lambda} - \frac{\ell n^2\lambda}{2\lambda^2} + \frac{\ell n\lambda}{\lambda^2} + \frac{\pi^2}{2\lambda^2} + i\pi\left(1 - \frac{1}{\lambda} - \frac{\ell n\lambda}{\lambda^2} + \frac{1}{\lambda^2}\right) + 0\left(\frac{\ell n^3\lambda}{\lambda^3}\right) \quad (11)$$

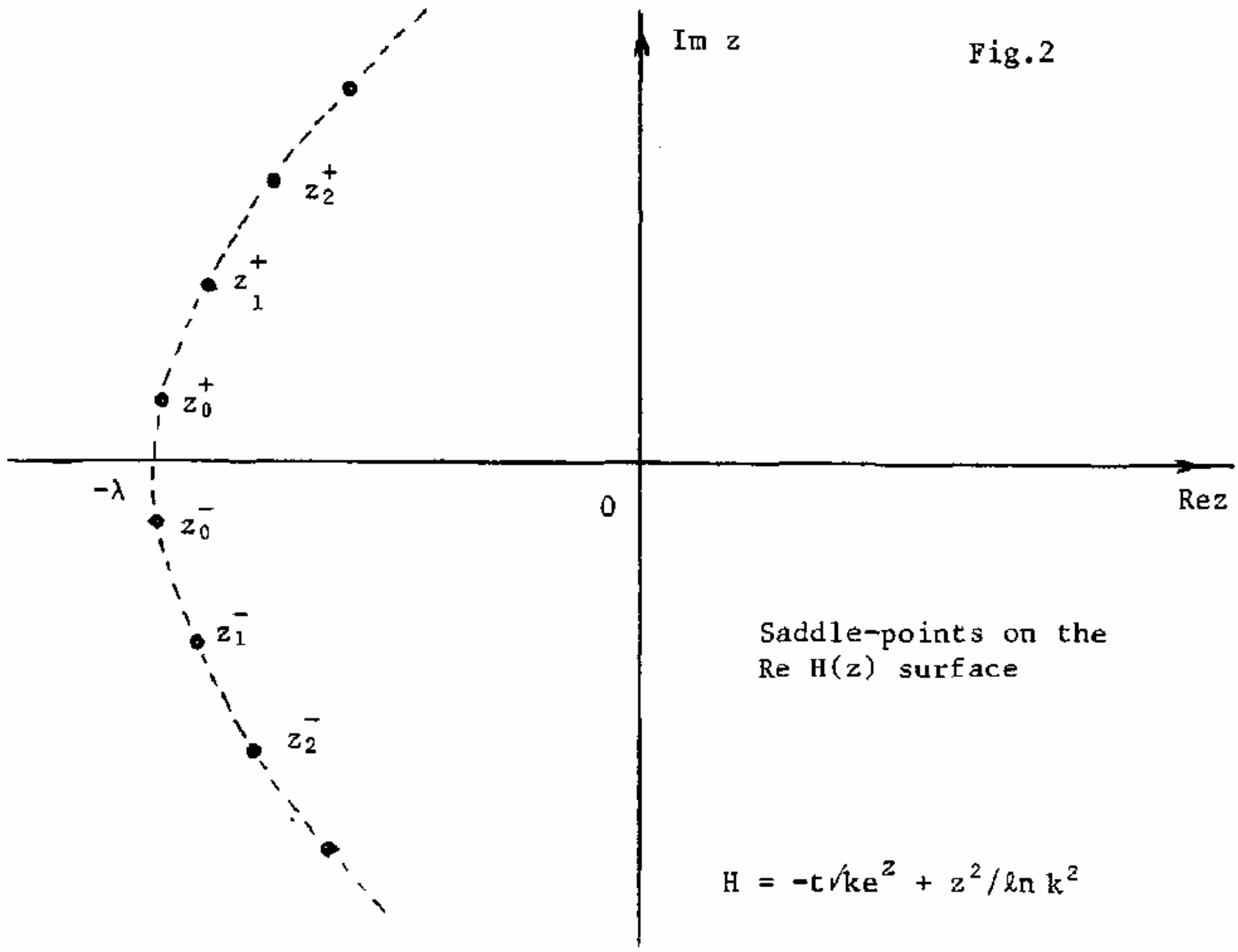
Other roots (which we label $z_m^{\pm}$) are determined by using other values of $\ell n(-\lambda)$, namely

