Sequential Fusion Estimation for Multi-Rate Complex Networks with Uniform Quantization: A Zonotopic Set-Membership Approach

Zhongyi Zhao, Zidong Wang, and Lei Zou

***Abstract*—In this paper, the sequential fusion estimation problem is investigated for multi-rate complex networks (MRCNs) with uniformly quantized measurements. The process and measurement noises, which are unknown-yet-bounded (UYB), are restrained into a family of zonotopes, and the multiple sensors are allowed to have different sampling periods. To facilitate digital transmissions, the sensor measurements are uniformly quantized before being sent to the remote estimator. The purpose of this paper is to design a sequential set-membership estimator such that, in the simultaneous presence of UYB noises, multi-rate samplings, and uniform quantization effects, the estimation error (after each measurement update) is confined to a zonotope with minimum F-radius at each time instant. By introducing certain virtual measurements, the MRCNs are first transformed into single-rate ones exhibiting switching phenomenon. Then, by utilizing the properties of zonotopes, the desired zonotopes are derived that contain the estimation error dynamics after each measurement update. Subsequently, the gain matrices of the sequential estimator are derived by minimizing the F-radii of these zonotopes, and the uniform boundedness is analyzed for the F-radius of the zonotope containing the estimation error after all measurement updates. Furthermore, sufficient conditions are derived to ensure the existence of the desired uniform upper/lower bounds. Finally, an illustrate example is proposed to show the effectiveness of the proposed sequential fusion estimation method.**

***Index Terms*—Multi-rate complex networks, sequential fusion estimation, set-membership state estimation, unknown-yet-bounded noises, uniform quantization, zonotopes.**

I. INTRODUCTION

Dynamics analysis of complex networks (CNs) has long been an active research topic in systems and control community owing to the fact that CNs are particularly suitable in modelling large-scale systems made up of various coupled dynamic units. Examples of these large-scale systems include, but are not limited to, sensor networks, power grids and artificial neural networks. Till now, tremendous research interest has been drawn onto various dynamics analysis problems (e.g. stability, synchronization, consensus, pinning control and state estimation) for CNs, and a large number of excellent results have recently been published in the literature, see [7], [11], [12], [18], [19], [36], [37], [39], [40], [42], [43], [47] for some representative findings.

Most existing results concerning the CNs have implicitly assumed that the sampling rates of the network and its sensor measurements are the same, but this assumption is often unrealistic since the system components with diverse physical features might have inherently *different* sampling rate [13], [29], [41], [49], [51], [54], and this necessitates the need to study the so-called multi-rate CNs (MRCNs). On the other hand, the state estimation scheme for CNs has proven to be practically significant since the information of certain node states, which is crucial for accomplishing certain tasks, are often unavailable because of the huge network scale and restricted resources. So far, a great deal of research attention has been paid to the state estimation problem for CNs with many algorithms available in the literature, see e.g., [36], [43], [55] and the references therein.

The state estimation approaches for CNs can be roughly categorized into distributed and centralized ones where, for a distributed estimation scheme, the estimation is carried out on each node by using the local and neighboring sensing information. As for the centralized scheme, the measurement information of all nodes is collected by a central processing unit (the estimator) and then processed to generate the state estimates by augmenting the original state and the measurement into a unified vector. Until now, the centralized estimation schemes for CNs have attracted considerable research attention due to their capability in providing globally optimal estimates under certain performance criteria [37], [43], [55].

Multi-sensor information fusion (MSIF) has been well recognized as an effective state estimation technique for multi-sensor systems [3], [4], [9], [14]–[16], [38] with successful applications in guidance, target tracking, robotics, and integrated navigation [3]–[5], [31], [53]. For the centralized fusion that provides the state estimates by employing all original measurement information, one way is to augment the system measurements (also called *parallel fusion*) as discussed previously, and another more prevalent way is the so-called *sequential fusion* that aims to collect the measurement infor-

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mation by the central processing unit (fusion center) and then process the information in a sequential order.

Comparing to its parallel fusion counterpart, the sequential fusion method could achieve similar estimation accuracy yet with much higher computational efficiency [30], [31], [52], [53]. When it comes to CNs, the fusion estimation problem is especially important because of the large amount of sensors deployed and the demand of fusing sensor data for uncertainty reduction. It is worth noting that, to the best of the authors' knowledge, the fusion estimation problem for CNs has not received adequate research attention yet, let along the consideration of the sequential nature of the fusion scheme for mitigating computational complexity, and the main motivation of this paper is therefore to shorten such a gap.

Traditional fusion algorithms, which have been specifically developed to tackle random and/or energy-bounded noises, might be inapplicable to handle *unknown-yet-bounded* (UYB) noises that are frequently encountered in practical systems [8], [25], [27], [34], [48]. In this case, a particularly suitable way is to fuse the measurement information of the CNs based on the *set-membership* state estimation (SMSE) whose aim is to give a compact set containing the real system state at each time instant. Note that the SMSE problems have drawn much research interest for various complex systems undergoing UYB noises, see e.g. [8], [25], [26], [34] and the references therein.

Zonotopes, which are convex polytopes that can be represented as the Minkowski sum of finite line segments, have recently been well utilized in the SMSE problems because such zonotopes can be ideally employed as compact sets that restrain the system states [1], [2], [6], [20], [22]–[24], [26], [45], [46]. By using zonotopes in SMSE problems, we would be able to balance the estimation accuracy and the computational burden. Specifically, in calculating the Minkowski sum and linear transformation (two widely utilized operations in SMSE), the loss of accuracy could be avoided when using the zonotopic SMSE method [17], [24], [45]. Moreover, the order reduction technique of zonotopes could reduce the complexity of operations in a significant way [17], [26].

The phenomenon of signal quantization is a common occurrence in digital communication as a result of the limited transmission capacity of the digital channels. In the context of networked control systems, the impacts from signal quantizations onto the overall system performance have been extensively examined in the literature, and most results have been concerned with the uniform quantization scheme that appears very often in engineering practice, see [21], [35], [50] for some representative results. Nevertheless, pertaining to the sequential fusion estimation problem for MRCNs, the signal quantization issue has not received adequate research attention yet and this constitutes another motivation for our current investigation.

Summarizing the discussions made so far, in this paper, we are interested in dealing with the sequential fusion estimation problem for MRCNs suffering from UYB noises. In doing so, we are facing three substantial difficulties identified as follows: 1) how to deal with the complexities brought by the multi-rate sampling and the uniform quantization schemes in analyzing the estimation performance? 2) how to design the parameters of the desired sequential estimator in a recursive way? and 3) how to tackle the boundedness analysis problem of the F-radius of the zonotope confining the estimation error (after all measurement updates) for concerned MRCNs?

Corresponding to the challenges discussed above, the contributions of this paper are highlighted from the following four aspects: 1) the sequential fusion estimation problem is, for the first time, investigated for MRCNs under the framework of zonotopic SMSE; 2) the gain parameters of the sequential estimator are designed such that the F-radii of zonotopes confining estimation errors are minimized at each time instant; 3) a sequential fusion algorithm is proposed, which is implemented in a recursive manner and hence suitable for online applications; and 4) sufficient conditions are obtained to ensure that the F-radius of the zonotope confining the estimation error (after the last measurement update among the sequential processes) is uniformly bounded.

The remainder of this paper is organized as follows. In Section II, the sequential estimator is formulated for MRCNs with uniform quantization effects. In Section III, under the zonotopes-based fusion criterion, the zonotopes are first derived that restrain the estimation error dynamics after each measurement update, and the parameters of the sequential estimator are then designed. Moreover, the uniform boundedness of the estimation error after all measurement updates is analyzed. Section IV provides a numerical example. Finally, the conclusion is drawn in Section V.

Notations: $\mathbb{N}$ and $\mathbb{N}^+$ represent the set $\{0,1,2,\cdots\}$ and $\{1,2,3,\cdots\}$, respectively. $\mathbb{R}^{i_1\times i_2}$ is the set of $i_1\times i_2$ real matrices. $\mathbb{R}^{i_1}$ and $\mathbb{R}$ are special cases of $\mathbb{R}^{i_1\times i_2}$ with $i_2=1$ and $i_1=i_2=1$, respectively. I and 0 represent identity matrix and zero matrix of proper dimensions, respectively. $\mathrm{diag}\{*\}$ represents a block-diagonal matrix. For a column vector $\xi=\begin{bmatrix}\xi_1 & \cdots & \xi_n\end{bmatrix}^T\in\mathbb{R}^n$, $\mathrm{diag}_v\{\xi\}$ denotes the diagonal matrix $\mathrm{diag}\{\xi_1,\cdots,\xi_{\mathrm{n}}\}$. $\lambda_{\max}\{\cdot\}$ denote the maximum eigenvalue of the square matrix "$\cdot$". For a matrix X, $|X|$ represents the element-to-element absolute value operation. Y^{-1} and $\mathrm{tr}\{\mathrm{Y}\}$ represent the inverse and the trace of square matrix Y, respectively. Z^T refers to the transpose of matrix Z. $\mathbf{1}\triangleq\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$ is a column vector of proper dimension. For a vector $z\in\mathbb{R}^{n_z}$, $\|z\|_1$, $\|z\|_2$, and $\|z\|_\infty$ represent the 1-norm, 2-norm, and infinite norm of z, respectively. $\mathrm{mod}(\delta_1,\delta_2)$ stands for the remainder on division of δ_1 by δ_2 with δ_i $(i=1,2)$ being positive integers. For sets $\mathcal{H}_1,\mathcal{H}_2\subset\mathbb{R}^m$ and a matrix $H\in\mathbb{R}^{n\times m}$, one has $\mathcal{H}_1\oplus\mathcal{H}_2\triangleq\{h_1+h_2:h_1\in\mathcal{H}_1,h_2\in\mathcal{H}_2\}$ and $H\odot\mathcal{H}_1\triangleq\{Hh_1:h_1\in\mathcal{H}_1\}$, where "$\odot$" is granted a higher precedence than "$\oplus$". Given a center vector $h\in\mathbb{R}^n$ and a generator matrix $H\in\mathbb{R}^{n\times m}$, $\langle h,H\rangle\triangleq\{h+Hz: z\in\mathbb{R}^m,\ \|z\|_\infty\leq 1\}$ represents a zonotope of order m [44].

II. PRELIMINARIES AND PROBLEM FORMULATION

As is shown in Fig. 1, we consider the sequential fusion estimation problem for a class of MRCNs with uniformly quantized measurements. The considered MRCNs are assumed to have $\mathcal{N}$ nodes with the measurements of the first q nodes being available. The available measurement information is first uniformly quantized and then transmitted to a sequential estimator to generate the state estimates. In the following, we shall introduce respectively the MRCNs, the uniform quantization mechanism and the sequential estimator in details.

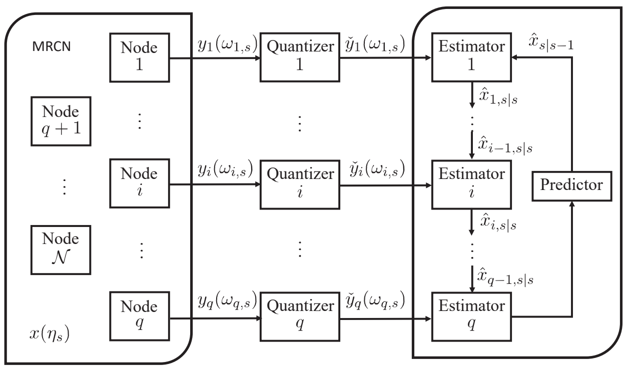

Fig. 1: Block diagram of the sequential fusion estimation problem for MRCN with uniform quantization.

A. System Model

Consider a class of MRCNs with $\mathcal{N}$ nodes, in which the i-th node has the following dynamics:

$$\begin{cases} x_i(\eta_{s+1}) = G_i(\eta_s)x_i(\eta_s) + \sum_{j=1}^{\mathcal{N}} A_{ij}(\eta_s)x_j(\eta_s) \\ \qquad\qquad + B_i(\eta_s)w_i(\eta_s) \\ z_i(\eta_s) = M_i(\eta_s)x_i(\eta_s) \\ x_i(\eta_0) \in \langle c_i(\eta_0), E_i(\eta_0)\rangle, \ i = 1, 2, \cdots, \mathcal{N} \end{cases} \tag{1}$$

where η_s is the s-th updating instant of the system state; $x_i(\eta_s) \in \mathbb{R}^{n_{x_i}}$ and $z_i(\eta_s) \in \mathbb{R}^{n_{z_i}}$ represent the state vector and the signal to be estimated, respectively; $x_i(\eta_0)$ is the initial condition which belongs to a known zonotope $\langle c_i(\eta_0), E_i(\eta_0)\rangle$ with center $c_i(\eta_0) \in \mathbb{R}^{n_{x_i}}$ and generator matrix $E_i(\eta_0) \in \mathbb{R}^{n_{x_i} \times n_{x_i}}$; $w_i(\eta_s) \in \mathbb{R}^{n_{w_i}}$ stands for the process noise; $G_i(\eta_s)$, $B_i(\eta_s)$ and $M_i(\eta_s)$ are known matrices with proper dimensions; and $A_{ij}(\eta_s)$ $(i, j = 1, 2, \cdots, \mathcal{N})$ are known matrices that characterize the mutual coupling among the MRCNs' nodes.

