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NEAR-BEST APPROXIMATIONS TO SOME PROBLEMS IN APPLIED MATHEMATICS.

by

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1. Introduction

Consider the general problem of finding $u \in U$ such that

$$Au = A^{-1} f \tag{1.1}$$

where $f \in F$ and A is an operator from the space U to the space F.

If A^{-1} exists then the formal solution of the problem is

$$u = A\text{-}1f \ . \tag{1.2}$$

In practice A^{-1} is not, in general, known and the solution u of (1.1) is usually approximated by one of the following two methods:

(a) The operator A is approximated by some other operator $\bar{A}$, which is easier to invert than A, and the solution u of (1.1) is approximated by A^{--1} f(e.g. descrete methods are of this type).

(b) The right hand side f of (1.1) is approximated by some $\bar{f}$ for which $A^{-1}\bar{f}$ is known, and the solution $u = A^{-1}$ f is approximated by $\bar{u} = A^{-1}\bar{f}$.

We shall refer to method (b) as an expansion method since usually u is taken to be of the form $\bar{u} = \sum_{j=1}^{n} \alpha_j u_j$.

Intuitively one expects an expansion method to be more efficient since, in this case, the operator A in (1.1) is preserved. Also, usually much more is known about approximations in the spaces U and F than about approximations of operators from U to F. However, in practice the expansion approximations are not very successful. The common approach in an expansion method is to obtain the approximation to $\bar{u}$ by making $f = A\bar{u}$ a good approximation to $f = Au$ in some sense (or norm) in F. This approach usually neglects completely the important role played by A^{-1} in transforming the

Abstract

Expansion approximations are considered for the solution of Fredholm integral equations of the second kind, of two-point boundary value problems and of harmonic mixed boundary value problems. In each of these three cases the relation between the expansion approximations and the approximation of a certain integral is investigated. This relation leads to definitions of pointwise and of global "near-best" approximations whose errors are given in terms of the error functional of a "best" quadrature formula.

approximation from the space F to the space U, and that seems to be the weakness in the expansion method. In general A^{-1} is not considered simply because it is not known. However, in many problems the formal form of A^{-1} is known and some of its properties can be found directly from A. In many cases this might help in choosing the type of approximation f to f that provides some 'best' approximation $\bar{u}$ to u.

In this work we try to include available information about A into the choice of the best desirable type of approximation $\bar{f}$ to f. We show that in most cases we cannot find a single approximation $\bar{f}$ to f that will yield a 'best' approximation to u unless we add some correction term to $\bar{u}$. This correction term seems to be crucial to the success of the expansion method.

We describe this approach by considering the solution of Fredholm integral equations of the second kind. We then apply it to two-point boundary value problems and to harmonic mixed boundary value problems. For each case we obtain the appropriate correction term and perform some numerical example using the corrected expansion approximation.

The theoretical motivation to the new approach is based on the simple approximation theorem presented in the following section.

2. A simple approximation theorem

Let H be the set of the integrable functions in some given continuity class of functions on [a,b], and consider the quadrature and expansion methods for approximating the integral

$$\int_a^b h(t)dt \quad , \; h \in H \; . \tag{2.1}$$

Here the expansion method consists of approximating h(t), $t \in [a,b]$,

by a linear combination $\bar{h}(t) = \sum_{j=1}^{n} \alpha_j h_j(t)$ of functions hj ∈ II

j = 1, 2, ..., n whose integrals, $a_j = \int_a^b h_j(t)dt$, are known. The

nth order expansion approximation to (2.1) is then defined as $\sum_{j=1}^{n} \alpha_j a_j$.

We are interested in the relation between the nth order expansion approximation and the nth order quadrature approximation

$\sum_{i=1}^{n} w_i h(t_i)$ of (2.1) In particular, we are interested in the relation between the best nth order approximations of each type.

Definition 2.1. An approximation which is exact on a subset Φ of H is called a best-Φ approximation.

Best quadrature approximations are usually obtained by choosing special wights w_i^* and abscissae t_i^* such that the quadrature rule $\sum_{i=1}^{n} w_i^* h(t_i^*)$ integrates exactly 2n basis functions $\phi_1, \phi_2, \ldots, \phi_{2n}$ in H. This quadrature approximation is then a best-Φ approximation where $\Phi = \text{span}\{\phi_1, \phi_2, \ldots, \phi_{2n}\}$.

Theorem 2.1 Given a best-Φ nth order quadrature approximation, $\sum_{i=1}^{n} w_i^* h(t_i^*)$ and n functions $h_1, h_2, \ldots, h_n \in \Phi$, such that for any h Î H there exists $\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*$ that satisfy

$$\sum_{j=1}^{n} \alpha_j^* h_j(t_i^*) = h(t_i^*), \qquad i = 1, 2, \ldots, n, \tag{2.2}$$

then the nth order expansion approximation $\sum_{j=1}^{n} \alpha_j^* \int_a^b h_i(t)dt$ to (2.1) is identical to the best-Φ nth order quadrature approximation.

i . e.

$$\sum_{j=1}^{n} \alpha_j^* \int_a^b h_j(t)dt = \sum_{i=1}^{n} w_i^* h(t_i^*) \quad , \quad h \in H \, . \tag{2.3}$$

Proof . The proof is straightforward. However, for later use , we present it by considering the error

$$E(h) \equiv \int_a^b h(t)dt - \sum_{j=1}^{n} \alpha_j^* \int_a^b h_j(t)dt \tag{2.4}$$

of the expansion approximation.

Application of the best-Φ quadrature formula to the integrals in (2.4) gives

$$E(h) = \sum_{j=1}^{n} w_i^* [h(t_i^*) - \sum_{j=1}^{n} \alpha_j^* h(t_j^*)] + e(h - \sum_{j=1}^{n} \alpha_j^* h_j) \tag{2.5}$$

where $e(f) \equiv \int_a^b f(t)dt - \sum_{i=1}^{n} w_i^* f(t_i^*)$ is the error of the best -Φ quadrature approximation.

The first term on the right hand side of (2.5) vanishes by (2.2), and the second reduces to e(h) since $e(h_j) = 0$ $j = 1, 2, ..., n$. Q.E.D.

Thus, with independent basis functions $h_j \in \Phi$ one can, in general, use (2.2) to obtain a best-Φ approximation. For a general set of independent functions h_1., h_2 , ..., $h_n \in H$ we use the same definition of the a_j*'s to define a near-best approximation.

Definition 2.2. Given a best nth order quadrature formula $\sum_{i=1}^{n} w_i^* h(t_i^*)$ and n independent functions, h_1 , h_2 , …. $h_n \in H$, the near-best nth order expansion approximation to (2.1) is defined by $\sum_{j=1}^{n} \alpha_j^* \int_a^b h_j(t)dt$

where the $\alpha_j^{*'s}$ are chosen such that $\sum_{j=1}^{n} \alpha_j^* h_j$ collocates h at the $t_j^{*'s}$.

Corollary 2.1. Let e be the error functional of a best nth order quadrature formula, and let E be the error functional of the associated near-best expansion approximation, then

$$E(h) = e(h - \sum_{j=1}^{n} \alpha_j^* h_j) . \tag{2.6}$$

The proof follows using expression (2.5) of E(h).

The above discussion seems to have no useful application to the problem of approximating integrals of given functions. However, as we show in the following sections, it can be very useful when considering expansion approximations in general.

3. Integral equations of the second kind

3.1 Connection between expansion approximation and the resolvent function.

Consider the integral equation of the second kind

$$\left.\begin{aligned} u(x) - \lambda \int_a^b K(x, t)u(t)\, dt &= f(x) \\ \text{or, in operator form,} \qquad & \\ Lu = u - \lambda Ku &= f . \end{aligned}\right\} \tag{3.1}$$

For any λ which is not an eigenvalue of K equation (3.1) has a unique solution which can be represented in terms of the resolvent function Γ of K as

$$u(x) = \int_a^b \Gamma(x, t; \lambda) f(t)dt$$

or, in operator form, as

$$u = \Gamma f. \qquad \textbf{(3.2)}$$

The problem of approximating u(x) is therefore equivalent to the problem of approximating the integral $\int_a^b \Gamma(x,t;\lambda) f(t)dt$. Usually Γ is not known and for this reason this integral cannot be approximated directly by a quadrature approximation. However, we might be able to approximate it by the expansion approximation of the previous section.

Let f belong to some continuity class F on [a,b] and, for a given x, let H_x be the set of all the functions $h(t) = r(x,t;\lambda)\Phi(t)$ with $\Phi \in F$.