$$\ell n\lambda \pm (2m+1)\pi i \text{ , } m = 0, 1, 2, ... \quad (12)$$

These other roots may be obtained from (11) by simply replacing $\pi$ by $\pm(2m+1)\pi$. This means that (at least for $\lambda >> m$) the roots vary with m approximately as

$$z_m^{\pm} \sim -A + (2m+1)^2 B \pm i(2m+1)c \quad , \quad (13)$$

where A,B and C are positive and depend essentially on $\lambda$ alone. Thus the complex roots of equation (8) not too far from the real axis lie (approximately) on a parabola in the complex z-plane, with its vertex near $-\lambda$ on the real axis (see figure 2).



At a saddle-point $z_m$ we have

$$H(z) = -t\sqrt{ke}^{\,z_m} + z_m^2/\ell n k^2 = z_m(2-z_m)/\ell n k^2 \quad ,$$

since $-t\sqrt{ke^z} = z/\ell n k^2$ for $z = z_m$. Using this expression, we obtain for the values of H(z) at the conjugate pair of saddle-points $z_m^+, z_m^-$ :

$$\ell n k^2 H(z) = \lambda^2 - 2\lambda\ell n\lambda + 2\lambda - 2\ell n\lambda + \ell n^2\lambda - (2m+1)^2\pi^2$$

$$\pm (2m+1)\pi i(-\lambda+\ell n\lambda-1) + o(1) \;,\; \lambda \to \infty, \qquad (14)$$

where $\lambda = \ell n(t\sqrt{k\ell n\, k})$ Thus conjugate pairs of saddle-points are at equal heights on the ReH(z) surface, since only even powers of ±(2m+1) will occur in the real part. Also, the pair nearest the real axis (the pair for which m = 0) are the highest. However, the pairs further from the real axis (for which |m| is larger) are only "numerically" lower (by the amount $-(2m+1)^2\pi^2$ approximately), that is, by an amount which is essentially independent of $\lambda$.