Without loss of generality, we assume that only the measurements from the first q $(q < \mathcal{N})$ nodes of the MRCNs (1) are accessible, where different nodes have different sampling rates. In this situation, the measurement of the i-th $(i \in \{1, 2, \cdots, q\})$ node is described as follows:

$$y_i(\omega_{i,s}) = C_i(\omega_{i,s})x_i(\omega_{i,s}) + D_i(\omega_{i,s})v_i(\omega_{i,s}) \tag{2}$$

where $\omega_{i,s}$ is the sampling time instant (dependent on the i-th node); $y_i(\omega_{i,s}) \in \mathbb{R}^{n_{y_i}}$ is the measurement output; $v_i(\omega_{i,s}) \in \mathbb{R}^{n_{v_i}}$ is the measurement noise; and $C_i(\omega_{i,s})$ and $D_i(\omega_{i,s})$ are known matrices of proper dimensions.

Assumption 1: The UYB process noise $w_i(\eta_s)$ satisfies

$$w_i(\eta_s) \in \langle 0, W_i(\eta_s)\rangle \tag{3}$$

with $W_i(\eta_s) \in \mathbb{R}^{n_{w_i} \times n_{w_i}}$ being a known matrix for $i = 1, 2, \cdots, \mathcal{N}$. Similarly, the UYB measurement noises $v_i(\omega_{i,s})$ $(i = 1, 2, \cdots, q)$ satisfy

$$v_i(\omega_{i,s}) \in \langle 0, V_i(\omega_{i,s})\rangle \tag{4}$$

with $V_i(\omega_{i,s}) \in \mathbb{R}^{n_{v_i} \times n_{v_i}}$ being a known matrix for $i = 1, 2, \cdots, q$.

Assumption 2: The updating period for system state is $h \triangleq \eta_{s+1} - \eta_s$, and the sampling period for the measurement output of the i-th node is $b_i h \triangleq \omega_{i,s+1} - \omega_{i,s}$, where b_i is a positive integer. In addition, $\eta_0 = 0$ and $\omega_{i,0} = \bar{\omega}_i \in \{0, h, 2h, \cdots\}$ $(i = 1, 2, \cdots, q)$.

According to Assumption 2, the sequence of sampling instants of node i can be denoted as

$$\mathcal{S}_i \triangleq \{\bar{\omega}_i + s b_i h : s = 0, 1, \cdots\}. \tag{5}$$

Remark 1: In many existing references concerning the multi-sensor multi-rate fusion, a common assumption is that the system and all sensors have the same initial sampling instants, i.e.,

$$\eta_0 = \omega_{1,0} = \omega_{2,0} = \cdots = \omega_{q,0} = 0.$$

This assumption, however, is often unrealistic especially for large-scale multi-sensor systems (CNs) because it is generally impossible to find a time instant at which the system state is updated while all sensor nodes are simultaneously sampled. In view of this, in Assumption 2, the initial sampling instants $\omega_{i,0}$ $(i = 1, 2, \cdots, q)$ are allowed to be different for different nodes. With the information of $\omega_{i,0}$ and the sampling period $b_i h$ of the i-th node, the set of sampling instants S_i can be obtained, which will then be used to convert the system (1) to a single-rate one.

B. Quantized Transmission

Let us first introduce the transmission model where the measurements $y_i(\omega_{i,s})$ $(i = 1, 2, \cdots, q)$ are quantized before being transmitted to the remote estimator. In this paper, we consider the effects of the uniform quantization mechanism.

For $y_i(\omega_{i,s})$, let its quantized signal be $\breve{y}_i(\omega_{i,s})$, i.e., $\breve{y}_i(\omega_{i,s}) = \mathscr{Q}_i(y_i(\omega_{i,s}))$, where $\mathscr{Q}_i(\cdot)$ represents the operation of the uniform quantization on signal "$\cdot$". When the saturation level in the quantization is sufficiently large, the quantized signal $\breve{y}_i(\omega_{i,s})$ can be modeled by

$$\breve{y}_i(\omega_{i,s}) = \mathscr{Q}_i(y_i(\omega_{i,s})) = \begin{bmatrix} \vartheta_i \mathscr{R}(\frac{y_i^{(1)}(\omega_{i,s})}{\vartheta_i}) \\ \vartheta_i \mathscr{R}(\frac{y_i^{(2)}(\omega_{i,s})}{\vartheta_i}) \\ \vdots \\ \vartheta_i \mathscr{R}(\frac{y_i^{(n_{y_i})}(\omega_{i,s})}{\vartheta_i}) \end{bmatrix} \tag{6}$$

where ϑ_i is the quantizing level; $y_i^{(l)}(\omega_{i,s})$ represents the l-th component of the vector $y_i(\omega_{i,s})$; and $\mathscr{R}(\cdot)$ stands for the function rounding a number to its nearest integer.

Denote $\Delta_i(\omega_{i,s}) \triangleq \breve{y}_i(\omega_{i,s}) - y_i(\omega_{i,s})$ as the quantization error. It follows from (6) that

$$\|\Delta_i(\omega_{i,s})\|_\infty \le \frac{\vartheta_i}{2}, \ i = 1, 2, \cdots, q. \tag{7}$$

For node i, define

$$\check{C}_i(\eta_s) \triangleq \begin{cases} C_i(\eta_s), & \text{if } \eta_s \in \mathcal{S}_i \\ 0_{n_{y_i} \times n_{x_i}}, & \text{otherwise} \end{cases},$$

$$\check{D}_i(\eta_s) \triangleq \begin{cases} D_i(\eta_s), & \text{if } \eta_s \in \mathcal{S}_i \\ 0_{n_{y_i} \times n_{v_i}}, & \text{otherwise} \end{cases},$$

$$\check{v}_i(\eta_s) \triangleq \begin{cases} v_i(\eta_s), & \text{if } \eta_s \in \mathcal{S}_i \\ 0_{n_{v_i} \times 1}, & \text{otherwise} \end{cases},$$

$$\check{\Delta}_i(\eta_s) \triangleq \begin{cases} \Delta_i(\eta_s), & \text{if } \eta_s \in \mathcal{S}_i \\ 0_{n_{y_i} \times 1}, & \text{otherwise} \end{cases}.$$

Then, the quantized output (6) can be rewritten as

$$\check{y}_i(\eta_s) = \check{C}_i(\eta_s)x_i(\eta_s) + \check{\Delta}_i(\eta_s) + \check{D}_i(\eta_s)\check{v}_i(\eta_s). \quad (8)$$

Define $x(\eta_s) \triangleq \begin{bmatrix} x_1^T(\eta_s) & x_2^T(\eta_s) & \cdots & x_{\mathcal{N}}^T(\eta_s) \end{bmatrix}^T$ and $z(\eta_s) \triangleq \begin{bmatrix} z_1^T(\eta_s) & z_2^T(\eta_s) & \cdots & z_{\mathcal{N}}^T(\eta_s) \end{bmatrix}^T$. By using (8), the original MRCNs (1) can be rewritten in the following compact form

$$\begin{cases} x(\eta_{s+1}) = \Big(G(\eta_s) + A(\eta_s)\Big)x(\eta_s) + B(\eta_s)w(\eta_s) \\ z(\eta_s) = M(\eta_s)x(\eta_s) \\ \check{y}_i(\eta_s) = \mathscr{C}_i(\eta_s)x(\eta_s) + \check{\Delta}_i(\eta_s) \\ \qquad + \check{D}_i(\eta_s)\check{v}_i(\eta_s), i = 1, 2, \cdots, q \\ x(\eta_0) \in \langle \check{c}_{0|0}, E_{0|0} \rangle \end{cases}, \quad (9)$$

where

$$A(\eta_s) \triangleq \begin{bmatrix} A_{11}(\eta_s) & \cdots & A_{1\mathcal{N}}(\eta_s) \\ \vdots & \ddots & \vdots \\ A_{\mathcal{N}1}(\eta_s) & \cdots & A_{\mathcal{N}\mathcal{N}}(\eta_s) \end{bmatrix},$$

$$G(\eta_s) \triangleq \mathrm{diag}\{G_1(\eta_s), G_2(\eta_s), \cdots, G_{\mathcal{N}}(\eta_s)\},$$

$$B(\eta_s) \triangleq \mathrm{diag}\{B_1(\eta_s), B_2(\eta_s), \cdots, B_{\mathcal{N}}(\eta_s)\},$$

$$M(\eta_s) \triangleq \mathrm{diag}\{M_1(\eta_s), M_2(\eta_s), \cdots, M_{\mathcal{N}}(\eta_s)\},$$

$$E_{0|0} \triangleq \mathrm{diag}\{E_1(\eta_0), E_2(\eta_0), \cdots, E_{\mathcal{N}}(\eta_0)\},$$

$$\check{c}_{0|0} \triangleq \begin{bmatrix} c_1^T(\eta_0) & c_2^T(\eta_0) & \cdots & c_{\mathcal{N}}^T(\eta_0) \end{bmatrix}^T,$$

$$w(\eta_s) \triangleq \begin{bmatrix} w_1^T(\eta_s) & w_2^T(\eta_s) & \cdots & w_{\mathcal{N}}^T(\eta_s) \end{bmatrix}^T,$$

$$\mathscr{C}_i(\eta_s) \triangleq \begin{bmatrix} \underbrace{0 \cdots 0}_{i-1} & \check{C}_i(\eta_s) & \underbrace{0 \cdots 0}_{q-i} & \underbrace{0 \cdots 0}_{\mathcal{N}-q} \end{bmatrix}.$$

Combining (3), (4) with the definition of zonotopes, we have

$$w(\eta_s) \in \langle 0, W(\eta_s) \rangle \quad (10)$$

with

$$W(\eta_s) \triangleq \mathrm{diag}\{W_1(\eta_s), W_2(\eta_s), \cdots, W_{\mathcal{N}}(\eta_s)\},$$

and

$$\check{v}_i(\eta_s) \in \langle 0, V_i(\eta_s) \rangle, \ i = 1, 2, \cdots, q \quad (11)$$

with $V_i(\eta_s) \triangleq 0_{n_{v_i} \times n_{v_i}}$ when $\eta_s \notin \mathcal{S}_i$. Moreover, when $\eta_s \in \mathcal{S}_i$, it follows from (7) that

$$\left\| \left(\frac{\vartheta_i}{2} \right)^{-1} \check{\Delta}_i(\eta_{s+1}) \right\|_\infty \leq 1,$$

which together with the definition of zonotopes gives rise to

$$\left(\frac{\vartheta_i}{2} \right)^{-1} \check{\Delta}_i(\eta_{s+1}) \in \langle 0, I \rangle \quad (12)$$

Remark 2: By using the pseudo measurement approach, we convert the MRCN (1)-(2) into a single-rate system. In such a conversion, one needs to judge whether the relationship $\eta_s \in \mathcal{S}_i$ holds or not, which can be easily checked by looking at

$$\mathrm{mod}(\eta_s - \bar{\omega}_i, b_i h) = 0 \wedge \eta_s \geq \bar{\omega}_i$$

where "$\wedge$" denotes the logical relationship "and". Note that the pseudo measurement approach has been widely utilized in converting multi-rate systems into single-rate systems. With this method, the state estimate can be obtained at each updating instant of the system state with avoidance of the augmentation of system state.