The application of the expansion method, for approximating $\int_a^b h(t)dt, h \in H_X$ requires a set of functions $h_i. \in H_x$ for which the integrals $\int_a^b h(t)dt$ are known. In general, it is not difficult to find functions u_j such that $f_j = Lu_j = u_j - lKu. \in F$. Then, the functions $h_i \equiv r(x,-;1)f. \in H_x$, and their integrals are simply

$$\int_a^b h_j(t)dt = \int_a^b \Gamma(x,t;\lambda) f_j(t)dt = u_j(x). \qquad (3.3)$$

Thus, the expansion method for approximating u(x) from (3.2) consists of determining an approximation of the form $\sum_{j=1}^{n} \alpha_j h_j(t)$ to $h(t) \equiv \Gamma(x, t; \lambda) f(t)$ $t \in [a,b]$, and then of approximating the integral of h by

$$\int_a^b \Gamma(x,t;\lambda)f(t) \approx \sum_{j=1}^{n} \alpha_j \int_a^b h_j(t)dt \,. \qquad (3.4)$$

But the approximation $h(t) \approx \sum_{j=1}^{n} \alpha_j h_j(t)$ reduces to the approximation $f(t) \approx \sum_{j=1}^{n} \alpha_j u_j(x)$. after removing the common factor $r(x,t;l)$, and the approximation (3.4) reduces by (3.3) to

$$u(x) \approx \sum_{j=1}^{n} \alpha_j u_j^*(x) \,. \qquad (3.5)$$

Therefore, the expansion method for approximating $\int_a^b h(t)dt$ is equivalent to the expansion method for approximating the solution of (3-1), as defined in the introduction,with $\bar{f} = \sum_{j=1}^{n} \alpha_j f_j$ and

$$\bar{u} = \sum_{j=1}^{n} \alpha_j u_j = L^{-1}\bar{f} \,.$$

Although G is not known, it is possible, in many cases, to classify its continuity properties and thus to find H_x . Therefore, in general, it is possible to find an appropriate best quadrature Formula, $\sum_{i=1}^{n} w_i^* h(t_i^*)$, for approximating $\int_a^b h(t)\, dt$, $h \in H_x$. It is unlikely that this best quadrature approximation is exact for the h_j's used in the above expansion method. Thus it is not possible to make full use of the approximation theorem 2.1 and obtain a best expansion approximation. However,we can use definition 2.2 and obtain a near-best approximation

$$\int_a^b h(t)dt \approx \sum_{j=1}^{n} \alpha_j \int_a^b h_j(t)dt \,, \qquad (3.6)$$

choosing the a_j's so that

$$\sum_{j=1}^{n} \alpha_j h_j(t_i^*) = h(t_i^*) , \qquad i = 1, 2, \ldots, n . \tag{3.7}$$

In terms of the approximation to the solution of the integral equation (3-1) this is equivalent to the approximation (3.5) with the α_j.'s chosen so that

$$\sum_{j=1}^{n} \alpha_j f_j(t_i^*) = f(t_i^*) , i = 1 , 2, \ldots, n \tag{3.8}$$

Definition 3.1. A near-best nth order expansion approximation, $\bar{u}(x) = \sum_{j=1}^{n} \alpha_j u_j(x)$, to the solution u(x) of the integral equation (3.1) at a given x is that which collocates the integral equation at the abscissae t_i^*, i = 1, 2, ..., n, of a best nth order quad rature formula on H_x .

Theorem 3.1- The error in the near-best nth order expansion approximation $\bar{u}(x)$ to u(x) is given by

$$u(x) - \bar{u}(x) = e_X (h - \sum_{j=1}^{n} \alpha_j h_j) \tag{3.9}$$

where e_x is the error functional of the associated best nth order quadrature formula on H_x .

The proof follows at once from corollary 2.1 applied to the near-best approximation to $\int_a^b h(t)\,dt = u(x)$.

The major shortcoming of the above near-best expansion approximation is that, in general, due to the singularities of T, the class H_x

depends strongly on x (e.g. H_x can be the class of infinitely smooth functions on [a,b] apart from jump discontinuity, or a logarithmic singularity, at t = x). Such a situation implies that for different values of x there are different best nth order quadrature formulae. Therefore, a near-best expansion approximation to u(x) for some given x, is, in general, expected to lose a great deal of its efficiency when used for other values of x.

We remark that similar arguments hold for expansion approximations to the solution of operator equations in general.

3.2 A global near-best approximation.

It is of course possible to obtain near-best approximations to u(x), for particular values of x, provided that the continuity properties of r(x,t;1) are known. However, as we show below, a global near-best approximation to u over [a,b] is attainable provided that the dominant singularities of can be removed in the following way.

Assume that Γ can be expressed as the sum of a 'singular' part S and a 'smooth' remainder R, i.e. in the form

$$r(x,t;\lambda) = S(x,t;\lambda) + R(x,t;\lambda) , \qquad (3\text{-}10)$$

and let H be the set of all the functions $h(t) = R(x,t;\lambda)\,\phi(t)$, $t \in [a,b]$, $\phi \in F$. In order to obtain an nth order global near-best approximation to u we require that S is a known function, and that there exists a best nth order quadrature formula, $\sum_{i=1}^{n} w_i^* h(t_i^*)$, independent of x, approximating $\int_a^b h(t)\,dt$, $h \in H$. A method for determining an appropriate S is described in the next section. Here we assume that S is known. We also assume that for any $f \in F$ the integral $\int_a^b S(x,t;\lambda) f(t)\,dt$ can be approximated to any desired accuracy.

Then the problem of approximating

$$u(x) = \int_a^b \Gamma(x,t;\lambda) f(t)\,dt$$

is reduced to the problem of approximating

$$u_R(x) \equiv \int_a^b R(x,t;\lambda) f(t)\,dt \ .$$

We apply the expansion method for the approximation of $\int_a^b h(t)\,dt, h \in H$ by taking $h_i(t) = R(x,t;\lambda) f_i(t)$ where, as in the previous section, $f_i = u_i - \lambda K u_i$ and the integrals

$$\int_a^b h_j(t) = \int_a^b R(x,t;\lambda) f_j(t) = u_j(x) - \int_a^b S(x,t;\lambda) f_j(t)\,dt \ , \qquad (3.11)$$

of the oasis functions, are assumed to be known.

By definition 2.2 the near-best nth order expansion approximation to $u_R(x)$ is $\sum_{j=1}^{n} \alpha_j \int_a^b h_j(t)\,dt$ where the α_j's satisfy

$$\sum_{j=1}^{n} \alpha_j h_j(t_i^*) = h(t_i^*) \ , \ i = 1, 2, \ldots, n \ ,$$

where $h(t) = R(x,t;\lambda) f(t)$, or equivalently,

$$\sum_{j=1}^{n} \alpha_j f_j(t_i^*) = f(t_i^*) \ , \ i = 1, 2, \ldots, n \ . \qquad (3.12)$$

The t_i^*'s in (3.12) are the abscissae of the best nth order quadrature formula on H and thus, by assumption, they are independent of x. Hence, using the near-best expansion approximation to $U_R(X)$ and (3.11) we can define a global near-best approximation to u as follows:

Definition 3.2. An nth order global near-best approximation $\bar{u}_c$ to the solution u of the integral equation 3.1 is defined as

$$\bar{u}_c(x) = \sum_{j=1}^{n} \alpha_j u_j(x) + \int_a^b S(x,t;\lambda)[f(t) - \sum_{j=1}^{n} \alpha_j f_j(t)]\,dt \qquad (3.13)$$

where the a_i-'s are determined by (3.12), i.e. the expansion approximation $\sum_{j=1}^{n} \alpha_j u_j$ collocates the integral equation at t_i^*, i = 1, 2, ..., n, the abscissae of the best nth order quadrature formula on H.

We also refer to $\bar{u}_c$ as the corrected expansion approximation and to $\int_a^b S(x, t;\lambda)[f(t) - \sum_{j=1}^{n} \alpha_j f_j(t)]dt$ as the correction term of the expansion approximation.

Theorem 3.2. The error in the corrected expansion approximation $\bar{u}_c$ to u is given by

$$u(x) - \bar{u}_c(x) = e(h - \sum_{j=1}^{n} \alpha_j h_j) \qquad (3.14)$$

where e is the error functional of the best nth order quadrature formula on H.

The proof follows by applying corollary 2.1 to the near-best approximation to $u_R(x)$.