<u>Global behaviour on the ReH(z) surface</u>

Recall that $$H(z) = -t\sqrt{ke^z} + z^2/\ell n\, k^2 \qquad (7)$$

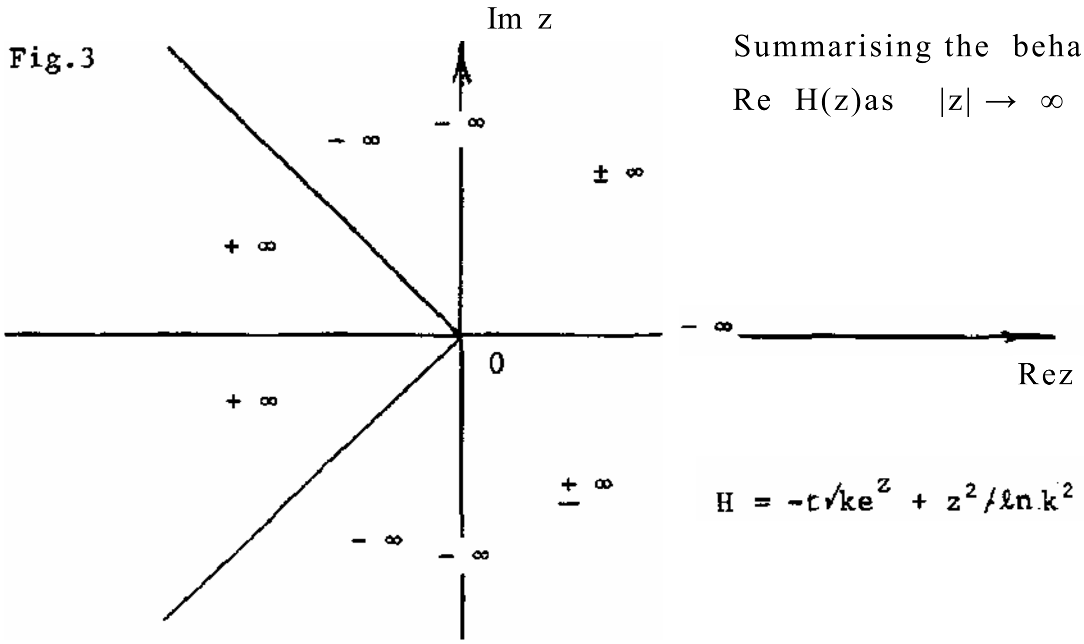
Firstly, we observe that H(z) is an entire function, that is, it has no singularities in the finite z-plane. This means that all steepest paths from saddle-points and all level contours on the ReH(z) surface must end at infinity. The steepest descent paths must go off to infinity in directions for which ReH(z) $\to$ as $|z| \to \infty$ , so first we establish these directions from the behaviour of ReH($re^{i\theta}$) as $r \to \infty$ for different values of $\theta$.
With $z = re^{i\theta}$ we have

$$ReH(re^{i\theta}) = -t\sqrt{ke^{r\cos\theta}}\cos(r\sin\theta) + \frac{1}{2}r^2\cos 2\theta/\ell n\, k \qquad (15)$$

(i) For <u>$\cos\theta \le 0$</u>, $ReH \sim \frac{1}{2}r^2\cos 2\theta\ell n\, k$ , so $ReH \to -\infty$ as $r \to \infty$ for $\cos 2\theta < 0$ i.e. $\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ or $\frac{5}{4}\pi < \theta < \frac{7}{4}\pi$ , which combined with $\frac{1}{2}\pi \le \theta \le \frac{3}{2}\pi$ yields $\frac{1}{2}\pi \le \theta < \frac{3}{4}\pi$ or $\frac{5}{4}\pi < \theta \le \frac{3}{2}\pi$.

(ii) <u>For $\cos\theta \le 0$</u>, $ReH \sim -t\sqrt{ke^{r\cos\theta}}\cos(r\sin\theta)$ , $r \to \infty$ . If $\theta$ is fixed ReH will oscillate with increasing amplitude as $r \to \infty$. However, if $\theta. = \arcsin(C/r)$, where C is a constant such that $-\frac{1}{2}\pi < C < \frac{1}{2}\pi$ , then $ReH \to \infty$ as $r \to \infty$ . So $ReH \to \infty$ as $r \to \infty$ along certain paths asympotic to the positive real axis. (In between these paths there will be paths along which $ReH \to +\infty$). In particular, $ReH \to \infty$ along the positive real axis.

Fig.3




Summarising the behaviour of Re H(z)as $|z| \rightarrow \infty$

$H = -t\sqrt{k}e^{z} + z^2/\ell n\, k^2$

The situation is summarised in figure 3.Thus steepest descent paths must go off to infinity in the sectors $\frac{1}{2}\pi \le \arg z < \frac{1}{2}\pi$ , $\frac{5}{4}\pi < \arg z \le \frac{3}{2}\pi$ , or along suitable paths asymptotic to the positive real axis. We are now in a position to sketch a contour map for the ReH(z) surface, and put in the steepest paths from the saddle-points. This is shown in figure 4. Only four saddle-points are shown.Other complex conjugate pairs of saddle-points may be added inexactly the same way as the outside pair have been incorporated into the contour map.