C. The Estimator

In this paper, the following sequential estimator is constructed for system (9):

$$\begin{cases} \hat{x}_{s+1|s} = \Big(G(\eta_s) + A(\eta_s)\Big)\hat{x}_{s|s} \\ \hat{x}_{1,s+1|s+1} = \hat{x}_{s+1|s} + K_{1,s+1}\tilde{y}_{1,s+1} \\ \tilde{y}_{1,s+1} = \check{y}_1(\eta_{s+1}) - \mathscr{C}_1(\eta_{s+1})\hat{x}_{s+1|s} \\ \hat{x}_{i,s+1|s+1} = \hat{x}_{i-1,s+1|s+1} \\ \qquad + K_{i,s+1}\tilde{y}_{i,s+1}, \ i = 2, 3, \cdots, q \\ \tilde{y}_{i,s+1} = \check{y}_i(\eta_{s+1}) - \mathscr{C}_i(\eta_{s+1})\hat{x}_{i-1,s+1|s+1} \\ \hat{x}_{s+1|s+1} = \hat{x}_{q,s+1|s+1} \\ \hat{z}_{s+1|s+1} = M(\eta_{s+1})\hat{x}_{s+1|s+1} \\ \hat{x}_{0|0} = \hat{c}_{0|0} \end{cases} \quad (13)$$

where $\hat{x}_{s+1|s}$, $\hat{x}_{i,s+1|s+1}$ and $\hat{x}_{s+1|s+1}$ are the prediction at time instant η_s, the estimate of $x(\eta_{s+1})$ after the i-th measurement update and the estimate of $x(\eta_{s+1})$ after the q-th measurement update, respectively; $\hat{z}_{s+1|s+1}$ is the estimate of $z(\eta_{s+1})$; $\hat{c}_{0|0}$ is a known vector; and $K_{i,s+1}$ $(i = 1, 2, \cdots, q)$ are the estimator parameters to be designed.

Let the one-step prediction error, the estimation error after the i-th measurement update, the estimation error after the q-th measurement update and the estimation error of the signal $z(\eta_{s+1})$ be $e_{s+1|s} \triangleq x(\eta_{s+1}) - \hat{x}_{s+1|s}$, $e_{i,s+1|s+1} \triangleq x(\eta_{s+1}) - \hat{x}_{i,s+1|s+1}$, $e_{s+1|s+1} \triangleq x(\eta_{s+1}) - \hat{x}_{s+1|s+1}$ and $\tilde{z}_{s+1|s+1} \triangleq z(\eta_{s+1}) - \hat{z}_{s+1|s+1}$, respectively. According to (9) and (13), we have

$$\begin{cases} e_{s+1|s} = \mathscr{A}_s e_{s|s} + B(\eta_s)w(\eta_s) \\ e_{1,s+1|s+1} = \Lambda_{1,s+1}e_{s+1|s} - K_{1,s+1}\check{\Delta}_1(\eta_{s+1}) \\ \qquad - K_{1,s+1}\check{D}_1(\eta_{s+1})\check{v}_1(\eta_{s+1}) \\ e_{i,s+1|s+1} = \Lambda_{i,s+1}e_{i-1,s+1|s+1} - K_{i,s+1}\check{\Delta}_i(\eta_{s+1}) \\ \qquad - K_{i,s+1}\check{D}_i(\eta_{s+1})\check{v}_i(\eta_{s+1}), \ i = 2, \cdots, q \\ e_{s+1|s+1} = e_{q,s+1|s+1} \\ \tilde{z}_{s+1|s+1} = M(\eta_{s+1})e_{s+1|s+1} \\ e_{0|0} \in \langle c_{0|0}, E_{0|0} \rangle \end{cases} \quad (14)$$

where

$$\mathscr{A}_s \triangleq G(\eta_s) + A(\eta_s),$$
$$\Lambda_{i,s+1} \triangleq I - K_{i,s+1}\mathscr{C}_i(\eta_{s+1}),$$
$$c_{0|0} \triangleq \check{c}_{0|0} - \hat{c}_{0|0}.$$

D. Problem Statement

Definition 1: Let a set of zonotopes

$$\mathcal{E} \triangleq \{\langle c_{i,s|s}, E_{i,s|s}\rangle : i = 1, 2, \cdots, q; s \in \mathbb{N}^+\}$$

be given. The estimation error system (14) is said to satisfy the $\mathcal{E}$-dependent constraint if

$$e_{i,s|s} \in \langle c_{i,s|s}, E_{i,s|s}\rangle$$

holds for all $s \in \mathbb{N}^+$ and $i = 1, 2, \cdots, q$.

Definition 2: [44] For a zonotope $\langle c, \Lambda\rangle \subset \mathbb{R}^n$, its F-radius is defined as

$$\|\Lambda\|_F \triangleq \sqrt{\mathrm{tr}\{\Lambda^T\Lambda\}}. \tag{15}$$

The objectives of this paper are to:

1) find a set of zonotopes $\mathcal{E} = \{\langle c_{i,s|s}, E_{i,s|s}\rangle : i = 1, 2, \cdots, q; s \in \mathbb{N}^+\}$ such that the estimation error system (14) satisfies the $\mathcal{E}$-dependent constraint;
2) minimize the F-radius of $\langle c_{i,s|s}, E_{i,s|s}\rangle$ by choosing appropriate estimator parameter $K_{i,s}$ for $i = 1, 2, \cdots, q$;
3) establish sufficient conditions ensuing that the F-radius of $\langle c_{q,s|s}, E_{q,s|s}\rangle$ is uniformly bounded.

III. MAIN RESULTS

The following lemma is useful for analyzing the $\mathcal{E}$-dependent constraint.

Lemma 1: [17] Let zonotopes $\langle \pi_1, \Pi_1\rangle, \langle \pi_2, \Pi_2\rangle \subset \mathbb{R}^n$ and a matrix $L \in \mathbb{R}^{l\times n}$ be given. The following relationships hold:

$$\langle \pi_1, \Pi_1\rangle \oplus \langle \pi_2, \Pi_2\rangle = \left\langle \pi_1 + \pi_2, \begin{bmatrix}\Pi_1 & \Pi_2\end{bmatrix}\right\rangle, \tag{16}$$
$$L \odot \langle \pi_1, \Pi_1\rangle = \langle L\pi_1, L\Pi_1\rangle, \tag{17}$$
$$\langle \pi_1, \Pi_1\rangle \subset \langle \pi_1, \mathrm{diag}_v\{|\Pi_1|\mathbf{1}\}\rangle. \tag{18}$$

A. Analysis on $\mathcal{E}$-Dependent Constraint

To analyze the $\mathcal{E}$-dependent constraint, we give the following theorem.

Theorem 1: Consider the system (9) and the sequential estimator (13) with given parameters $K_{i,s+1}$ $(i = 1, 2, \cdots, q)$. Assume that the estimation error $e_{s|s}$ satisfies

$$e_{s|s} \in \langle c_{s|s}, E_{s|s}\rangle. \tag{19}$$

Then, the one-step prediction error $e_{s+1|s}$, the estimation errors $e_{i,s+1|s+1}$ $(i = 1, 2, \cdots, q)$, $e_{s+1|s+1}$ and $\tilde{z}_{s+1|s+1}$ satisfy

$$\begin{aligned}&e_{s+1|s}\\ &\in \left\langle \mathscr{A}_s c_{s|s}, \begin{bmatrix}\mathscr{A}_s E_{s|s} & B(\eta_s)W(\eta_s)\end{bmatrix}\right\rangle\\ &\triangleq \langle c_{s+1|s}, E_{s+1|s}\rangle,\end{aligned} \tag{20}$$

$$\begin{aligned}&e_{1,s+1|s+1}\\ &\in \left\langle \Lambda_{1,s+1}c_{s+1|s}, \left[\Lambda_{1,s+1}E_{s+1|s}\right.\right.\\ &\quad \left.\left. -\tfrac{\vartheta_1}{2}K_{1,s+1} \quad -K_{1,s+1}\check{D}_1(\eta_{s+1})V_1(\eta_{s+1})\right]\right\rangle\\ &\triangleq \langle c_{1,s+1|s+1}, E_{1,s+1|s+1}\rangle,\end{aligned} \tag{21}$$

$$\begin{aligned}&e_{i,s+1|s+1}\\ &\in \langle \Lambda_{i,s+1}c_{i-1,s+1|s+1}, [\Lambda_{i,s+1}E_{i-1,s+1|s+1}\\ &\quad -\tfrac{\vartheta_i}{2}K_{i,s+1} \quad -K_{i,s+1}\check{D}_i(\eta_{s+1})V_i(\eta_{s+1})]\rangle\\ &\triangleq \langle c_{i,s+1|s+1}, E_{i,s+1|s+1}\rangle,\ i = 2, 3, \cdots, q,\end{aligned} \tag{22}$$

$$\begin{aligned}&e_{s+1|s+1}\\ &\in \langle c_{q,s+1|s+1}, E_{q,s+1|s+1}\rangle\\ &\triangleq \langle c_{s+1|s+1}, E_{s+1|s+1}\rangle,\end{aligned} \tag{23}$$

$$\begin{aligned}&\tilde{z}_{s+1|s+1}\\ &\in \langle M(\eta_{s+1})c_{s+1|s+1}, M(\eta_{s+1})E_{s+1|s+1}\rangle\\ &\triangleq \mathcal{Z}_{s+1}.\end{aligned} \tag{24}$$

Proof: In this proof, we aim to show (20)-(24) based upon (19).

It follows from (10), (14) and (19) that

$$\begin{aligned}e_{s+1|s} &= \mathscr{A}_s e_{s|s} + B(\eta_s)w(\eta_s)\\ &\in \mathscr{A}_s \odot \langle c_{s|s}, E_{s|s}\rangle \oplus B(\eta_s) \odot \langle 0, W(\eta_s)\rangle.\end{aligned} \tag{25}$$

Applying (16)-(17) to (25), we have (20) readily. Furthermore, in light of (12) and (17), we obtain

$$\begin{aligned}&\check{\Delta}_i(\eta_{s+1})\\ &= \left(\frac{\vartheta_i}{2}I\right)\left(\frac{\vartheta_i}{2}\right)^{-1}\check{\Delta}_i(\eta_{s+1})\\ &\in \left\langle 0, \frac{\vartheta_i}{2}I\right\rangle.\end{aligned} \tag{26}$$

It follows from (17) and (26) that

$$\begin{aligned}&-K_{1,s+1}\check{\Delta}_1(\eta_{s+1})\\ &\in (-K_{1,s+1}) \odot \left\langle 0, \frac{\vartheta_1}{2}I\right\rangle\\ &= \left\langle 0, -\frac{\vartheta_1}{2}K_{1,s+1}\right\rangle.\end{aligned} \tag{27}$$

With (11), (14) and (27) in mind, we obtain that

$$\begin{aligned}e_{1,s+1|s+1} &\in \Lambda_{1,s+1} \odot \left\langle c_{s+1|s}, E_{s+1|s}\right\rangle \oplus \left\langle 0, -\frac{\vartheta_1}{2}K_{1,s+1}\right\rangle\\ &\quad \oplus (-K_{1,s+1}\check{D}_1(\eta_{s+1})) \odot \langle 0, V_1(\eta_{s+1})\rangle\\ &= \left\langle c_{1,s+1|s+1}, E_{1,s+1|s+1}\right\rangle,\end{aligned} \tag{28}$$

which is consistent with (21). Similarly, (22) can be obtained easily.

Utilizing (14) and (17) again, we see that (23)-(24) are true, and the proof is now complete. ■

In the following, based on Theorem 1, we proceed to give zonotopes with which the estimation error $e_{i,s|s}$ satisfies the $\mathcal{E}$-dependent constraint.

Theorem 2: Consider the system (9) and the sequential estimator (13) with given parameters $K_{i,s+1}$ $(i = 1, 2, \cdots, q)$. Let the sequence of zonotopes $\mathcal{Z}_s$ $(s \in \mathbb{N}^+)$ be given by

$$\mathcal{Z}_s = \left\langle M(\eta_s)c_{s|s}, M(\eta_s)E_{s|s}\right\rangle, \tag{29}$$

$$\langle c_{s|s}, E_{s|s}\rangle = \left\langle c_{q,s|s}, E_{q,s|s}\right\rangle, \tag{30}$$

$$\langle c_{i,s|s}, E_{i,s|s}\rangle = \Big\langle \Lambda_{i,s}c_{i-1,s|s}, [\Lambda_{i,s}E_{i-1,s|s} \quad -\tfrac{\vartheta_i}{2}K_{i,s} \quad -K_{i,s}\check{D}_i(\eta_s)V_i(\eta_s)]\Big\rangle,\ i=q,\cdots,2, \tag{31}$$

$$\langle c_{1,s|s}, E_{1,s|s}\rangle = \Big\langle \Lambda_{1,s}c_{s|s-1}, [\Lambda_{1,s}E_{s|s-1} \quad -\tfrac{\vartheta_1}{2}K_{1,s} \quad -K_{1,s}\check{D}_1(\eta_s)V_1(\eta_s)]\Big\rangle, \tag{32}$$

$$\langle c_{s|s-1}, E_{s|s-1}\rangle = \Big\langle \mathscr{A}_{s-1}c_{s-1|s-1}, \big[\mathscr{A}_{s-1}E_{s-1|s-1} \quad B(\eta_{s-1})W(\eta_{s-1})\big]\Big\rangle \tag{33}$$

with given initial condition $\langle c_{0|0}, E_{0|0}\rangle$. Then, the estimation error system (14) satisfies the $\mathcal{E}$-dependent constraint. Moreover, $\tilde{z}_{s|s} \in \mathcal{Z}_s$ holds for all $s \in \mathbb{N}$.