Usually, the best nth order quadrature approximation *on* H is more accurate than the best nth order quadrature approximation on H_x of section 3.1. Comparing results 3.9 and 3.14, this indicates another possible advantage of the global near-best approximation $\bar{u}_c$ over the near-best expansion approximation $\bar{u}$.

We note that the same correction terra is expected to also 'correct' other expansion approximations to u(x) (e.g. those obtained by the least squares method or the Galerkin method). However, when the $t_i^{*'}$s t_i*'s can be found, it is simpler and usually better to work with the above corrected collocation approximation $\bar{u}_c$.

3.3 The singularities of Γ.

In order to apply the above approximations we have to be able to analyse the continuity properties of the resolvent kernel r(x,t;λ).

In operator form we have the relations $u - \lambda Ku = f$ and $u = \Gamma f$. Hence

$$\begin{aligned} \Gamma f &= u \\ &= f + \lambda Ku \\ &= f + \lambda K(f + \lambda Ku) \\ &\vdots \\ &= f + \lambda Kf + \lambda^2 K^2 f + \ldots + \lambda^m K^m f + \lambda^{m+1} K^{m+1} u \ . \end{aligned}$$

Replacing u by Γ f in the last term of the above expression we obtain the operator identity

$$\Gamma = I + \lambda K + \lambda^2 K^2 + \ldots + \lambda^m K^m + \lambda^{m+1} K^{m+1} \Gamma \qquad (3\text{-}15)$$

for Γ. In (3.15) the product of two operators A and B, with kernels A(x,t) and B(x,t) respectively, is the operator AB with the kernel

$$C(x,t) = \int_a^b A(x,y) B(y,t)\, dy \ , \qquad (3.16)$$

and I is the identity operator.

Integral operators are smoothing operators, and thus, in general, $K^{j+1}(x,t)$ is smoother than $K^{j}(x,t)$. Therefore, the most severe singularities of $r(x,t;\lambda)$ are imbeded in the first terms of the expression (3.15). The identity operator I contributes a $\delta(x-t)$ singularity to $r(x,t;\lambda)$ where δ is the Dirac-δ function.

We consider the important class of kernels $K(x,t)$ which can be represented as

$$K(x,t) = K_1(x,t) + H(x-t) \sum_{j=0}^{\infty} k_j^{(1)} \frac{(x-t)^j}{j!} \tag{3.17}$$

where $K_1 \in C^{\infty}([a,b]x[a,b])$, H is the Heaviside step function and $k_j^{(1)}$ is the jump in the jth derivative of K, i.e.

$$k_j^{(1)} = \frac{\partial^j K}{\partial x^j}(t,+t) - \frac{\partial^j K}{\partial x^j}(t,-t) . \tag{3.18}$$

Using the equality

$$\int_a^b H(x-y) \frac{(x-y)^k}{k!} H(y-t) \frac{(y-t)^{\ell}}{\ell!} dy = H(x-t) \frac{(x-t)^{k+\ell+1}}{(k+\ell+1)!} \tag{3.19}$$

it can be proved, by induction on n, that the kernel of the operator K^n can be represented as

$$K^n(x,t) = K_n(x,t) + H(x-t) \sum_{j=n-1}^{\infty} k_j^{(n)} \frac{(x-t)^j}{j!} \tag{3.20}$$

where $K_n \in C^{\infty}([a,b]x[a,b])$ and the $k_i^{(n)}$'s are obtained recursively from the $kj^{(1)}$'s, by

$$\begin{cases} k_j^{(m)} = 0 & j < m-1 \\ k_j^{(m)} = \sum_{i=1}^{j-1} k_i^{(1)} k_{j-i-1}^{(m-1)} & j \geq m-1 \end{cases} \tag{3.21}$$

From the expression (3.15) of G and the relations (3.20) it can be shown that $r(x,t;\lambda)$ is infinitely smooth on [a,b]x[a,b] apart from a d(x-t) singularity and jump discontinuities on t = x.

The near-best expansion approximation $\bar{u}(x)$ is defined by means of a best quadrature formula on H_x, where H_x is the set of all the functions $h(t) = G(x,t;1)f(t)$, $t \in [a,b]$, $f \in F$. Let $F = C^{\infty}[a,b]$, thus H_x is the class of infinitely smooth functions on [a,b] apart from a $\delta(x-t)$ discontinuity and jump discontinuities at t = x. A best nth order quadrature formula on this H_x can be obtained by a combination of two Gaussian quadrature formulae, one on [a,x] and the other on [x,b], of combined order n-1, together with a special treatment for the point t = x to take care of the $\delta(x-t)$ singularity.

Let $e_k^{[c,d]}$ denote the error functional of the kth order Gaussian quadrature formula (G.q.f.) on [c,d]. Then, for a given x we compose the best nth order quadrature formula on H_x by means of a k_x th order G.q.f. on [a,x] and an $(n-k_x-1)$th order G.q.f. on [x,b], where k_x is chosen so that $e_{k_x}^{[a,x]}(h) \approx O(e_{n-k_x-1}^{[x,b]}(h))$. The near-best expansion approximation is then defined by collocation at the k_x Gaussian points in [a,x], at x, and at the $n-k_x-1$ Gaussian points in [x,b].

Corollary 3.1. The error of the near-best expansion approximation $\bar{u}(x) = \sum_{j=1}^{n} \alpha_j u_j(x)$ to the solution u(x) of the integral equation 3.1. with a kernel of the form (3.17) and $f \in C^{\infty}[a,b]$ is

$$u(x) - \bar{u}(x) = e_{k_x}^{[a,x]}(h - \sum_{j=1}^{n} \alpha_j h_j) + e_{n-k_x-1}^{[x,b]}(h - \sum_{j=1}^{n} \alpha_j h_j) \qquad (3.22)$$

where h and the h_j's are as defined in section 3.1.

For a global near—best approximation we need to find a suitable 'singular' part S, of G, as defined in section 3.2. A possible S can be induced from (3.15), by taking the kernel of the operator

$$S_m = I + \lambda K + \lambda^2 K^2 + \ldots + \lambda^m K^m \qquad (3.23)$$

where m can be chosen so that the 'remainder'

$$R_m = \lambda^{m+1} K^{m+1} \Gamma \qquad (3.24)$$

has a sufficiently smooth kernel. For kernels of the form (3.17) we can use (3.20) to show that the kernel of R has at least m-1 continuous derivatives. However, for kernels of the form (3.17), a more practical S than (3.23) can be obtained as follows: the kernel of each K^n in (3.23) is replaced by its representation (3.20), and only the terms which contribute to the jump discontinuities in the derivatives $\frac{\partial^j S}{\partial x^j}$, $j = 0,1, \ldots, m-1$, are retained to give

$$\bar{S}_m(x,t;\lambda) = \delta(x-t) + H(x-t) \sum_{j=0}^{m-1} \left[\sum_{i=1}^{m} \lambda^i k_j(i) \right] \frac{(x-t)^j}{j!} \qquad (3.25)$$

This is $\bar{S}_m$ far easier to compute than the kernel of S_m, and the kernel of its 'remainder' $\bar{R}_m \equiv \Gamma - \bar{S}_m$, like that of R_m, has at least m-1 continuous derivatives.

The set H associated with a 'remainder' R is defined in the previous section as the set of the functions $h(t) = R(x,t;\lambda) f(t)$, $t \in [a,b]$, $f \in F$. It follows that if $F = C^{\infty}[a,b]$, then for both $R = R$ and $R = \bar{R}_m$ $H \subseteq C^{(m-1)}[a,b]$. For an nth order global near-best approximation we take $m = 2n$. Then, $H \subseteq C^{2n-1}[a,b]$ and as a best nth order quadrature formula on H we take the nth order G.q.f. on [a,b]. The global near-best approximation $\bar{u}_c$ is then defined as

$$\bar{u}_c(x) = \sum_{j=1}^{n} \alpha_j u_j(x) + \int_a^b \bar{S}_{2n}(x,t;\lambda) \left[f(t) - \sum_{j=1}^{n} \alpha_j f_j(t) \right] dt , \qquad (3.26)$$

where the a_j's are determined by (3.12) with the t_i^*'s taken as the n Gaussian points in [a,b],

The choice of Gaussian points as collocation points is also suggested in other works; see e.g. Pruess [5]. However, the motivation there is based on the attempt of making $f - \sum_{j=1}^{n} \alpha_j f_j$ nearly orthogonal to all polynomials of degree < n. The motivation used in the present paper for collocation at Gaussian points reveals the necessity of the extra correction term to the collocation approximation. With this correction term, we have the following promising result:

Corollary 3.2. The error in the global near-best approximation $\bar{u}_c$ to u is given by

$$u(x) - \bar{u}_c(x) = e_n^{[a,b]}(h - \sum_{j=1}^{n} \alpha_j h_j) \tag{3.27}$$

where h, h_1 , h_2 , ..., $h_n \in C^{(2n-1)}[a,b]$ and $e_n[a,b]'$ is the error functional of the nth order G.q..f. on [a,b].

