LIMIT THEOREMS FOR A CLASS OF UNBOUNDED OBSERVABLES WITH AN APPLICATION TO "SAMPLING THE LINDELÖF HYPOTHESIS"

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Abstract. We prove the Central Limit Theorem (CLT), the first order Edgeworth Expansion and a Mixing Local Central Limit Theorem (MLCLT) for Birkhoff sums of a class of unbounded heavily oscillating observables over a family of full-branch piecewise C^2 expanding maps of the interval. As a corollary, we obtain the corresponding results for Boolean-type transformations on $\mathbb{R}$. The class of observables in the CLT and the MLCLT on $\mathbb{R}$ include the real part, the imaginary part and the absolute value of the Riemann zeta function. Thus obtained CLT and MLCLT for the Riemann zeta function are in the spirit of the results of Lifschitz & Weber [30] and Steuding [41] who have proven the Strong Law of Large Numbers for *sampling the Lindelöf hypothesis*.

Contents

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1. Introduction

The study of the statistical properties of dynamical systems has a long and rich history, dating back to the works of Maxwell and Boltzmann that introduced the *ergodic hypothesis*. In fact, a whole facet of ergodic theory, which originated with the (almost simultaneous) publication of the well-known ergodic theorems of Birkhoff and von Neumann in the early 1930s providing evidence to ergodic hypothesis, is concerned with establishing limit laws such as the Central Limit Theorem (CLT), and Large Deviation Principles (LDPs) for sufficiently chaotic dynamical systems. These limit laws describe the behaviour of a dynamical system over a long period of time and can provide important insights into the properties of the system.

Expanding maps of the unit interval are the most elementary class of dynamical systems that exhibit chaotic behaviour and there is a vast literature on limit theorems for Birkhoff sums of expanding maps. For example, in [37], the CLT is established for observables with bounded variation (BV) over piecewise uniformly expanding maps whose inverse derivative is also BV. We refer the reader to [6] for a review of limit theorems for transformations of an interval. In [10], Edgeworth expansions describing the error terms in the CLT are established in the case of BV observables over C^2 covering uniformly expanding maps. Since the observables are BV, this result is limited to bounded observables.

One standard technique of establishing limit theorems for dynamical systems is the Nagaev-Guivarc'h spectral approach which was first introduced by Nagaev in the Markovian setting in [35] and later adapted to deterministic dynamical systems by Guivarc'h in [16]. The key idea is to code the characteristic function using iterated twisted transfer operator (one can think of this as the deterministic counterpart of the dual of the Markov operator in the Markovian setting) and to analyze the the spectral data of these family of operators in a suitable Banach space, see [15] for details. Though transfer operator techniques to handle unbounded observables are available, see for example, [20, 4, 32, 11], they have not been applied to obtain limit theorems for uniformly expanding maps of the interval. In this paper, we introduce a class of Banach spaces that are not contained in L^∞ for which the conditions introduced in [20, 11] can be verified. In particular, we establish the CLT, its first order correction – the order 1 Edgeworth expansion, and a Mixing Local Central Limit Theorem (MLCLT) for the Birkhoff sums of a class of *unbounded heavily oscillating observables* over a family of full-branch piecewise C^2 uniformly expanding maps of the interval.

While providing a class of elementary examples where the theory developed in [11] for limit theorems for unbounded observables can be applied, these results pave the way to obtain further results on *sampling the Lindelöf hypothesis* which is a line of research in analytic number theory that deals with understanding the properties of the Riemann zeta function on the critical strip. We elaborate on this below.

Let $\zeta : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ be the Riemann zeta function defined by $\zeta(s) = \sum n^{-s}, \Re(s) > 1$ and by analytic continuation elsewhere except $s = 1$. The Lindelöf hypothesis states that the Riemann zeta function does not grow too quickly on the critical line $\Re z = 1/2$. More precisely, it is conjectured that

$$\zeta_{1/2}(t) := \zeta\left(\frac{1}{2} + it\right) = \mathcal{O}(t^\varepsilon), \quad t \to \pm\infty$$

for all $\varepsilon > 0$, i.e., $\lim_{t\to\pm\infty} |\zeta_{1/2}(t)|/t^\varepsilon < \infty$. To date, the best estimates are due to Bourgain in [3] where it is proved that this is true for all $\varepsilon > 13/84 \approx 0.154$. It is worth noting that the Riemann hypothesis implies the Lindelöf hypothesis and the latter is a substitute for the former in some applications.

Since the conjecture is related to the value distribution of $\zeta_{1/2}(t)$ as $t \to \pm\infty$, to study ergodic averages of $\zeta_{1/2}$ when sampled over the orbits of heavy-tailed stochastic processes was initiated by Lifschitz and Weber in [30]. In particular, they prove that when $\{Y_j\}_{j\ge 0}$ are independent Cauchy distributed random variables and $X_k = \sum_{j=0}^{k-1} Y_j$ (the Cauchy random walk), then for all $b > 2$,

$$\frac{1}{n}\sum_{k=0}^{n-1} \zeta_{1/2}(X_k) = 1 + o\left(\frac{(\log n)^b}{\sqrt{n}}\right),\ n \to \infty,$$

almost surely, where we denote $a_n = o(b_n)$ if $\lim_{n\to\infty} |a_n|/b_n = 0$. This work was later generalized by Shirai, see [39], where X_k was taken to be a symmetric α-stable process with $\alpha \in [1, 2]$. Since X_k are heavy tailed, i.e., $\mathbb{E}(|X_k|^p) = \infty$ when $p = \lceil \alpha \rceil$, the α-stable process samples large values with high probability. So, this result illustrates that the values of $\zeta_{1/2}(t)$ are small on average even for large values of $|t|$.

Similarly, in the deterministic setting, the Birkhoff sums

$$\sum_{k=0}^{n-1} \zeta_{1/2}(\phi^k x) \tag{1.1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ the Boolean-type transformation given by $\phi(0) = 0$ and

$$\phi(x) = \frac{1}{2}\left(x - \frac{1}{x}\right), \quad x \neq 0$$

are studied in [41]. Since ϕ preserves the ergodic probability measure $\mathrm{d}\mu = \frac{\mathrm{d}x}{\pi(1+x^2)}$ (the law of a standard Cauchy random variable) and $\zeta_{1/2}$ is integrable with respect to μ, it follows from Birkhoff's point-wise ergodic theorem that for almost every (a.e.) $x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \zeta_{1/2}(\phi^k x) = \int \zeta_{1/2}(x)\frac{\mathrm{d}x}{\pi(1+x^2)} = \zeta_{1/2}(3/2) - 8/3 \approx -0.054\,. \tag{1.2}$$

This too illustrates that most of the values of $\zeta_{1/2}$ are not too large, and hence, provides evidence in favour of the Lindelöf hypothesis.

Sampling the Lindelöf hypothesis has two other theoretical underpinnings. On the one hand, it is known that the Lindelöf hypothesis is true if and only if for all $m \in \mathbb{N}$ and for a.e. $x \in \mathbb{R}$, the following limit exists

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} |\zeta_{1/2}(\phi^k x)|^{2m} = \int |\zeta_{1/2}(x)|^{2m}\frac{\mathrm{d}x}{\pi(1+x^2)}.$$

On the other hand, the Riemann hypothesis is true if and only if for a.e. $x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \log|\zeta_{1/2}((\phi^k x)/2)| = 0.$$

In both cases, evidence can be gathered numerically, see [41, Theorems 4.1 and 4.2] for details.

The results by Steuding have also been generalized, both by replacing ζ and replacing ϕ: in [8], Elaissaoui and Guennoun used $\log|\zeta|$ as the observable and a slight variation of ϕ as the transformation, and in [29], Lee and Suriajaya studied different classes of meromorphic functions such as Dirichlet $L-$functions or Dedekind ζ functions while taking ϕ to be an affine version of the Boolean-type transformation. Maugmai and Srichan gave further generalizations of these results, see [34]. It must also be mentioned that these transformations ϕ have been studied earlier in a solely ergodic theoretic context by Ishitani(s) in [22, 23].

To further understand the value distribution of the Birkhoff averages given by (1.1) around their asymptotic mean $A = \zeta_{1/2}(3/2) - 8/3$, and in turn, the values of $\zeta_{1/2}$, the crucial next step is the study of the CLT and MLCLT. In [40], the second author establishes the CLT: There exists $\sigma^2 > 0$ such that

$$\frac{1}{\sqrt{n}}\left(\sum_{k=0}^{n-1}\zeta_{1/2}(\phi^k(\cdot)) - nA\right) \xrightarrow{d} \mathcal{N}(0,\sigma^2) \tag{1.3}$$

where $\xrightarrow{d}$ denotes the convergence in distribution and $\mathcal{N}(0,\sigma^2)$ is the centered normal random variable with variance σ^2. However, there was a critical mistake in the proof: the normed vector space considered there in order to study the spectrum of the transfer operator is not complete. In this paper, we not only correct this mistake but also establish a MLCLT for (1.1). Further, we provide conditions for the 1st order Edgeworth expansion to hold. Even though the state of the art on $\zeta_{1/2}$ is not sufficient to verify these conditions, a slight improvement of results in [3] will provide us what is required.

The proofs of the CLT, MLCLT and Edgeworth expansion are based on two key ideas: the spectral techniques introduced in [11] and the existence of a smooth conjugacy between the doubling map on the unit interval and ϕ. In fact, we consider an increasing sequence of Banach spaces on each of which the twisted transfer operators corresponding to full-branch C^2 expanding maps satisfy Doeblin-Fortet Lasota-Yorke (DFLY) inequalities and other good spectral properties, prove limit theorems for the expanding maps, and finally, deduce the limit theorems for ϕ via the conjugacy. In doing so, we introduce a novel class of Banach spaces that can be used to study Birkhoff sums of unbounded and highly oscillatory observables. Further, the class of dynamical systems we consider is sufficiently rich. The restriction to full branch maps was done in order to simplify the computations.

The Banach spaces introduced in [4, 32] are seemingly more general than the Banach spaces we introduce. In fact, in our case, the observables can have non-removable discontinuities only at the fixed points of the map. However, to obtain results for sampling the Lindelöf hypothesis, we have to consider observables $\chi : (0,1) \to \mathbb{R}$ such that

$$|\chi| \lesssim x^{-a}(1-x)^{-a} \quad \text{and} \quad \max\{|\chi'(x+)|, |\chi'(x-)|\} \lesssim x^{-b}(1-x)^{-b}$$

for some $a, b > 0$. In particular, we consider real and imaginary parts of

$$\zeta_{1/2} \circ \xi : (0,1) \to \mathbb{R} \quad \text{where} \quad \xi(x) = \cot(\pi x) .$$

But it is not clear whether such observables or even more elementary observables like $x^{-c}\sin(1/x)$, $c > 0$ belong to Banach spaces in the literature [33]. It is worth mentioning that observables with a non removable singularity at the fixed point are particularly interesting: once an orbit lands close to a fixed point, a few subsequent iterates might stay relatively close to the fixed point and the Birkhoff sum might be very large locally. Alternatively, such situations can cause the system behave qualitatively different from the independently and identically distributed (IID) setting, see for example, [28, Theorem 1.10].

The structure of the paper is as follows: Section 2 is dedicated to preliminaries and main results: in Section 2.1, we introduce the relevant notation and common definitions that we will use throughout the paper, in Section 2.2, we state precisely the class of expanding maps we consider, Section 2.3 we introduce our Banach spaces, in Section 2.4, we state our main results for the interval maps, and in Section 2.5, we state the corresponding results for the Boolean transformation on $\mathbb{R}$ and their implications to sampling the Lindelöf hypothesis. In Section 3, we recall known abstract results in [20, 11] tailored (with justifications) to our setting. The desirable properties of the Banach spaces we introduce are discussed in Section 4 and the spectral properties of twisted transfer operators acting on these spaces including the DFLY inequality are established in Section 5. In Section 6, we collect the proofs of our main results. In particular, the proofs of the limit theorems for interval maps appear in Section 6.1 and in Section 6.2 we prove the corresponding results for the Boolean-type transformation. Finally, we have relegated some technical results to the Appendices.

2. Main Results

2.1. **Preliminaries.** Let X be a metric space with a reference Borel probability measure m, and let $T : X \to X$ be a non-singular dynamical system, i.e., for all $U \subseteq X$ Borel subsets $m(U) = 0$ holds if and only if $m(T^{-1}U) = 0$ holds. We denote by $\mathcal{M}_1(X)$ the set of Borel probability measures on X. Let $\nu \in \mathcal{M}_1(X)$. For $p \geq 1$, by $L^p(\nu)$, we denote the standard Lebesgue spaces with respect to ν, i.e.,

$$L^p(\nu) = \{h : X \to X \,|\, h \,\text{is Borel measurable}, \, \nu(|h|^p) < \infty\}$$

where the notation $\nu(h)$ refers to the integral of a function h with respect to a measure ν and the corresponding norm is denoted by $\|\cdot\|_{L^p(\nu)}$. When $\nu = m$, we often write, L^p instead of $L^p(m)$ and $\|\cdot\|_p$ instead of $\|\cdot\|_{L^p(m)}$.

For us, an observable is a real valued function $f \in L^2$ and we consider the Birkhoff sums (also commonly referred to as ergodic sums),

$$S_n(f,T) = \sum_{k=0}^{n-1} f \circ T^k \tag{2.1}$$

which we denote by $S_n(f)$ when the dynamical system T is fixed.

We say $\widehat{T} : L^1 \to L^1$ is the transfer operator of $\widehat{T}$ with respect to m, if for all $f \in L^1$ and $f^* \in L^\infty$,

$$m(\widehat{T}(f) \cdot f^*) = m(f \cdot f^* \circ T). \tag{2.2}$$

Let $\mathfrak{m} \in \mathcal{M}_1(X)$ be absolutely continuous with respect to m with density $\rho_{\mathfrak{m}}$. Then, from (2.2), it follows that

$$\mathbb{E}_{\mathfrak{m}}(e^{isS_n(f)}) = m\left(\widehat{T}^n_{is}(\rho_{\mathfrak{m}})\right) \tag{2.3}$$

where $\mathbb{E}_{\mathfrak{m}}$ is the expectation with respect to the law of S_n where the initial point x is distributed according to $\mathfrak{m}$ and

$$\widehat{T}_{is}(\cdot) = \widehat{T}(e^{isf}\,\cdot), \;\; s \in \mathbb{R}\,, \tag{2.4}$$

see, for example, [19, Chapter 4]. Eventually, we are interested in the asymptotics of quantities of the form $\mathfrak{m}(S_n(f) \leq z_n)$ and $\mathbb{E}_{\mathfrak{m}}(V_n(S_n(f)))$ as $n \to \infty$ where $z_n \in \mathbb{R}$ and $V_n : \mathbb{R} \to \mathbb{R}$ are from a suitable class of functions, and to obtain these asymptotics we exploit the relation (2.3).

We denote

$$A(f,t) = \lim_{n\to\infty} \mathbb{E}_{\mathfrak{m}}\left(\frac{S_n(f,T)}{n}\right) \quad \text{and} \quad \sigma^2(f,T) = \lim_{n\to\infty} \mathbb{E}_{\mathfrak{m}}\left(\frac{S_n(f,T) - n\,A(f,T)}{\sqrt{n}}\right)^2$$

for the asymptotic mean and the asymptotic variance of Birkhoff sums, $S_n(f)$, respectively. Then, it can be seen that, under the assumptions we impose on T in Section 2.2, A and σ^2 are independent of the choice of $\mathfrak{m}$; see, for example, [12, Lemma 3.4]. In particular, under our assumptions there will be a unique absolutely continuous invariant measure (acip), say $\bar{\mathfrak{m}}$, then $A(f,T) = \bar{\mathfrak{m}}(f)$. So, we can focus on zero average observables by considering $\overline{f} := f - A$ instead of f.

We call f to be $T-$*cohomologous to a constant* if there exist $\ell \in L^2$ and a constant c such that

$$f = \ell \circ T - \ell + c$$

and $T-$*coboundary* if there exists $\ell \in L^2$ such that

$$f = \ell \circ T - \ell\,.$$

We say f is *non-arithmetic* if it is not $T-$cohomologous in L^2 to a sublattice-valued function, i.e., if there exists no triple (γ, B, θ) with $\gamma : X \to \mathbb{R}$, B a closed proper subgroup of $\mathbb{R}$ and a constant θ such that $f + \gamma - \gamma \circ T \in \theta + B$.

Given a Banach space $\mathcal{B}_1$, the $\mathbb{C}-$valued continuous linear functionals are denoted by $\mathcal{B}_1'$ and given another Banach space $\mathcal{B}_2$, $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ denotes the space of bounded linear operators from $\mathcal{B}_1$ to $\mathcal{B}_2$. When $\mathcal{B}_1 = \mathcal{B}_2$, we write $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_1)$ as $\mathcal{L}(\mathcal{B}_1)$. When $\mathcal{B}_1 \subset \mathcal{B}_2$, $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ denotes continuous embedding of Banach spaces, i.e., there exists $\mathfrak{c} > 0$ such that $\| \cdot \|_{\mathcal{B}_2} \leq \mathfrak{c} \| \cdot \|_{\mathcal{B}_1}$.

Given a set $D \subseteq X$, its complement $X \setminus D$ is denoted by D^c, and $\mathring{D}$ denotes its interior. Given a function $f : D \to \mathbb{R}$ set $f_+ := \max\{f, 0\}$ and $f_- := \max\{-f, 0\}$. Given $g : D \to \mathbb{R}$, $g \lesssim f$ denotes that there exists constant $K > 0$ such that $g(x) \leq K f(x)$, for all $x \in D$. Let Q_1, Q_2 be $\mathbb{R}_0^+$ valued functionals acting on a class of functions $\mathfrak{G}_1$ and $\mathfrak{G}_2$, the inequality $Q_1(g) \lesssim Q_2(h)$ for all $g \in \mathfrak{G}_1$ and $h \in \mathfrak{G}_2$ is written to denote that there exists K independent of the choices of g and h such that $Q_1(g) \leq K Q_2(h)$. Finally, given two numbers $a, b \in \mathbb{R}$, $a \approx b$ means that $0 \leq a - b \leq 10^{-3}$.

We denote the standard Gaussian density and the corresponding distribution function by

$$\mathfrak{n}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{and} \quad \mathfrak{N}(x) = \int_{-\infty}^{x} \mathfrak{n}(y)\, \mathrm{d}y\,,$$

respectively.

2.2. **The classes of dynamical systems.** Let $I = [0, 1]$ and λ the Lebesgue measure (on $\mathbb{R}$) and λ_I its restriction to I. We use λ_I as the reference measure on I and let $I = \bigcup_{j=0}^{k-1} [c_j, c_{j+1}]$ be a partition of I with $c_0 = 0$ and $c_k = 1$. We consider the class of maps $\psi : I \to I$ satisfying the following conditions.

(1) There are $\psi_{j+1} : [c_j, c_{j+1}] \to I$ such that for all j, $\psi_{j+1} \in C^2$, $|\psi_{j+1}'| > 1$, $\text{Range}(\psi_{j+1}) = I$ and

$$\psi_{j+1}|_{(c_j, c_{j+1})} = \psi|_{(c_j, c_{j+1})}\,.$$

(2) For all j, the derivative of ψ_{j+1}^{-1} is uniformly $\vartheta-$Hölder, i.e., there exists c such that for all j, for all $\varepsilon > 0$, for all $z \in I$ and for all $x, y \in B_\varepsilon(z) := [z - \varepsilon, z + \varepsilon] \cap [0, 1]$,

$$|(\psi_{j+1}^{-1})'(x) - (\psi_{j+1}^{-1})'(y)| \leq c |(\psi_{j+1}^{-1})'(z)| \varepsilon^{\vartheta}\,.$$

Remark 2.1. The full branch assumption was made in order to simplify our calculations. This does not exclude the doubling map - the interval map studied in [40] to further analyze the situation studied in [41].