Proof: In this proof, we first use mathematical induction to prove that the estimation error system (14) satisfies the $\mathcal{E}$-dependent constraint. That is,

$$e_{i,s|s} \in \langle c_{i,s|s}, E_{i,s|s}\rangle$$

holds for all $i \in \{1,2,\cdots,q\}$ and $s \in \mathbb{N}^+$.

When $s=1$, with the initial condition $e_{0|0} \in \langle c_{0|0}, E_{0|0}\rangle$ and (30)-(33), we know from Theorem 1 that $e_{i,1|1} \in \langle c_{i,1|1}, E_{i,1|1}\rangle$ is true for $i = 1,2,\cdots,q$. Assume that $e_{i,s|s} \in \langle c_{i,s|s}, E_{i,s|s}\rangle$ is satisfied at time instant s. Similarly, we can obtain from Theorem 1 and (30)-(33) that $e_{i,s+1|s+1} \in \langle c_{i,s+1|s+1}, E_{i,s+1|s+1}\rangle$ holds for $i=1,2,\cdots,q$, which implies that the estimation error system (14) satisfies the $\mathcal{E}$-dependent constraint.

After proving that the estimation error system (14) satisfies the $\mathcal{E}$-dependent constraint, it follows from (14) and (30) that

$$e_{s|s} \in \langle c_{s|s}, E_{s|s}\rangle, \forall s \in \mathbb{N}. \tag{34}$$

According to (34) and taking (14) and (17) into account, we can easily obtain that $\tilde{z}_{s|s} \in \mathcal{Z}_s$ holds for all $s \in \mathbb{N}$. This ends the proof. ■

Remark 3: In Theorem 1, based on the condition that the estimation error $e_{s|s}$ resides within a known zonotope, we obtain zonotopes containing the one-step prediction error $e_{s+1|s}$, confining estimation errors $e_{i,s+1|s+1}$ $(i=1,2,\cdots,q)$, and restraining the estimation errors $e_{s+1|s+1}$ and $\tilde{z}_{s+1|s+1}$. Resting on Theorem 1, we further give zonotopes ensuring that the estimation error system (14) satisfies the $\mathcal{E}$-dependent constraint. It should be pointed out that the generator matrix of the zonotope $\langle c_{i,s+1|s+1}, E_{i,s+1|s+1}\rangle$ is closely related to the quantization level ϑ_i. Generally speaking, the F-radius of the zonotope $\langle c_{i,s+1|s+1}, E_{i,s+1|s+1}\rangle$ would become greater with the increase of ϑ_i.

B. Design of Sequential Estimator Parameters

In this subsection, we shall deal with the estimator design problem.

Theorem 3: Assume that the parameter $K_{i,s+1}$ of the sequential estimator (13) is designed as

$$K_{i,s+1} = Q_{i-1,s+1}\mathscr{C}_i^T(\eta_{s+1})\Phi_{i,s+1}^{-1} \tag{35}$$

where

$$Q_{i-1,s+1} \triangleq \begin{cases} E_{i-1,s+1|s+1}E_{i-1,s+1|s+1}^T, & i \geq 2 \\ E_{s+1|s}E_{s+1|s}^T, & i=1 \end{cases},$$

$$\Phi_{i,s+1} \triangleq \mathscr{C}_i(\eta_{s+1})Q_{i-1,s+1}\mathscr{C}_i^T(\eta_{s+1}) + \frac{\vartheta_i^2}{4}I + \check{D}_i(\eta_{s+1})V_i(\eta_{s+1})V_i^T(\eta_{s+1})\check{D}_i^T(\eta_{s+1}).$$

Then, the estimation error system (14) satisfies the $\mathcal{E}$-dependent constraint. Moreover, the F-radius of the zonotope $\langle c_{i,s+1|s+1}, E_{i,s+1|s+1}\rangle$ is minimized.

Proof: It is easy to see from Theorem 2 that, with the estimator parameter (35), the estimation error system (14) satisfies the $\mathcal{E}$-dependent constraint. Hence, it remains to show that the estimator parameter (35) minimizes the F-radius of the zonotope $\langle c_{i,s+1|s+1}, E_{i,s+1|s+1}\rangle$. From (31), we have

$$\begin{aligned}
&\|E_{i,s+1|s+1}\|_F^2 \\
&= \mathrm{tr}\Big\{\Lambda_{i.s+1}Q_{i-1,s+1}\Lambda_{i.s+1}^T + K_{i,s+1}\Big(\frac{\vartheta_i^2}{4}I \\
&\quad + \check{D}_i(\eta_{s+1})V_i(\eta_{s+1})V_i^T(\eta_{s+1})\check{D}_i^T(\eta_{s+1})\Big)K_{i,s+1}^T\Big\} \\
&= \mathrm{tr}\Big\{K_{i,s+1}\Phi_{i,s+1}K_{i,s+1}^T - K_{i,s+1}\mathscr{C}_i(\eta_{s+1})Q_{i-1,s+1} \\
&\quad - Q_{i-1,s+1}\mathscr{C}_i^T(\eta_{s+1})K_{i,s+1}^T + Q_{i-1,s+1}\Big\}.
\end{aligned} \tag{36}$$

Applying the completion-of-the-square method to (36) gives

$$\begin{aligned}
&\|E_{i,s+1|s+1}\|_F^2 \\
&= \mathrm{tr}\Big\{\Big(K_{i,s+1}^T - \Phi_{i,s+1}^{-1}\mathscr{C}_i(\eta_{s+1})Q_{i-1,s+1}\Big)^T\Phi_{i,s+1} \\
&\quad \times \Big(K_{i,s+1}^T - \Phi_{i,s+1}^{-1}\mathscr{C}_i(\eta_{s+1})Q_{i-1,s+1}\Big) \\
&\quad - Q_{i-1,s+1}\mathscr{C}_i^T(\eta_{s+1})\Phi_{i,s+1}^{-1}\mathscr{C}_i(\eta_{s+1})Q_{i-1,s+1} \\
&\quad + Q_{i-1,s+1}\Big\},
\end{aligned} \tag{37}$$

which implies that the parameter given by (35) indeed minimizes the F-radius of $\langle c_{i,s+1|s+1}, E_{i,s+1|s+1}\rangle$. The proof is now complete. ■

C. Sequential Fusion Estimation Algorithm

As a summary of obtained results on the analysis of the $\mathcal{E}$-dependent constraint and the design of sequential estimator parameters, a sequential fusion estimation algorithm is proposed in Algorithm 1.

Remark 4: It can be seen from Theorem 2 that, with the execution of Algorithm 1, the number of columns of $E_{q,s+1|s+1}$ increases steadily. If not handled properly, such an increase would result in heavy computational burden. To deal with this issue, in Algorithm 1, we adopt the order reduction technique (see (18) of Lemma 1). The essence of this technique is to utilize a low-order zonotope to contain a high-order zonotope at the cost of sacrificing certain accuracy. With the order reduction technique, the required number of floating-point-operations of Algorithm 1 is bounded by $\mathcal{O}\left(\left(\sum_{i=1}^{\mathcal{N}} n_{x_i}\right)^2\left(M_q + \sum_{i=1}^{\mathcal{N}} n_{w_i} + \sum_{i=1}^{q}(n_{y_i}+n_{v_i})\right)\right)$. Though it provides an effective way of saving computational cost, the order reduction technique of zonotopes would render the boundedness analysis of F-radius of $\langle c_{q,s|s}, E_{q,s|s}\rangle$ more difficult, and this motivates our further investigation in the next subsection.

Algorithm 1: Sequential fusion estimation algorithm

Input: Initial conditions $\hat{x}_{0|0}$, $\langle c_i(\eta_0), E_i(\eta_0)\rangle$, $(i = 1, 2, \cdots, \mathcal{N})$.

Output: $\hat{z}_{s+1|s+1}$, $\bar{z}_{s+1}$, $\underline{z}_{s+1}$.

1 **Initialization**: Give the maximum simulation times $s_{\max}$, the positive integer M_q, the quantizing levels ϑ_i $(i = 1, 2, \cdots, q)$, the zonotope $\langle c_{0|0}, E_{0|0}\rangle$. Set $s = 0$, $\bar{x}_0 = \hat{x}_{0|0} + c_{0|0} + |E_{0|0}|\mathbf{1}$ and $\underline{x}_0 = \hat{x}_{0|0} + c_{0|0} - |E_{0|0}|\mathbf{1}$

2 **for** $s \le s_{\max}$ **do**

3   Calculate $\hat{x}_{s+1|s}$ and $\langle c_{s+1|s}, E_{s+1|s}\rangle$ by (13) and (33), respectively. Set $i = 1$;

4   **for** $i \le q$ **do**

5   Obtain $K_{i,s+1}$ by (35) ;

6   Update $\hat{x}_{i,s+1|s+1}$ by (13) ;

7   **if** $i = 1$ **then**

8   calculate $c_{i,s+1|s+1}$ and $E_{i,s+1|s+1}$ by (32) ;

9   **else**

10   calculate $c_{i,s+1|s+1}$ and $E_{i,s+1|s+1}$ by (31) ;

11   **if** *the number of columns of* $E_{q,s+1|s+1}$ *is greater than* M_q **then**

12   set $E_{q,s+1|s+1} = \mathrm{diag}_v\{|E_{q,s+1|s+1}|\mathbf{1}\}$;

13   Output $\hat{x}_{s+1|s+1} = \hat{x}_{q,s+1|s+1}$ and $\langle c_{s+1|s+1}, E_{s+1|s+1}\rangle = \langle c_{q,s+1|s+1}, E_{q,s+1|s+1}\rangle$;

14   Output $\hat{z}_{s+1|s+1} = M(\eta_{s+1})\hat{x}_{s+1|s+1}$ and $\langle M(\eta_{s+1})c_{s+1|s+1}, M(\eta_{s+1})E_{s+1|s+1}\rangle$;

15   Output the "bounds" $\bar{z}_{s+1}$ and $\underline{z}_{s+1}$ by $\bar{z}_{s+1} = M(\eta_{s+1})c_{s+1|s+1} + |M(\eta_{s+1})E_{s+1|s+1}|\mathbf{1}$ and $\underline{z}_{s+1} = M(\eta_{s+1})c_{s+1|s+1} - |M(\eta_{s+1})E_{s+1|s+1}|\mathbf{1}$, respectively ;

D. Boundedness Analysis of F-radius of $\mathcal{Z}_s$

In the following, we shall consider the boundedness of the F-radius of $\mathcal{Z}_s = \langle M(\eta_s)c_{s|s}, M(\eta_s)E_{s|s}\rangle$ (calculated by Algorithm 1).

For convenience, we introduce the following notations:

$$
\begin{aligned}
\mathrm{rs}\{E_{q,s|s}\} &\triangleq \mathrm{diag}_v\{|E_{q,s|s}|\mathbf{1}\},\\
\mathcal{Q}_{q,s} &\triangleq \mathrm{rs}\{E_{q,s|s}\}\left(\mathrm{rs}\{E_{q,s|s}\}\right)^T,\\
\mathscr{V}_{i,s} &\triangleq V_i(\eta_s)V_i^T(\eta_s),\\
\mathscr{V}_s &\triangleq \mathrm{diag}\{\mathscr{V}_{1,s}, \mathscr{V}_{2,s}, \cdots, \mathscr{V}_{q,s}\},\\
\Psi_{i,s} &\triangleq \frac{\vartheta_i^2}{4}I + \check{D}_i(\eta_s)\mathscr{V}_{i,s}\check{D}_i^T(\eta_s),\\
\Upsilon &\triangleq \frac{1}{4}\mathrm{diag}\{\vartheta_1^2 I_{n_{y_1}}, \vartheta_2^2 I_{n_{y_2}}, \cdots, \vartheta_q^2 I_{n_{y_q}}\},\\
\mathscr{D}_s &\triangleq \mathrm{diag}\{\check{D}_1(\eta_s), \check{D}_2(\eta_s), \cdots, \check{D}_q(\eta_s)\},\\
\mathscr{C}_s &\triangleq \begin{bmatrix}\mathscr{C}_1^T(\eta_s) & \mathscr{C}_2^T(\eta_s) & \cdots & \mathscr{C}_q^T(\eta_s)\end{bmatrix}^T,\\
n_x &\triangleq \sum_{i=1}^{\mathcal{N}} n_{x_i},\ \ n_z \triangleq \sum_{i=1}^{\mathcal{N}} n_{z_i},\\
\bar{n} &\triangleq \sum_{i=1}^{\mathcal{N}} n_{w_i} + \sum_{i=1}^{q}(n_{y_i} + n_{v_i}).
\end{aligned}
$$

Let r_k be the k-th time instant in the execution of the order reduction (*Step 6* of Algorithm 1). The set of time instants (when the order reduction is performed), denoted as $\mathscr{K}_{\mathrm{re}}$, can be given by

$$\mathscr{K}_{\mathrm{re}} = \{r_k : k = 1, 2, \cdots\}.$$

To analyze the uniform boundedness of the F-radius of $\mathcal{Z}_s$, we make the following assumption.