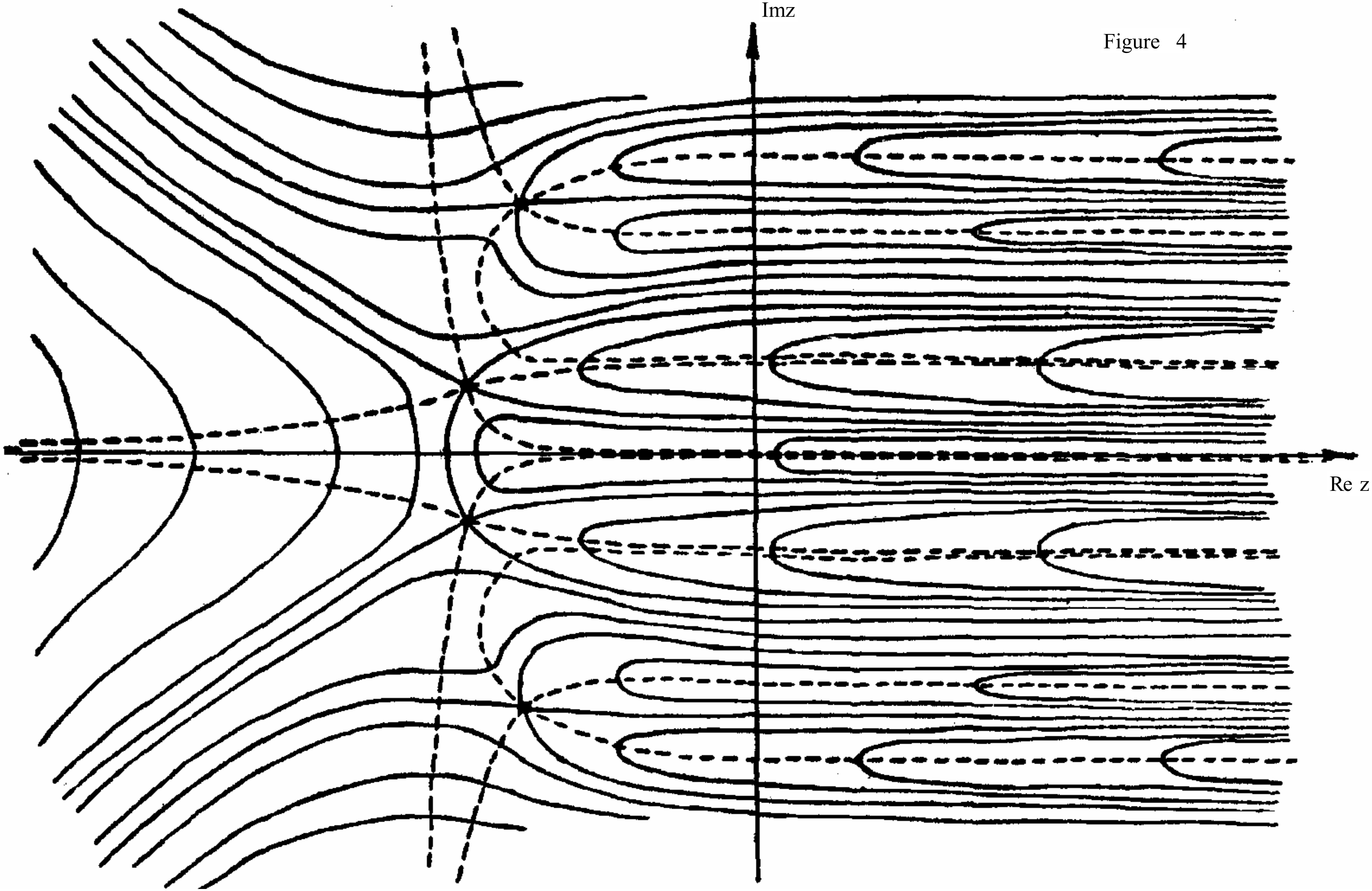



Figure 4

Level contours and steepest paths through saddle points on the $\mathrm{Re}(-t/\sqrt{k}e^{z} + z/\ell nk)$ surface