Since these maps are C^2, Markov and topologically mixing, each map has one and only one acip and it is exact [13, Theorem 6.1.1]. We denote this acip by π. Since ψ_{j+1}' are C^1, there exists $\eta_+ < \infty$ such that

$$\max_j \|\psi_{j+1}'\|_\infty = \eta_+.$$

Also, since $|\psi_{j+1}'| > 1$, there exists $\eta_- > 1$ such that

$$\max_j \|(\psi_{j+1}^{-1})'\|_\infty = 1/\eta_-.$$

Without loss of generality we assume that $\psi' > 0$ and we have

$$\widehat{\psi}_{is}(\varphi)(x) = \sum_{j=0}^{k-1} \frac{e^{is\chi(\psi_{j+1}^{-1}x)}}{\psi'(\psi_{j+1}^{-1}x)} \varphi(\psi_{j+1}^{-1}x)\,, \tag{2.5}$$

see, for example, [19] for a proof of this fact.

2.3. **The Banach spaces.** For a measurable function $f\colon I \to \mathbb{C}$ and a Borel subset S of I, we define the oscillation on S by

$$\operatorname{osc}(f,S) \coloneqq \operatorname{osc}(\Re f, S) + \operatorname{osc}(\Im f, S),$$

where $\Re f$ and $\Im f$ refer to real and imaginary parts of f, respectively and we set $\operatorname{osc}(f,\emptyset) \coloneqq 0$. Also, note that up to a constant this is equivalent to the more intuitive definition

$$\overline{\operatorname{osc}}(f,S) \coloneqq \operatorname*{ess\,sup}_{x,y\in S} |f(x) - f(y)|\,.$$

This can be easily seen. We have $|f(x)-f(y)| \leq |\Re f(x) - \Re f(y)| + |\Im f(x) - \Im f(y)|$, and thus, $\overline{\operatorname{osc}}(f,S) \leq \operatorname{osc}(f,S)$. On the other hand, we have $\operatorname{osc}(f,S) \leq 2\max\{\overline{\operatorname{osc}}(\Re f,S), \overline{\operatorname{osc}}(\Im f,S)\} \leq 2\,\overline{\operatorname{osc}}(f,S)$. In what follows, we use osc as the standard definition.

For $\alpha\in\mathbb{R}$, define, R_α, an operator on the space of measurable functions by

$$R_\alpha f(x) \coloneqq \begin{cases} x^\alpha\cdot(1-x)^\alpha\cdot f(x) & \text{if } |f(x)|<\infty \\ 0 & \text{otherwise,}\end{cases}$$

denote by $B_\varepsilon(x)$ the ε-ball around x in I, and define a seminorm

$$|f|_{\alpha,\beta} \coloneqq \sup_{\varepsilon\in(0,\varepsilon_0]} \varepsilon^{-\beta}\int \operatorname{osc}\left(R_\alpha f, B_\varepsilon(x)\right)\,\mathrm{d}\lambda_I(x)\,,$$

where ε_0 is sufficiently small (to be chosen later). Let

$$\|\cdot\|_{\alpha,\beta,\gamma} \coloneqq \|\cdot\|_\gamma + |\cdot|_{\alpha,\beta}$$

and set

$$L^\gamma \coloneqq \left\{f\colon I\to\mathbb{C}\colon\ \|f\|_\gamma<\infty\right\},\qquad \mathsf{V}_{\alpha,\beta,\gamma} \coloneqq \left\{f\colon I\to\mathbb{C}\colon\ \|f\|_{\alpha,\beta,\gamma}<\infty\right\}.$$

Finally, by $\mathsf{V}'_{\alpha,\beta,\gamma}$ we denote the set of $\mathbb{C}-$valued continuous linear functionals on $\mathsf{V}_{\alpha,\beta,\gamma}$.

Remark 2.2. It is shown in Appendix A that for $\alpha\in[0,1)$, $\beta\in(0,1]$ and $\gamma\geq1$, $\mathsf{V}_{\alpha,\beta,\gamma}$ is a Banach space. Similar real Banach spaces were considered in [25, 2, 27]. In all these cases, their spaces correspond to our spaces with $\alpha=0$, and hence, are embedded in L^∞; see Lemma A.4.

Due to the dampening operation R_α, which was first introduced in [40], the functions in $\mathsf{V}_{\alpha,\beta,\gamma}$ may be unbounded and oscillate heavily near 0 and 1. We remark that depending on the application one could consider different damping operators and use the ideas presented here to prove limit theorems.

2.4. **Results for the unit interval.** Now, we are ready to state the limit theorems for $S_n(\chi) \coloneqq S_n(\chi,\psi)$ over dynamical systems ψ defined as in Section 2.2. Though we do not state this explicitly, it will later turn out that the χ specified in the following theorems belongs to an appropriate $\mathsf{V}_{\alpha,\beta,\gamma}$.

We first state the CLT in the stationary case.

Theorem 2.3. *Suppose χ is continuous and the right and left derivatives of χ exist on $\mathring{I}$, χ is not a coboundary and there exist constants $a,b>0$ such that*

$$|\chi| \lesssim x^{-a}(1-x)^{-a} \quad \text{and} \quad \max\{|\chi'(x+)|,|\chi'(x-)|\} \lesssim x^{-b}(1-x)^{-b}\,. \tag{2.6}$$

Assume

$$a < \min\left\{\vartheta,\frac{1}{b},\frac12\right\}\cdot\min\left\{1,\frac{\log\eta_-}{\log\eta_+}\right\}. \tag{2.7}$$

Then, the following Central Limit Theorem holds:

$$\pi\left(\frac{S_n(\chi)-n\,\pi(\chi)}{\sigma\sqrt{n}}\leq x\right) - \mathfrak{N}(x) = o(1), \quad \text{for all}\quad x\in\mathbb{R}\quad\text{as}\quad n\to\infty\,. \tag{2.8}$$

Now, we discuss sufficient conditions for the MLCLT.

Theorem 2.4. *Suppose χ is continuous and the right and left derivatives of χ exist on $\mathring{I}$, χ is not arithmetic and there exist constants $a, b > 0$ such that (2.6) and (2.7) are true. Then, $S_n(\chi)$ satisfies the following MLCLT:for all $0 < \alpha_0 < \alpha_1 < \beta$, $M \geq 1$, $U \in \mathsf{V}_{\alpha_0,\beta,M}$, $V : \mathbb{R} \to \mathbb{R}$ a compactly supported continuous function, $\mathfrak{m} \in \mathcal{M}_1(I)$ being absolutely continuous wrt λ_I, and $W \in L^1$ such that $(W \cdot \mathfrak{m}) \in \mathsf{V}'_{\alpha_1,\beta,M}$, we have*

$$\lim_{n\to\infty} \sup_{\ell\in\mathbb{R}} \left| \sigma\sqrt{2\pi n}\, \mathbb{E}_{\mathfrak{m}}(U \circ \psi^n\, V(S_n(\overline{\chi}) - \ell)\, W) - e^{-\frac{\ell^2}{2n\sigma^2}}\, \mathbb{E}_{\mathfrak{m}}(W)\, \mathbb{E}_{\pi}(U) \int V(x)\, \mathrm{d}x \right| = 0\,. \tag{2.9}$$

Remark 2.5. In particular, it is possible to choose $\mathfrak{m} = \pi$ for all $W \in L^{\bar{M}}$ where $M^{-1} + \bar{M}^{-1} = 1$. In fact, under our assumptions, there exists $\rho \in \mathrm{BV}$ such that $\pi = \rho\lambda_I$; see, for example, [31]. Therefore, $|W \cdot \pi(h)| = \left|\int (hW)\rho\, \mathrm{d}\lambda_I\right| \leq \|\rho\|_\infty |Wh|_{L^1} \leq \|\rho\|_\infty \|W\|_{\bar{M}} \|h\|_M \leq C\|h\|_{\alpha_1,\beta,M}$ with $C = \|\rho\|_\infty \|W\|_{\bar{M}}$, and hence, $W \cdot \pi \in \mathsf{V}'_{\alpha_1,\beta,M}$ as required.

Next, we discuss the first order asymptotics of the CLT with no assumptions on the stationarity. In particular, under the conditions of the theorem, we have the CLT for initial measures that are not necessarily invariant.

Theorem 2.6. *Suppose χ is continuous and the right and left derivatives of χ exist on $\mathring{I}$, χ is arithmetic and there exist constants $a, b > 0$ such that (2.6) and*

$$3\min\{2a, \max\{a, a+b-2\}\} < \min\left\{\vartheta, \frac{1}{b}, \frac{1}{2}\right\} \cdot \min\left\{1, \frac{\log\eta_+}{\log\eta_-}\right\}. \tag{2.10}$$

are true. Then, $S_n(\chi)$ satisfies the first order Edgeworth expansion, i.e., for all $\mathfrak{m} \in \mathcal{M}_1(I)$ being absolutely continuous wrt λ_I there exists a quadratic polynomial P whose coefficients depend on the first three asymptotic moments of $S_n(\chi)$ but not on n such that

$$\sup_{x\in\mathbb{R}} \left| \mathfrak{m}\left(\frac{S_n(\chi) - n\,\pi(\chi)}{\sigma\sqrt{n}} \leq x\right) - \mathfrak{N}(x) - \frac{P(x)}{\sqrt{n}}\mathfrak{n}(x)\right| = o(n^{-1/2}), \quad \text{as} \quad n \to \infty\,.$$

Remark 2.7. Note that from (2.10) and (2.6) with the corresponding choices of a and b it follows that $\chi \in L^3$. So, $\mathbb{E}_{\mathfrak{m}}(|S_n(\chi)|^3) < \infty$ for each n. Our proof shows that the third asymptotic moment

$$\lim_{n\to\infty} \mathbb{E}_{\mathfrak{m}} \left(\frac{S_n(\chi) - n\,\pi(\chi)}{\sqrt{n}}\right)^3$$

does, indeed, exist.

Finally, we provide a concrete example of a class of observables that satisfies our conditions.

Example 2.8. *Let $\chi(x) = x^{-c}\sin(1/x)$ and define $\tilde{\eta} = \min\left\{1, \frac{\log\eta_-}{\log\eta_+}\right\}$.*

(1) *If $0 \leq c < \min\left\{\sqrt{1+\tilde{\eta}} - 1, \vartheta\tilde{\eta}\right\}$, then $S_n(\chi)$ satisfies the CLT and MLCLT.*
(2) *If $0 \leq c < \min\{\sqrt{1+\tilde{\eta}/6} - 1, \vartheta\tilde{\eta}/6\}$, then $S_n(\chi)$ admits the first order Edgeworth Expansion.*

If ψ is the doubling map, i.e. $\psi(x) = 2x \mod 1$, then the conditions simplify in the following way:

(1a) *If $c < \sqrt{2} - 1\,(\approx 0.414)$, then $S_n(\chi)$ satisfies the CLT and MLCLT.*
(2a) *If $c < \sqrt{7/6} - 1\,(\approx 0.080)$, then $S_n(\chi)$ admits the first order Edgeworth Expansion.*

2.5. **The application to the Boolean-type transformation.** Recall the Boolean-type transformation $\phi\colon \mathbb{R}\to\mathbb{R}$ defined as

$$\phi(x) := \begin{cases} \frac{1}{2}\left(x-\frac{1}{x}\right) & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases} \tag{2.11}$$

and $\mu\in\mathcal{M}_1(\mathbb{R})$ defined by

$$\mathrm{d}\mu\left(x\right) := \frac{1}{\pi\cdot(x^2+1)}\mathrm{d}\lambda(x). \tag{2.12}$$

We are interested in limit theorems for Birkhoffs sums $\widetilde{S}_n(h) := S_n(h,\phi)$ where $h:\mathbb{R}\to\mathbb{R}$. To study these systems we go back to an easier system which fulfills all our properties of the last section.

Let $\psi\colon I\to I$ be given by $\psi\left(x\right) := 2x \mod 1$ and $\xi\colon I\to\mathbb{R}$ be given by $\xi\left(x\right) := \cot\left(\pi x\right)$. Note that ξ is almost surely bijective. An elementary calculation yields that the dynamical systems $(\mathbb{R},\mathcal{B}_{\mathbb{R}},\mu,\phi)$ and $(I,\mathcal{B}_I,\lambda_I,\psi)$ are isomorphic via ξ, i.e.

$$\left(\phi\circ\xi\right)\left(x\right) = \left(\xi\circ\psi\right)\left(x\right),$$

for all $x\in I$ and additionally ξ and ξ^{-1} are measure preserving, i.e. for all $B\in\mathcal{B}_{\mathbb{R}}$ it holds that $\mu\left(B\right)=\lambda_I\left(\xi^{-1}B\right)$ and for all $B\in\mathcal{B}_I$ it holds that $\lambda_I\left(B\right)=\mu\left(\xi B\right)$. To simplify the notation, we define $\widetilde{\sigma}^2 := \sigma^2(h,\phi)$.

Hence, instead of studying the Birkhoff sum $\sum_{n=0}^{N-1}\left(h\circ\phi^n\right)(x)$ with $x\in\mathbb{R}$ we can study the sum $\sum_{n=0}^{N-1}\left(h\circ\xi\circ\psi^n\right)(y)$, for $y\in I$. Since the transformations ϕ and ψ are isomorphic we conclude that

$$\mu\left(\sum_{n=0}^{N-1}\left(h\circ\phi^n\right)(x)\in B\right) = \lambda_I\left(\sum_{n=0}^{N-1}\left(h\circ\xi\circ\psi^n\right)(y)\in B\right), \tag{2.13}$$

for all sets $B\in\mathcal{B}_{\mathbb{R}}$. Formally, we define $\chi\colon I\to\mathbb{R}$ by $\chi\left(x\right) := \left(h\circ\xi\right)\left(x\right)$ and consider then the Birkhoff sum $S_n(\chi)$. Then our task reduces to transferring the conditions we have for χ to conditions for h.

Let $\mathfrak{F}$ be the class of functions $h:\mathbb{R}\to\mathbb{R}$ such that the left and right derivatives exist and there exist $u,v\geq 0$ fulfilling

$$h\left(x\right)\lesssim\left|x\right|^u \qquad \text{and} \qquad \max\left\{\left|h'\left(x-\right)\right|,\left|h'\left(x+\right)\right|\right\}\lesssim\left|x\right|^v \tag{2.14}$$

and $u(2+v)<1$. Analogously to $\overline{f}$, we define $\overline{h}=h-\mu(h)$.

Under the non-coboundary condition on ϕ, we have the CLT:

Proposition 2.9. *Suppose $h\in\mathfrak{F}$ is not $\phi-$cohomologous to a constant. Then, the following CLT holds:*

$$\mu\left(\frac{\widetilde{S}_n(h)-n\,\mu(h)}{\widetilde{\sigma}\sqrt{n}}\leq x\right)-\mathfrak{N}(x)=o(1), \quad \textit{for all} \quad x\in\mathbb{R} \quad \textit{as} \quad n\to\infty \tag{2.15}$$

with $\widetilde{\sigma}^2\in(0,\infty)$.

Under a non-arithmeticity condition on ϕ, we have the MLCLT:

Proposition 2.10. *Let $h\in\mathfrak{F}$ be non-arithmetic. Let $0<\alpha_0<\alpha_1<\beta$ and $M\geq 1$. Then, the following MLCLT holds: for $V:\mathbb{R}\to\mathbb{R}$ compactly supported and continuous, U such that*

$U\circ\xi\in\mathsf{V}_{\alpha_0,\beta,M}$, W such that $W\circ\xi\in L^1$ for all $\mathfrak{m}\in\mathcal{M}_1(\mathbb{R})$ being absolutely continuous with respect to λ such that $(W\circ\xi\cdot\xi_*\mathfrak{m})\in\mathsf{V}'_{\alpha_1,\beta,M}$, we have

$$\lim_{n\to\infty}\sup_{\ell\in\mathbb{R}}\left|\sigma\sqrt{2\pi n}\,\mathbb{E}_{\mathfrak{m}}(U\circ\psi^n\,V(\widetilde{S}_n(\overline{h})-\ell)\,W)-e^{-\frac{\ell^2}{2n\widetilde{\sigma}^2}}\,\mathbb{E}_{\mathfrak{m}}(W)\,\mathbb{E}_{\mu}(U)\int V(x)\,\mathrm{d}x\right|=0\,. \tag{2.16}$$

Since $|\Re\,\zeta(s+i\,\cdot)|^a, |\Im\,\zeta(s+i\,\cdot)|^a, |\zeta(s+i\,\cdot)|^a\in\mathfrak{F}$ for some suitable choices of s and a, we obtain two corollaries that improve the existing results on sampling the Lindelöf hypothesis.

Corollary 2.11. *Let $s\in(3-2\sqrt{2},1)$ and define $h:\mathbb{R}\to\mathbb{R}$ as follows.*

- *$h(x)=\Re\,\zeta(s+ix)$,*
- *$h(x)=\Im\,\zeta(s+ix)$, or*
- *$h(x)=|\zeta(s+ix)|$*

where $\zeta:\mathbb{C}\to\mathbb{C}$ is the Riemann zeta function. If h is not $\phi-$cohomologous to a constant, then the CLT, (2.15) holds. Moreover, if h is non-arithmetic, then the MLCLT, (2.16), holds.

Remark 2.12. See [40, Section 2.5] for a discussion where it is shown using numerics that for $\zeta_{1/2}$ all of the above choices of h are not coboundaries. Similarly, for a fixed value of s, one can numerically check whether h is not a $\psi-$coboundary by calculating the sum of values of $\chi=h\circ\xi$ over some appropriate periodic orbit of the doubling map and showing that it is not equal to 0.

Corollary 2.13. *Let $h:\mathbb{R}\to\mathbb{R}$ be as follows.*

- *$h=|\Re\,\zeta_{1/2}|^a$,*
- *$h=|\Im\,\zeta_{1/2}|^a$, or*
- *$h=|\zeta_{1/2}|^a$*

where $1\le a<84/13(\sqrt{2}-1)\,(\approx 2.677)$. If h is not $\phi-$cohomologous to a constant, then the CLT, (2.15) holds. Moreover, if h is non-arithmetic, then the MLCLT, (2.16), holds.

Remark 2.14. On the one hand, the Lindelöf hypothesis states that $|\zeta_{1/2}(x)|\lesssim x^{\varepsilon}$ holds for all $\varepsilon>0$, and hence, if it is true, the above statement has to hold for any $a>0$.

On the other hand, sampling $|\zeta(s+i\phi^k(x))|^a$ with larger values of a and obtaining normally distributed samples provide further evidence that the Lindelöf hypothesis is indeed true.

Finally, we state a set of sufficient conditions that implies the Edgeworth Expansions for ϕ.

Proposition 2.15. *Let $h:\mathbb{R}\to\mathbb{R}$ be such that the left and right derivatives exist and there exist $u,v\ge 0$ fulfilling (2.14) and*

$$\min\{2u(2+v),(u+v)(2+v)\}<1/3 \tag{2.17}$$

and h is not arithmetic. Then there exists a quadratic polynomial P whose coefficients depend on the first three asymptotic moments of $\widetilde{S}_n(h)$ but not on n such that for all $\mathfrak{m}\in\mathcal{M}_1(\mathbb{R})$ being absolutely continuous with respect to λ we have

$$\sup_{x\in\mathbb{R}}\left|\mathfrak{m}\left(\frac{\widetilde{S}_n(h)-n\,\mu(h)}{\widetilde{\sigma}\sqrt{n}}\le x\right)-\mathfrak{N}(x)-\frac{P(x)}{\sqrt{n}}\mathfrak{n}(x)\right|=o(n^{-1/2}),\quad \text{as}\quad n\to\infty\,. \tag{2.18}$$

Remark 2.16. The condition (2.17) forces that $0\le u<1$ and $u<v$.