Assumption 3: There exist positive scalars $\underline{a}$, $\bar{a}$, $\bar{c}$, $\underline{w}$, $\bar{w}$, and $\underline{\gamma}$ such that the following inequalities hold for each time instant $s \in \mathbb{N}$:

$$
\begin{aligned}
\underline{a}I \le \mathscr{A}_s^T\mathscr{A}_s, \mathscr{A}_s\mathscr{A}_s^T \le \bar{a}I, \mathscr{C}_s^T\mathscr{C}_s \le \bar{c}I,\\
\underline{w}I \le B(\eta_s)W(\eta_s)W^T(\eta_s)B^T(\eta_s) \le \bar{w}I,\\
\underline{\gamma}I \le \Upsilon + \mathscr{D}_{s+1}\mathscr{V}_{s+1}\mathscr{D}_{s+1}^T.
\end{aligned}
$$

Defining $Q_{q,s} \triangleq E_{q,s|s}E_{q,s|s}^T$, the F-radius of $\mathcal{Z}_s$ is equal to $\sqrt{\mathrm{tr}\{M(\eta_s)Q_{q,s}M^T(\eta_s)\}}$. Hence, in the following, we shall focus on analyzing the uniform boundedness of $Q_{q,s}$. Now, let us first give the uniform lower bound of $Q_{q,s}$.

Theorem 4: Under Assumption 3, there exists a lower bound

$$\underline{\iota} \triangleq (\underline{w}^{-1} + \bar{c}\underline{\gamma}^{-1})^{-1}$$

such that the generator matrix $E_{q,s|s}$ of the zonotope $\langle c_{q,s|s}, E_{q,s|s}\rangle$ satisfies

$$Q_{q,s} = E_{q,s|s}E_{q,s|s}^T \ge \underline{\iota}I \tag{38}$$

for every $s > 0$.

Proof: With the estimator parameter (35), the relationship

$$
\begin{aligned}
Q_{i,s+1}^{-1} &= \Big(Q_{i-1,s+1} - Q_{i-1,s+1}\mathscr{C}_i^T(\eta_{s+1})\Phi_{i,s+1}^{-1}\\
&\qquad \times \mathscr{C}_i(\eta_{s+1})Q_{i-1,s+1}\Big)^{-1}\\
&= Q_{i-1,s+1}^{-1} + \mathscr{C}_i^T(\eta_{s+1})\Psi_{i,s+1}^{-1}\mathscr{C}_i(\eta_{s+1})
\end{aligned}
\tag{39}
$$

holds for $i = 1, 2, \cdots, q$. Letting $i = q$ in (39), we iterate (39) for q times to yield

$$
\begin{aligned}
Q_{q,s+1}^{-1} &= Q_{0,s+1}^{-1} + \sum_{i=1}^{q}\mathscr{C}_i^T(\eta_{s+1})\Psi_{i,s+1}^{-1}\mathscr{C}_i(\eta_{s+1})\\
&= Q_{0,s+1}^{-1} + \mathscr{C}_{s+1}^T(\Upsilon + \mathscr{D}_{s+1}\mathscr{V}_{s+1}\mathscr{D}_{s+1}^T)^{-1}\mathscr{C}_{s+1}.
\end{aligned}
\tag{40}
$$

Recalling $Q_{0,s+1} = E_{s+1|s}E_{s+1|s}^T$ and $E_{s+1|s} = \begin{bmatrix}\mathscr{A}_sE_{s|s} & B(\eta_s)W(\eta_s)\end{bmatrix}$, we obtain from Assumption 3 that

$$
\begin{aligned}
Q_{0,s+1} &= \mathscr{A}_sQ_{q,s}\mathscr{A}_s^T + B(\eta_s)W(\eta_s)W^T(\eta_s)B^T(\eta_s)\\
&\ge B(\eta_s)W(\eta_s)W^T(\eta_s)B^T(\eta_s)\\
&\ge \underline{w}I
\end{aligned}
\tag{41}
$$

if $s \notin \mathscr{K}_{\mathrm{re}}$, and

$$
\begin{aligned}
Q_{0,s+1} &= \mathscr{A}_s\mathcal{Q}_{q,s}\mathscr{A}_s^T + B(\eta_s)W(\eta_s)W^T(\eta_s)B^T(\eta_s)\\
&\ge B(\eta_s)W(\eta_s)W^T(\eta_s)B^T(\eta_s)\\
&\ge \underline{w}I
\end{aligned}
\tag{42}
$$

if $s \in \mathscr{K}_{\mathrm{re}}$. Therefore, we have

$$Q_{0,s+1} \ge \underline{w}I \tag{43}$$

for all $s \in \mathbb{N}$.

Combining (40), (43) with Assumption 3, we arrive at

$$\begin{aligned} Q_{q,s+1}^{-1} &= Q_{0,s+1}^{-1} + \mathscr{C}_{s+1}^T(\Upsilon + \mathscr{D}_{s+1}\mathscr{V}_{s+1}\mathscr{D}_{s+1}^T)^{-1}\mathscr{C}_{s+1} \\ &\le \underline{w}^{-1}I + \mathscr{C}_{s+1}^T(\Upsilon + \mathscr{D}_{s+1}\mathscr{V}_{s+1}\mathscr{D}_{s+1}^T)^{-1}\mathscr{C}_{s+1} \\ &\le \underline{w}^{-1}I + \bar{c}\underline{\gamma}^{-1}I \\ &= \underline{\iota}^{-1}I, \end{aligned} \tag{44}$$

which implies (38), and the proof is now complete. ■

After acquiring a uniform lower bound of $Q_{q,s}$ in Theorem 4, we move onto the study of the upper bound of $Q_{q,s}$.

Theorem 5: Under Assumption 3, there exists an upper bound

$$\iota_s \triangleq \begin{cases} \bar{a}^{s-r_k}\lambda_{\max}\{\mathcal{Q}_{q,r_k}\} + \bar{w}\sum_{\ell=0}^{s-1-r_k}\bar{a}^\ell, \\ \qquad \text{if } s \in \cup_{k=1}^{+\infty}\{r_k+1, r_k+2, \cdots, r_{k+1}\} \\ \bar{a}^s\lambda_{\max}\{Q_{q,0}\} + \bar{w}\sum_{\ell=0}^{s-1}\bar{a}^\ell, \\ \qquad \text{if } s \in \{0, 1, \cdots, r_1\} \end{cases}$$

at each time instant $s \in \mathbb{N}$ such that $Q_{q,s}$ satisfies

$$Q_{q,s} \le \iota_s I. \tag{45}$$

Proof: For $s \in \cup_{k=1}^{+\infty}\{r_k+1, r_k+2, \cdots, r_{k+1}\}$, we have from (40) that

$$\begin{aligned} &Q_{q,s} \le Q_{0,s} \\ &= \begin{cases} \mathscr{A}_{s-1}Q_{q,s-1}\mathscr{A}_{s-1}^T + B(\eta_{s-1})W(\eta_{s-1}) \\ \quad \times W^T(\eta_{s-1})B^T(\eta_{s-1}), \ s \notin \cup_{k=1}^{+\infty}\{r_k+1\} \\ \mathscr{A}_{s-1}\mathcal{Q}_{q,s-1}\mathscr{A}_{s-1}^T + B(\eta_{s-1})W(\eta_{s-1}) \\ \quad \times W^T(\eta_{s-1})B^T(\eta_{s-1}), \ s \in \cup_{k=1}^{+\infty}\{r_k+1\} \end{cases}. \end{aligned} \tag{46}$$

In light of (46) and $B(\eta_{s-1})W(\eta_{s-1})W^T(\eta_{s-1})B^T(\eta_{s-1}) \le \bar{w}I$ (see Assumption 3), we further derive

$$\begin{aligned} &Q_{q,s} \\ &\le \begin{cases} \mathscr{A}_{s-1}Q_{q,s-1}\mathscr{A}_{s-1}^T + \bar{w}I, \ s \notin \cup_{k=1}^{+\infty}\{r_k+1\} \\ \mathscr{A}_{s-1}\mathcal{Q}_{q,s-1}\mathscr{A}_{s-1}^T + \bar{w}I, \ s \in \cup_{k=1}^{+\infty}\{r_k+1\} \end{cases}. \end{aligned} \tag{47}$$

Then, iterating (47) for $s-r_k$ times and utilizing $\mathscr{A}_s\mathscr{A}_s^T \le \bar{a}I$, we obtain

$$\begin{aligned} &Q_{q,s} \\ &\le \mathscr{A}_{s-1}Q_{q,s-1}\mathscr{A}_{s-1}^T + \bar{w}I \\ &\le \mathscr{A}_{s-1}\mathscr{A}_{s-2}Q_{q,s-2}\mathscr{A}_{s-2}^T\mathscr{A}_{s-1}^T \\ &\quad + \bar{w}\mathscr{A}_{s-1}\mathscr{A}_{s-1}^T + \bar{w}I \\ &\le \cdots \\ &\le \mathscr{A}_{s-1}\mathscr{A}_{s-2}\cdots\mathscr{A}_{r_k}\mathcal{Q}_{q,r_k}\mathscr{A}_{r_k}^T\cdots\mathscr{A}_{s-2}^T\mathscr{A}_{s-1}^T \\ &\quad + \bar{w}I + \bar{w}\mathscr{A}_{s-1}\mathscr{A}_{s-1}^T \\ &\quad + \cdots + \bar{w}\mathscr{A}_{s-1}\cdots\mathscr{A}_{r_k+1}\mathscr{A}_{r_k+1}^T\cdots\mathscr{A}_{s-1}^T \\ &\le \bar{a}^{s-r_k}\lambda_{\max}\{\mathcal{Q}_{q,r_k}\}I \\ &\quad + \bar{w}I + \bar{w}\bar{a}I + \cdots + \bar{w}\bar{a}^{s-1-r_k}I \\ &= \bar{a}^{s-r_k}\lambda_{\max}\{\mathcal{Q}_{q,r_k}\}I + \bar{w}\sum_{\ell=0}^{s-1-r_k}\bar{a}^\ell I \\ &= \iota_s I. \end{aligned} \tag{48}$$

The rest of the proof follows immediately when $s \in \{0, 1, \cdots, r_1\}$. ■

Theorem 5 provides a time-varying upper bound of $Q_{q,s}$ for each $s \in \mathbb{N}$. Next, based on partial results of Theorem 5, we shall give a sufficient condition that ensures the existence of uniform upper bounds of $Q_{q,s}$.