The main contribution of the correction term is

$$\int_a^b \delta(x - t)[f(t) - \sum_{j=1}^{n} \alpha_j f_j(t)\, dt = f(x) - \sum_{j=1}^{n} \alpha_j f_j(x)$$

due to the first term in $\bar{S}_{2n}$. The approximation

$$\sum_{j=1}^{n} \alpha_j u_j(x) + f(x) - \sum_{j=1}^{n} \alpha_j f_j(x)$$

can be interpreted as a one stage Neumann iteration from the 'point' $\sum_{j=1}^{n} \alpha_j u_j$. It is mentioned in Baker [1] that this simple correction frequently improves an expansion approximation.

If in (3.26) the $\bar{S}_{2n}$ is replaced by the kernel of S_m of (3.23) then $\bar{u}_c$ is simply the mth Neumann iterate starting from the 'point' $\sum_{j=1}^{n} \alpha_j u_j$.Itis important to notice the difference between the present motivation of obtaining this approximation and the simpler motivation based on Neumann iterations. Thus, the motivation used here for obtaining the corrected approximation is based only upon the fact that the kernel K^{J+1} (x,t) is smoother than the kernel K^J(x,t) and does not require the convergence of the Neumann series.

It can be shown that corollary 3.2 holds for the 2nth Neumann iterate for $\sum_{j=1}^{n} \alpha_j u_j$ .Therefore, even for a divergent Neumann series we expect the first 2n iterations to improve the collocation approximation $\sum_{j=1}^{n} \alpha_j u_j$,although further iterations might destroy this improvement.

Repeated Neumann iterations are not commonly used because of the numerical complexity involved in computing the kernels k^j (x,t). However, for kernels of the form (3.17) a corrected approximation, using the computationally simple function $\bar{S}_{2n}$, plays the same role as the 2nth Neumann iterate. For other classes of kernels it might be more difficult to find a computationally simple form for S. However, the results obtained with the kernels of the form (3.17) indicate that the study of other classes of kernels deserves strong consideration.

3.4 Numerical example - Integral equations of the second kind.

We consider the Fredholm equation (3.1) with $f \in C^{\infty}[a,b]$ and with kernel

$$K(x,t) = \begin{cases} x(1-t) & 0 = x = t = 1 \\ t(1-x) & 0 = t = x = 1 \end{cases} \qquad (3\text{-}28)$$

This kernel can he rewritten as

$$K(x,t) = x(1-t) - (x-t)H(x-t) \quad , \tag{3.29}$$

i.e. in the form (3.17) with K (x,t) = x(l-t), $k_0^{(1)} = 0$, $k_1^{(1)} = -1$ and $k_i^{(1)} = 0 \quad i \geq 2$. Hence we can use (3.21) and (3.25) to compute the 'singular' part $\bar{S}_m$ of G.

To demonstrate the power of the corrected expansion approximation we consider corrections to a low order expansion approximation, in this case 4th order. The expansion approximation $\bar{u}$ is taken as the third degree polynomial which collocates the integral equation at the 4 Gaussian points on [a,b]. In the corrected expansion approximation $\bar{u}_c$, defined by (3.26), the computation of the f_j's is performed analytically and the correction term is approximated by using Simpson's rule with $h = \frac{b-a}{100}$.

To investigate the influence of particular singularities of $\Gamma(x,t;\lambda)$ on the correction term we compute a sequence of approximations, $\bar{u}^{-m}$,m=0, 1,2, ..., corresponding to correction terms with $\bar{S}_m(x,t;\lambda)$ Then $\bar{u}_c^{(m)}$ takes care of the jump discontinuities in the first m-1 derivatives of $\Gamma(x,t;\lambda)$. We note that only $k_1^{(1)}$ 0 and thus $\bar{S}_{2i+1} = \bar{S}_{2i}$ and $\bar{u}_c^{(2i+1)} = \bar{u}_c^{(2i)}$.

Consider the particular case of $\lambda = 1$ and f(x) = x of (3.1), i.e. the equation

$$u(x) - \int_0^1 K(x,t)\,u(t)\,dt = x \quad , \tag{3.30}$$

which has the exact solution u(x) = sinx/sinl.

In Table 3.1 we give results at the points x = 0.0 (0.2)1.0 computed from the collocation approximation (x) and the corrected approximations

$\bar{u}_c^{(0)}$ (x), $\bar{u}_c^{(2)}$ (x) and $\bar{u}_c^{(4)}$ (x). These are compared with values computed from the analytic solution u(x).

Table 3-1

X	0.0	0.2	0.4	0.6	0.8
$\bar{u}(x)$	-0.000271694	0.236219378	0.462707145	0.670937092	0.852654706
$\bar{u}_c^{(0)}(x)$	0.000000000	0.236098504	O.462781577	0.671017224	0.852503761
$\bar{u}_c^{(2)}(x)$	0.000000000	0.236097644	0.462782799	0.6710l8304	0.852502461
$\bar{u}_c^{(4)}(x)$	0.000000000	0.236097651	O.462782835	0.671018328	0.852502439
u(x)	0.000000000	0.237097660	O.462782852	0.671018352	0.8525021467

x	1.0
$\bar{u}(x)$	0.999605471
$\bar{u}_c^{(0)}(x)$	1.000000000
$\bar{u}_c^{(2)}(x)$	1.000000000
$\bar{u}_c^{(4)}(x)$	0.999999967
u(x)	1.000000000

We note that the basis for obtaining the near-best approximations is the representation (3.2) of the solution of (3.1). In fact, similar near-best approximations can be defined for any problem whose solution has a representation of the form (3.2). Two such problems are considered in the following chapters.

4. Two-point boundary value problem

4.1 Expansion approximations and the Green's function

In this section we consider expansion approximations to the solution of the rath order linear differential equation

$$Lu(x) = p_0(x)u^{(m)}(x) + p_1(x)u^{(m-1)}(x) + \ldots + p_m(x)u(x) = f(x), \quad x \in [a,b], \tag{4.1}$$

where $p_j \in C^{(n-j)}$ [a,b] and $P_0(x)$ does not vanish on [a,b], subject to the m homogeneous, lineary independent, boundary conditions

$$\sum_{j=1}^{m} a_{ij}u^{(j-1)}(a) + \sum_{j=1}^{m} b_{ij}u^{(j-1)}(b) = 0 \quad i = 1,2,\ldots, m. \tag{4.2}$$

The solution of this boundary value problem can be expressed by means of the Green's function G(x,t) of the problem (4.1) - (4.2) as

$$u(x) = \int_a^b G(x, t)f(t)dt \quad . \tag{4.3}$$

This representation is similar to the representation (3.2) of the solution of the integral equation (3.1). Hence, by replacing $\Gamma(x,t;\lambda)$ by G(x,t), the results obtained in sections 3.1 and 3.2 can be adapted to deal with expansion approximations to the solution of two-point boundary value problems.

Let f belong to a continuity class F on [a,b], and consider expansion approximations, $\sum_{j=1}^{n} \alpha_j u_j$, to the solution u of the problem (4.1) - (4.2), with u_i's such that $f_i = Lu_i \in F$, j = 1, 2, ..., n. Here, for a given x, we define H_x as the set of the functions $h(t) = G(x,t)\phi(t)$ with $\phi \in F$.

Definition 4.1. A near-best (m+n)th order expansion approximation, $\bar{u}(x) = \sum_{j=1}^{m+n} \alpha_j u_j(x)$, t.o the nolution u(x) of the boundary value problem (4.1) - (4.2), at a given $x \in [a,b]$, is that which satisfies the m boundary conditions (4.2) and collocates the differential equation (4.1) at the abscissae t_i^*, i = 1, 2, ..., n, of a best nth order quadrature formula for approximating $\int_a^b h(t)\,dt, h \in H_X$.

Theorem 4.1. The error in the near-best expansion approximation u(x) to u(x) is given by

$$u(x) - \bar{u}(x) = e_X\left(h - \sum_{j=1}^{m+n} \alpha_j h_j\right) \tag{4.4}$$

where

$$h(t) = G(x,t)f(t) \in H_x ,$$

$$h.(t) = G(x,t)f\text{-}(t) \in H_x ,$$

and e_x is the error functional of the best nth order quadrature formula on H_x.