Remark 2.17. The state of the art is not sufficient to conclude that the Riemann zeta function, or more precisely $\Re\zeta_{1/2}$, $\Im\zeta_{1/2}$ and $|\zeta_{1/2}|$, satisfy the conditions of the theorem. However, our theorem shows that if the Lindelöf hypothesis is true, then the first order Edgeworth expansion has to hold.

3. Review of Abstract Results for Limit Theorems

One known technique used to establish limit theorems for ergodic sums with unbounded observables is a combination of the Keller-Liverani perturbation result (see [26]) applied to a sequence of Banach spaces as in [20, 11, 36]. There exist elementary criteria for the CLT and the MLCLT to hold. We state them below as propositions adapted from [20, Corollary 2.1, Theorem 5.1] to our setting.

Proposition 3.1. *Let $T : X \to X$ be a dynamical system that has an ergodic invariant probability measure $\bar{\mathfrak{m}}$. Let $f \in L^2(\bar{\mathfrak{m}})$ be such that $\bar{\mathfrak{m}}(f) = 0$ and $\sum_{n\geq 0} \widehat{T}^n(f)$ converges in $L^2(\bar{\mathfrak{m}})$. Then, we have the following CLT.*

$$\lim_{n\to\infty} \bar{\mathfrak{m}}\left(\frac{S_n(f)}{\sqrt{n}} \leq x\right) = \mathfrak{N}\left(\frac{x}{\sigma}\right), \quad \text{for all} \quad x \in \mathbb{R} \quad \text{as} \quad n \to \infty, \tag{3.1}$$

where $\sigma^2 = \sigma^2(f,T)$ can be written as

$$\sigma^2 = \mathbb{E}_{\bar{\mathfrak{m}}}(f^2) + 2\sum_{k=1}^{\infty} \mathbb{E}_{\bar{\mathfrak{m}}}(f \cdot f \circ T^k) \in [0, \infty).$$

Here $\sigma = 0$ if and only if f is a T-coboundary and in this case $\mathfrak{N}(x/\sigma) := \mathbf{1}_{[0,\infty)}$ and $\frac{S_n(f)}{\sqrt{n}} \to \delta_0$ in distribution as $n \to \infty$.

Proof. This follows due to Gordin [14]. See [20, Corollary 2.1, Proposition 2.4] for details. □

Proposition 3.2. *Let $T : X \to X$ be a non-singular dynamical system wrt a probability measure m. Suppose T has an ergodic invariant probability measure $\bar{\mathfrak{m}}$ absolutely continuous wrt m and that there exist two, not necessarily distinct, Banach spaces $\mathcal{X}$ and $\mathcal{X}^{(+)}$ such that*

$$\mathcal{X} \hookrightarrow \mathcal{X}^{(+)} \hookrightarrow L^1(\pi) \tag{3.2}$$

each containing $\mathbf{1}_X$ and satisfying the following:

(I) *For all $s \in \mathbb{R}$, $\widehat{T}_{is} \in \mathcal{L}(\mathcal{X}) \cap \mathcal{L}(\mathcal{X}^{(+)})$.*
(II) *The map $s \mapsto \widehat{T}_{is} \in \mathcal{L}(\mathcal{X}, \mathcal{X}^{(+)})$ is continuous on $\mathbb{R}$.*
(III) *Either $\mathcal{X} = \mathcal{X}^{(+)}$, or there exist $\kappa \in (0,1)$ and $\delta > 0$ such that for all*
$$z \in D_\kappa := \{z \in \mathbb{C} | |z| > \kappa, |z-1| > (1-\kappa)/2\},$$
and for all $s \in (-\delta, \delta)$ we have
$$(z\mathsf{Id} - \widehat{T}_{is})^{-1} \in \mathcal{L}(\mathcal{X}) \quad \text{and} \quad \sup_{|s|<\delta} \sup_{z\in D_\kappa} \|(z\mathsf{Id} - \widehat{T}_{is})^{-1}\|_{\mathcal{X}\to\mathcal{X}} < \infty.$$
(IV) $\lim_{n\to\infty} \|\widehat{T}^n(\cdot) - \bar{\mathfrak{m}}(\cdot)\mathbf{1}_X\|_{\mathcal{X}_0\to\mathcal{X}_0} = 0$.
(V) *The CLT, (3.1) holds with $\sigma > 0$.*
(VI) *For all $s \neq 0$, the spectrum of the operators $\widehat{T}_{is}$ acting on $\mathcal{X}$ is contained in the open unit disc, $\{z \in \mathbb{C} \mid |z| < 1\}$.*

Then, for all $U \in \mathcal{X}$, $V : \mathbb{R} \to \mathbb{R}$ a compactly supported continuous function, $\mathfrak{m} \in \mathcal{M}_1(X)$ being absolutely continuous wrt m and $W \in L^1$ such that $(W \cdot \mathfrak{m}) \in \mathcal{X}^{(+)\prime}$, we have

$$\lim_{n\to\infty} \sup_{\ell\in\mathbb{R}} \left| \sigma\sqrt{2\pi n}\, \mathbb{E}_{\mathfrak{m}}(U \circ T^n\, V(S_n(\overline{\chi}) - \ell)\, W) - e^{-\frac{\ell^2}{2n\sigma^2}}\, \mathbb{E}_{\mathfrak{m}}(U)\, \mathbb{E}_{\bar{\mathfrak{m}}}(W) \int V(x)\, \mathrm{d}x \right| = 0. \tag{3.3}$$

Proof. This follows from a modified version of [20, Theorem 5.1]. The condition (CLT) there is assumed here in (V).

Also, the Condition $(\widetilde{K})$ there follows from our assumptions (I) through (IV) because $(K1)$ is (IV), $(\widetilde{K1})$ is (II), and finally, $(\widetilde{K2})$ can be replaced by (III) (see Remark 3.4).

Our assumptions (II) and (VI) yield that on any compact set $K \subset \mathbb{R} \setminus \{0\}$, there exist $\rho \in (0,1)$ and $C_K > 0$ such that

$$\sup_{s\in K} \|\widehat{T}_{is}^n\|_{\mathcal{X}\to\mathcal{X}^+} \le C_K \rho^n,$$

for all $n \in \mathbb{N}$ (see, for example, [11, Proposition 1.13] for a proof). This replaces the non-lattice condition (S) there.

So, for all $U \in \mathcal{X}$, $V : \mathbb{R} \to \mathbb{R}$ a compactly supported continuous function and $W \in L^1$ such that $(W \cdot \mathfrak{m}) \in \mathcal{X}^{(+)\,\prime}$, we have the MLCLT due to [20, Theorem 5.1]. □

Finally, we state a result that gives us sufficient conditions for the first order Edgeworth expansion. It is adapted from [20, 11] to our setting (compare with [20, Proposition 7.1, Propostion A.1] and [11, Corollary 1.8, Proposition 1.12]).

Proposition 3.3. *Let $T : X \to X$ be a non-singular dynamical system wrt a probability measure m. Suppose T has an ergodic invariant probability measure $\bar{\mathfrak{m}}$ absolutely continuous wrt m and that there exists a sequence of, not necessarily distinct, Banach spaces*

$$\mathcal{X}_0 \hookrightarrow \mathcal{X}_0^{(+)} \hookrightarrow \mathcal{X}_1 \hookrightarrow \mathcal{X}_1^{(+)} \hookrightarrow \mathcal{X}_2 \hookrightarrow \mathcal{X}_2^{(+)} \hookrightarrow \mathcal{X}_3 \hookrightarrow \mathcal{X}_3^{(+)} \tag{3.4}$$

each containing $\mathbf{1}_X$, $\mathcal{X}_3^{(+)} \hookrightarrow L^1$ and satisfying the following:

(I) *For each space $\mathcal{C}$ in (3.4), $s \in \mathbb{R}$, $\widehat{T}_{is} \in \mathcal{L}(\mathcal{C})$.*
(II) *For all $a = 0,1,2,3$, the map $s \mapsto \widehat{T}_{is} \in \mathcal{L}(\mathcal{X}_a, \mathcal{X}_a^{(+)})$ is continuous on $\mathbb{R}$.*
(III) *For all $a = 0,1,2$, the map $s \mapsto \widehat{T}_{is} \in \mathcal{L}(\mathcal{X}_a^{(+)}, \mathcal{X}_{a+1})$ is C^1 on $(-\delta,\delta)$.*
(IV) *Either all spaces in (3.4) are equal, or there exist $\kappa \in (0,1)$ and $\delta > 0$ such that for all*

$$z \in D_\kappa := \{z \in \mathbb{C} \mid |z| > \kappa, |z-1| > (1-\kappa)/2\},$$

for all $s \in (-\delta,\delta)$ and for each space $\mathcal{C}$ in (3.4),

$$(z\mathsf{Id} - \widehat{T}_{is})^{-1} \in \mathcal{L}(\mathcal{C}) \quad \text{and} \quad \sup_{|s|<\delta} \sup_{z\in D_\kappa} \|(z\mathsf{Id} - \widehat{T}_{is})^{-1}\|_{\mathcal{C}\to\mathcal{C}} < \infty.$$

(V) *$\widehat{T}$ has a spectral gap of $(1-\kappa)$ on each space $\mathcal{C}$ in (3.4).*
(VI) *For all $s \neq 0$, the spectrum of the operators $\widehat{T}_{is}$ acting on either $\mathcal{X}_0$ or $\mathcal{X}_0^{(+)}$ is contained in the open unit disc, $\{z \in \mathbb{C} \mid |z| < 1\}$.*
(VII) *The sequence*

$$\left\{\sum_{k=0}^{n-1} \overline{f} \circ T^k\right\}_{n\in\mathbb{N}}$$

where $\overline{f} := f - A$ has an L^2-weakly convergent subsequence.
(VIII) *f is not $T-$cohomologous to a constant.*

Then for all $\mathfrak{m} \in \mathcal{M}_1(X)$ being absolutely continuous wrt m, there exists a quadratic polynomial P whose coefficients depend on the first three asymptotic moments of $S_n(\chi)$ such that the following asymptotic expansion holds;

$$\sup_{x\in\mathbb{R}} \left| \bar{\mathfrak{m}}\left(\frac{S_n(\overline{f})}{\sigma\sqrt{n}} \le x\right) - \mathfrak{N}(x) - \frac{P(x)}{\sqrt{n}}\mathfrak{n}(x)\right| = o(n^{-1/2}), \quad \text{as} \quad n \to \infty. \tag{3.5}$$

Remark 3.4. In [20] and [11], instead of the condition (IV) above, the following stronger condition of a uniform DFLY inequality is assumed.

> Either all spaces in (3.4) are equal, or there exist $\widetilde{C} > 0$, $\widetilde{\kappa}_1 \in (0,1)$ and $p_0 \ge 1$ such that, for every $\mathcal{C}$ in (3.4),

$$\forall h \in \mathcal{C}, \quad \sup_{|s|<\delta} \|\widehat{T}_{is}^n h\|_{\mathcal{C}} \le \widetilde{C}\left(\widetilde{\kappa}_1^n \|h\|_{\mathcal{C}} + \|h\|_{L^{p_0}(\bar{\nu})}\right). \tag{3.6}$$

However, the proof of the key theorem, [11, Proposition 1.11], is based on [20, Proposition A, Corollary 7.2] which uses the hypothesis $\mathcal{D}(m)$ in [20, Appendix A] that contains the much weaker condition (IV) instead of condition (3.6). Therefore, all the results in [11] based on [11, Proposition 1.11] including [11, Proposition 1.12] remain true with this replacement. We refer the reader to [20] for more details.

Remark 3.5. For an elementary illustration of the proof of the CLT based on the classical Nagaev-Guivarc'h approach, we refer the reader to [15] where the C^2 regularity of $s \mapsto \widehat{T}_{is}$ along with the spectral gap of $\widehat{T}$ on a single Banach space (instead of a chain) is used. This corresponds to the C^2 regularity of the characteristic function in the IID case. When it comes to the MLCLT in the IID setting, a non-lattice assumption is necessary. In our case, the equivalent assumption is (VI).

Proof of Proposition 3.3. We apply results in [11] restricted to a single dynamical system with $r = 1$ there, i.e., when Assumptions (0) and (A)[1](1-2) in [11, Section 1.2] are trivially true. This case is, thus, similar to the $r = 1$ case of [11, Proposition 1.12] which implies [11, Corollary 1.8] which, in turn, gives the first order Edgeworth expansion. This is because our assumptions above imply Assumptions (A)[1] and (B) in [11, Section 1.2], *except* for (A)[1](4) which is equivalent to (3.6). However, as discussed in Remark 3.4, [11, Corollary 1.8] remains true because the key ingredient of the proof in [11] is our assumption (IV) (implied by the much stronger (A)[1](4)). ☐

4. Multiplication in $\mathsf{V}_{\alpha,\beta,\gamma}$

4.1. **Multiplication by $e^{is\chi}$.** In this section, we prove some properties of multiplication by $e^{is\chi}$ in $\mathsf{V}_{\alpha,\beta,\gamma}$ that are necessary for our proofs.

Observe that the spaces $\mathsf{V}_{\alpha,\beta,\gamma}$, as opposed to spaces usually used in ergodic theory such as L^∞, BV$[0,1]$ or $C^1[0,1]$, are not Banach algebras. Hence, $s \mapsto \widehat{\psi}_{is} \in \mathcal{L}(\mathsf{V}_{\alpha,\beta,\gamma})$ may not be continuous. The following lemma will allow us to establish its continuity as a function from $\mathbb{R}$ to $\mathcal{L}(\mathsf{V}_{\alpha_1,\beta_1,\gamma_1}, \mathsf{V}_{\alpha_2,\beta_2,\gamma_2})$ for some good choices of indices.

Lemma 4.1. *Suppose $g \in \mathsf{V}_{\alpha_1,\beta_1,\gamma_1}$, $h \in \mathsf{V}_{\alpha_2,\beta_2,\gamma_2}$ and $\alpha_3 = \alpha_1 + \alpha_2$, $\beta_3 \le \min\{\beta_1\,, \beta_2\}$ and $\gamma_3 \le (\gamma_1^{-1} + \gamma_2^{-1})^{-1}$. Then,*

$$\|gh\|_{\alpha_3,\beta_3,\gamma_3} \lesssim \|g\|_{\alpha_1,\beta_1,\gamma_1} \|h\|_{\alpha_2,\beta_2,\gamma_2}$$

with the proportionality constant independent of g and h but dependent on $\alpha_j, \beta_j, \gamma_j$, $j = 1, 2, 3$.

Proof. First, suppose g and h are real valued. Then

$$\operatorname{osc}(R_\alpha u, B_\varepsilon(x)) = \operatorname{osc}(R_\alpha u_-, B_\varepsilon(x)) + \operatorname{osc}(R_\alpha u_+, B_\varepsilon(x))\,. \tag{4.1}$$

By applying [38, Proposition 3.2 (iii)] to the positive and negative parts of g,

$$\begin{aligned}
&\operatorname{osc}(R_{\alpha_3}(gh), B_\varepsilon(x)) \\
&\quad = \operatorname{osc}(R_{\alpha_3}(g_+ - g_-)h, B_\varepsilon(x)) \\
&\quad = \operatorname{osc}(R_{\alpha_1}(g_+ - g_-) \cdot R_{\alpha_2} h, B_\varepsilon(x)) \\
&\quad \le \operatorname{osc}(R_{\alpha_1} g_+ \cdot R_{\alpha_2} h, B_\varepsilon(x)) + \operatorname{osc}(R_{\alpha_1} g_- \cdot R_{\alpha_2} h, B_\varepsilon(x)) \\
&\quad \le \sum_{r=\pm} \Big(\operatorname{osc}(R_{\alpha_1} g_r, B_\varepsilon(x)) \cdot \operatorname{ess\,sup} |R_{\alpha_2} h| + \operatorname{osc}(R_{\alpha_2} h, B_\varepsilon(x)) \cdot \operatorname{ess\,sup} |R_{\alpha_1} g_r| \Big) \\
&\quad \le \operatorname{osc}(R_{\alpha_1} g, B_\varepsilon(x)) \operatorname{ess\,sup} |R_{\alpha_2} h| + 2 \cdot \operatorname{osc}(R_{\alpha_2} h, B_\varepsilon(x)) \operatorname{ess\,sup} |R_{\alpha_1} g|\,.
\end{aligned}$$

If g is complex valued, using the definition of osc, we have

$$\begin{aligned}
&\operatorname{osc}(R_{\alpha_3}(gh), B_\varepsilon(x)) \\
&\quad \le \operatorname{osc}(R_{\alpha_1} g, B_\varepsilon(x)) \operatorname{ess\,sup} |R_{\alpha_2} h| + 2 \cdot \operatorname{osc}(R_{\alpha_2} h, B_\varepsilon(x)) (\operatorname{ess\,sup} |R_{\alpha_1} \Re g| + \operatorname{ess\,sup} |R_{\alpha_1} \Im g|)\,,
\end{aligned}$$

$$\leq \operatorname{osc}(R_{\alpha_1} g, B_\varepsilon(x)) \operatorname{ess\,sup} |R_{\alpha_2} h| + 2\sqrt{2} \cdot \operatorname{osc}(R_{\alpha_2} h, B_\varepsilon(x)) \operatorname{ess\,sup} |R_{\alpha_1} g| .$$

If h is not real valued, repeating the argument for the real and imaginary parts of h, we obtain

$$\begin{aligned}&\operatorname{osc}(R_{\alpha_3}(gh), B_\varepsilon(x))\\ &\quad\leq 2\sqrt{2} \cdot \operatorname{osc}(R_{\alpha_1} g, B_\varepsilon(x)) \operatorname{ess\,sup} |R_{\alpha_2} h| + 2\sqrt{2} \cdot \operatorname{osc}(R_{\alpha_2} h, B_\varepsilon(x)) \operatorname{ess\,sup} |R_{\alpha_1} g| .\end{aligned}$$

Now, we use the inclusion of L^∞ in $\mathsf{V}_{0,\beta_r,1}$ where $r = 1, 2$ to conclude that

$$\begin{aligned}&\int \operatorname{osc}(R_{\alpha_3}(gh), B_\varepsilon(x)) \,\mathrm{d}\lambda_I(x)\\ &\quad\lesssim \int \operatorname{osc}(R_{\alpha_1} g, B_\varepsilon(x)) \,\mathrm{d}\lambda_I(x) \cdot \|R_{\alpha_2} h\|_{0,\beta_2,1} + \int \operatorname{osc}(R_{\alpha_2} h, B_\varepsilon(x)) \,\mathrm{d}\lambda_I(x) \cdot \|R_{\alpha_1} g\|_{0,\beta_1,1}\\ &\quad\lesssim \varepsilon^{\beta_1} |g|_{\alpha_1,\beta_1} (|h|_{\alpha_2,\beta_2} + \|R_{\alpha_2} h\|_1) + \varepsilon^{\beta_2} |h|_{\alpha_2,\beta_2} (|g|_{\alpha_1,\beta_1} + \|R_{\alpha_1} g\|_1).\end{aligned}$$

This gives us that for all $\varepsilon \in (0, 1]$,

$$\begin{aligned}&\varepsilon^{-\beta_3} \int \operatorname{osc}(R_{\alpha_3}(gh), B_\varepsilon(x)) \,\mathrm{d}\lambda_I(x)\\ &\quad\lesssim |g|_{\alpha_1,\beta_1} |h|_{\alpha_2,\beta_2} + |g|_{\alpha_1,\beta_1} \|h\|_{\gamma_2} + |h|_{\alpha_2,\beta_2} |g|_{\alpha_1,\beta_1} + |h|_{\alpha_2,\beta_2} \|g\|_{\gamma_1} .\end{aligned}$$

Taking the supremum over ε and combining with $\|gh\|_{\gamma_3} \leq \|g\|_{\gamma_1} \|h\|_{\gamma_2}$ implies the result. □

Due to the linearity of the operator $\widehat{\psi}$, in order to show regularity of $s \mapsto \widehat{\psi}_{is} = \psi(e^{is\chi} \times \cdot)$, it is enough to show the regularity of the one parameter group of multiplication operators $s \mapsto e^{is\chi} \times \cdot$. Our next lemma provides general conditions that guarantees this.