Theorem 6: Under Assumption 3, assume that there exist positive scalars $\bar{\iota}$ and κ such that the following inequalities hold for all $s \in \mathbb{N}$:

$$n_x + \bar{n}\left(\max_{i=1,2,\cdots,q}\{\bar{\omega}_i\} + \kappa + 1\right) < M_q, \tag{49}$$

$$\iota_s \le \bar{\iota}, \ (s - \max_{i=1,2,\cdots,q}\{\bar{\omega}_i\} < \kappa - 1) \tag{50}$$

$$\begin{aligned} &\sum_{p=s+1-\kappa}^{s+1} \sigma^{p-s-1}\Omega^T(p, s+1)\mathscr{C}_p^T(\Upsilon + \mathscr{D}_p\mathscr{V}_p\mathscr{D}_p^T)^{-1}\mathscr{C}_p \\ &\times\Omega(p, s+1) \ge (\bar{\iota})^{-1}I, \ (s - \max_{i=1,2,\cdots,q}\{\bar{\omega}_i\} \ge \kappa - 1) \end{aligned} \tag{51}$$

where

$$\sigma \triangleq 1 + \underline{a}^{-1}\underline{\iota}^{-1}\bar{w},$$

$$\Omega(p, s+1) \triangleq \begin{cases} \mathscr{A}_p^{-1}\mathscr{A}_{p+1}^{-1}\cdots\mathscr{A}_s^{-1}, & p < s+1 \\ I, & p = s+1 \end{cases}.$$

Then, $Q_{q,s}$ satisfies

$$Q_{q,s} \le \bar{\bar{\iota}}I \tag{52}$$

for all $s \in \mathbb{N}$, where

$$\begin{aligned} \bar{\bar{\iota}} &\triangleq \max\left\{\bar{\iota}\max_{\ell=0,1,2}\left\{\sigma^\ell\bar{a}^\ell\right\}, \sigma_M\bar{\iota}\max_{\ell=0,1,\cdots,\kappa-1}\left\{\sigma^\ell\bar{a}^{\ell+1}\right\}\right\}, \\ \sigma_M &\triangleq \tilde{\theta} + \bar{w}\underline{a}^{-1}\underline{\iota}^{-1}, \ \tilde{\theta} \triangleq \underline{\iota}^{-1}\bar{\iota}n_x(M_q + \bar{n})^2. \end{aligned}$$

Proof: We first prove the following two inequalities which will be utilized in the subsequent proof:

$$Q_{q,s+1}^{-1} \ge Q_{0,s+1}^{-1}, \ \forall s \in \mathbb{N}, \tag{53}$$

$$Q_{0,s+1}^{-1} \ge \sigma^{-1}\mathscr{A}_s^{-T}Q_{q,s}^{-1}\mathscr{A}_s^{-1}, \ \forall s \notin \mathscr{K}_{\text{re}}. \tag{54}$$

It follows immediately from (40) that (53) holds. Let us now prove (54).

In accordance with the condition $0 < \underline{a}I \le \mathscr{A}_s^T\mathscr{A}_s$ given in Assumption 3, it can be seen that $\mathscr{A}_s$ is invertible. Thus, we have from $B(\eta_s)W(\eta_s)W^T(\eta_s)B^T(\eta_s) \le \bar{w}I$ and $0 < \underline{a}I \le \mathscr{A}_s^T\mathscr{A}_s$ that

$$\mathscr{A}_s^{-1}B(\eta_s)W(\eta_s)W^T(\eta_s)B^T(\eta_s)\mathscr{A}_s^{-T} \le \underline{a}^{-1}\bar{w}I. \tag{55}$$

If $s \notin \mathscr{K}_{\text{re}}$, we obtain from the definition of $Q_{0,k+1}$, (55) and Theorem 4 that

$$\begin{aligned} Q_{0,s+1} &= \mathscr{A}_sQ_{q,s}\mathscr{A}_s^T + B(\eta_s)W(\eta_s)W^T(\eta_s)B^T(\eta_s) \\ &= \mathscr{A}_s(\mathscr{A}_s^{-1}B(\eta_s)W(\eta_s)W^T(\eta_s)B^T(\eta_s)\mathscr{A}_s^{-T} \\ &\quad + Q_{q,s})\mathscr{A}_s^T \\ &\le \mathscr{A}_s(Q_{q,s} + \underline{a}^{-1}\underline{\iota}^{-1}\bar{w}Q_{q,s})\mathscr{A}_s^T \\ &= \sigma\mathscr{A}_sQ_{q,s}\mathscr{A}_s^T \end{aligned} \tag{56}$$

which, together with (53), indicates that when $s \notin \mathscr{K}_{\text{re}}$, the following

$$Q_{q,s+1}^{-1} \ge Q_{0,s+1}^{-1} \ge (\sigma\mathscr{A}_sQ_{q,s}\mathscr{A}_s^T)^{-1} \tag{57}$$

is true. It is obvious that the correctness of (54) can be ensured by (57).

In the following, let us prove this theorem based on (53) and (54). With the set $\mathscr{K}_{\rm re}$, we can divide $\mathbb{N}$ (the set of all natural numbers) into the following several subsets:

$$\mathbb{N} = \{0, 1, \cdots, r_0 + \kappa\} \cup \cup_{\ell=0}^{r_{k+1}-r_k-\kappa-1}\{r_{k+1} - \ell : k = 0, 1, \cdots\} \cup \cup_{\ell=1}^{\kappa}\{r_k + \ell : k = 1, 2, \cdots\} \quad (58)$$

where $r_0 \triangleq \max_{i=1,2,\cdots,q}\{\bar{\omega}_i\}$. Next, we divide the rest of the proof into the following three cases.

<u>*Case 1*</u>: $s \in \{0, 1, \cdots, r_0 + \kappa\}$. In this case, we can see from (49) that at time instant s, the order reduction is not performed, which means that (54) is satisfied.

From (45) and (50), it can be seen that

$$Q_{q,s} \le \iota_s I \le \bar{\iota} I, \ \ s = 0, 1, \cdots, r_0 + \kappa - 2. \quad (59)$$

As for $s \in \{r_0 + \kappa - 1, r_0 + \kappa\}$, we have from (53)-(54) that

$$Q_{q,r_0+\kappa-1} \le \sigma \mathscr{A}_{r_0+\kappa-2} Q_{q,r_0+\kappa-2} \mathscr{A}^T_{r_0+\kappa-2} \le \bar{\iota}\sigma\bar{a} I, \quad (60)$$

and

$$Q_{q,r_0+\kappa} \le \sigma \mathscr{A}_{r_0+\kappa-1} Q_{q,r_0+\kappa-1} \mathscr{A}^T_{r_0+\kappa-1} \le \bar{\iota}\sigma^2\bar{a}^2 I. \quad (61)$$

With (59), (60), (61), and the definition of $\bar{\bar{\iota}}$, we know that (52) is true for $s \in \{0, 1, \cdots, r_0 + \kappa\}$.

<u>*Case 2*</u>: $s \in \cup_{\ell=0}^{r_{k+1}-r_k-\kappa-1}\{r_{k+1} - \ell : k = 0, 1, \cdots\}$. In this case, when $s \in \{r_k + \kappa + 1 : k = 1, 2, \cdots\}$, it is obvious that $s - \kappa - 1 = r_k \in \mathscr{K}_{\rm re}$.

For $s \in \{r_0 + \kappa + 1\} \cup \cup_{\ell=0}^{r_{k+1}-r_k-\kappa-2}\{r_{k+1} - \ell : k = 0, 1, \cdots\}$, utilizing (40) and (54), we have

$$\begin{aligned}
& Q^{-1}_{q,s} \\
&= Q^{-1}_{0,s} + \mathscr{C}^T_s(\Upsilon + \mathscr{D}_s\mathscr{V}_s\mathscr{D}^T_s)^{-1}\mathscr{C}_s \\
&\ge \sigma^{-1}\mathscr{A}^{-T}_{s-1} Q^{-1}_{q,s-1}\mathscr{A}^{-1}_{s-1} \\
&\quad + \mathscr{C}^T_s(\Upsilon + \mathscr{D}_s\mathscr{V}_s\mathscr{D}^T_s)^{-1}\mathscr{C}_s \\
&\ge \sigma^{-2}\mathscr{A}^{-T}_{s-1}\mathscr{A}^{-T}_{s-2} Q^{-1}_{q,s-2}\mathscr{A}^{-1}_{s-2}\mathscr{A}^{-1}_{s-1} \\
&\quad + \mathscr{C}^T_s(\Upsilon + \mathscr{D}_s\mathscr{V}_s\mathscr{D}^T_s)^{-1}\mathscr{C}_s \\
&\quad + \sigma^{-1}\mathscr{A}^{-T}_{s-1}\mathscr{C}^T_{s-1}(\Upsilon + \mathscr{D}_{s-1}\mathscr{V}_{s-1}\mathscr{D}^T_{s-1})^{-1}\mathscr{C}_{s-1}\mathscr{A}^{-1}_{s-1} \\
&\ge \cdots \\
&\ge \sigma^{-\kappa-1}\Omega^T(s-1-\kappa, s) Q^{-1}_{q,s-1-\kappa}\Omega(s-1-\kappa, s) \\
&\quad + \sum_{p=s-\kappa}^{s}\sigma^{p-s}\Omega^T(p, s)\mathscr{C}^T_p(\Upsilon + \mathscr{D}_p\mathscr{V}_p\mathscr{D}^T_p)^{-1}\mathscr{C}_p\Omega(p, s) \\
&\ge \sum_{p=s-\kappa}^{s}\sigma^{p-s}\Omega^T(p, s)\mathscr{C}^T_p(\Upsilon + \mathscr{D}_p\mathscr{V}_p\mathscr{D}^T_p)^{-1}\mathscr{C}_p\Omega(p, s)
\end{aligned} \quad (62)$$

which, together with (51), ensures (52).

As for $s \in \{r_k + \kappa + 1 : k = 1, 2, \cdots\}$, similarly, we also obtain from (40) and (54) that

$$\begin{aligned}
& Q^{-1}_{q,r_k+\kappa+1} \\
&= Q^{-1}_{0,r_k+\kappa+1} + \mathscr{C}^T_{r_k+\kappa+1} \\
&\qquad \times (\Upsilon + \mathscr{D}_{r_k+\kappa+1}\mathscr{V}_{r_k+\kappa+1}\mathscr{D}^T_{r_k+\kappa+1})^{-1}\mathscr{C}_{r_k+\kappa+1} \\
&\ge \sigma^{-1}\mathscr{A}^{-T}_{r_k+\kappa} Q^{-1}_{q,r_k+\kappa}\mathscr{A}^{-1}_{r_k+\kappa} \\
&\quad + \mathscr{C}^T_{r_k+\kappa+1}(\Upsilon + \mathscr{D}_{r_k+\kappa+1}\mathscr{V}_{r_k+\kappa+1}\mathscr{D}^T_{r_k+\kappa+1})^{-1}\mathscr{C}_{r_k+\kappa+1} \\
&\ge \cdots \\
&\ge \sigma^{-\kappa}\Omega^T(r_k+1, r_k+\kappa+1) Q^{-1}_{q,r_k+1}\Omega(r_k+1, r_k+\kappa+1) \\
&\quad + \sum_{p=r_k+2}^{r_k+\kappa+1}\sigma^{p-r_k-\kappa-1}\Omega^T(p, r_k+\kappa+1)\mathscr{C}^T_p \\
&\qquad \times (\Upsilon + \mathscr{D}_p\mathscr{V}_p\mathscr{D}^T_p)^{-1}\mathscr{C}_p\Omega(p, r_k+\kappa+1).
\end{aligned} \quad (63)$$

Substituting

$$Q^{-1}_{q,r_k+1} = Q^{-1}_{0,r_k+1} + \mathscr{C}^T_{r_k+1}(\Upsilon + \mathscr{D}_{r_k+1}\mathscr{V}_{r_k+1}\mathscr{D}^T_{r_k+1})^{-1}\mathscr{C}_{r_k+1} \quad (64)$$

into (63), we have

$$\begin{aligned}
& Q^{-1}_{q,r_k+\kappa+1} \\
&\ge \sigma^{-\kappa}\Omega^T(r_k+1, r_k+\kappa+1) Q^{-1}_{q,r_k+1}\Omega(r_k+1, r_k+\kappa+1) \\
&\quad + \sum_{p=r_k+2}^{r_k+\kappa+1}\sigma^{p-r_k-\kappa-1}\Omega^T(p, r_k+\kappa+1)\mathscr{C}^T_p \\
&\qquad \times (\Upsilon + \mathscr{D}_p\mathscr{V}_p\mathscr{D}^T_p)^{-1}\mathscr{C}_p\Omega(p, r_k+\kappa+1) \\
&= \sigma^{-\kappa}\Omega^T(r_k+1, r_k+\kappa+1) Q^{-1}_{0,r_k+1}\Omega(r_k+1, r_k+\kappa+1) \\
&\quad + \sum_{p=r_k+1}^{r_k+\kappa+1}\sigma^{p-r_k-\kappa-1}\Omega^T(p, r_k+\kappa+1)\mathscr{C}^T_p \\
&\qquad \times (\Upsilon + \mathscr{D}_p\mathscr{V}_p\mathscr{D}^T_p)^{-1}\mathscr{C}_p\Omega(p, r_k+\kappa+1) \\
&\ge \sum_{p=r_k+1}^{r_k+\kappa+1}\sigma^{p-r_k-\kappa-1}\Omega^T(p, r_k+\kappa+1)\mathscr{C}^T_p \\
&\qquad \times (\Upsilon + \mathscr{D}_p\mathscr{V}_p\mathscr{D}^T_p)^{-1}\mathscr{C}_p\Omega(p, r_k+\kappa+1).
\end{aligned} \quad (65)$$

It follows now from (51) and (65) that (52) holds when $s \in \{r_k + \kappa + 1 : k = 1, 2, \cdots\}$.