We note that definition 4.1 can be used to define a near-best approximation ū(x) even when the conditions (4.2) are replaced by non-homogeneous linear two-point boundary conditions. Theorem 4.1 then holds for the case of non-homogeneous linear boundary conditions.

Typically the Green's function for an mth order boundary value problem has the form

$$G(x,t) = G_1(x,t) + H(x-t) \sum_{j=m-1}^{\infty} g_j \frac{(x-t)^j}{j!} \tag{4.5}$$

where $G_1 \in C^\infty([a,b]x[a,b])$. Therefore, as in the case of integral

equations, the class H_x depends strongly on x and, in general, a particular near-best expansion approximation $\bar{u}(x)$ has the 'near-best' property (4.4) only for that particular x for which it is defined. For example, if $F = C^{\infty}[a,b]$ and G is of the form (4.5), then H_x is the class of infinitely smooth functions on [a,b] apart from a jump discontinuity in the (m-l)th derivative at t - x. Thus, as in section 3.3, a best nth order quadrature formula on H_x can be composed of a k_x th order G.q.f. on [a,x] and an $(n-k_x -1)$ th order G.q.f. on Cx,b]. The corresponding near-best expansion approximation then has an error of the form (3.22).

One way of making a single expansion approximation efficient for several values, $x_0, x_1, \ldots, x_k$, of x is by replacing the class H_x, in definition 4.1, by a wider class $H(x_0,x_1,\ldots,x_k)$ such that $H_{xi} \leq H_{(x0,x1\ldots xk)}$. $i = 0, 1, \ldots, k$. For example, in the case $F = C^{\infty}[a,b]$ and G of the form (4.5), $H(x_0, x_1, \ldots x_k)$ may be taken as the class of infinitely smooth functions with only jump discontinuities at $x_0, x_1, \ldots, x_k$, in the (m-1)th and higher order derivatives. For equidistant points, $x_i = a + i\frac{b-a}{k}$, i = 0, 1, ... ,k, a best (km) th order quadrature formula on $H(x_0,x_1,\ldots,x_k)$ can be composed of k nth order G.q.f.'s on $[x_i,x_{i+1}]$, i = 0, 1, ..., k-1. The corresponding (kn+m) order near-best approximation, $\bar{u}_k = \sum_{j=1}^{kn+m} \alpha_j u_j$ is that which satisfies the m boundary conditions and collocates the differential equation at the n Gaussian points at each of the k intervals $[x_i,x_{i+1}]$, i = 0, 1, ..., k-1. It can be shown that

$$u(x_i) - \bar{u}_k(x_i) = \sum_{r=0}^{k-1} e_n[x_r, x_{r+1}]\,(h - \sum_{j=1}^{kn+m} \alpha_j h_j) \qquad (4.6)$$

where h(t) = G(x.,t)f(t) and h.(t) = G(x.,t)f_j(t}.

The result (4.6) is supported by a result of De-Boor and Swartz [2] for the more general case of collocation approximation to the solution of mth order nonlinear ordinary differential equations with m linear side conditions using piecewise polynomials. Their result, applied to the simpler linear case (4.1) can be stated as follows:

"If $\bar{u}_k \in C^{(2n+m)}[a,b]$ satisfies the m boundary conditions (4.2) and collocates the differential equation (4.1) at the n Gaussian points in each of the intervals $[x_i, x_{i+1}]$, i = 0, 1, ..., k-1 then,

$$\left|u(x) - \bar{u}_k(x)\right| = 0(\Delta^{n+m}) \tag{4.7}$$

where $A = \max_{0 \le i \le k-1} (x_{i+1} - x_i)$. Furthermore,

$$\left|u^{(j)}(x_i) - \bar{u}_k^{(j)}(x_i)\right| = 0(\Delta^{2n}), \quad j = 0, 1, \ldots, m-1, \quad 1 = 0, 1, \ldots k". \tag{4.8}$$

It is suggested in [2] that a global approximation of order 2n can be obtained by interpolation from the values $\bar{u}_k, (x_i)$, i = 0, 1, ...,k Here, as in chapter 3, a global near-best approximation to the solution of the linear boundary value problem (4.1) - (4.2) is obtained by correcting the expansion approximation.

4.2 A global near-best approximation

If throughout section 3.2 the integral operator L = u - λKu and the resolvent r(x,t;λ) are replaced respectively by the differential operator L of (4.1) and the Green's function G(x,t), it becomes clear that a global near-best approximation to the solution u of (4.1) - (4.2) can be obtained provided that the dominant singularities of G(x,t) can be removed. The singularities of the Green's function are in general easier to establish than those of the resolvent kernel T. This is due to the fact that the Green's function is actually defined by its singularities. Thus, regarded as a function of x with t fixed G(x,t) is defined as follows:

(i) $LG(x,t) = 0$ for $x \neq t$.

(ii) G satisfies the homogeneous boundary conditions (4.2)

(iii)

$$\frac{\partial^j}{\partial x^j} G(t^+, t) - \frac{\partial^j}{\partial x^j} G(t^-, t) = \begin{cases} 0 \text{ for } j = 0,1,..., m-2 \\ \frac{1}{p_o(t)} \text{ for } j = m-1 . \end{cases} \quad (4.9)$$

From (4.9) we see that G has m-2 continuous derivatives. Therefore, the effect of the singularities of G, upon expansion approximations to the solution of the problem (4.1) - (4.2), is reduced as the order m of the problem increases. This fact can be concluded from the result (4.7).

If p_0 , p_1 , ,.., $p_m \in C$ [a,b] then $G(.,t) \in C^\infty$ ([a,t)u(t,b]) and the jump discontinuities in the derivatives of G can be obtained recursively as follows :

Let

$$g_j(t) = \frac{\partial^j}{\partial x^j} G(t^+, t) - \frac{\partial^j}{\partial x^j} G(t^-, t) \quad (4.10)$$

From the definition of G we have that

$$p_0(x)\frac{\partial^m}{\partial x^m} G(x,t) + p_1(x)\frac{\partial^{m-1}}{\partial x^{m-1}} G(x,t) + ... + p_m(x) G(x,t) = 0, \quad (4.11)$$
$$x \neq t$$

Hence

$$p_0(t)g_m(t) + p_1(t)g_{m-1}(t) + ... + p_m(t)g_0(t) = 0 ,$$

and, using (4.9), we have that

$$g_m(t) = \frac{-p_1(t)}{p_0^2(t)} . \quad (4.12)$$

When $j \geq m+1$ the values of g.(t) can be found, in a similar way, by using the relations obtained by differentiating equation (4.11).

In fact we are interested in the singularities of G(x,t) as a function of t. However, removal of the jump discontinuities of $\frac{\partial^j G}{\partial x^j}$, $j = m-1, ..., k$ ensures, in general, the continuity of $\frac{\partial^j G}{\partial x^j}$, $j = m\text{-}1, ..., k$.

The function

$$S_k(x,t) \equiv H(x-t) \sum_{j=m-1}^{k} g_j(t) \frac{(x-t)^j}{j!} \qquad (4.13)$$

has the same jump discontinuity in its jth derivative, $0 \leq j \leq k$, with respect to x as G(x,t). Hence the 'remainder'

$$R_k(x,.t) \equiv G(x,t) - S_k(x_s t) \qquad (4.14)$$

has at least k continuous derivatives with respect to both x and t.

In order te obtain a global near-best approximation we deal, as in section 3.2 ,, with expansion approximations to $\int_a^b h(t)\,dt$, for $h \in H$, where

$$H = \{h \quad h(t) = R_k(x,t)\phi(t), \quad \phi \in F\} \quad .$$

That is, we choose k large enough so that there exists a best nth order quadrature formula $\sum_{i=1}^{n} w_I^* h(t_i^*)$, independent of x, approximating $\int_a^b h(t)\,dt$, $h \in H$, and define the global (m+n)th order near-best

approximation as follows:

Definition *k.2.* An (m+n)th order global near-best approximation $\bar{u}_c$ to the solution u of the problem (4.1) - (4.2) is defined as

$$\bar{u}_c(x) = \sum_{j=1}^{m+n} \alpha_j u_j(x) + \int_a^b S_k(x,t)[f(t) - \sum_{j=1}^{m+n} \alpha_j f_j(t)]dt, \qquad (415)$$

where the expansion approximation $\sum_{j=1}^{m+n} \alpha_j u_j$ satisfies the m boundary conditions (4.2) and collocates the differential equation (4.1) at the abscissae $t_i^*, i = 1, 2, \ldots, n$ of the best nth order quadrature formula on H.