Lemma 4.2. *Let $0 \leq \alpha_0, \beta \leq 1$ and $\gamma_0 \geq 1$. For each $s \in \mathbb{R}$, consider the multiplication operator, $H_s(\cdot) = e^{is\chi} \times \cdot$, on $\mathsf{V}_{\alpha_0,\beta,\gamma_0}$.*

(1) *Suppose there is $\bar{\beta} \geq \beta$ such that, for all $s \in \mathbb{R}$, $|e^{is\chi}|_{0,\bar{\beta}} < \infty$. Then, for all $s \in \mathbb{R}$, $H_s \in \mathcal{L}(\mathsf{V}_{\alpha_0,\beta,\gamma_0})$.*

(2) *Suppose, in addition to the conditions in (1), there exists $0 < \alpha^* < \beta$ such that*

$$\lim_{s \to 0} |1 - e^{is\chi}|_{\alpha^*,\beta} = 0. \tag{4.2}$$

Put $\alpha_1 = \alpha_0 + \alpha^*$ and $\gamma_1 \leq \gamma_0$. Then $s \mapsto H_s \in \mathcal{L}(\mathsf{V}_{\alpha_0,\beta,\gamma_0}, \mathsf{V}_{\alpha_1,\beta,\gamma_1})$ is continuous.

(3) *Suppose, in addition to the conditions in (1) and (2), there exist $0 < \alpha^{**} < \beta$ and $\gamma \geq 1$ such that*

$$\lim_{s \to 0} \left| \frac{e^{is\chi} - 1 - is\chi}{s} \right|_{\alpha^{**},\beta} = 0 \quad \text{and} \quad \|\chi\|_\gamma < \infty . \tag{4.3}$$

Put $\alpha_2 = \alpha_0 + \max\{\alpha^*, \alpha^{**}\}$ and $\gamma_2 \leq (\gamma_1^{-1} + \gamma^{-1})^{-1}$. Then, the function $s \mapsto H_s \in \mathcal{L}(\mathsf{V}_{\alpha_0,\beta,\gamma_0}, \mathsf{V}_{\alpha_2,\beta,\gamma_2})$ is differentiable with the derivative

$$H'_s(\cdot) = (i\chi) e^{is\chi} \times \cdot \ .$$

(4) *Suppose, the conditions in (1), (2) and (3) are true. Put $\alpha_3 = \alpha_2 + \alpha^*$ and $\gamma_3 \leq \gamma_2$. Then $s \mapsto H_s \in \mathcal{L}(\mathsf{V}_{\alpha_0,\beta,\gamma_0}, \mathsf{V}_{\alpha_3,\beta,\gamma_3})$ is continuously differentiable.*

Remark 4.3. It would be possible to have some more flexibility on the parameter β and change it for different spaces. However, we only use the version of the lemma as stated which also keeps a simpler notation.

Proof of Lemma 4.2.

Proof of (1):

We note that for all $g \in \mathsf{V}_{\alpha_0,\beta,\gamma_0}$, $\|H_s(g)\|_{\gamma_0} = \|g\|_{\gamma_0}$ and due to [38, Proposition 3.2 (iii)],

$$\begin{aligned}\operatorname{osc}(R_{\alpha_0}(e^{is\chi}g), B_\varepsilon(x)) &\le \operatorname{osc}(R_{\alpha_0}(e^{is\chi}g_+), B_\varepsilon(x)) + \operatorname{osc}(R_{\alpha_0}(e^{is\chi}g_-), B_\varepsilon(x)) \\ &\lesssim \operatorname{osc}(R_{\alpha_0}g, B_\varepsilon(x)) + \operatorname{osc}(e^{is\chi}, B_\varepsilon(x)) \cdot \operatorname{ess\,sup}(|R_{\alpha_0}g|) \\ &\lesssim \operatorname{osc}(R_{\alpha_0}g, B_\varepsilon(x)) + \operatorname{osc}(e^{is\chi}, B_\varepsilon(x))\|g\|_{\alpha_0,\beta,\gamma_0} \text{ and} \\ \varepsilon^{-\beta}\operatorname{osc}(R_{\alpha_0}(e^{is\chi}g), B_\varepsilon(x)) &\lesssim \varepsilon^{-\beta}\operatorname{osc}(R_{\alpha_0}g, B_\varepsilon(x)) + \varepsilon^{-\beta}\operatorname{osc}(e^{is\chi}, B_\varepsilon(x))\|g\|_{\alpha_0,\beta,\gamma_0}.\end{aligned}$$

The first $\lesssim$ is due to adding up the positive and negative part of g the second is due to the inclusion $\mathsf{V}_{0,\beta,\gamma_0} \hookrightarrow L^\infty$. Integrating and taking the supremum over ε, we have

$$|H_s(g)|_{\alpha_0,\beta} \lesssim |g|_{\alpha_0,\beta} + |e^{is\chi}|_{0,\beta}\|g\|_{\alpha_0,\beta,\gamma_0}$$

which gives

$$\|H_s(g)\|_{\alpha_0,\beta,\gamma_0} \le (1 + |e^{is\chi}|_{0,\beta})\|g\|_{\alpha_0,\beta,\gamma_0}. \tag{4.4}$$

Therefore, for all s, H_s maps $\mathsf{V}_{\alpha_0,\beta,\gamma_0}$ to itself, and is a bounded linear operator on $\mathsf{V}_{\alpha_0,\beta,\gamma_0}$.

Proof of (2):

We note that, $H_t g - H_s g = (\mathsf{Id} - H_{s-t})H_t g$ and if $g \in \mathsf{V}_{\alpha_0,\beta,\gamma_0}$ then $H_t g \in \mathsf{V}_{\alpha_0,\beta,\gamma_0}$. Hence, due to Lemma 4.1, it is enough to prove that

$$\lim_{s\to 0}\|\mathsf{Id} - H_s\|_{\mathsf{V}_{\alpha_0,\beta,\gamma_0}\to\mathsf{V}_{\alpha_1,\beta,\gamma_1}} = 0.$$

To this end, let $g \in \mathsf{V}_{\alpha_0,\beta,\gamma_0}$ be such that $\|g\|_{\alpha_0,\beta,\gamma_0} \le 1$. Then,

$$\lim_{s\to 0}\|(\mathsf{Id} - H_s)g\|_{\gamma_1}^{\gamma_1} = \lim_{s\to 0}\int |(1 - e^{is\chi})g|^{\gamma_1}\, \mathrm{d}\lambda_I = 0$$

by the dominated convergence theorem. Moreover, by [38, Proposition 3.2 (iii)]

$$\begin{aligned}&\operatorname{osc}(R_{\alpha_1}(\mathsf{Id} - H_s)g, B_\varepsilon(x)) \\ &\quad = \operatorname{osc}(R_{\alpha^*}(1 - e^{is\chi})R_{\alpha_0}g, B_\varepsilon(x)) \\ &\quad \lesssim \operatorname{osc}(R_{\alpha_0}g, B_\varepsilon(x)) \cdot \operatorname{ess\,sup}|R_{\alpha^*}(1 - e^{is\chi})| + \operatorname{osc}(R_{\alpha^*}(1 - e^{is\chi}), B_\varepsilon(x)) \cdot \operatorname{ess\,sup}|R_{\alpha_0}g|,\end{aligned}$$

where $\lesssim$ is due to the fact that we have to consider the positive and negative part of g separately. Because $\mathsf{V}_{0,\beta,1} \hookrightarrow L^\infty$, we have

$$\begin{aligned}\varepsilon^{-\beta}\operatorname{osc}(R_{\alpha_1}(\mathsf{Id} - H_s)g, B_\varepsilon(x)) &\lesssim \varepsilon^{-\beta}\operatorname{osc}(R_{\alpha_0}g, B_\varepsilon(x))(|1 - e^{is\chi}|_{\alpha^*,\beta} + \|R_{\alpha^*}(1 - e^{is\chi})\|_1) \\ &\quad + \varepsilon^{-\beta}\operatorname{osc}(R_{\alpha^*}(1 - e^{is\chi}), B_\varepsilon(x))\|g\|_{\alpha_0,\beta,\gamma_0}.\end{aligned}$$

Integrating, taking the sup over ε, and finally, using $\|g\|_{\alpha_0,\beta,\gamma_0} \le 1$, we get

$$|(\mathsf{Id} - H_s)g|_{\alpha_1,\beta} \lesssim |1 - e^{is\chi}|_{\alpha^*,\beta} + \|R_{\alpha^*}(1 - e^{is\chi})\|_1.$$

By the bounded convergence theorem $\lim_{s\to 0}\|R_{\alpha^*}(1 - e^{is\chi})\|_1 = 0$. Therefore,

$$\lim_{s\to 0}|(\mathsf{Id} - H_s)g|_{\alpha_1,\beta} = 0.$$

Hence, we have the continuity of $s \mapsto H_s$.

Proof of (3):

First, we show that for all $g \in \mathsf{V}_{\alpha_1,\beta_1,\gamma_1}$ such that $\|g\|_{\alpha_1,\beta_1,\gamma_1} \le 1$,

$$\lim_{h\to 0}\Big\|\Big(\frac{H_{s+h} - H_s - H_s'h}{h}\Big)g\Big\|_{\alpha_2,\beta,\gamma_2} = \lim_{h\to 0}\Big\|\Big(\frac{H_h - \mathsf{Id} - i\chi h}{h}\Big)H_s g\Big\|_{\alpha_2,\beta,\gamma_2} = 0.$$

Due to Lemma 4.1, it is enough to show that

$$\lim_{h\to 0}\Big\|\Big(\frac{H_h-\mathsf{Id}-i\chi h}{h}\Big)\mathbf{1}\Big\|_{\alpha^{**},\beta,\gamma}=\lim_{h\to 0}\Big\|\frac{e^{ih\chi}-1-i\chi h}{h}\Big\|_{\alpha^{**},\beta,\gamma}=0.$$

From the dominated convergence theorem, we have

$$\lim_{h\to 0}\Big\|\frac{e^{ih\chi}-1-i\chi h}{h}\Big\|_{\gamma}=0.$$

The assumption (4.3) completes the proof of differentiability.

Finally, picking $h\neq 0$ sufficiently close to 0, applying the estimate in part (1), part (2) with $\gamma_1=\gamma_0$, and Lemma 4.1, we note that for all $g\in\mathsf{V}_{\alpha_1,\beta,\gamma_1}$ and for all s,

$$\begin{aligned}\|H_s'(g)\|_{\alpha_2,\beta,\gamma_2}&=\Big\|\frac{e^{ih\chi}-1-ih\chi}{h}e^{is\chi}g+\frac{1}{h}(1-e^{ih\chi})e^{is\chi}g\Big\|_{\alpha_2,\beta,\gamma_2}\\&\leq\Big\|\frac{e^{ih\chi}-1-ih\chi}{h}e^{is\chi}g\Big\|_{\alpha_2,\beta,\gamma_2}+\frac{1}{h}\|(1-e^{ih\chi})e^{is\chi}g\|_{\alpha_2,\beta,\gamma_2}\\&\leq\Big\|\frac{e^{ih\chi}-1-ih\chi}{h}\Big\|_{\alpha^{**},\beta,\gamma}\|H_s(g)\|_{\alpha_0,\beta,\gamma_0}+\frac{1}{h}\|(1-e^{ih\chi})\|_{\alpha^*,\beta,\gamma}\|H_s(g)\|_{\alpha_0,\beta,\gamma_0}\\&\lesssim\left(\Big\|\frac{e^{ih\chi}-1-ih\chi}{h}\Big\|_{\alpha^{**},\beta,\gamma}+\|(1-e^{ih\chi})\|_{\alpha^*,\beta,\gamma}\right)(1+|e^{is\chi}|_{0,\beta})\|g\|_{\alpha_0,\beta,\gamma_0}\end{aligned}$$

So, H_s' is, in fact, a bounded linear operator in $\mathcal{L}(\mathsf{V}_{\alpha_0,\beta,\gamma_0},\mathsf{V}_{\alpha_2,\beta,\gamma_2})$.

Proof of (4): Since $\mathsf{V}_{\alpha_2,\beta,\gamma_2}\hookrightarrow\mathsf{V}_{\alpha_3,\beta,\gamma_3}$, we have that $s\to H_s\in\mathcal{L}(\mathsf{V}_{\alpha_0,\beta,\gamma_0},\mathsf{V}_{\alpha_3,\beta,\gamma_3})$ is differentiable. So, we need to check whether $s\to H_s'$ is continuous. Note that for all $g\in\mathsf{V}_{\alpha_0,\beta,\gamma_0}$ and for all $s>0$, $H_s'(g)\in\mathsf{V}_{\alpha_2,\beta,\gamma_2}$ and for all $h>0$

$$\begin{aligned}\|(H_{s+h}'-H_s')g\|_{\alpha_3,\beta,\gamma_3}&=\|(e^{ih\chi}-1)H_s'(g)\|_{\alpha_3,\beta,\gamma_3}\\&\lesssim\|(H_h-H_0)\mathbf{1}\|_{\alpha^*,\beta,\gamma_0}\|H_s'(g)\|_{\alpha_2,\beta,\gamma_2}\to 0,\end{aligned}$$

as $h\to 0$ due to part (2). Hence, we have the continuity of the derivative. □

4.2. Sufficient conditions for Lemma 4.2.

We limit our scope by providing sufficient conditions for the assumptions in Lemma 4.2.

Lemma 4.4. *Let $\bar{\beta}>0$. Suppose χ is continuous and the right and left derivatives of χ exist on $\mathring{I}$. If there exists a constant $b\in[0,1/\bar{\beta})$ such that*

$$\max\{|\chi'(x+)|,|\chi'(x-)|\}\lesssim x^{-b}(1-x)^{-b}\tag{4.5}$$

then

$$|e^{is\chi}|_{0,\bar{\beta}}<\infty$$

holds for all $s>0$.

Proof. We have

$$|e^{is\chi}|_{0,\bar{\beta}}\leq\sup_{\varepsilon\in(0,\varepsilon_0]}\int_0^{1/2}\frac{\operatorname{osc}(e^{is\chi},B_\varepsilon(x))}{\varepsilon^{\bar{\beta}}}\,\mathrm{d}\lambda_I(x)+\sup_{\varepsilon'\in(0,\varepsilon_0]}\int_{1/2}^{1}\frac{\operatorname{osc}(e^{is\chi},B_{\varepsilon'}(x))}{\varepsilon'^{\bar{\beta}}}\,\mathrm{d}\lambda_I(x).$$

We will only estimate the first summand as the estimation of the second follows analogously. Using the definition $\operatorname{osc}(h,A)=\operatorname{osc}(\Re h,A)+\operatorname{osc}(\Im h,A)$ we note that for any measurable set A we have

$$\operatorname{osc}\left(e^{is\chi},A\right)\leq\min\{4,4s/\pi\operatorname{osc}(\chi,A)\}.\tag{4.6}$$

By (4.5) there exists $C > 0$ such that for all $\varepsilon > 0$ and all $x \in [\varepsilon, 1/2]$ we have

$$\operatorname{osc}(e^{is\chi}, B_\varepsilon(x)) \le \frac{8|s|\varepsilon}{\pi} \sup_{y \in B_\varepsilon(x)} \max\{|\chi'(y+)|, |\chi'(y-)|\} \le \frac{8C|s|\varepsilon}{\pi}(x-\varepsilon)^{-b}.$$

We have that $8C|s|\varepsilon/\pi(x-\varepsilon)^{-b} \le 4$ if and only if

$$x \le \left(\frac{2C|s|\varepsilon}{\pi}\right)^{1/b} + \varepsilon =: \gamma_\varepsilon.$$

Hence, we split the integral on $[0, 1/2]$ into two, one on $[0, \gamma_\varepsilon]$ and the other on $[\gamma_\varepsilon, 1/2]$. For the first range, we use the first bound in (4.6) and for the second range, we use the second bound. Then,

$$\begin{aligned}\sup_{\varepsilon \in (0,\varepsilon_0]} \int_0^{1/2} \frac{\operatorname{osc}(e^{is\chi}, B_\varepsilon(x))}{\varepsilon^{\bar\beta}} \mathrm{d}\lambda_I(x) &\le \sup_{\varepsilon \in (0,\varepsilon_0]} \left(4\gamma_\varepsilon \varepsilon^{-\bar\beta} + \int_{\gamma_\varepsilon}^{1/2} \frac{8C|s|\varepsilon^{1-\bar\beta}}{\pi}(x-\varepsilon)^{-b} \mathrm{d}\lambda_I(x)\right) \\ &\le \sup_{\varepsilon \in (0,\varepsilon_0]} 4\gamma_\varepsilon\varepsilon^{-\bar\beta} + \sup_{\varepsilon \in (0,\varepsilon_0]} \int_{\gamma_\varepsilon}^{1/2} \frac{8C|s|\varepsilon^{1-\bar\beta}}{\pi}(x-\varepsilon)^{-b}\mathrm{d}\lambda_I(x). \end{aligned} \tag{4.7}$$

For the first summand, we have

$$\sup_{\varepsilon \in (0,\varepsilon_0]} 4\gamma_\varepsilon \varepsilon^{-\bar\beta} \le 8 \sup_{\varepsilon \in (0,\varepsilon_0]} \max\left\{\left(\frac{2C|s|}{\pi}\right)^{1/b} \varepsilon^{1/b-\bar\beta}, \varepsilon^{1-\bar\beta}\right\} < \infty,$$

which follows from the fact that $b < 1/\bar\beta$ and $\bar\beta \le 1$. For the second summand of (4.7), we have

$$\begin{aligned} &\sup_{\varepsilon \in (0,\varepsilon_0]} \int_{\gamma_\varepsilon}^{1/2} \frac{8C|s|\varepsilon^{1-\bar\beta}}{\pi}(x-\varepsilon)^{-b}\mathrm{d}\lambda_I(x) \\ &\quad \le \frac{8C|s|}{\pi} \sup_{\varepsilon \in (0,\varepsilon_0]} \varepsilon^{1-\bar\beta} \int_{\left(\frac{2Cs\varepsilon}{\pi}\right)^{1/b}}^{1/2} x^{-b} \mathrm{d}\lambda_I(x) \\ &\quad \le \begin{cases} \frac{8C|s|}{\pi} \sup_{\varepsilon \in (0,\varepsilon_0]} \frac{\varepsilon^{1-\bar\beta}}{|1-b|} \max\left\{\frac{1}{2}, \left(\frac{2C|s|\varepsilon}{\pi}\right)^{1/b}\right\}^{1-b} & b \ne 1 \\ \frac{8C|s|}{\pi} \sup_{\varepsilon \in (0,\varepsilon_0]} \varepsilon^{1-\bar\beta} \log\left(\frac{\pi}{2C|s|\varepsilon}\right) & b = 1 \end{cases} \\ &\quad = \frac{8C|s|}{\pi} \max\left\{\frac{\varepsilon_0^{1-\bar\beta}}{2^{1-b}|1-b|}, \left(\frac{2C|s|}{\pi}\right)^{1/b-1} \frac{\varepsilon_0^{1/b-\bar\beta}}{|1-b|}, \varepsilon_0^{1-\bar\beta} \log\left(\frac{\pi}{2C|s|\varepsilon_0}\right)\right\} < \infty, \end{aligned}$$

which again follows from the fact that $\bar\beta \le 1$ and $b < 1/\bar\beta$. □

Remark 4.5. The above lemma combined with Corollary 5.2 gives a sufficient condition on χ for the operator H_s, and hence, $\widehat{\psi}_{is}$ to be a bounded linear operator on $V_{\alpha,\beta,\gamma}$ for all $\alpha \ge 0$, $\beta \le \bar\beta$ and $\gamma \ge 1$.