Summarizing above discussions, we know that (52) is true for $s \in \cup_{\ell=0}^{r_{k+1}-r_k-\kappa-1}\{r_{k+1} - \ell : k = 0, 1, \cdots\}$.

<u>*Case 3*</u>: $s \in \cup_{\ell=1}^{\kappa}\{r_k + \ell : k = 1, 2, \cdots\}$. In this case, there must exist a time instant $s^* \in \{s-1, s-2, \cdots, s-\kappa\}$ satisfying $s^* \in \mathscr{K}_{\rm re}$.

By resorting to the definition of $\mathcal{Q}_{q,r_k}$, we have

$$\mathcal{Q}_{q,r_k} = {\rm diag}\{\|\vec{e}^T_{1,r_k}\|^2_1, \cdots, \|\vec{e}^T_{n_x,r_k}\|^2_1\} \quad (66)$$

where $\vec{e}_{\imath,r_k} \triangleq \begin{bmatrix} \underbrace{0 \ \cdots \ 0}_{\imath-1} & 1 & \underbrace{0 \ \cdots \ 0}_{n_x-\imath} \end{bmatrix} E_{q,r_k|r_k}$ for $\imath = 1, 2, \cdots, n_x$. Noticing that the dimension of $\vec{e}^T_{\imath,r_k}$ is not greater than $M_q + \bar{n}$, it follows from (66) and $\|\vec{e}^T_{\imath,r_k}\|_1 \le (M_q + \bar{n})\|\vec{e}^T_{\imath,r_k}\|_\infty \le (M_q + \bar{n})\|\vec{e}^T_{\imath,r_k}\|_2$ that

$$\mathcal{Q}_{q,r_k} \le (M_q + \bar{n})^2{\rm diag}\{\|\vec{e}^T_{1,r_k}\|^2_2, \cdots, \|\vec{e}^T_{n_x,r_k}\|^2_2\}. \quad (67)$$

From the definition of Q_{q,r_k}, it is easy to see that

$${\rm diag}\{\|\vec{e}^T_{1,r_k}\|^2_2, \cdots, \|\vec{e}^T_{n_x,r_k}\|^2_2\} \le {\rm tr}\{Q_{q,r_k}\}I, \quad (68)$$

which together with (67) gives

$$\mathcal{Q}_{q,r_k} \le (M_q + \bar{n})^2{\rm tr}\{Q_{q,r_k}\}I. \quad (69)$$

According to (69) and $Q_{q,r_k} \leq \bar{\iota} I$ (proved in <u>*Case 2*</u>), we have

$$\mathcal{Q}_{q,r_k} \leq \bar{\iota} n_x (M_q + \bar{n})^2 I = \tilde{\theta} \underline{\iota} I. \tag{70}$$

Based on (70) and $Q_{q,s} \geq \underline{\iota} I$ for each $s > 0$ (proved in Theorem 4), we can obtain

$$\mathcal{Q}_{q,r_k} \leq \tilde{\theta} Q_{q,r_k}. \tag{71}$$

Adopting a similar line with the method of obtaining (56), we have from (71) and $Q_{q,r_k} \leq \bar{\iota} I$ (proved in <u>*Case 2*</u>) that

$$\begin{aligned}
&Q_{0,r_k+1} \\
=& \mathscr{A}_{r_k} \mathcal{Q}_{q,r_k} \mathscr{A}_{r_k}^T + B(\eta_{r_k}) W(\eta_{r_k}) W^T(\eta_{r_k}) B^T(\eta_{r_k}) \\
\leq& \tilde{\theta} \mathscr{A}_{r_k} Q_{q,r_k} \mathscr{A}_{r_k}^T + B(\eta_{r_k}) W(\eta_{r_k}) W^T(\eta_{r_k}) B^T(\eta_{r_k}) \\
=& \mathscr{A}_{r_k} (\mathscr{A}_{r_k}^{-1} B(\eta_{r_k}) W(\eta_{r_k}) W^T(\eta_{r_k}) B^T(\eta_{r_k}) \mathscr{A}_{r_k}^{-T} \\
&+ \tilde{\theta} Q_{q,r_k}) \mathscr{A}_{r_k}^T \\
\leq& \mathscr{A}_{r_k} (\tilde{\theta} Q_{q,r_k} + \underline{a}^{-1} \underline{\iota}^{-1} \bar{w} Q_{q,r_k}) \mathscr{A}_{r_k}^T \\
=& \sigma_M \mathscr{A}_{r_k} Q_{q,r_k} \mathscr{A}_{r_k}^T \\
\leq& \bar{\iota} \sigma_M \bar{a} I.
\end{aligned} \tag{72}$$

Utilizing (40) and (72), we further have

$$\begin{aligned}
&Q_{q,r_k+1}^{-1} \\
=& Q_{0,r_k+1}^{-1} + \mathscr{C}_{r_k+1}^T (\Upsilon + \mathscr{D}_{r_k+1} \mathscr{V}_{r_k+1} \mathscr{D}_{r_k+1}^T)^{-1} \mathscr{C}_{r_k+1} \\
\geq& Q_{0,r_k+1}^{-1} \\
\geq& (\bar{\iota} \sigma_M \bar{a})^{-1} I,
\end{aligned} \tag{73}$$

which gives

$$Q_{q,r_k+1} \leq \bar{\iota} \sigma_M \bar{a} I. \tag{74}$$

For $s \in \cup_{\ell=2}^{\kappa} \{r_k + \ell : k = 1, 2, \cdots\}$, it is obvious that $s \notin \mathscr{K}_{\rm re}$. Thus, from (53), (54), (74) and Assumption 3, we derive

$$\begin{cases}
Q_{q,r_k+2} \leq \sigma \mathscr{A}_s Q_{q,r_k+1} \mathscr{A}_s^T \leq \sigma_M \bar{\iota} \sigma \bar{a}^2 I, \\
Q_{q,r_k+3} \leq \sigma \mathscr{A}_s Q_{q,r_k+2} \mathscr{A}_s^T \leq \sigma_M \bar{\iota} \sigma^2 \bar{a}^3 I, \\
\qquad \vdots \\
Q_{q,r_k+\kappa} \leq \sigma \mathscr{A}_s Q_{q,r_k+\kappa-1} \mathscr{A}_s^T \leq \sigma_M \bar{\iota} \sigma^{\kappa-1} \bar{a}^{\kappa} I.
\end{cases} \tag{75}$$

It follows from (74) and (75) that (52) is satisfied when $s \in \cup_{\ell=1}^{\kappa} \{r_k + \ell : k = 1, 2, \cdots\}$.

According to the above analysis, we can conclude that under conditions (49)-(51), (52) is satisfied for all $s \in \mathbb{N}$. The proof is now complete. ■

By using Theorem 6, we have the following corollary about the uniform boundedness of the F-radius of $\mathcal{Z}_s$.

Corollary 1: Under Assumption 3, assume that

1) there exist positive scalars $\bar{m}$ and $\underline{m}$ such that

$$\underline{m} I \leq M(\eta_s) M^T(\eta_s) \leq \bar{m} I; \tag{76}$$

2) there exist positive scalars $\bar{\iota}$ and κ such that (49)-(51) hold.

Then, the F-radius of $\mathcal{Z}_s$ satisfies

$$\sqrt{\underline{m} \cdot \underline{\iota} n_z} \leq \| M(\eta_s) E_{s|s} \|_F \leq \sqrt{\bar{m} \bar{\bar{\iota}} n_z} \tag{77}$$

for all $s \in \mathbb{N}^+$.

Proof: Recall that

$$\mathcal{Z}_s = \langle M(\eta_s) c_{s|s}, M(\eta_s) E_{s|s} \rangle,$$

with which we have

$$\| M(\eta_s) E_{s|s} \|_F = \sqrt{\mathrm{tr}\{ M(\eta_s) Q_{q,s} M^T(\eta_s) \}}. \tag{78}$$

According to conditions 1) and 2) of this corollary, we obtain from Theorems 4 and 6 that

$$\underline{m} \cdot \underline{\iota} I_{n_z} \leq M(\eta_s) Q_s M^T(\eta_s) \leq \bar{m} \bar{\bar{\iota}} I_{n_z}. \tag{79}$$

In view of (78) and (79), the proof of this corollary follows directly. ■

Remark 5: In Theorem 6, a sufficient condition is given to guarantee that $Q_{q,s}$ is uniformly bounded. Resting on this condition, a criterion is then proposed in Corollary 1 to ensure that the F-radius of $\mathcal{Z}_{s|s}$ is uniformly bounded. There are two main factors that complicate the boundedness analysis, i.e., the order reduction and the multi-rate sampling. In general, the utilization of order reduction, while beneficial in reducing computational burden, would degrade the estimation accuracy. In other words, a smaller M_q (which means that the order reduction is performed more frequently), would lead to a worse estimation accuracy. The effects of the order reduction to the uniform boundedness of the F-radius of $\mathcal{Z}_{s|s}$ are reflected in (49) where the value of M_q required to ensure the uniform boundedness is provided. Also, the order reduction would affect the uniform upper bound significantly which can be seen in <u>*Case 3*</u> of the proof of Theorem 6. The effects brought by the multi-rate sampling are mainly reflected in (50)-(51).

Remark 6: The usage of the order reduction technique would make it difficult to obtain a criterion guaranteeing the existence of a uniform upper bound of $Q_{q,s}$ by directly using existing analysis methods (e.g., the uniform observability condition [10]). In this paper, based on the method proposed in [28], we further solve the technical problem caused by the order reduction (see <u>*Case 2*</u> and <u>*Case 3*</u> of the proof of Theorem 6 for details) by using some matrix inequality techniques. It is worth noting that the obtained uniform upper bound $\bar{\bar{\iota}} I$ of $Q_{q,s}$ might be conservative due to the usage of inequality techniques, leading to a large uniform upper bound of the F-radius of $\mathcal{Z}_s$. The main role of the existence of such a uniform upper bound is to ensure that the proposed sequential fusion estimation algorithm is non-divergent. Moreover, the proposed analysis method on such a bound represents one of the first few attempts to handle the uniform boundedness analysis problem in zonotopic SMSE for time-varying systems. On the other hand, when looking for a tighter uniform upper bound of $Q_{q,s}$ becomes a concern, some other inequalities with less conservatism could be used, which constitutes one of future research topics.

Remark 7: So far, we have solved the sequential fusion estimation for MRCNs with uniform quantization effects. Compared with existing results on fusion estimation of CNs, the main novelties of this paper are indicated as follows: 1) the considered sequential fusion estimation problem is new for MRCNs with UYB noises; 2) under the zonotopes-based fusion criterion, the sequential estimator is designed such that the F-radius of the zonotope (containing the estimation error after each measurement update) is minimized; and 3) sufficient criteria are established to guarantee the uniform boundedness of the F-radius restraining the estimation error calculated after all measurement updates.

IV. ILLUSTRATIVE EXAMPLE

In this section, we present a numerical example to demonstrate the validity of the proposed fusion estimation scheme.