The expression for the error of this global near-best approximation is identical to that given in (3.14) for the global near-best approximation to the solution of the integral equation. That is, corresponding to Theorem 3.2 we have:

Theorem *k.2.* The error in the (m+n)th order global near-best approximation $\bar{u}_c$ to u is given by

$$u(x) - \bar{u}_c(x) = e(h - \sum_{j=1}^{m+n} \alpha_j h_j) \qquad (4.16)$$

where $h(t) = R_k(x,t)f(t) \in H$, $h_j(t) = R_k(x,t)f_j(t) \in H$, $j = 1, 2, \ldots, m+n$ and e is the error functional of the best nth order quadrature formula on H.

For the case $F = C^{\infty}[a,b]$ and $k = 2n-1$ it follows that $H \leq c^{(2n-1)}[a,b]$. Thus, the collocation points t_i^*, in definition 4.2,may be chosen as the n Gaussian points in [a,b]. The error in the resulting approximation is therefore given by (4.16) with $e = e_n^{[a,b]}$, the error functional of the nth order G.q.f. on [a,b],

It is interesting to observe that collocation at Gaussian points is in fact performed when solving two-point boundary value problems by the T-method of Lanczos [3], using a basis of Legendre polynomials.

4.3 Numerical example — Two-point boundary value problem

We consldex the second order boundary value problem

$$u'' + u = 2x(1-x)e^{x} \; ; \quad u(1) = u(2) = 0 \qquad (4.17)$$

which has the exact solution $u(x) = (x-1)(x-2)e$.

We take $\bar{u}$ to be the fifth degree polynomial which satisfies the boundary conditions and collocates the differential equation at the 4 Caussian points in [1,2].

Using the procedure described in section 4.2 we find that the jumps in the derivatives of G(x, t) are $g_0(t) = 0$, $g_1(t) = 1$, $g_2(t) = 0$, $g_3(t) = 1$, $g_4(t) = 0$, $g_5(t) = 1$, $g_6(t) = 0$, $g_7(t) = -1$ etc. Thus, we use (4.13) and (4.15) with k = 7 to obtain a global near-best approxima.tion $\bar{u}_c$ whose error is given by (4.16) with $e = e_4^{[1,2]}$.

To demonstrate the effect of the singularities of G(x,t) upon expansion approximations to u we compare in Table 4.1 the results obtained from $\bar{u}$ with those obtained from the corresponding corrected approximations $\bar{u}_c$, The integral in the definition(4. 15)of $\bar{u}_c$ is approximated *by* using Simpson's rule with .h = 0.01.

Table 4.1

6th order expansion approximation $\bar{u}(x)$ and the corresponding corrected approximation $\bar{u}_c(x)$ compared with the exact solution u(x) of (4.17) x = 0.2(0.2)0.8.

x	0.2	0.4	0.6	0.8
$\bar{u}(x)$	-0.5314406	-0.9729560	-1.1884627	-0.9682479
$\bar{u}_C(x)$	-0.5312215	-0.9732502	-1.1887293	-0.9679444
$\bar{u}(x)$	-0.5312187	-0.9732480	-1.1887278	-0.9679436

We no-be that the improvement obtained by the corrected approximation is less impressive for two-point boundary value problems than for integral equations. This is due to the fact that the singularities of G(x,t) are usually less severe than those of $\Gamma(x,t;\lambda)$.

5. Harmonic boundary value problems

5.1 Expansion approximations and the Green's function

Let $\Omega \subseteq^{R2}$ be a simply-connected domain with boundary $\partial\Omega$ and consider the two-dimensional boundary value problem in which the function u(x,y) satisfies Laplace's equation

$$\Delta(x,y) = 0 \quad , \quad (x,y) \in \Omega \tag{5.1}$$

and mixed boundary conditions of the form

$$\begin{cases} u(x,y) = f(x,y) & , \quad (x,y) \in \partial\Omega_1 \\ \dfrac{\partial}{\partial n} u(x,y) = g(x,y) & , \quad (x,y) \in \partial\Omega_2 . \end{cases} \tag{5.2}$$

In (5.-2) $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$, $\partial\Omega_1 \neq \Phi$ and $\dfrac{\partial}{\partial n}$ denotes the derivative in the direction of the outward normal to the boundary.

Near-best approximations to the solution of this problem may be obtained in much the same way as for the two previous problems, by using the Green's function representation of the solution. The Green's function of the problem (5.1) - (5.2) is defined as

$$G(x,y;x_0,y_0) = -\frac{1}{2\pi}\log r + h(x,y) \tag{5.3}$$

where $r = C(x-x)^2 + (y-y)^2]^2$, h(x,y) is a regular harmonic function on Ω, and G satisfies the homogeneous boundary conditions

$$\begin{cases} G(x,y;x_0, y_0) = 0 & (x,y) \in \partial\Omega_1 \\ \dfrac{\partial}{\partial n} G(x,y;x_0, y_0) = 0 & (x,y) \in \partial\Omega_2 . \end{cases} \tag{5.4}$$

The solution of the problem (5.1) -(5.2) is then given by

$$u(x_0,y_0) = -\int_{\partial\Omega_1} \frac{\partial}{\partial n} G(x,y;x_0,y_0)\, f(x,y)\,ds + \int_{\partial\Omega_2} G(x,y;x_0,y_0) g(x,y)\,ds \tag{5.5}$$

Let f belong to a continuity class F on $\partial\Omega_1$ and let g belong to a continuity class G on $\partial\Omega_2$. We consider expansion approximations $S a_j u_j$ to u where u_j, $j = 1,2,\ldots$ are harmonic functions such that $f_j \equiv u_j\Big|_{\partial\Omega_1} \in$ and $g_j \equiv \dfrac{\partial u_j}{\partial n}\Big|_{\partial\Omega_2} \in G$.

For a given $(x_o, y_o) \in W$ we define the set of functions $H(x_0, y_0)$ on Ω as

$$H(x_0,y_0 = \left\{ h \,\middle|\, h(x,y) = \begin{cases} \dfrac{\partial}{\partial n} G(x,y;x_0,y_0)\varphi\varphi(xy) & , \varphi \in F, (x,y) \in \partial\Omega_1 \\ G(x,y;x_0,y_0)\psi\psi(xy) & , \psi \in F, (x,y) \in \partial\Omega_2 \end{cases} \right\}$$

and, in complete analogy to the two previous cases, we define a near-best approximation to $u(x_0, y_0)$ by means of the abscissae of a best quadrature formula. In this case the quadrature formula is for approximating $\int_{\partial\Omega} h(x,y)\,ds$, $h \in H(x_0,y_0)$.

Let $\sum_{i=1}^{n} w_i^* h(t_i^*)$ be a best nth order quadrature formula on $H_{(x_0,y_0)}$, and let $e_{(x_0,y_0)}$ be the error functional of this formula.

Then we have:

Definition 5-1. A near-best nth order expansion approximation, $\bar{u}(x_0,y_0) = \sum_{j=1}^{n} \alpha_j u_j(x_0,y_0)$, to the solution $u(x_o, y_o)$ of the harmonic boundary value problem (5.1) -(5.2) at a given point $(x, y) \in \Omega$ is that which matches the boundary conditions (5.2) at the abscissae t_i^*, i = 1,2, ... n.

Theorem 5.1. The error in the near-best expansion approximation $\bar{u}(x_0, y_0)$ to $u(x_0, y_0)$ is given by

$$u(x_0,y_0) - \bar{u}(x_0,y_0) = e_{(x_0,y_0)}\left(h - \sum_{j=1}^{n} \alpha_j h_j\right) \tag{5.6}$$

where

$$h(x,y) = \begin{cases} \frac{\partial}{\partial n} G(x,y;x_0,y_0)\, f(x,y) & (x,y) \quad \partial\Omega_1 \in H(x_0,y_0) \\ G(x,y;x_0,y_0)\, g(x,y) & (x,y) \quad \partial\Omega_2 \end{cases}$$

and

$$h_j(x,y) = \begin{cases} \frac{\partial}{\partial n} G(x,y;x_0,y_0)\, f_j(x,y) & (x,y) \quad \partial\Omega_1 \in H(x_0,y_0) \\ G(x,y;x_0,y_0)\, g_j(x,y) & (x,y) \quad \partial\Omega_2 \end{cases}$$

5.2. The singularities of the Green's function of harmonic problems.

The application of the near-best approximation, of definition 5.1, at a given point (x_0, y_0) requires knowledge of the set $H(x_0, y_0)$, i.e. it requires knowledge of the behaviour of $\frac{\partial}{\partial n} G(x,y;x_0,y_0)$ on $\in \Omega$ and of $G(x,y;x, y)$ on $\partial\Omega_2$. ln general the problem of finding G is in itself as difficult as that of finding u. However, in many cases, sufficient information about G is available to permit the determination of the set $H_{(x_0,y_0)}$ and, in some cases, to provide a global near-test approximation to u.