## Deformation,of the original path of integration

Since the integrand is an entire function of z , we have no singularities in the finite part of the plane to worry about. We can pull the $-i\infty$ to $i\infty$ contour along to the left until it passes through the two saddle-points nearest the real axis.Next we bend the path so that it goes off to infinity along steepest descent paths in the sectors $\frac{1}{2}\pi \leq \arg z < \frac{1}{2}\pi$ and $\frac{3}{4}\pi < \arg z \leq \frac{3}{2}$(Probably plumb in the middle of them), Finally, we pull it to the right along the real exis until the upper and lower parts of it go off to infinity along the steepest descent paths from the two saddle-points which are asymptotic to the positive real axis. It is very important to note that we cannot deform the path to go through any other pair of saddle-points along steepest descent paths from them,because we cannot continuously deform the contour to go along the appropriate path asymptotic to the positive real axis, since it is separated from its conjugate twin by indefinitely high ridges. This is important, because equation (14) shows that the contribution to the integral from other pairs of saddle-points is only "numerically"smaller (rather than asymptotically smaller) than the contribution from the pair nearest the real axis. The fact that it is impossible to deform the contour to pass along steepest paths through other saddle-points means we do not have totry and"add-up" any infinite sum of contributions, or worry about apparent arbitrariness concerning which pairof saddle-points we should be using.The required contour is shown in figure 5.

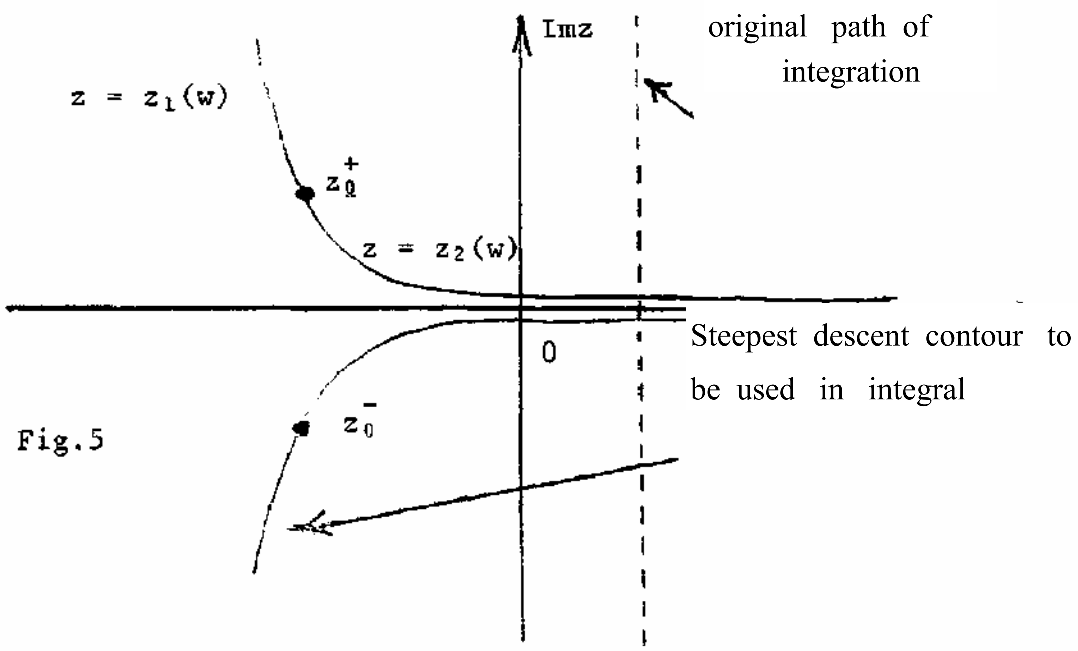



Fig.5

For large $\lambda$ (large t) the value of the integral will be dominated by the behaviour of the integrand in the neighbourhood of the saddle-points $z_0^{\pm}$.