Consider an MRCN with three sensors, in which the state updating period is $h = 1$, and the positive integers b_1, b_2 and b_3 (representing the multiples of the state updating period) are set as $b_1 = 2$, $b_2 = 3$, $b_3 = 4$. The initial transmission time instants of three sensors are set as $\bar{\omega}_1 = 1$, $\bar{\omega}_2 = 2$, and $\bar{\omega}_3 = 1$. The parameters of the MRCN are given as follows:

$$G_1(\eta_s) = \begin{bmatrix} 1.05 & 0.12 \\ -0.14 & -0.2 \end{bmatrix},\ G_2(\eta_s) = \begin{bmatrix} 0.1 & 0.4 \\ -0.1 & -0.15 \end{bmatrix},$$
$$G_3(\eta_s) = \begin{bmatrix} 0.11 & 0.1 \\ -0.1 & 0.1 \end{bmatrix},\ G_4(\eta_s) = -0.2I,$$
$$A(\eta_s) = \begin{bmatrix} -0.3 & 0.1 & 0.1 & 0.1 \\ 0.1 & -0.21 & 0.01 & 0.1 \\ 0.1 & 0.01 & -0.11 & 0 \\ 0.1 & 0.1 & 0 & -0.2 \end{bmatrix} \otimes \mathrm{diag}\{0.1, 0.11\},$$
$$B_1(\eta_s) = \mathrm{diag}\{0.1, 0.2\},\ B_2(\eta_s) = \mathrm{diag}\{0.15, 0.1\},$$
$$B_3(\eta_s) = \mathrm{diag}\{0.3, 0.2\},\ B_4(\eta_s) = \mathrm{diag}\{0.15, 0.2\},$$
$$M_1(\eta_s) = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix},\ M_2(\eta_s) = \begin{bmatrix} 0.2 & 0.5 \end{bmatrix},$$
$$M_3(\eta_s) = \begin{bmatrix} 0.15 & 0.2 \end{bmatrix},\ M_4(\eta_s) = \begin{bmatrix} 0.3 & 0.2 \end{bmatrix},$$
$$C_1(\omega_{1,s}) = \begin{bmatrix} 0.1 & 0.2 + 0.01\sin(\frac{\pi}{6}\omega_{1,s}) \end{bmatrix},\ D_1(\omega_{1,s}) = 1,$$
$$C_2(\omega_{2,s}) = \begin{bmatrix} 0.12 & 0.1 \\ 0.2 & 0.15 \end{bmatrix},\ D_2(\omega_{2,s}) = I,$$
$$C_3(\omega_{3,s}) = \begin{bmatrix} 0.13 & 0.2 \\ 0.31 & 0.17 \end{bmatrix},\ D_3(\omega_{3,s}) = I$$

where "$\otimes$" denotes the Kronecker product.

In this example, the quantizing levels are set to be $\vartheta_1 = \vartheta_2 = \vartheta_3 = 0.1$. Moreover, the external noises are chosen as

$$w_1(\eta_s) = \begin{bmatrix} 0.1\cos(0.1\eta_s) \\ 0.1\sin(0.1\eta_s) \end{bmatrix},\ w_2(\eta_s) = \begin{bmatrix} 0.1\sin(0.1\eta_s) \\ 0.1\cos(0.1\eta_s) \end{bmatrix},$$
$$w_3(\eta_s) = \begin{bmatrix} 0.1\cos(0.1\eta_s) \\ 0.1\sin(0.1\eta_s) \end{bmatrix},\ w_4(\eta_s) = \begin{bmatrix} 0.1\sin(0.1\eta_s) \\ 0.1\cos(0.1\eta_s) \end{bmatrix},$$
$$v_1(\omega_{1,s}) = 0.1\cos(0.1\omega_{1,s}),\ v_2(\omega_{2,s}) = \begin{bmatrix} 0.1\sin(0.1\omega_{2,s}) \\ 0.08\cos(0.1\omega_{2,s}) \end{bmatrix},$$
$$v_3(\omega_{3,s}) = \begin{bmatrix} 0.1\sin(0.1\omega_{3,s}) \\ 0.09\cos(0.1\omega_{3,s}) \end{bmatrix},$$

from which we have $W_1(\eta_s) = W_2(\eta_s) = W_3(\eta_s) = W_4(\eta_s) = 0.1I$, $V_1(\omega_{1,s}) = 0.1$, $V_2(\omega_{2,s}) = 0.1I$, and $V_3(\omega_{3,s}) = 0.1I$. Furthermore, the initial values are set as

$$x_1(\eta_0) = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix}^T,\ x_2(\eta_0) = \begin{bmatrix} -0.1 & 0.1 \end{bmatrix}^T,$$
$$x_3(\eta_0) = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}^T,\ x_4(\eta_0) = \begin{bmatrix} -0.05 & 0.05 \end{bmatrix}^T,$$

by which we obtain that $\langle c_i(\eta_0), E_i(\eta_0)\rangle = \langle 0, 0.1I\rangle$ $(i = 1, 2, 3, 4)$.

Let the simulation steps be 800 and the allowed maximum number of columns of $E_{s|s}$ be 200 (i.e., $M_q = 200$). The initial value of $\hat{x}_{s|s}$ is set as $\hat{c}_{0|0} = 0$. By means of the MATLAB software, the estimator parameters can be obtained recursively. Based on the calculated estimator parameters, the simulation results are obtained according to Algorithm 1. To be specific, Figs. 2-5 display the elements $z^{(j)}(\eta_s)$ $(j = 1, 2, 3, 4)$ of the signal to be estimated, their estimates computed by (13), and the information of their upper bounds and lower bounds (the estimated signal is denoted as $z(\eta_s) = \begin{bmatrix} z^{(1)}(\eta_s) & z^{(2)}(\eta_s) & z^{(3)}(\eta_s) & z^{(4)}(\eta_s) \end{bmatrix}^T$). It can be observed that the proposed sequential estimation algorithm (i.e., Algorithm 1) performs indeed well.

According to above given system parameters, we have $\underline{a} = 0.0034$, $\bar{a} = 1.0843$, $\bar{c} = 0.1727$, $\underline{w} = 0.0001$, $\bar{w} = 0.0009$, $\underline{\gamma} = 0.1$, $\underline{m} = 0.05$ and $\bar{m} = 0.29$, by which we can see that Assumption 3 is satisfied. Accordingly, we know from Theorem 4 that $Q_{q,s}$ has a uniform lower bound $\underline{\iota}I$ with the calculated $\underline{\iota}$ being 9.9983×10^{-5}. Furthermore, it can be checked that, when $M_q = 200$, $\bar{\iota} = 3.1706 \times 10^9$, and $\kappa = 6$, (49)-(51) are satisfied. Therefore, we have from Theorem 6 that $Q_{q,s}$ also has a uniform upper bound $\bar{\bar{\iota}}I$ with the calculated $\bar{\bar{\iota}}$ being 7.4805×10^{45}. Moreover, it is obvious that Assumption 3 and the conditions 1) and 2) of Corollary 1 are satisfied simultaneously, and therefore the F-radius of $\mathcal{Z}_s$ is uniformly bounded according to Corollary 1. The uniform upper and lower bounds of the F-radius of $\mathcal{Z}_s$ are plotted in Fig. 6, from which it can be confirmed that $\|M(\eta_s)E_{q,s|s}\|_F$ stays within the calculated bounds. All simulation results show the effectiveness of the proposed fusion estimation method and validate the correctness of the obtained results on boundedness analysis.

V. CONCLUSION

In this paper, we have studied the sequential fusion estimation problem for MRCNs with uniformly quantized measurements under the zonotopic SMSE framework. With the aid of virtual measurements, the MRCNs have been transformed into single-rate switched ones. By virtue of the properties of zonotopes, desired zonotopes have been derived such that the estimation error after each measurement update satisfies the pre-defined $\mathcal{E}$-dependent constraint. The sequential estimator parameters have been then computed by minimizing the F-radii of these zonotopes. In addition, sufficient criteria have been proposed to guarantee the uniform boundedness of the F-radius of the zonotope restraining the estimation error after all measurement updates. Finally, a numerical example has been proposed to illustrate the effectiveness of the proposed sequential fusion estimation method.

In addition, related topics for further research work include the extension of our results to other complex systems such as neural networks [18], switched systems [32] and nonlinear systems [33].

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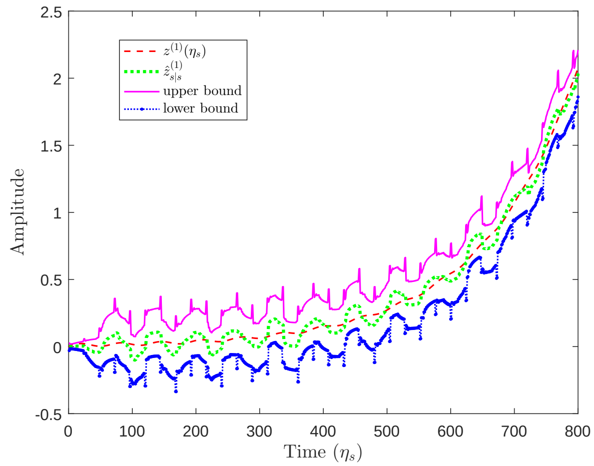

Fig. 2: $z^{(1)}(\eta_s)$, its estimate and its bounds.

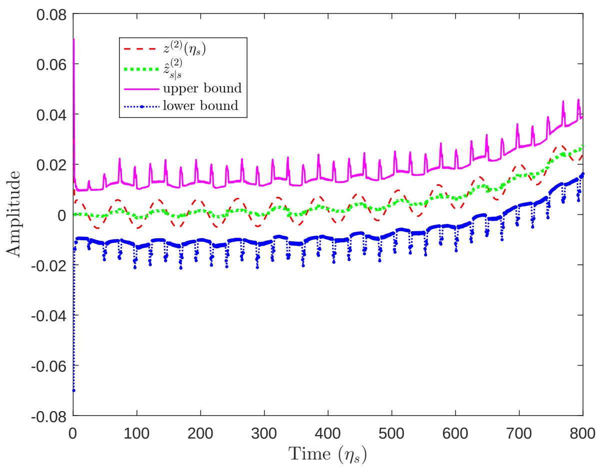

Fig. 3: $z^{(2)}(\eta_s)$, its estimate and its bounds.

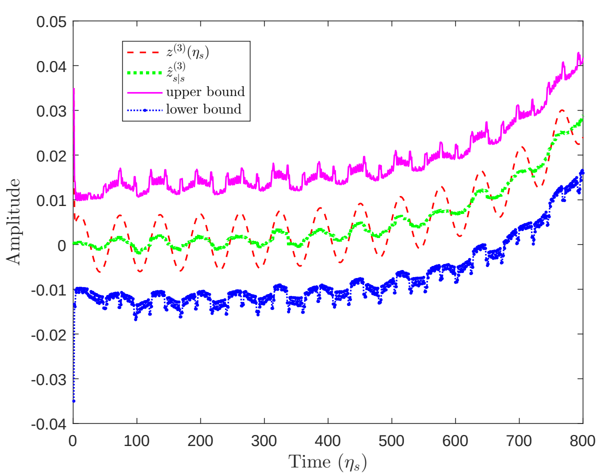

Fig. 4: $z^{(3)}(\eta_s)$, its estimate and its bounds.

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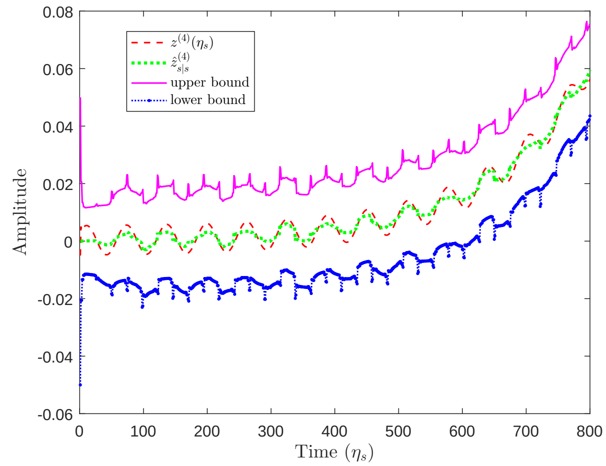

Fig. 5: $z^{(4)}(\eta_s)$, its estimate and its bounds.

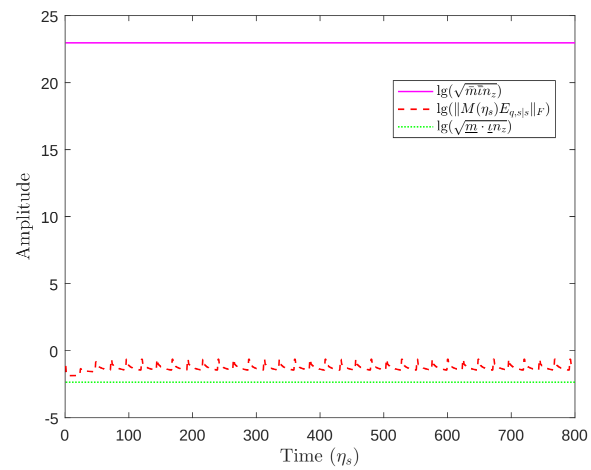

Fig. 6: $\lg(\|M(\eta_s)E_{q,s|s}\|_F)$, its uniform upper bound and its uniform lower bound.

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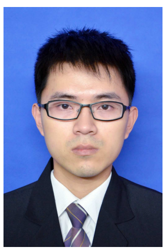

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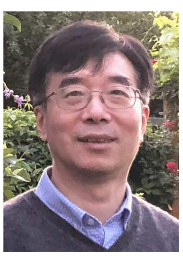

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