The following lemma gives a sufficient condition on χ for the operator valued function $s \mapsto H_s$, and hence, $s \mapsto \widehat{\psi}_{is}$ to be continuous.

Lemma 4.6. *Suppose $|\chi|_{\alpha,\beta} < \infty$ with $0 \le \alpha \le \beta < 1/(1+\alpha)$ and there exists $b \in [0, 1/\beta)$ such that (4.5) holds. Then, for all $\alpha^* \in (0,1)$*

$$\lim_{s \to 0} |1 - e^{is\chi}|_{\alpha^*,\beta} = 0.$$

Proof. We will do the calculation only for the real part $\Re(1-e^{is\chi}) = 1-\cos(s\chi)$ and the calculations for the imaginary part $\Im(1-e^{is\chi}) = -\sin(s\chi)$ follow analogously and we mention these estimates briefly. Furthermore, we use the splitting of the positive and negative part as in (4.1). Also, since $\Re(1-e^{is\chi})_- = 0$, it does not contribute to the estimates.

For $\delta \in (0, \varepsilon_0)$ to be specified later depending on ε and s, we have

$$|\Re(1-e^{is\chi})_+|_{\alpha^*,\beta} = \sup_{\varepsilon\le\varepsilon_0} \frac{\int \operatorname{osc}(R_{\alpha^*}\Re(1-e^{is\chi})_+, B_\varepsilon(x))\,\mathrm{d}\lambda_I(x)}{\varepsilon^\beta}$$

$$\le \sup_{\varepsilon\le\varepsilon_0} \frac{\int \operatorname{osc}(R_{\alpha^*}\Re(1-e^{is\chi})_+\mathbf{1}_{[0,\delta+\varepsilon]}, B_\varepsilon(x))\,\mathrm{d}\lambda_I(x)}{\varepsilon^\beta} \tag{4.8}$$

$$+ \sup_{\varepsilon\le\varepsilon_0} \frac{\int \operatorname{osc}(R_{\alpha^*}\Re(1-e^{is\chi})_+\mathbf{1}_{(\delta+\varepsilon,1-\delta-\varepsilon)}, B_\varepsilon(x))\,\mathrm{d}\lambda_I(x)}{\varepsilon^\beta} \tag{4.9}$$

$$+ \sup_{\varepsilon\le\varepsilon_0} \frac{\int \operatorname{osc}(R_{\alpha^*}\Re(1-e^{is\chi})_+\mathbf{1}_{[1-\delta-\varepsilon,1]}, B_\varepsilon(x))\,\mathrm{d}\lambda_I(x)}{\varepsilon^\beta}, \tag{4.10}$$

where we assume that s and ε_0 are so small that $\delta+\varepsilon < 1-\delta-\varepsilon$.

We start by estimating the middle summand (4.9). [38, Proposition 3.2(ii)] yields

$$\begin{aligned}
&\operatorname{osc}(R_{\alpha^*}\Re(1-e^{is\chi})_+\mathbf{1}_{(\delta+\varepsilon,1-\delta-\varepsilon)}, B_\varepsilon(x)) \\
&\quad\le \operatorname{osc}(R_{\alpha^*}\Re(1-e^{is\chi})_+, (\delta+\varepsilon, 1-\delta-\varepsilon)\cap B_\varepsilon(x))\mathbf{1}_{(\delta+\varepsilon,1-\delta-\varepsilon)}(x) \\
&\qquad + 2\left[\operatorname*{ess\,sup}_{(\delta+\varepsilon,1-\delta-\varepsilon)\cap B_\varepsilon(x)} R_{\alpha^*}\Re(1-e^{is\chi})_+\right]\mathbf{1}_{B_\varepsilon((\delta+\varepsilon,1-\delta-\varepsilon))\cap B_\varepsilon((\delta+\varepsilon,1-\delta-\varepsilon)^c)}(x)\,.
\end{aligned} \tag{4.11}$$

We first investigate the first summand of (4.11). For the following, we set

$$D(\delta,\varepsilon,x) := (\delta+\varepsilon, 1-\delta-\varepsilon)\cap B_\varepsilon(x). \tag{4.12}$$

For $x \in (\delta+\varepsilon, 1-\delta-\varepsilon)$,

$$\begin{aligned}
\operatorname{osc}(R_{\alpha^*}(1-\cos(s\chi)), D(\delta,\varepsilon,x)) &\le 2\varepsilon \sup_{D(\delta,\varepsilon,x)} [R_{\alpha^*}(1-\cos(s\chi))]' \\
&\le 2\varepsilon\left[\sup_{D(\delta,\varepsilon,x)} |(R_{\alpha^*}\mathbf{1})'|\,(1-\cos(s\chi)) + \sup_{D(\delta,\varepsilon,x)} (R_{\alpha^*}\mathbf{1})\,|(1-\cos(s\chi))'|\right].
\end{aligned} \tag{4.13}$$

Both of the above calculations follow analogously for the imaginary part with $|\sin(s\chi)|$ instead of $1-\cos(s\chi)$.

We set $\delta = \delta(\varepsilon, s) = \varepsilon^\kappa \cdot |s|^\iota$ with $\kappa \in (0,1)$ and $\iota > 0$ to be specified later. Since $|\chi|_{\alpha,\beta} < \infty$ implies that $R_\alpha\chi$ is essentially bounded, we can conclude that there exists $K(\chi) \in (0,\infty)$ such that $|\chi(x)| \le K(\chi)\cdot x^{-\alpha}(1-x)^{-\alpha}$ almost everywhere. Recall that there is $C>0$ such that $\max\{|1-\cos(x)\}|, |\sin(x)|\} \le C|x|$. Combining this with $(R_{\alpha^*}\mathbf{1})' = \alpha^*(x^{\alpha^*-1}(1-x)^{\alpha^*} + x^{\alpha^*}(1-x)^{\alpha^*-1})$, we have

$$\sup_{D(\delta,\varepsilon,x)} |(R_{\alpha^*}\mathbf{1})'| \max\{1-\cos(s\chi)\,, |\sin(s\chi)|\} \lesssim \frac{|s|}{(x-\varepsilon)^{1+\alpha-\alpha^*}}\,, \tag{4.14}$$

when $x \le 1/2$. The estimates for $x \ge 1/2$ follows from replacing $(x-\varepsilon)$ by $(1-x+\varepsilon)$, and the final estimates remain unchanged. So, we restrict our attention to the former case.

It follows that

$$\lim_{s\to0}\sup_{\varepsilon\in(0,\varepsilon_0]} \frac{1}{\varepsilon^\beta}\int_{\delta+\varepsilon}^{1/2} 2\varepsilon \sup_{D(\delta,\varepsilon,x)}\Big(|(R_{\alpha^*}\mathbf{1})'|(1-\cos(s\chi))\Big)\mathbf{1}_{(\delta,1-\delta)}(x)\,\mathrm{d}\lambda_I(x)$$

$$\lesssim \lim_{s\to 0} |s| \sup_{\varepsilon\in(0,\varepsilon_0]} \varepsilon^{1-\beta} \int_{\delta+\varepsilon}^{1/2} (x-\varepsilon)^{\alpha^*-1-\alpha}\, \mathrm{d}\lambda_I(x)$$

$$\lesssim \lim_{s\to 0} |s| \sup_{\varepsilon\in(0,\varepsilon_0]} \varepsilon^{1-\beta} \int_{\delta}^{1/2-\varepsilon} x^{\alpha^*-1-\alpha}\, \mathrm{d}\lambda_I(x)$$

$$\lesssim \begin{cases} \lim_{s\to 0} \varepsilon_0^{1-\beta+\kappa(\alpha^*-\alpha)} \lim_{s\to 0} |s|^{1+\iota(\alpha^*-\alpha)} & \alpha^*<\alpha \\ \varepsilon_0^{1-\beta}(|\log(1/2-\varepsilon_0)|+\kappa|\log(\varepsilon_0)|)\lim_{s\to 0}|s| + \iota\varepsilon_0^{1-\beta}\lim_{s\to 0}|s|\,|\log|s|| & \alpha^*=\alpha \\ \varepsilon_0^{1-\beta}\lim_{s\to 0}|s| & \alpha^*>\alpha \end{cases}$$

$$= 0 \tag{4.15}$$

provided that under the condition $\alpha^*<\alpha$ we have

$$\begin{aligned} 1-\beta+\kappa(\alpha^*-\alpha) > 0 &\iff \kappa < (1-\beta)/(\alpha-\alpha^*)\,, \\ \iota\,(\alpha^*-\alpha)+1>0 &\iff \iota < 1/(\alpha-\alpha^*)\,. \end{aligned} \tag{4.16}$$

Analogously, under the same conditions,

$$\lim_{s\to 0} \sup_{\varepsilon\in(0,\varepsilon_0]} \frac{1}{\varepsilon^\beta} \int 2\varepsilon \sup_{D(\delta,\varepsilon,x)} \Big(|(R_{\alpha^*}\mathbf{1})'\,| (\sin(s\chi))_{\pm} \Big) \mathbf{1}_{(\delta+\varepsilon,1-\delta-\varepsilon)}(x)\, \mathrm{d}\lambda_I(x) = 0.$$

To estimate the second summand of (4.13), we use $(1-\cos(s\chi))' = \sin(s\chi)\cdot s\chi'$, $(\sin(s\chi))' = \cos(s\chi)\cdot s\chi'$, $|\cos(s\chi)|\le 1$, $|\sin(s\chi)|\le 1$, and our assumption about χ'. Then, we have

$$\sup_{D(\delta,\varepsilon,x)} \max\left\{ (R_{\alpha^*}\mathbf{1})\,|(1-\cos(s\chi))'|\,, (R_{\alpha^*}\mathbf{1})|(\sin(s\chi)^{\pm})'| \right\} \lesssim \begin{cases} |s|(x-\varepsilon)^{\alpha^*-b} & \alpha^*<b \\ |s|\cdot 1 & \alpha^*\ge b \end{cases}$$

for $x\le 1/2$. Also, note that for $x\le 1/2$ and the estimate for $x\ge 1/2$ is the same with $(x-\varepsilon)$ replaced by $(1-x+\varepsilon)$. Thus, if $\alpha^*<b$

$$\lim_{s\to 0} \sup_{\varepsilon\in(0,\varepsilon_0]} \frac{1}{\varepsilon^\beta} \int_{\delta+\varepsilon}^{1/2} 2\varepsilon \sup_{D(\delta,\varepsilon,x)} \Big((R_{\alpha^*}\mathbf{1})\,|(1-\cos(s\chi))'| \Big) \mathbf{1}_{(\delta+\varepsilon,1-\delta-\varepsilon)}(x)\, \mathrm{d}\lambda_I(x)$$

$$\lesssim \lim_{s\to 0} \sup_{\varepsilon\in(0,\varepsilon_0]} \varepsilon^{1-\beta}|s| \int_{\delta}^{1/2-\varepsilon} x^{\alpha^*-b} \mathrm{d}\lambda_I(x)$$

$$\lesssim \begin{cases} \varepsilon_0^{1-\beta+\kappa(1+\alpha^*-b)} \lim_{s\to 0}|s|^{1+\iota(1+\alpha^*-b)} & b>1+\alpha^* \\ \varepsilon_0^{1-\beta}(|\log(1/2-\varepsilon_0)|+\kappa|\log(\varepsilon_0)|)\, \lim_{s\to 0}|s| + \iota\varepsilon_0^{1-\beta}\lim_{s\to 0}|s|\,|\log|s|| & b=1+\alpha^* \\ \varepsilon_0^{1-\beta}\, \lim_{s\to 0}|s| & b<1+\alpha^* \end{cases}$$

$$= 0\,, \tag{4.17}$$

where, in the case of $b>1+\alpha^*$, we have assumed that

$$\begin{aligned} 1-\beta+\kappa(1+\alpha^*-b)>0 &\iff \kappa<(1-\beta)/(b-1-\alpha^*)\,, \\ 1+\iota(1+\alpha^*-b)>0 &\iff \iota<1/(b-1-\alpha^*)\,. \end{aligned} \tag{4.18}$$

The $\alpha^*\ge b$ case is similar to the $b<1+\alpha^*$ case above. Analogously, under the same assumptions on κ and ι, we obtain

$$\lim_{s\to 0} \sup_{\varepsilon\in(0,\varepsilon_0]} \frac{1}{\varepsilon^\beta} \int 2\varepsilon \sup_{D(\delta,\varepsilon,x)} \Big(R_{\alpha^*}\mathbf{1}\, \left|(\sin(s\chi)_{\pm})'\right| \Big) \mathbf{1}_{(\delta+\varepsilon,1-\delta-\varepsilon)}(x)\, \mathrm{d}\lambda_I(x) = 0\,.$$

Hence, combining (4.15) and (4.17), we can conclude

$$\lim_{s\to 0} \sup_{\varepsilon\in(0,\varepsilon_0]} \frac{1}{\varepsilon^\beta} \int \operatorname{osc}\left(R_{\alpha^*}\Re(1-e^{is\chi})_+, D(\delta,\varepsilon,x)\right) \mathbf{1}_{(\delta,1-\delta)}(x)\, \mathrm{d}\lambda_I(x) = 0. \tag{4.19}$$

Also, the analogous result for the imaginary part, $\Im(1-e^{is\chi})_\pm$, follows.

Next, we will estimate the second summand in (4.11). We note that

$$B_\varepsilon((\delta+\varepsilon,1-\delta-\varepsilon))\cap B_\varepsilon((\delta+\varepsilon,1-\delta-\varepsilon)^c)=B_\varepsilon(\delta+\varepsilon)\cup B_\varepsilon(1-\delta-\varepsilon)$$

and hence,

$$\mathbf{1}_{B_\varepsilon((\delta+\varepsilon,1-\delta-\varepsilon))\cap B_\varepsilon((\delta+\varepsilon,1-\delta-\varepsilon)^c)}=\mathbf{1}_{B_\varepsilon(\delta+\varepsilon)\cup B_\varepsilon(1-\delta-\varepsilon)}. \tag{4.20}$$

It follows that

$$\sup_{D(\delta,\varepsilon,x)} R_{\alpha^*}(1-\cos(s\chi))\lesssim\begin{cases}|s|(x-\varepsilon)^{\alpha^*-\alpha} & \alpha^*<\alpha\\ |s|(x+\varepsilon)^{\alpha^*-\alpha} & \alpha^*\geq\alpha.\end{cases} \tag{4.21}$$

Due to the symmetry around $x=1/2$, we obtain

$$\begin{aligned}
&\lim_{s\to0}\sup_{\varepsilon\in(0,\varepsilon_0]}\frac{1}{\varepsilon^\beta}\int\sup_{D(\delta,\varepsilon,x)}R_{\alpha^*}\Re(1-e^{is\chi})_+\cdot\mathbf{1}_{B_\varepsilon((\delta+\varepsilon,1-\delta-\varepsilon))\cap B_\varepsilon((\delta+\varepsilon,1-\delta-\varepsilon)^c)}(x)\,\mathrm{d}\lambda_I(x)\\
&\quad=\lim_{s\to0}\sup_{\varepsilon\in(0,\varepsilon_0]}\frac{1}{\varepsilon^\beta}\left(\int_\delta^{\delta+2\varepsilon}+\int_{1-\delta-2\varepsilon}^{1-\delta}\right)\sup_{D(\delta,\varepsilon,x)}R_{\alpha^*}(1-\cos(s\chi))\,\mathrm{d}\lambda_I(x)\\
&\quad\lesssim\lim_{s\to0}\sup_{\varepsilon\in(0,\varepsilon_0]}|s|\varepsilon^{-\beta}\int_\delta^{\delta+2\varepsilon}\max\{(\delta+\varepsilon)^{\alpha^*-\alpha},(\delta+2\varepsilon)^{\alpha^*-\alpha}\}\,\mathrm{d}\lambda_I(x)\\
&\quad\lesssim\begin{cases}\lim_{s\to0}\varepsilon_0^{1-\beta-\kappa(\alpha-\alpha^*)}|s|^{1-\iota(\alpha-\alpha^*)} & \alpha^*<\alpha\\ \lim_{s\to0}\varepsilon_0^{1-\beta}|s| & \alpha^*\geq\alpha\end{cases}\\
&\quad=0
\end{aligned} \tag{4.22}$$

where, in the case of $\alpha^*<\alpha$, we assume that

$$\begin{aligned}1-\beta-\kappa(\alpha-\alpha^*)>0&\iff\kappa<(1-\beta)/(\alpha-\alpha^*)\,,\\ 1-\iota(\alpha-\alpha^*)>0&\iff\iota<1/(\alpha-\alpha^*)\,.\end{aligned} \tag{4.23}$$

Combining this with (4.11) and (4.19) yields that the summand (4.9) tends to zero for $s\to0$ and the same is true for the imaginary part, $\Im(1-e^{is\chi})_\pm$, because the same assumptions on κ and ι along with $|\sin(x)|\lesssim|x|$ and (4.21) yield

$$\lim_{s\to0}\sup_{\varepsilon\in(0,\varepsilon_0]}\frac{1}{\varepsilon^\beta}\int\operatorname{osc}\left((\sin(s\chi))_\pm,D(\delta,\varepsilon,x)\right)\mathbf{1}_{(\delta+\varepsilon,1-\delta-\varepsilon)}(x)\,\mathrm{d}\lambda_I(x)=0\,,$$

$$\lim_{s\to0}\sup_{\varepsilon\in(0,\varepsilon_0]}\frac{1}{\varepsilon^\beta}\int\sup_{D(\delta,\varepsilon,x)}R_{\alpha^*}(\sin(s\chi))_\pm\cdot\mathbf{1}_{B_\varepsilon((\delta+\varepsilon,1-\delta-\varepsilon))\cap B_\varepsilon((\delta+\varepsilon,1-\delta-\varepsilon)^c)}\,\mathrm{d}\lambda_I(x)=0\,.$$

Finally, we investigate into the first summand (4.8). As the calculation for the summand (4.10) is very similar, we will only give the details for (4.8). We split the integral into

$$\begin{aligned}
&\lim_{s\to0}\sup_{\varepsilon\leq\varepsilon_0}\frac{1}{\varepsilon^\beta}\int\operatorname{osc}(R_{\alpha^*}\Re(1-e^{is\chi})\mathbf{1}_{[0,\delta+\varepsilon]},B_\varepsilon(x))\,\mathrm{d}\lambda_I(x)\\
&\quad=\lim_{s\to0}\sup_{\varepsilon\leq\varepsilon_0}\frac{1}{\varepsilon^\beta}\left(\int_{[0,\delta)}+\int_{[\delta,\delta+2\varepsilon]}\right)\operatorname{osc}(R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[0,\delta+\varepsilon]},B_\varepsilon(x))\,\mathrm{d}\lambda_I(x).
\end{aligned} \tag{4.24}$$

For the first summand of (4.24), we write

$$\bar{D}(\delta,\varepsilon,x):=[0,\delta+\varepsilon]\cap B_\varepsilon(x) \tag{4.25}$$

and we note that $\Re(1-e^{is\chi})\in(0,2)$ and

$$\operatorname{osc}(R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[0,\delta+\varepsilon]},B_\varepsilon(x))\leq2\cdot\sup\nolimits_{\bar{D}(\delta,\varepsilon,x)}R_{\alpha^*}\mathbf{1}\leq2R_{\alpha^*}\mathbf{1}(x+\varepsilon)\leq2(x+\varepsilon)^{\alpha^*}. \tag{4.26}$$

Now, we have

$$\lim_{s\to 0}\sup_{\varepsilon\le\varepsilon_0}\frac{1}{\varepsilon^\beta}\int_0^\delta \operatorname{osc}(R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[0,\delta+\varepsilon]},B_\varepsilon(x))\,\mathrm{d}\lambda_I(x)=0 \tag{4.27}$$

under the condition

$$\iota>0 \quad \text{and} \quad \kappa>\frac{\beta}{1+\alpha^*} \tag{4.28}$$

due to Lemma C.1 in Appendix C.