Contributionto the integral from the neighbourhood of the saddle-points $z_0^+, z_0^-$.

Wemake thechangeofvariable

$$w = H(z_0^+) - H(z) \tag{16}$$

in the integral- Along the steepest descent paths from the saddle-point at $z_0^+$, ImH(z) is constant, so $H(z_0^+) - H(z)$ stays real and varies monotonically from zero (at the saddle-point) to $+\infty$ as $z \to \infty$ along the path in either direction. The two solutions of equation (16) for z as a function of the positive real variable w we denote by $Z_1(w)$ and $Z_2(w)$ (see figure 5).

As $w \to \infty$ $\arg Z_2 \to 0$ and $\arg z_1 \to$ a limit between $\frac{1}{2}$ and $\frac{3}{4}$ .Both $z_1(w)$ and $Z_2(w)$ will have expansions in powers of w of the form

$$z(w) = z_0^+ + \sum_{r=1}^{\infty} a_r w^{\frac{1}{2}r} \tag{17}$$

Denoting the part of the deformed contour in the upper half-plane by C , we have

$$\int_C e^{H(Z)} dz = e^{H(z_0^+)} \int_0^{\infty} e^{-w} \left\{ \frac{dz_1}{dw} - \frac{dz_2}{dw} \right\} dw \tag{18}$$

From (16), expanding H(z) about the point $z_0^+$, we have after cancelling the $H(z_0^+)$ terms and using the fact that $H'(z_0^+) = 0$,

$$w = -\frac{1}{2}(z - z_0^+)^2 \; H''(z_0^+) \; - \frac{1}{6}(z - z_0^+)^3 \; H'''(z_0^+) - \ldots \tag{19}$$

Inserting the expansion (17) for z in terms of w we get the identity

$$w = -\frac{1}{2} H''(z_0^+)\,(a_1 w^{\frac{1}{2}} + a_2 w + \ldots)^2 \; - \frac{1}{6} H'''(z_0^+)\,(a_1 w^{\frac{1}{2}} + \ldots)^3 - \ldots$$

Equating the coefficients of w gives

$$1 = -\frac{1}{2} H''(z_0^+) a_1^2$$

so that

$$a_1 = \pm \sqrt{\{-2/H''(z_0^+)\}} \tag{20}$$

To find which sign applies to $z_1$ and which to $Z_2$ in the expansion (17), we must examine the argument of the complex quantity $H''(z_0^+)$. Writing

$$H''(z_0^+) = Re^{i\phi}/nk \tag{21}$$

we find from (11)

$$R = \lambda + 1 + \frac{\ell n^2 \lambda}{2\lambda} + \frac{\pi^2}{2\pi} + 0\left(\frac{\ell n^3 \lambda}{\lambda^3}\right), \lambda \to \infty \tag{22}$$

$$\phi = \frac{-\pi}{\lambda} - \frac{\pi \ell n \lambda}{\lambda^2} + \frac{2\pi}{\lambda^2} + 0\left(\frac{\ell n^3 \lambda}{\lambda^3}\right) \quad ,\lambda \to \infty \quad . \tag{23}$$

Thus, taking only dominant terms,

$$H''(z_0^+) \sim \lambda e^{-i\pi/\lambda}/\ell n\, k, \lambda \to \infty$$

so that

$$a_1 \sim \surd\{2\ell n\, k/\lambda\} exp(\pm \frac{\pi}{2} i - \frac{\pi}{2\lambda} i) \ , \ \ \lambda \to \infty \tag{24}$$

Thus for $\lambda \to \infty$ and $w \to 0$

$$z(w) \sim z_0^+ + \surd\{2\ell n\ k/\lambda\} exp(\pm \frac{\pi}{2} i + \frac{\pi}{2\lambda} i) w^{\frac{1}{2}}$$