We recall that $G(x,y;x_0, y_0)$ is defined as a harmonic function with a logarithmic singularity at (x_0, y_0). This singularity is, in general, the main source of the singularities of G and $\frac{\partial G}{\partial n}$ near the boundary $\partial\Omega$. We shall not discuss here other singularities that may occur due to a special geometry of $\partial\Omega$.

In general, apart from the singularity at (x_0, y_0) there is also at least one singular point of G and $\frac{\partial G}{\partial n}$ outside ft, at a 'mirror-image' of (x_0, y_0) with respect to $\partial\Omega$. For example, the Green's function of the Dirichlet problem in the half plane $y \geq 0$ is

$$G(x,y;x_0,y_0) = -\frac{1}{2\pi}\{\log[(x-x_0)^2 + (y-y_0)^2]^{\frac{1}{2}} - \log[(x-x_0)^2 + (y+y_0)^2]^{\frac{1}{2}}\}. \tag{5.7}$$

Hence, $\frac{\partial G}{\partial n}$ is singular at (x_0, y_0) and also at its image $(x_0, -y_0)$.

Another classical example is the Green's function of the Dirichlet problem in the disc $x^2 + y^2 \le R^2$, which is given by

$$G(x,y;x_0,y_0) = -\frac{1}{2\pi}\{\log r - \log[\,(x-\frac{R^2x_0}{r^2})^2 + (y-\frac{R^2Y_0}{r^2})\,]^{\frac{1}{2}}\}\ , \qquad (5.8)$$

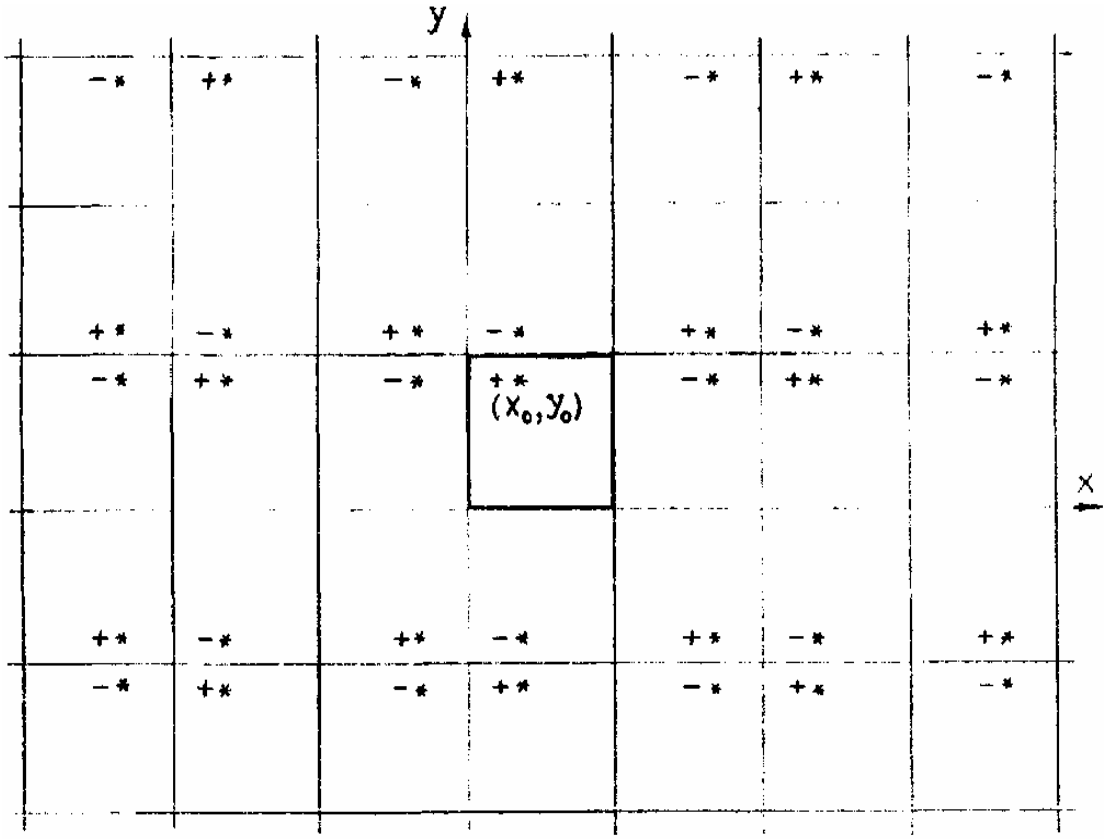
where r = [(x-x)^2 +(y-y)^2]½. In this case, $\frac{\partial G}{\partial n}$ has a singularity at the point $(\frac{R^2x_0}{r^2}, \frac{R^2y_0}{r^2})$ which is the image of $(x_0,y_{0)}$ with respect to the circle $x^2+y^2 = R^2$.

The method of images can be used to investigate the singularities G and $\frac{\partial G}{\partial n}$ for some problems defined in domains whos of e boundaries are composed of straight line segments and circular arcs. To illustrate this we consider the simple, yet non-trivial, case of the Dirichlet problem in the unit square Ω = {{x,y)|0 <x, y< 1 }.

Reflecting (x ,y) ∂Ω , with respect to the four sides of the square as mirrors, and assigning appropriate signs to the images so that their combined contribution vanishes on ∂Ω.

Figure 5.1

The singularities of the Green's function of the Dirichlet problem in the unit square.



We obtain the representation

$$G(x,y;y_0) = \frac{1}{4\pi} \sum_{i,j=-\infty}^{n} \{ \log [(x-x_0+2i)^2 + (y-y_0+2j)^2] \qquad (5.9)$$
$$- \log [(x+x_0+2i)^2 + (y-y_0+2j)^2]$$
$$- \log [(x-x_0+2i)^2 + (y+y_0+2j)^2]$$
$$+ \log [(x+x+2i)^2 + (y+y+2j)^2]\} ,$$

for the Green's function. Although this representation is not convenient for the actual computation of G, it can be used to investigate the singularities of $\frac{\partial G}{\partial n}$ near $\partial\Omega$. In fact G can be written as

$$G(x,y;x_0,y_o) = S(x,y;x_0,y_o) + R(x,y;x_0,y_o)$$

where (5.10)

$$S(x,y;x_0,y_0 = \frac{1}{4\pi} \{ \log [(x-x_0)^2 + (y-y_0)^2] - \log t(x+x_0)^2+(y-y_0)^2] \qquad (5.11)$$
$$- \log [(x-x_0)^2+(y+y_0)^2 3 + \log [(x+x_0)^2+(y+y_0)^2]$$
$$- \log C(x+x_o-2)^2+(y-y_o)^2] + \log [(x+x-2)^2+(y+y_0)^2]$$
$$- \log [(x-x_0)^2 +(y+y_0-2)^2]+ \log[(x+x_0)^2+(y+y_0-2)^2]$$
$$+ \log [(x+x_0-2)^2+ (y+y_0-2)^2],$$

and $\frac{\partial}{\partial n} R(x,y;x_0,y_0)$ is regular in the neighbourhood of $-1 \le x,y \le 2$ of .

The above example may serve as a model for the qualitative nature of the class $H(x_0,y_0)$ Thus, apart from a singularity at (x_0,y_0), which can always be removed, there are also additional singularities located outside Ω, which approach $\partial\Omega$ as (x_0,y_0) approaches $\partial\Omega$. These singularities play the same role as that of the discontinuities of $\Gamma(x,t;\lambda)$ and G(x,t) discussed in chapters 3 and 4. That is, their presence makes the best quadrature rules on $H(X_0,Y_0)$ depend strongly upon (x_0,y_0) and for this reason, the 'near-best' property (5.6) of the near-best approximation $\bar{u}$ holds only in a small neighbourhood of the point (x_0,y_0) for which it is defined.