In order to estimate the second summand of (4.24), we first note that for $x\in[\delta,\delta+2\varepsilon]$,

$$\sup_{\bar D(\delta,\varepsilon,x)} R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[\delta,\delta+\varepsilon]}\lesssim |s|\sup_{y\in\bar D(\delta,\varepsilon,x)\cap[\delta,\delta+2\varepsilon]} y^{\alpha^*-\alpha}\le\begin{cases}|s| & \alpha^*\ge\alpha\\ |s|\cdot\delta^{\alpha^*-\alpha} & \alpha^*<\alpha\,.\end{cases} \tag{4.29}$$

Hence,

$$\begin{aligned}
&\lim_{s\to 0}\sup_{\varepsilon\le\varepsilon_0}\frac{1}{\varepsilon^\beta}\int_{[\delta,\delta+2\varepsilon]}\operatorname{osc}\left(R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[\delta,\delta+\varepsilon]},B_\varepsilon(x)\right)\,\mathrm{d}\lambda_I(x)\\
&\quad\lesssim \lim_{s\to 0}\sup_{\varepsilon\le\varepsilon_0}\frac{1}{\varepsilon^\beta}\int_{[\delta,\delta+2\varepsilon]}\sup_{\bar D(\delta,\varepsilon,x)}\left(R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[\delta,\delta+\varepsilon]}\right)\,\mathrm{d}\lambda_I(x)\\
&\quad\lesssim\begin{cases}\varepsilon_0^{1-\beta}\lim_{s\to 0}|s| & \alpha^*\ge\alpha\\ \varepsilon_0^{1-\beta+\kappa(\alpha^*-\alpha)}\lim_{s\to 0}|s|^{1+\iota(\alpha-\alpha^*)} & \alpha^*<\alpha\end{cases}\\
&\quad=0
\end{aligned} \tag{4.30}$$

provided that, in the case of $\alpha^*<\alpha$,

$$\begin{aligned}
1-\beta+\kappa\,(\alpha^*-\alpha)>0 &\iff \kappa<(1-\beta)/(\alpha-\alpha^*)\,,\\
1+\iota(\alpha^*-\alpha)>0 &\iff \iota<1/(\alpha-\alpha^*).
\end{aligned} \tag{4.31}$$

Next, by [38, Prop. 3.2(ii)] we have for $x\in(\delta,\delta+2\varepsilon]$

$$\begin{aligned}
&\operatorname{osc}\left(R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[0,\delta]},B_\varepsilon(x)\right)\\
&\quad\le\operatorname{osc}\left(R_{\alpha^*}(1-\cos(s\chi)),B_\varepsilon(x)\cap[0,\delta]\right)\mathbf{1}_{[0,\delta]}(x)+2\operatorname*{ess\,sup}_{B_\varepsilon(x)\cap[0,\delta]}R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[\delta-\varepsilon\vee 0,\delta+\varepsilon]}(x)\\
&\quad\le 0+2\operatorname*{ess\,sup}_{[\delta-\varepsilon\vee 0,\delta]}R_{\alpha^*}\mathbf{1}\le 2\delta^{\alpha^*}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{s\to 0}\sup_{\varepsilon\le\varepsilon_0}\frac{1}{\varepsilon^\beta}\int_{[\delta,\delta+2\varepsilon]}\operatorname{osc}\left(R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[0,\delta]},B_\varepsilon(x)\right)\,\mathrm{d}\lambda_I(x)\\
&\quad\lesssim\lim_{s\to 0}\sup_{\varepsilon\le\varepsilon_0}\frac{1}{\varepsilon^\beta}\int_{[\delta,\delta+2\varepsilon]}\delta^{\alpha^*}\,\mathrm{d}\lambda_I(x)\lesssim\lim_{s\to 0}\sup_{\varepsilon\le\varepsilon_0}\varepsilon^{1-\beta+\kappa\alpha^*}s^{\iota\alpha^*}=0
\end{aligned}$$

under (4.28). This together with (4.30) imply

$$\lim_{s\to 0}\sup_{\varepsilon\le\varepsilon_0}\frac{1}{\varepsilon^\beta}\int_{[\delta,\delta+2\varepsilon]}\operatorname{osc}\left(R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[0,\delta+2\varepsilon]},B_\varepsilon(x)\right)\,\mathrm{d}\lambda_I(x)=0.$$

Combining this with (4.24) and (4.27) implies that (4.8) tends to zero for s tending to zero. The same is true for the imaginary part, $\Im(1-e^{is\chi})_\pm$, as $\Im(1-e^{is\chi})_\pm\le 1$.

Finally, we discuss here possible values of α^* and the implicit requirements on b that ensure the existence of $\iota>0$ and $\kappa>0$ used in the proof. There are four cases.

Note that, in the case of $\alpha^* < \alpha$ and $b > 1+\alpha^*$, under (4.16), (4.18), (4.23), (4.28) and (4.31), we have

$$\frac{\beta}{\alpha^*+1} < \kappa < \min\left\{\frac{1-\beta}{\alpha-\alpha^*}, \frac{1-\beta}{b-1-\alpha^*}, 1\right\},$$
$$0 < \iota < \min\left\{\frac{1}{\alpha-\alpha^*}, \frac{1}{b-1-\alpha^*}\right\}.$$

First, we see that the conditions on ι are always fulfilled, because

$$0 < \frac{1}{\alpha-\alpha^*} \quad \text{and} \quad 0 < \frac{1}{b-1-\alpha^*}.$$

Similarly, considering the inequalities that guarantee the existence of κ, we have

$$\alpha > \alpha^* > \max\{\alpha\beta+\beta-1, \beta b-1\}$$

is necessary and sufficient. Note that due to $\beta < \min\{1/b, 1/(\alpha+1)\}$ we have $\alpha\beta+\beta-1<0$, and also, $\beta b-1<0$. So, $0<\alpha^*<\min\{\alpha, b-1\}$ which is equivalent to $\alpha^*<\alpha$ and $b>1+\alpha^*$.

In the case of $\alpha^* < \alpha$ and $b \le 1+\alpha^*$, (4.18) poses no restrictions. So, under (4.16), (4.23), (4.28) and (4.31) we have $b-1<\alpha^*<\alpha$ and $b\le 1+\alpha$ which is equivalent to our assumptions $\alpha^*<\alpha$ and $b\le 1+\alpha^*$.

In the case of $\alpha^* \ge \alpha$, (4.16), (4.23), and (4.31) pose no restrictions. So, when $b<1+\alpha^*$, we have $\alpha^* > \max\{\alpha\,, b-1\}$ and when $b>1+\alpha^*$, we have $\alpha<\alpha^*<b-1$ and $b>1+\alpha$ and we don't obtain any additional restrictions either. ☐

The next lemma of this section gives a sufficient condition on χ for the operator valued function $s \mapsto H_s$, and hence, $s \mapsto \widehat{\psi}_{is}$ to be differentiable.

Lemma 4.7. *Suppose $|\chi|_{\alpha,\beta}<\infty$ with $0\le\alpha\le\beta<1/(1+\alpha)$ and there exists $b\in[0,1/\beta)$ such that (4.5) holds. Then, for all $\alpha^*>\min\{2\alpha, \max\{\alpha, \alpha+b-2\}\}$ we have*

$$\lim_{s\to 0}\left|\frac{e^{is\chi}-1-is\chi}{s}\right|_{\alpha^*,\beta} = 0\,.$$

Proof. The proof follows very similar to the proof of the previous lemma and we will stick to the same notation. Again, we will do the calculations only for the non-negative real part, only noting some differences for the imaginary part. We have

$$\operatorname{osc}\left(R_{\alpha^*}\Re\left(\frac{e^{is\chi}-1-is\chi}{s}\right), B_\varepsilon(x)\right) = \frac{1}{s}\operatorname{osc}\left(R_{\alpha^*}(1-\cos(s\chi)), B_\varepsilon(x)\right)$$

and

$$\operatorname{osc}\left(R_{\alpha^*}\Im\left(\frac{e^{is\chi}-1-is\chi}{s}\right), B_\varepsilon(x)\right) = \frac{1}{s}\operatorname{osc}\left(R_{\alpha^*}(\sin(s\chi)-s\chi), B_\varepsilon(x)\right).$$

As in (4.8) to (4.10), we have for $\delta\in(0,\varepsilon_0)$ (to be specified later and depending on s and ε) that

$$\left|\Re(e^{is\chi}-1-is\chi)\right|_{\alpha^*,\beta} = \sup_{\varepsilon\le\varepsilon_0}\frac{\int \operatorname{osc}(R_{\alpha^*}(1-\cos(s\chi)), B_\varepsilon(x))\,\mathrm{d}\lambda_I(x)}{s\varepsilon^\beta}$$

$$\le \sup_{\varepsilon\le\varepsilon_0}\frac{\int \operatorname{osc}(R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[0,\delta+\varepsilon]}, B_\varepsilon(x))\,\mathrm{d}\lambda_I(x)}{s\varepsilon^\beta} \tag{4.32}$$

$$+ \sup_{\varepsilon\le\varepsilon_0}\frac{\int \operatorname{osc}(R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{(\delta+\varepsilon,1-\delta-\varepsilon)}, B_\varepsilon(x))\,\mathrm{d}\lambda_I(x)}{s\varepsilon^\beta} \tag{4.33}$$

$$+ \sup_{\varepsilon\le\varepsilon_0}\frac{\int \operatorname{osc}(R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[1-\delta-\varepsilon,1]}, B_\varepsilon(x))\,\mathrm{d}\lambda_I(x)}{s\varepsilon^\beta}, \tag{4.34}$$

and similarly, for the imaginary part.

Now, we start by estimating the middle term (4.33), and as in (4.11), we use [38, Proposition 3.2(ii)] to obtain

$$
\begin{aligned}
&\operatorname{osc}(R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{(\delta+\varepsilon,1-\delta-\varepsilon)}, B_\varepsilon(x)) \\
&\quad \le \operatorname{osc}(R_{\alpha^*}(1-\cos(s\chi)), D(\delta,\varepsilon,x))\mathbf{1}_{(\delta+\varepsilon,1-\delta-\varepsilon)}(x) \\
&\qquad + 2\left[\sup_{D(\delta,\varepsilon,x)} R_{\alpha^*}(1-\cos(s\chi))\right]\mathbf{1}_{B_\varepsilon((\delta+\varepsilon,1-\delta-\varepsilon))\cap B_\varepsilon((\delta+\varepsilon,1-\delta-\varepsilon)^c)}(x)\,.
\end{aligned}
\tag{4.35}
$$

For $x \in (\delta+\varepsilon, 1-\delta-\varepsilon)$,

$$
\begin{aligned}
\operatorname{osc}(R_{\alpha^*}(1-\cos(s\chi)), D(\delta,\varepsilon,x)) &\le 2\varepsilon \sup_{D(\delta,\varepsilon,x)} |[R_{\alpha^*}(1-\cos(s\chi))]'| \\
&\le 2\varepsilon\left[\sup_{D(\delta,\varepsilon,x)} |(R_{\alpha^*}\mathbf{1})'|\,(1-\cos(s\chi)) + \sup_{D(\delta,\varepsilon,x)} (R_{\alpha^*}\mathbf{1})|(1-\cos(s\chi))'|\right].
\end{aligned}
\tag{4.36}
$$

Both of the above calculations follow analogously for the imaginary part.

For the following, as in the previous proof, we set $\delta = \delta(\varepsilon, s) = \varepsilon^\kappa \cdot |s|^\iota$ with $\kappa \in (0,1)$, $\iota > 0$ and recall that there is $C > 0$ such that $\max\{|1-\cos(x)|, |\sin(x) - x|\} \le C|x|^2$. The latter fact and $(R_{\alpha^*}\mathbf{1})' = \alpha^*(x^{\alpha^*-1}(1-x)^{\alpha^*} + x^{\alpha^*}(1-x)^{\alpha^*-1})$, imply that

$$
\sup_{D(\delta,\varepsilon,x)} |(R_{\alpha^*}\mathbf{1})'| \cdot \max\{1-\cos(s\chi), |\sin(s\chi) - (s\chi)|\} \lesssim \frac{|s|^2}{(x-\varepsilon)^{1+2\alpha-\alpha^*}}\,,
\tag{4.37}
$$

when $x \le 1/2$. The estimates for $x \ge 1/2$ follows from replacing $(x-\varepsilon)$ by $(1-x+\varepsilon)$, and the final estimates remain unchanged. So, we restrict our attention to the former case.

This implies that the contribution of the first term in (4.36) is

$$
\begin{aligned}
&\lim_{s\to 0} \sup_{\varepsilon\in(0,\varepsilon_0]} \frac{1}{|s|\varepsilon^\beta}\int_{\delta+\varepsilon}^{1/2} 2\varepsilon \sup_{D(\delta,\varepsilon,x)} |(R_{\alpha^*}\mathbf{1})'|\,(1-\cos(s\chi))\mathbf{1}_{(\delta+\varepsilon,1-\delta-\varepsilon)}(x)\,\mathrm{d}\lambda_I(x) \\
&\lesssim \lim_{s\to 0} |s| \sup_{\varepsilon\in(0,\varepsilon_0]} \varepsilon^{1-\beta}\int_\delta^{1/2-\varepsilon} x^{\alpha^*-1-2\alpha}\,\mathrm{d}\lambda_I(x) \\
&\lesssim \begin{cases} \varepsilon_0^{1-\beta+\kappa(\alpha^*-2\alpha)} \lim_{s\to 0} |s|^{1+\iota(\alpha^*-2\alpha)} = 0\,, & \alpha^* < 2\alpha \\ \varepsilon_0^{1-\beta}(|\log(1/2-\varepsilon_0)| + \kappa|\log(\varepsilon_0)|)\lim_{s\to 0}|s| + \iota\varepsilon_0^{1-\beta}\lim_{s\to 0}|s|\,|\log|s|| & \alpha^* = 2\alpha \\ \varepsilon_0^{1-\beta}\lim_{s\to 0}|s| & \alpha^* > 2\alpha \end{cases} \\
&= 0
\end{aligned}
$$

provided that, in the $\alpha^* < 2\alpha$ case,

$$
\begin{aligned}
1-\beta+\kappa(\alpha^*-2\alpha) > 0 &\iff \kappa < (1-\beta)/(2\alpha-\alpha^*) \\
1+\iota(\alpha^*-2\alpha) > 0 &\iff \iota < 1/(2\alpha-\alpha^*)\,,
\end{aligned}
\tag{4.38}
$$

and similarly,

$$
\lim_{s\to 0} \sup_{\varepsilon\in(0,\varepsilon_0]} \frac{1}{s\varepsilon^\beta}\int 2\varepsilon \sup_{D(\delta,\varepsilon,x)} |(R_{\alpha^*}\mathbf{1})'|\,(\sin(s\chi) - s\chi)_\pm \mathbf{1}_{(\delta+\varepsilon,1-\delta-\varepsilon)}(x)\,\mathrm{d}\lambda_I(x) = 0\,.
$$

Next, we estimate the second summand of (4.36). Using $(1-\cos(s\chi))' = \sin(s\chi)\cdot s\chi'$, $|\sin(s\chi)| \le |s\chi|$, and our assumption about χ and χ' we have

$$
\sup_{D(\delta,\varepsilon,x)} (R_{\alpha^*}\mathbf{1})\,|(1-\cos(s\chi))'| \lesssim \begin{cases} |s|^2(x-\varepsilon)^{\alpha^*-(\alpha+b)} & \alpha^* < \alpha+b \\ |s|^2\cdot 1 & \alpha^* \ge \alpha+b\,. \end{cases}
$$

Also, note that the estimate for $x \le 1/2$ and for $x \ge 1/2$ are the same with $(x-\varepsilon)$ replaced by $(1-x+\varepsilon)$. Thus, when $\alpha^* < \alpha + b$

$$\begin{aligned}
&\lim_{s\to 0} \sup_{\varepsilon\in(0,\varepsilon_0]} \frac{\varepsilon}{s\varepsilon^\beta} \int_{\delta+\varepsilon}^{1/2} \sup_{D(\delta,\varepsilon,x)} (R_{\alpha^*}\mathbf{1})\, |(1-\cos(s\chi))'| \mathbf{1}_{(\delta+\varepsilon,1-\delta-\varepsilon)}(x)\,\mathrm{d}\lambda_I(x)\\
&\lesssim \lim_{s\to 0} \sup_{\varepsilon\in(0,\varepsilon_0]} \varepsilon^{1-\beta}|s| \int_{\delta}^{1/2-\varepsilon} x^{\alpha^*-(\alpha+b)} \mathrm{d}\lambda_I(x)\\
&\lesssim \begin{cases} \varepsilon_0^{1-\beta+\kappa(1+\alpha^*-\alpha-b)} \lim_{s\to 0}|s|^{1+\iota(1+\alpha^*-\alpha-b)} & \alpha+b > 1+\alpha^*\\ \varepsilon_0^{1-\beta}(|\log(1/2-\varepsilon_0)| + \kappa|\log(\varepsilon_0)|)\lim_{s\to 0}|s| + \iota\varepsilon_0^{1-\beta}\lim_{s\to 0}|s|\,|\log|s|| & \alpha+b = 1+\alpha^*\\ \varepsilon_0^{1-\beta}\lim_{s\to 0}|s| & \alpha+b<1+\alpha^* \end{cases}\\
&= 0\,.
\end{aligned}$$

where, in the case of $\alpha + b > 1 + \alpha^*$, we have assumed that

$$\begin{aligned} 1-\beta+\kappa(1+\alpha^*-\alpha-b) > 0 &\iff \kappa < (1-\beta)/(\alpha+b-1-\alpha^*)\,,\\ 1+\iota(1+\alpha^*-\alpha-b) > 0 &\iff \iota < 1/(\alpha+b-1-\alpha^*)\,. \end{aligned} \tag{4.39}$$

Analogously, under the same assumptions on κ and ι, we obtain

$$\lim_{s\to 0} \sup_{\varepsilon\in(0,\varepsilon_0]} \frac{\varepsilon}{s\varepsilon^\beta} \int \sup_{D(\delta,\varepsilon,x)} R_{\alpha^*}\mathbf{1}(x) \left|((\sin(s\chi)-s\chi)_\pm)'\right| \mathbf{1}_{(\delta+\varepsilon,1-\delta-\varepsilon)}(x)\,\mathrm{d}\lambda_I(x) = 0$$

because $|(\sin(s\chi)-s\chi)'| = |\cos(s\chi)-1|\cdot|s\chi'|$ and $|\cos(s\chi)-1| \le |s\chi|$.