Thus one steepestdescentpathleaves $z_0^+$ at an angle $\frac{\pi}{2}(1+\lambda^{-1})$ (with reference to the real axis): this must be $z_2(w)$; the other leaves at an angle $-\frac{\pi}{2}(1-\lambda^{-1})$: this must be $z_1(w)$, So, with $a_1$ as defined in (20) we must take

$$z_1(w) \ = z_0^+ + \surd\{-2/H''(z_0^+)\} w^{\frac{1}{2}} + \ldots \quad ,$$

$$z_2(w) \ = z_0^+ - \surd\{-2/H''(z_0^+)\} w^{\frac{1}{2}} + \ldots \quad ,$$

Thus,

$$\frac{dz_1}{dw} \ \frac{dz_2}{dw} = \surd\{-2/H''(z_0^+) \ w^{-\frac{1}{2}} + \ldots$$

so that, from (18), using just the first term of the expansion as above,

$$\int_c e^{H(z)} dz \sim e^{H(z_0^+)} / \left\{\frac{-2}{H''(z_0^+)}\right\} \int_0^\infty e^{-w} w^{-\frac{1}{2}} dw \ , \ \ \lambda \to \infty \tag{25}$$

The rigorous proof that this does, indeed, produce the first term in an asymptotic expansion for the integral as $\lambda \to \infty$, relies on a generalisation of the usual "Laplace Method" or "Saddle-point method" which deals only with integrals of the form

$$\int_c e^{\lambda h(z)} dz \ , \quad \lambda \to \infty$$

in which h(z) is <u>independent</u> of $\lambda$ . This generalisation may be found in Evgrafoy[3] . Performing the integration in (25), we arrive at the result

$$\int_c e^{H(Z)} dz \sim e^{H(z_0^+)} \sqrt{\left\{\frac{-2\pi}{H''(z_0^+)}\right\}} \ , \ \lambda \to \infty \tag{26}$$

and we use this to estimate the contribution from the neighbourhood of the saddle-point at $z_0^+$

The work now just consists in evaluating the expression on the right hand side of (26) to a suitable order. If z is a saddle-point, where $H'(z) = 0$, we find, using $ze^{-z} \quad - e^{\lambda}$ ,

$$H = z(z-2)/\ell nk^2 \quad , \quad H'' = (1-z)/\ell nk \quad . \tag{27}$$

Writing $z = \alpha + i\beta$ and using (26) we get for the contribution to the original integral (6) for y(t) from a saddle-point at $\alpha + i\beta$

$$\exp\{(\alpha^2 2\alpha - \beta^2)/\ell n\, k^2 - \tfrac{1}{2}\ell nR\}\exp i\,\{-\tfrac{1}{2}\varphi - \beta(1-\alpha)/\ell n\, k\} \ , \tag{28}$$

where

$$\ell nR = \frac{1}{2}\ell n[(1-\alpha)^2 + \beta^2] \ , \ \varphi = \arctan(\beta rc - 1) \tag{29}$$

The saddle-points $z_0^{+}$ will produce complex conjugate contributions, and so, adding (28) and its complex conjugate we get the asymptotic estimate

$$2\exp\{\alpha^2 - 2\alpha + \beta^2)/\ell n\, k^2 - \tfrac{1}{2}\ell nR\}\cos\{\tfrac{1}{2}\varphi + \beta(1-\alpha)/\ell n\, k\} \ . \tag{30}$$