5.3 A global near-best approximation

A global near-best approximation to the solution of the harmonic boundary value problem (5.1) -(5.2) can be obtained, provided that all the singularities of the Green's function of the problem, in a sufficiently large neighbourhood of W, are known. Then, as in the cases of integral equations and of two-point boundary value problems, the singularities can be used to 'correct' the expansion approximation.

Let the Green's function of the harmonic boundary value problem (5.1) -(5.2) be given in the form (5.10) and, associated with the function R, define a set of functions H on W by,

$$H=\left\{h\left|h(x,y)=\begin{cases}\dfrac{\partial}{\partial n}R(x,y;x_0,y_0)\varphi\varphi(xy) & ,\ \varphi\in F \qquad (x,y)\in\partial\Omega_1\\ R(x,y;x_0,y_0)\psi\psi(xy) & ,\ \psi\in G \qquad (x,y)\in\partial\Omega_2\end{cases}\right.\right\} \tag{5.12}$$

In order to define an nth order global near-best approximation to u we require that S in (5•10) is a known function, and that there exists a best nth order quadrature formula, $\sum_{i=1}^{n} w_i^* h(t_i^*)$, independent of (x_0, y_0), approximating $\int_{\partial\Omega} h(x,y)ds$, $h\in H$. The last condition is in fact equivalent to the assumption that $R(x,y;x_0,y_0)$ is regular in a sufficiently large neighbourhood of W, independent of (x_0, y_0).

Definition 5.2. An nth order global near-best approximation, $\bar{u}_c$, to the solution u of the harmonic boundary value problem (5.1) - (5.2) is defined as

$$\bar{u}_c(x_0,y_0)=\sum_{j=1}^{n}\alpha_j u_j(x_0,y_0) \tag{5.13}$$

$$+\int_{\partial\Omega_1}\frac{\partial}{\partial n}S(x,y;x_0,y_0)[f(x,y)-\sum_{j=1}^{n}\alpha_j f_j(x,y)]ds$$

$$+\int_{\partial\Omega_1}S(x,y;x_0,y_0)[g(x,y)-\sum_{j=1}^{n}\alpha_j g_j(x,y)]ds,$$

where the expansion approximation $\sum_{j=1}^{n} \alpha_j u_j$ matches the boundary condition:; (5.2) at t_i^*, $i = 1,2,\ldots,n$, the abscissae of the best nth order quadrature formula on H. The two integrals in (5.13) constitute the correction term of the collocation approximation $\sum_{j=1}^{n} \alpha_j u_j$

Theorem 5.2. The error in the global near-best approximation (5.13) to u is given by

$$u(x_0, y_0) - \bar{u}_c(x_0, y_0) = e(h - \sum_{j=1}^{n} \alpha_j h_j) \tag{5.14}$$

where e is the error functional of the best nth order quadrature formula on H,

$$h(x,y) = \begin{cases} \frac{\partial}{\partial n} R(x,y;x_0,y_0) f(x,y) & (x,y) \in \partial\Omega_1 \\ R(x,y;x_0,y_0) g(x,y) & (x,y) \in \partial\Omega_2 \end{cases} \in H$$

and

$$h_j(x,y) = \begin{cases} \frac{\partial}{\partial n} R(x,y;x_0,y_0) f_j(x,y) & (x,y) \in \partial\Omega_1 \\ R(x,y;x_0,y_0) g_j(x,y) & (x,y) \in \partial\Omega_2 \end{cases} \in H$$

5.4 Numerical example - Harmonic problems

We consider the Dirichlet problem

$$\begin{cases} \Delta u(x,y) = 0 & (x,y) \in \Omega \\ u(x,0) = \phi!(x)\ , u(1,y) = \phi_2(y), u(x,1) = \phi_3(x), u(0,y) = \phi_4(y) \end{cases} \tag{5.15}$$

where W is the unit square and $\phi_i \in C^{\infty}[0,1]$,i = 1,4.

The singular part S of the Green's function of this problem, as given by (5.11), satisfies the conditions required to provide a global near-best approximations in definition 5.2 . Best quadrature formulae, on the corresponding set H defined by (5.12),

can be composed of four Gaussian formulae on the four sides of the square. Hence, a (4m)th order global near-best approximation may be defined by (5.13) using the m Gaussian points on each of the sides of the square as collocation points. The error in the resulting approximation may then be expressed as

$$u(x_0, y_0) - \bar{u}_c(x_0, y_0) = e_m^{[0,1]}(\phi)$$

where $\phi \in C^{\infty}[0,1]$ has no singularities near [0,1].

We consider 16th order approximations using the sixteen harmonic polynomials 1, $\mathrm{Re}(x+iy)^j$ $j=1,\ldots,8$, $\mathrm{Im}(x+iy)^j$ $j=2,\ldots,8$. The function y is omitted since the system of equations for the a_j.'s in (5.13) becomes singular if both x and y are included). The collocation points in definition 5.2 are taken as the four Gaussian points on each of the sides of the square, and the correction term $\int_{\partial\Omega} \frac{\partial}{\partial n} S(x, y; x_0, y_0)[f(x,y) - \sum_{j=1}^{16} \alpha_j f_j(x,y)]ds$ is approximated by using Simpson's rule with $h = 0.01$ on each side of the square. The singularities of S present some difficulties in the numerical computation of the correction term when (x_0, y_0) is near W. However, the effect of these singularities can be easily removed when $(x_0, y_0) \in \partial\Omega$ using the fact that on W the Green's function reproduces the boundary conditions. Since $\bar{u}_c$ is a harmonic function, we obtain bounds for $\max_{(x,y)\in\Omega} |u - \bar{u}_c|$ by evaluating the maximum error on $\partial\Omega$.

To demonstrate the importance of including the correction term we examine two cases in which polynomial approximations do not perform well.

Case 1. Problem (5-15) with $\phi_1(x) = \sin 5x$, $\phi_2(y) = \sin 5e^{5y}$, $\phi_3(x) = \sin 5xe^5$ and $\phi_4(y) = 0$. The solution of this is $u(x,y) = \sin 5xe^{5y}$

Collocation at the four Gaussian points on each side of the square gives a maximum error of ~ 1.55. Addition of the correction term reduces the maximum error to ~ 0.0073.

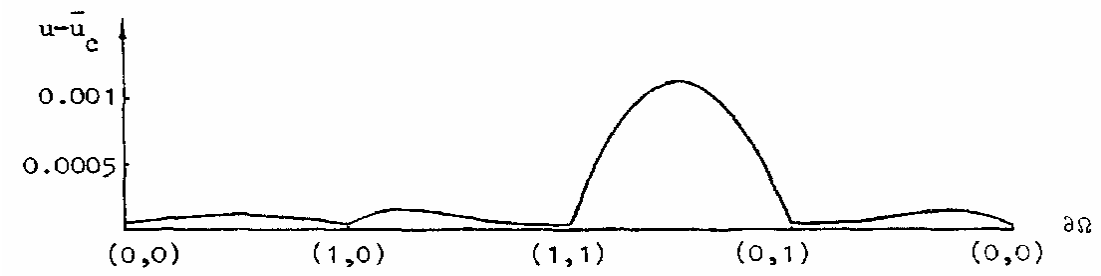
Case 2. Problem (5.15) with ($\phi_1 = \phi_2 = \phi_4 = 0$ and $\phi_3 = 1$.

In this case the boundary conditions are not continuous at the corners (0,1) and (1,1) of the square. Hence, any polynomial approximation has a maximum error of at least 0.5. However, the 16th order corrected approximation $\bar{u}_c$ gives maximum error of ~ 0.00125.

It turns out that the correction term cancels the oscillatory nature of the error in the collocation approximation- For example, in case 2, although the boundary conditions are discontinuous at the corners (0,1) and (1,1), the error in $\bar{u}_c$ is positive, non-oscillatory on the four sides of the square and continuous at the corners; see figure 5-2.

Figure 5.2

The error in the approximation $\bar{u}_c$ for the solution of the example in case 2.



This behaviour of the error in $\bar{u}_c$ on the boundary suggests that a good approximation to this error might be obtained by collocatior with harmonic polynomials. In fact this has been found to be very effective, e.g. an approximation to the error in the 16th order approximation $\bar{u}_c$ for case 2, obtained by collocating with the above harmonic polynomials at the four Gaussian points on each side of the square, reduces the error to $\sim 10^{-6}$.

We note that global near-best approximations have also been found to be very efficient in dealing with some mixed boundary value problems in rectangular domains. For details see [4].

Acknowledgments. The author wishes to thank Dr.N.Papamichael for his assistance in the preparation of this paper.

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