Next, we look at the second summand of (4.35). Using (4.20), our assumption about χ and the symmetry around $x = 1/2$, the corresponding integral over the second summand is dominated by

$$\begin{aligned}
&\lim_{s\to 0} \sup_{\varepsilon\in(0,\varepsilon_0]} \frac{2}{s\varepsilon^\beta} \left(\int_\delta^{\delta+2\varepsilon} + \int_{1-\delta-2\varepsilon}^{1-\delta} \right) \sup_{D(\delta,\varepsilon,x)} R_{\alpha^*}(1-\cos(s\chi))\,\mathrm{d}\lambda_I(x)\\
&\lesssim \lim_{s\to 0} \sup_{\varepsilon\in(0,\varepsilon_0]} |s|\varepsilon^{-\beta} \int_\delta^{\delta+2\varepsilon} \max\{(\delta+\varepsilon)^{\alpha^*-2\alpha}, (\delta+3\varepsilon)^{\alpha^*-2\alpha}\}\,\mathrm{d}\lambda_I(x)\\
&\lesssim |s|^{1-\iota(2\alpha-\alpha^*)} \lim_{s\to 0} \varepsilon_0^{1-\beta}|s|\\
&= 0\,.
\end{aligned}$$

Here, in the case of $\alpha^* < 2\alpha$, we have to assume additionally that

$$\begin{aligned} 1-\beta-\kappa(2\alpha-\alpha^*) > 0 &\iff \kappa < (1-\beta)/(2\alpha-\alpha^*)\,,\\ 1-\iota(2\alpha-\alpha^*) > 0 &\iff \iota < 1/(2\alpha-\alpha^*)\,. \end{aligned} \tag{4.40}$$

Analogously, under the same assumptions on κ and ι, using our assumption about χ, we have

$$\lim_{s\to 0} \sup_{\varepsilon\in(0,\varepsilon_0]} \frac{2}{s\varepsilon^\beta} \left(\int_\delta^{\delta+2\varepsilon} + \int_{1-\delta-2\varepsilon}^{1-\delta} \right) \sup_{D(\delta,\varepsilon,x)} R_{\alpha^*}(\sin(s\chi)-s\chi)_\pm\,\mathrm{d}\lambda_I(x) = 0.$$

Finally, we investigate (4.32). The estimations for (4.34) then follow analogously. We split the integral as in (4.24).

For the first integral, due to Lemma C.1 in Appendix C, we have

$$\lim_{s\to 0} \sup_{\varepsilon\le\varepsilon_0} \frac{1}{s\varepsilon^\beta} \int_{[0,\delta)} \operatorname{osc}(R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[0,\delta+\varepsilon]}, B_\varepsilon(x))\,\mathrm{d}\lambda_I(x) = 0 \tag{4.41}$$

provided that

$$\begin{aligned} \kappa(1+\alpha^*)-\beta > 0 &\iff \kappa > \beta/(1+\alpha^*)\,,\\ \iota - 1 > 0 &\iff \iota > 1\,. \end{aligned} \tag{4.42}$$

For the imaginary part, since we assumed $\alpha^* \geq \alpha$, we can use the following estimate.

$$\sup_{\bar D(\delta,\varepsilon,x)}|R_{\alpha^*}(\sin(s\chi)-s\chi)\mathbf{1}_{[0,\delta+\varepsilon]}| \lesssim |s|\sup_{\bar D(\delta,\varepsilon,x)}|R_{\alpha^*}\chi\mathbf{1}_{[0,\delta+\varepsilon]}| \lesssim |s|\sup_{\bar D(\delta,\varepsilon,x)}R_{\alpha^*-\alpha}\mathbf{1}_{[0,\delta+\varepsilon]} \lesssim |s|\,(x+\varepsilon)^{\alpha^*-\alpha}.$$

Then repeating the argument in Appendix C leading to Equation (C.2) with $\alpha^*-\alpha$ replacing α^*, we have that

$$\lim_{s\to 0}\sup_{\varepsilon\leq\varepsilon_0}\frac{1}{s\varepsilon^\beta}\int_{[0,\delta)}\operatorname{osc}(R_{\alpha^*}(\sin(s\chi)-s\chi)\mathbf{1}_{[0,\delta+\varepsilon]},B_\varepsilon(x))\,\mathrm{d}\lambda_I(x)=0$$

provided that

$$\begin{aligned}\kappa(1+\alpha^*-\alpha)-\beta>0 &\iff \kappa>\beta/(1+\alpha^*-\alpha)\,,\\ \iota-1>0 &\iff \iota>1\,.\end{aligned} \tag{4.43}$$

For the second integral, as in (4.29) but using (4.37) instead, we obtain for all $x\in(\delta,\delta+2\varepsilon]$

$$\sup_{\bar D(\delta,\varepsilon,x)}R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[\delta,\delta+\varepsilon]}\lesssim s^2\sup_{y\in\bar D(\delta,\varepsilon,x)\cap(\delta,\delta+2\varepsilon]}y^{\alpha^*-2\alpha}\leq\begin{cases}s^2 & \alpha^*\geq 2\alpha\\ s^2\cdot\delta^{\alpha^*-2\alpha} & \alpha^*<2\alpha\,.\end{cases}$$

Therefore,

$$\begin{aligned}&\lim_{s\to 0}\sup_{\varepsilon\leq\varepsilon_0}\frac{1}{s\varepsilon^\beta}\int_{(\delta,\delta+2\varepsilon]}\operatorname{osc}(R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[\delta,\delta+\varepsilon]},B_\varepsilon(x))\,\mathrm{d}\lambda_I(x)\\ &\lesssim\begin{cases}\varepsilon_0^{1-\beta}\lim_{s\to0}|s| & \alpha^*\geq 2\alpha\\ \varepsilon_0^{1-\beta+\kappa(\alpha^*-2\alpha)}\lim_{s\to0}|s|^{1+\iota(\alpha-2\alpha^*)} & \alpha^*<2\alpha\end{cases}\\ &=0\end{aligned}$$

provided that, in the case of $\alpha^*<2\alpha$,

$$\begin{aligned}1-\beta+\kappa\,(\alpha^*-2\alpha)>0 &\iff \kappa<(1-\beta)/(2\alpha-\alpha^*)\,,\\ 1+\iota(\alpha^*-2\alpha)>0 &\iff \iota<1/(2\alpha-\alpha^*)\,.\end{aligned} \tag{4.44}$$

Due to [38, Prop. 3.2(ii)] and our assumption that $\alpha^*>\alpha$ we have for $x\in(\delta,\delta+2\varepsilon]$,

$$\begin{aligned}&\operatorname{osc}\left(R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[0,\delta]},B_\varepsilon(x)\right)\\ &\quad\leq\operatorname{osc}\left(R_{\alpha^*}(1-\cos(s\chi)),B_\varepsilon(x)\cap[0,\delta]\right)\mathbf{1}_{[0,\delta]}(x)+2\operatorname*{ess\,sup}_{B_\varepsilon(x)\cap[0,\delta]}R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[\delta-\varepsilon\vee0,\delta+\varepsilon]}(x)\\ &\quad\leq 0+2|s|\operatorname*{ess\,sup}_{[\delta-\varepsilon\vee0,\delta]}R_{\alpha^*}\chi\leq 2|s|\delta^{\alpha^*-\alpha}.\end{aligned}$$

So,

$$\begin{aligned}&\lim_{s\to 0}\sup_{\varepsilon\leq\varepsilon_0}\frac{1}{s\varepsilon^\beta}\int_{(\delta,\delta+2\varepsilon]}\operatorname{osc}\left(R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[0,\delta]},B_\varepsilon(x)\right)\,\mathrm{d}\lambda_I(x)\\ &\qquad\lesssim\lim_{s\to 0}\sup_{\varepsilon\leq\varepsilon_0}\frac{1}{\varepsilon^\beta}\int_{(\delta,\delta+2\varepsilon]}\delta^{\alpha^*-\alpha}\,\mathrm{d}\lambda_I(x)\lesssim\lim_{s\to0}\varepsilon_0^{1-\beta+\kappa\alpha^*}s^{\iota(\alpha^*-\alpha)}=0\end{aligned}$$

under (4.42). So we have

$$\lim_{s\to 0}\sup_{s\varepsilon\leq\varepsilon_0}\frac{1}{\varepsilon^\beta}\int_{[\delta,\delta+2\varepsilon]}\operatorname{osc}\left(R_{\alpha^*}(1-\cos(s\chi))\mathbf{1}_{[0,\delta+2\varepsilon]},B_\varepsilon(x)\right)\,\mathrm{d}\lambda_I(x)=0.$$

Finally, we discuss here values of α^* and implicit restrictions on b that ensure the existence of $\iota>0$ and $\kappa\in(0,1)$ used in the proof. There are four key cases to consider.

(1) $\alpha < \alpha^* < 2\alpha$ and $\alpha + b > 1 + \alpha^*$: Under (4.38), (4.39), (4.40), (4.42), (4.43) and (4.44), we have

$$\frac{\beta}{\alpha^* - \alpha + 1} < \kappa < \min\left\{\frac{1-\beta}{2\alpha - \alpha^*}, \frac{1-\beta}{\alpha + b - 1 - \alpha^*}\right\}$$
$$1 < \iota < \min\left\{\frac{1}{2\alpha - \alpha^*}, \frac{1}{\alpha + b - 1 - \alpha^*}\right\}.$$

Considering the conditions for ι, we have $\alpha^* > 2\alpha - 1$ and $\alpha^* > \alpha + b - 2$. Since $\alpha > 2\alpha - 1$, the former is automatic. Next, considering each of the two inequalities that guarantee the existence of κ, we obtain that

$$\alpha^* > \max\{\beta - 1 + \beta\alpha + \alpha,\ \beta b + \alpha - 1\} = \alpha + \max\{\beta - 1 + \beta\alpha,\ \beta b - 1\}$$

is necessary. Note that $\beta - 1 + \beta\alpha < 0$ and $\beta b - 1 < 0$ because $\beta < \min\{1/b, 1/(1+\alpha)\}$. So $\alpha^* > \alpha$ is a sufficient choice. Combining everything, we have that

$$\max\{\alpha + b - 2, \alpha\} < \alpha^* < \min\{\alpha + b - 1, 2\alpha\}$$

is sufficient.

(2) $\alpha < \alpha^* < 2\alpha$ and $\alpha + b < 1 + \alpha^*$: (4.39) poses no extra restriction. So, under (4.38), (4.40), (4.42), (4.43) and (4.44), we have $\alpha^* > \alpha$ as before. Hence,

$$\max\{\alpha + b - 1, \alpha\} < \alpha^* < 2\alpha$$

is sufficient.

(3) $\alpha^* > 2\alpha$ and $\alpha + b > 1 + \alpha^*$: (4.38), (4.40) and (4.44) pose no extra restrictions. Under (4.39), (4.42) and (4.43) we have $\alpha^* < \alpha + b - 1$ and $\alpha^* > \beta b + \alpha - 1$. Since $\beta b + \alpha - 1 < \alpha < 2\alpha$, the latter is true. So,

$$2\alpha < \alpha^* < \alpha + b - 1$$

is sufficient.

(4) $\alpha^* > 2\alpha$ and $\alpha + b < 1 + \alpha^*$: (4.39) is not relevant, and both (4.42) and (4.43) pose no extra restrictions. Hence,

$$\max\{2\alpha, \alpha + b - 1\} < \alpha^*$$

is sufficient.

We obtain from (1) and (2) that $\max\{\alpha + b - 2, \alpha\} < \alpha^*$ is sufficient if $\alpha^* < 2\alpha$. From (3) and (4) we obtain that $\alpha^* > 2\alpha$ is sufficient if $\alpha^* > 2\alpha$. So,

$$\alpha^* > \min\{2\alpha, \max\{\alpha, \alpha + b - 2\}\}$$

is sufficient. □

Lemma 4.8. *Assume χ is continuous, the right and left derivatives of χ exist on $\mathring{I}$, and there exist $a \geq 0, b > 0$ such that*

$$|\chi(x)| \lesssim x^{-a}(1-x)^{-a} \quad \text{and} \quad \max\{|\chi'(x+)|, |\chi'(x-)|\} \lesssim x^{-b}(1-x)^{-b}, \tag{4.45}$$

then $\|\chi\|_{\alpha,\beta,\gamma} < \infty$ if

$$\begin{gathered} \alpha > a, \\ \beta < (1+\alpha-a)/(b-a) \quad \text{or} \quad b < a+1 \quad \text{and} \\ 1 \leq \gamma < 1/a. \end{gathered} \tag{4.46}$$

Proof. The first inequality of (4.45) implies $\chi \in L^\gamma$ with $1 \leq \gamma < 1/a$.

For simplicity we assume χ is differentiable. Otherwise, at a point where χ is not differentiable, both one-sided derivatives will exist and the following estimates do hold for them.

Now, we proceed as in the proof of Lemma 4.6, however, with $\delta = \varepsilon^\kappa$ to find the minimal α and maximal β such that $|R_\alpha \chi|_{0,\beta} < \infty$. Set $g := R_\alpha \chi$, then

$$g'(x) = \alpha(1-2x)R_{\alpha-1}\chi(x) + R_\alpha \chi'(x).$$

Choose ε sufficiently small and split the domain into three parts, $[0, \varepsilon^\kappa + \varepsilon), (\varepsilon^\kappa + \varepsilon, 1-\varepsilon-\varepsilon^\kappa)$ and $(1-\varepsilon-\varepsilon^\kappa, 1]$. Due to the symmetry of the bounds, we only focus on $[0, 1/2]$.

On $(\varepsilon^\kappa + \varepsilon, 1-\varepsilon-\varepsilon^\kappa)$, we use [38, Prop. 3.2(ii)] implying

$$\begin{aligned} &\operatorname{osc}\left(g\,\mathbf{1}_{(\varepsilon^\kappa+\varepsilon, 1-\varepsilon^\kappa-\varepsilon)}, B_\varepsilon(x)\right) \\ &\le \operatorname{osc}(g, D(\varepsilon^\kappa, \varepsilon, x))\mathbf{1}_{(\varepsilon^\kappa+\varepsilon, 1-\varepsilon^\kappa-\varepsilon)}(x) \;+\; 2\Big(\sup_{D(\varepsilon^\kappa,\varepsilon,x)} g\Big)\left(\mathbf{1}_{B_\varepsilon(\varepsilon^\kappa+\varepsilon)\cup B_\varepsilon(1-\varepsilon^\kappa-\varepsilon)}(x)\right) \end{aligned} \tag{4.47}$$

with D as in (4.12).

For the following we set $\tilde{\alpha} = \min\{\alpha - b + 1, \alpha - a\}$. Then the contribution from the first term to $|R_\alpha\chi|_{0,\beta}$ is (up to a constant) bounded by

$$\begin{aligned} \sup_{0<\varepsilon\le\varepsilon_0} \varepsilon^{1-\beta} \int_{\varepsilon^\kappa+\varepsilon}^{1/2} \sup_{D(\varepsilon^\kappa,\varepsilon,x)} g' \,\mathrm{d}\lambda_I(x) &\lesssim \sup_{0<\varepsilon\le\varepsilon_0} \varepsilon^{1-\beta} \int_{\varepsilon^\kappa+\varepsilon}^{1/2} (x-\varepsilon)^{-a+\alpha-1} + (x-\varepsilon)^{-b+\alpha} \,\mathrm{d}\lambda_I(x) \\ &\lesssim \sup_{0<\varepsilon\le\varepsilon_0} \varepsilon^{1-\beta} \int_{\varepsilon^\kappa}^{1/2} x^{\bar{\alpha}-1} \,\mathrm{d}\lambda_I(x) \\ &\lesssim \begin{cases} \sup_{0<\varepsilon\le\varepsilon_0} \varepsilon^{1-\beta+\kappa\bar{\alpha}} & \tilde{\alpha} < 1 \\ \sup_{0<\varepsilon\le\varepsilon_0} \varepsilon^{1-\beta}(\log(1/2) - \kappa\log(\varepsilon)) & \bar{\alpha} = 1 \\ \varepsilon_0^{1-\beta} & \tilde{\alpha} > 1\,. \end{cases} \end{aligned}$$

In the $\tilde{\alpha} \le 1$ case, we require that

$$1 - \beta + \kappa\tilde{\alpha} > 0 \iff \big(\kappa < (1-\beta)/(b-\alpha-1) \text{ or } b < \alpha+1\big)\,, \tag{4.48}$$

where we have made use of the fact $\alpha > a$. On the other hand, (4.48) is automatically fulfilled if $\tilde{\alpha} > 1$, so we don't have to distinguish the cases anymore.

Since $\alpha > a$ the contribution from the second term in (4.47) is bounded by

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \frac{1}{\varepsilon^\beta} \int_{\varepsilon^\kappa}^{\varepsilon^\kappa+2\varepsilon} \sup_{D(\varepsilon^\kappa,\varepsilon,x)} g \,\mathrm{d}\lambda_I(x) \lesssim \sup_{\varepsilon\in(0,\varepsilon_0]} \varepsilon^{-\beta} \int_{\varepsilon^\kappa}^{\varepsilon^\kappa+2\varepsilon} 1 \;\mathrm{d}\lambda_I(x) \lesssim \varepsilon_0^{1-\beta}\,.$$

Now, for $x \in [0, \varepsilon^\kappa)$ we use the following estimate

$$\sup_{\bar{D}(\varepsilon^\kappa,\varepsilon,x)} |g| \lesssim \sup_{\bar{D}(\varepsilon^\kappa,\varepsilon,x)} |R_{\alpha-a}\mathbf{1}_{[0,\varepsilon^\kappa+\varepsilon]}| \lesssim (x+\varepsilon)^{\alpha-a}$$

with $\bar{D}$ as in (4.25). Following the argument in Appendix C with $\alpha - a$ replacing α^* and *without* the $s \to 0$ limit but fixing $s = 1$, we have, since $\alpha - a + 1 > \beta$ automatically holds, that

$$\begin{aligned} \sup_{\varepsilon\le\varepsilon_0} \frac{1}{\varepsilon^\beta} \int_{[0,\delta)} \operatorname{osc}(g, B_\varepsilon(x)) \,\mathrm{d}\lambda_I(x) &\lesssim \sup_{\varepsilon\le\varepsilon_0} \frac{2\left((\varepsilon^\kappa+\varepsilon)^{\alpha-a+1} - \varepsilon^{\alpha-a+1}\right)}{(\alpha-a+1)\varepsilon^\beta} \\ &= \frac{2\left((\varepsilon_0^\kappa+\varepsilon_0)^{\alpha-a+1} - \varepsilon_0^{\alpha-a+1}\right)}{(\alpha-a+1)\varepsilon_0^\beta} \end{aligned}$$

provided that

$$\kappa(1+\alpha-a) - \beta > 0 \iff \kappa > \frac{\beta}{1+\alpha-a}\,.$$

So, together with (4.48) we require that there exists κ such that

$$\frac{\beta}{1+\alpha-a} < \kappa < \frac{1-\beta}{b-\alpha-1} \quad \text{or} \quad b < \alpha+1\,.$$

This is true if and only if

$$(b-a)\beta < 1+\alpha-a \quad \text{or} \quad b < \alpha+1\,.$$

□

5. Twisted Transfer Operators $\widehat{\psi}_{is}$

5.1. Properties of twisted transfer operators.

We first prove L^γ norm estimates for $\widehat{\psi}_{is}$.