$\alpha$ and $\beta$ are the real and imaginary parts of the root $z_0^{+}$ given by (11), so using these expressions for $\alpha$ and $\beta$ in (30) we obtain, after some work, the asymptotic formula

$$y(t) \sim \exp\left\{\ell n2 - \frac{\pi^2}{\ell n\, k^2}\right\} \exp\left\{\frac{(\lambda - \ell n\lambda\lambda^2 + 2\lambda - \ell n\, k\ell n\lambda}{\ell n\, k^2}\right\}\cos\pi\left\{\frac{\lambda - \ell n\lambda}{\ell n\, k}\right\} \tag{31}$$

as $\lambda \to \infty$, where 0(1) terms have been neglected in the exponents. Replacing $\lambda$ by $\ell n(t\sqrt{k}\ell nk)$ in (31) gives the required asymptotic formula for y(t) as $t \to \infty$ ;-

$$y(t) \sim At^k \ (\ell nt)^h \ \exp\{(\ell nt - \ell n\ell nt)^2/\ell nk^2\}\cos\pi\{(\ell nt - \ell n\ell nt + \ell n\ell n\, k)/\ell n\, k + \tfrac{1}{2}\} \tag{32}$$

where

$$h = -1 - \ell n\ell n\, k/\ell nk \ ,$$

$$K - \frac{1}{2} + (1 + \ell n\ell n\, k)/\ell nk \ ,$$

$$A = 2\{\exp\,[\,(\ell n\ell n\, k)^2 - \pi^2 + 2\ell n\ell n\, k]/\ell n\, k^2 + \tfrac{2}{0}\ell n\, k + \tfrac{1}{2}\ell n\ell n\, k + \tfrac{1}{2}\}$$

Once again, o(1) terms have been neglected, since the next term in the main asymptotic expansion would contribute such terms.

<u>The large zeros of the solution</u>

Zeros of y(t) occur for values of t satisfying (asymptotically)

$$\lambda - \ell n\lambda = (n + \tfrac{1}{2})\ell n\, k \ , \tag{33}$$

where n is a large positive integer and $\lambda = \ell n(t\sqrt{k}\ell n\, k)$ . To solve this equation we use the iterative procedure

$$\lambda^{(p+1)} = \ell n\lambda^{(p)} + (n + \tfrac{1}{2})\ell n\, k \ , \text{ with } \lambda^{(0)} = 1 \tag{34}$$

Thus

$$\lambda^{(1)} = n\ell n\,k + \frac{3}{2\ell n\,k},$$

$$\lambda^{(2)} = n\ell n\,k + \ell nn + \frac{1}{2}\ell n\,k + \ell n\ell nk + 0(1)\,,$$

$$\lambda^{(3)} = n\ell n\,k + \ell nn + \frac{1}{2}\ell n\,k + \ell n\ell nk + 0(1)$$

The terms of the iterates $\lambda^{(2)}$ and $\lambda^{(3)}$ are thus the same up to o(1) terms, so, substituting $\lambda = \ell n(t\sqrt{k}\ell nk)$, we deduce that the large zeros of y(t) are given by

$$t_n \sim nk^n \quad , \quad n \to \infty \tag{35}$$

The asymptotic behaviour of the ratios of successive zeros is thus given by

$$t_{n+1}/t_n \sim (1+\frac{1}{n})k, \quad n \to 8 \quad , \tag{36}$$

confirming Feldstein and Graftons'[(1)] conjecture that $t_{n+1}/t_n \to k$ as $n \to \infty$. Furthermore, we see from formula (36) that the ratios <u>decrease</u> to the value k , consistent with the numerical results in Feldstein and Graftons' paper.

<u>Rate of growth of the solution</u>

The most rapidly varying factor in the asymptotic formula (32) for y(t) is

$$\exp\{\ell n^2 t/\ell nk^2\} \quad ,$$

so the amplitude of the oscillations grows faster than any polynomial in t , but slower than any exponential, as $t \to \infty$ .

Appendix: a check on the asymptotic formula (31).

As a check on the result we evaluate y'(t) and y(t/k) from formula (31), neglecting 0(1) terms. Observing that

$$\lambda'(t) = 1/t, \lambda(t/k) = \lambda(t) - \ell nk$$

we find after some calculation,

$$y'(t) \sim \lambda y(t)/t\ell nk \quad ,$$

$$y(t/k) \sim -\lambda y\,(t)/t\ell n\,k\,, \quad t \to \infty$$

consistently with equation (1).

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