Lemma 5.1. *For all $\gamma > 1$, $s \in \mathbb{R}$ and $\varphi \in L^\gamma$, there exists a constant $C_\gamma > 1$ that depends only on ψ and γ such that*

$$\|\widehat{\psi}_{is}(\varphi)\|_1 \le \|\widehat{\psi}_{is}(\varphi)\|_\gamma \le C_\gamma \|\varphi\|_\gamma\,.$$

Proof. The first inequality follows from a direct application of Hölder's inequality. The second one is a straightforward application of Minkowski's inequality.

$$\begin{aligned}
\left(\int |\widehat{\psi}_{is}(\varphi)|^\gamma \,\mathrm{d}\lambda_I\right)^{1/\gamma} &\le \left(\int \widehat{\psi}(|\varphi|)^\gamma \,\mathrm{d}\lambda_I\right)^{1/\gamma} \\
&= \left(\int \left(\sum_{j=0}^{k-1} \frac{|\varphi|\circ\psi_{j+1}^{-1}}{|\psi'\circ\psi_{j+1}^{-1}|}\right)^\gamma \mathrm{d}\lambda_I\right)^{1/\gamma} \\
&\le \sum_{j=0}^{k-1} \left(\int \left(\frac{|\varphi|\circ\psi_{j+1}^{-1}}{|\psi'\circ\psi_{j+1}^{-1}|}\right)^\gamma \mathrm{d}\lambda_I\right)^{1/\gamma} \\
&= \sum_{j=0}^{k-1} \left(\int \left(\frac{|\varphi|}{|\psi'|}\right)^\gamma \mathbf{1}_{[c_j,c_{j+1}]}|\psi'| \,\mathrm{d}\lambda_I\right)^{1/\gamma} \\
&\le \frac{k}{\eta_-^{1-\gamma}} \left(\int |\varphi|^\gamma \,\mathrm{d}\lambda_I\right)^{1/\gamma}.
\end{aligned}$$

Put $C_\gamma = k\cdot\eta_-^{\gamma-1}$. Then

$$\|\widehat{\psi}_{is}(\varphi)\|_\gamma \le C_\gamma\|\varphi\|_\gamma\,.$$

□

Next, we have the following result on the required regularity of the transfer operators.

Corollary 5.2. *Let $0 \le \alpha_0, \alpha^*, \alpha^{**}, \beta \le 1$ and $\gamma_0, \gamma \ge 1$. Put*

$$\alpha_1 = \alpha_0 + \alpha^* \qquad \alpha_2 = \alpha_1 + \max\{\alpha^{**}, \alpha^*\}$$
$$1 \le \gamma_1 \le \gamma_0 \qquad 1 \le \gamma_2 \le (\gamma_1^{-1} + \gamma^{-1})^{-1}$$

and consider the chain of Banach spaces

$$\mathsf{V}_{\alpha_0,\beta,\gamma_0} \hookrightarrow \mathsf{V}_{\alpha_1,\beta,\gamma_1} \hookrightarrow \mathsf{V}_{\alpha_2,\beta,\gamma_2}\,. \tag{5.1}$$

Suppose that for all $s \in \mathbb{R}$, $|e^{is\chi}|_{0,\beta} < \infty$. Then

(1) *for $s \in \mathbb{R}$, $\widehat{\psi}_{is}$ is a bounded linear operator on each of the Banach spaces in (5.1).*

Suppose, in addition, that $\lim_{s\to 0} |1 - e^{is\chi}|_{\alpha^*,\beta} = 0$. Then

(2) *$s \mapsto \widehat{\psi}_{is}$ is continuous as a function from $\mathbb{R}$ to $\mathcal{L}(\mathsf{V}_{\alpha_0,\beta,\gamma_0}, \mathsf{V}_{\alpha_1,\beta,\gamma_1})$.*

Finally, suppose that

$$\lim_{s\to 0}\left|\frac{e^{is\chi}-1-is\chi}{s}\right|_{\alpha^{**},\beta}=0 \quad \textit{and} \quad \|\chi\|_\gamma<\infty.$$

Then,

(3) *$s\mapsto \widehat{\psi}_{is}$ is continuously differentiable as a function from $\mathbb{R}$ to $\mathcal{L}(\mathsf{V}_{\alpha_1,\beta,\gamma_1},\mathsf{V}_{\alpha_2,\beta,\gamma_2})$.*

Proof. Since $\widehat{\psi}$ is a bounded linear operator on each of the Banach spaces in (5.1) (in particular, due to the DFLY inequality below), the theorem follows from Lemma 4.2 and Lemma 5.1. □

5.2. **DFLY Inequalities.** In this section, we prove DFLY inequalities for the family $\widehat{\psi}_{is}$. First, we state and prove two preparatory lemmas. Throughout this section, we assume that χ is continuous and the right and left derivatives of χ exist on $\mathring{I}$ and that there exists a constant $b>0$ such that

$$\max\{|\chi'(x+)|,|\chi'(x-)|\}\lesssim x^{-b}(1-x)^{-b}\,. \tag{5.2}$$

Lemma 5.3. *Let $\alpha,\beta\in(0,1)$ and let $\bar\gamma\in[1,1/\alpha)$. Suppose the constant $b>0$ in (5.2) is such that*

$$\min\left\{\bar\gamma^{-1}+(\alpha-\beta)b\,,\,\bar\gamma^{-1}+\alpha-\beta b\right\}>0\,. \tag{5.3}$$

Then, there exists $C_{\varepsilon_0}>0$ independent of γ such that

$$\sup_{\varepsilon\in(0,\varepsilon_0]}\varepsilon^{-\beta}\left\|R_\alpha\operatorname{osc}\left(e^{is\chi},B_\varepsilon(\cdot)\right)\right\|_{\bar\gamma}\le C_{\varepsilon_0} \tag{5.4}$$

for all $s\in\mathbb{R}$.

Remark 5.4. We note that, if $b>1$, then $\bar\gamma^{-1}+\alpha-\beta b>0 \implies \bar\gamma^{-1}+(\alpha-\beta)b>0$, and if $b<1$ then $\bar\gamma^{-1}+(\alpha-\beta)b>0\implies\bar\gamma^{-1}+\alpha-\beta b>0$.

Proof of Lemma 5.3. Since $e^{is\chi}$ is 2π periodic in s, we will estimate

$$\sup_{s\in[0,2\pi]}\sup_{\varepsilon\in(0,\varepsilon_0]}\varepsilon^{-\beta}\|R_\alpha\operatorname{osc}(e^{is\chi},B_\varepsilon(\cdot))\|_{\bar\gamma}\,.$$

Note that

$$\begin{aligned}\sup_{\varepsilon\in(0,\varepsilon_0]}\|R_\alpha\operatorname{osc}(e^{is\chi},B_\varepsilon(\cdot))\|_{\bar\gamma}\cdot\varepsilon^{-\beta}\le{}&\sup_{\varepsilon\in(0,\varepsilon_0]}\left(\int_0^{1/2}\left(R_\alpha\operatorname{osc}(e^{is\chi},B_\varepsilon(x))\right)^{\bar\gamma}\,\mathrm{d}\lambda_I(x)\right)^{1/\bar\gamma}\cdot\varepsilon^{-\beta}\\&+\sup_{\varepsilon\in(0,\varepsilon_0]}\left(\int_{1/2}^1\left(R_\alpha\operatorname{osc}(e^{is\chi},B_\varepsilon(x))\right)^{\bar\gamma}\,\mathrm{d}\lambda_I(x)\right)^{1/\bar\gamma}\cdot\varepsilon^{-\beta}.\end{aligned}$$

We will only estimate the first summand as the estimation of the second follows analogously. Using the definition $\operatorname{osc}(h,A)=\operatorname{osc}(\Re h,A)+\operatorname{osc}(\Im h,A)$ and $|e^{it_1}-e^{it_2}|\le\min\{2,|t_1-t_2|\}$, we note that for any measurable set A we have $\operatorname{osc}\left(e^{is\chi},A\right)\le\min\{4,4s/\pi\operatorname{osc}(\chi,A)\}$. Due to (5.2) there exists $C>0$ such that for all $s>0$, for all $\varepsilon>0$ and all $x\in[\varepsilon,1/2]$ we have

$$\operatorname{osc}(e^{is\chi},B_\varepsilon(x))\le\frac{8|s|\varepsilon}{\pi}\sup_{y\in B_\varepsilon(x)}\max\{|\chi'(y+)|,|\chi'(y-)|\}\le\frac{8C|s|\varepsilon}{\pi}(x-\varepsilon)^{-b}.$$

We have that $8C|s|\varepsilon(x-\varepsilon)^{-b}/\pi\le 4$ if and only if

$$x\ge\left(\frac{2C|s|\varepsilon}{\pi}\right)^{1/b}+\varepsilon=:\gamma_\varepsilon>\varepsilon.$$

Since $\gamma_\varepsilon > \varepsilon$, on $[\gamma_\varepsilon, 1/2]$, we use $\frac{8C|s|\varepsilon}{\pi}(x-\varepsilon)^{-b}$, and on $[0,\gamma_\varepsilon)$, we use 4 as upper bounds for $\operatorname{osc}(e^{is\chi}, B_\varepsilon(x))$, to obtain

$$\begin{aligned}
&\sup_{\varepsilon\in(0,\varepsilon_0]}\left(\int_0^{1/2}\left(R_\alpha \operatorname{osc}(e^{is\chi}, B_\varepsilon(x))\right)^{\bar\gamma}\mathrm{d}\lambda_I(x)\right)^{1/\bar\gamma}\cdot\varepsilon^{-\beta}\\
&\quad\le \sup_{\varepsilon\in(0,\varepsilon_0]}\left(4\gamma_\varepsilon\sup_{[0,\gamma_\varepsilon]}R_\alpha\mathbf{1}\cdot\varepsilon^{-\beta}+\left(\int_{\gamma_\varepsilon}^{1/2}\left(\frac{8C|s|\varepsilon^{1-\beta}}{\pi}R_\alpha\mathbf{1}\cdot(x-\varepsilon)^{-b}\right)^{\bar\gamma}\mathrm{d}\lambda_I(x)\right)^{1/\bar\gamma}\right)\\
&\quad\le \sup_{\varepsilon\in(0,\varepsilon_0]}4\gamma_\varepsilon^{1+\alpha}\varepsilon^{-\beta}+\sup_{\varepsilon'\in(0,\varepsilon_0]}\frac{8C|s|\varepsilon^{1-\beta}}{\pi}\left(\int_{\gamma_\varepsilon}^{1/2}\left(x^\alpha(x-\varepsilon)^{-b}\right)^{\bar\gamma}\mathrm{d}\lambda_I(x)\right)^{1/\bar\gamma}. \qquad (5.5)
\end{aligned}$$

For the first summand of (5.5), we have that there exists $\tilde C_{\varepsilon_0}>0$ such that

$$\begin{aligned}
\sup_{\varepsilon\in(0,\varepsilon_0]}4\gamma_\varepsilon^{1+\alpha}\varepsilon^{-\beta}&\le 8\sup_{\varepsilon\in(0,\varepsilon_0]}\max\left\{\left(\frac{2C|s|}{\pi}\right)^{(1+\alpha)/b}\varepsilon^{(1+\alpha)/b-\beta},\varepsilon^{1+\alpha-\beta}\right\}\\
&\le \tilde C_{\varepsilon_0}(1+|s|^{(1+\alpha)/b})<\infty
\end{aligned}$$

which follows from the fact that $\beta<(1/\bar\gamma+\alpha)/b<(1+\alpha)/b$ and $\beta\le 1$.

For the second summand of (5.5), we use $\bar\gamma<1/\alpha$ and $(x+\varepsilon)^{\alpha\bar\gamma}\le x^{\alpha\bar\gamma}+\varepsilon^{\alpha\bar\gamma}$ to compute

$$\begin{aligned}
&\sup_{\varepsilon\in(0,\varepsilon_0]}\frac{8C|s|\varepsilon^{1-\beta}}{\pi}\left(\int_{\gamma_\varepsilon}^{1/2}\left(x^\alpha(x-\varepsilon)^{-b}\right)^{\bar\gamma}\mathrm{d}\lambda_I(x)\right)^{1/\bar\gamma}\\
&\le\frac{8C|s|}{\pi}\sup_{\varepsilon\in(0,\varepsilon_0]}\varepsilon^{1-\beta}\left(\int_{\left(\frac{2Cs\varepsilon}{\pi}\right)^{1/b}}^{1/2}(x+\varepsilon)^{\alpha\bar\gamma}x^{-b\bar\gamma}\mathrm{d}\lambda_I(x)\right)^{1/\bar\gamma}\\
&\le\frac{8C|s|}{\pi}\sup_{\varepsilon\in(0,\varepsilon_0]}\left(\varepsilon^{1-\beta}\left(\int_{\left(\frac{2Cs\varepsilon}{\pi}\right)^{1/b}}^{1/2}x^{\bar\gamma(\alpha-b)}\mathrm{d}\lambda_I(x)\right)^{1/\bar\gamma}+\varepsilon^{1+\alpha-\beta}\left(\int_{\left(\frac{2Cs\varepsilon}{\pi}\right)^{1/b}}^{1/2}x^{-b\bar\gamma}\mathrm{d}\lambda_I(x)\right)^{1/\bar\gamma}\right)\\
&\lesssim|s|\sup_{\varepsilon\in(0,\varepsilon_0]}\left(\varepsilon^{1-\beta}\max\left\{\frac12,\left(\frac{2Cs\varepsilon}{\pi}\right)^{1/b}\right\}^{1/\bar\gamma+\alpha-b}+\varepsilon^{1+\alpha-\beta}\max\left\{\frac12,\left(\frac{2Cs\varepsilon}{\pi}\right)^{1/b}\right\}^{1/\bar\gamma-b}\right)\\
&\lesssim|s|\sup_{\varepsilon\in(0,\varepsilon_0]}\left(\max\left\{\varepsilon^{1-\beta},|s|^{1/(\bar\gamma b)+\alpha/b-1}\varepsilon^{1/(\bar\gamma b)+\alpha/b-\beta}\right\}+\max\left\{\varepsilon^{1+\alpha-\beta},|s|^{1/(\bar\gamma b)-1}\varepsilon^{1/(\bar\gamma b)+\alpha-\beta}\right\}\right)\\
&\le\tilde C_{\varepsilon_0}|s|(1+|s|^{1/(\bar\gamma b)+\alpha/b-1}+|s|^{1/(\bar\gamma b)-1})
\end{aligned}$$

for some constant $\tilde C_{\varepsilon_0}>0$. This follows from the assumption that $1/(\bar\gamma b)+\alpha/b-\beta>0$ and $1/(\bar\gamma b)+\alpha-\beta>0$.

Finally, combining this with the first step and using symmetry, we have that

$$\begin{aligned}
\sup_{s\in[0,2\pi]}\sup_{\varepsilon\in(0,\varepsilon_0]}\|R_\alpha\operatorname{osc}(e^{is\chi},B_\varepsilon(\cdot))\|_{\bar\gamma}\cdot\varepsilon^{-\beta}&\le\tilde C_{\varepsilon_0}\sup_{s\in[0,2\pi]}(1+|s|+|s|^{1/(\bar\gamma b)+\alpha/b}+|s|^{1/(\bar\gamma b)}+|s|^{(1+\alpha)/b})\\
&\le C_{\varepsilon_0}
\end{aligned}$$

for some $C_{\varepsilon_0}>0$ which is independent of $\bar\gamma\ge 1$. □

For the following for all $j=0,\dots,k-1$, let $\bar R_{j+1}:[c_j,c_{j+1}]\to\mathbb{R}$ be given by

$$\bar R_{j+1}=\frac{(R_\alpha\mathbf{1})\circ\psi_{j+1}}{R_\alpha\mathbf{1}}$$

and the following lemma is independent of the choice of χ.

Lemma 5.5. *$\bar{R}_{j+1}$ is bounded*[1] *for all j. Further, let $0<\varepsilon<\delta$ and $\alpha\in(0,1)$. Then, for all j, there is a constant C which is independent of ε and δ such that*

$$\sup_{x\in[c_j+\delta+\varepsilon,c_{j+1}-\delta-\varepsilon]}\Big((R_\alpha\mathbf{1})(x)\sup_{B_\varepsilon(x)}|\bar{R}'_{j+1}|\Big)\le C\cdot\delta^{\alpha-1}. \tag{5.6}$$

Proof. First, we notice that for all j

$$\begin{aligned}\bar{R}_{j+1}(x)&=\frac{\psi_{j+1}(x)^\alpha(1-\psi_{j+1}(x))^\alpha}{x^\alpha(1-x)^\alpha}\le\max\left\{\frac{(\psi_{j+1}(x)-0)^\alpha}{x^\alpha},\frac{(1-\psi_{j+1}(x))^\alpha}{(1-x)^\alpha}\right\}\\&\le\max\left\{\frac{((x-0)\eta_+)^\alpha}{x^\alpha},\frac{((1-x)\eta_+)^\alpha}{(1-x)^\alpha}\right\}\le\eta_+^\alpha,\end{aligned}\tag{5.7}$$

where the first inequality holds true, because at most one of the arguments in the maximum can be larger than 1. Hence, for all j, $\bar{R}_{j+1}$ is bounded.

We know from (1) in the proof of Lemma B.1 that $\bar{R}'_1$ is bounded at 0 and $\bar{R}'_{k-1}$ is bounded at 1. We can infer from the representation in (B.2) that there exist $K'_3,K_3>0$ such that

$$|\bar{R}'_{j+1}(x)|\le\frac{K'_3}{(\psi_{j+1}(x)(1-\psi_{j+1}(x)))^{1-\alpha}}\le\frac{K_3}{((x-c_j)(c_{j+1}-x))^{1-\alpha}}, \tag{5.8}$$

for all $j\in\{1,\dots,k-2\}$. This can be deduced as follows: We assume we are in the interval $[\delta_0,1-\delta_0]$ with δ_0 as in (1) of the proof of Lemma B.1. Then the subtrahend of (B.2) has to be bounded as it only has a pole at 0 and 1. Furthermore, considering the minuend it is easy to notice that the factor $\alpha\psi'_{j+1}(x)(1-2\psi_{j+1}(x))/(x(1-x))^\alpha$ has to be bounded on $[\delta_0,1-\delta_0]$ as well. This leaves the remaining factor as in the middle term of (5.8).

In order to verify the second inequality we notice that $\psi_{j+1}(x)\in[\eta_-(x-c_j),\eta_+(x-c_j)]$ which follows from the fact that $\lim_{\varepsilon\to0}\psi_{j+1}(c_j+\varepsilon)=0$ and from the bound on the derivative. With a similar argumentation, using that $\lim_{\varepsilon\to0}\psi_{j+1}(c_{j+1}-\varepsilon)=1$ we obtain $1-\psi_{j+1}(x)\in[\eta_-(c_{j+1}-x),\eta_+(c_{j+1}-x)]$.

In addition, from the proof of Lemma B.1

$$|\bar{R}'_1(x)|\le\frac{K_3}{(c_1-x)^{1-\alpha}}\qquad\text{and}\qquad\bar{R}'_k(x)\le\frac{K_3}{(x-c_{k-1})^{1-\alpha}}.$$

Hence,

$$\sup_{x\in[c_j+\delta+\varepsilon,c_{j+1}-\delta-\varepsilon]}\Big((R_\alpha\mathbf{1})(x)\sup_{B_\varepsilon(x)}|\bar{R}'_{j+1}|\Big)\lesssim\begin{cases}\sup\frac{1}{[(x\pm\delta-c_j)(c_{j+1}-x\pm\delta)]^{1-\alpha}} & j\notin\{0,k-1\}\\ \sup\frac{1}{(c_1-x\pm\delta)^{1-\alpha}} & j=0\\ \sup\frac{1}{(x\pm\delta-c_{k-1})^{1-\alpha}} & j=k-1\end{cases}$$
$$\lesssim\delta^{\alpha-1}. \tag{5.9}$$

□

Now, we are ready to prove the main lemma.

Lemma 5.6. *Let $0\le\alpha<\beta<\min\{1/2,\vartheta,1/b\}$ be such that*

$$\kappa:=\frac{\eta_+^\alpha}{\eta_-^\beta}<1\,,\qquad\textit{and}\qquad\max\{|\chi'(x+)|,|\chi'(x-)|\}\lesssim x^{-b}(1-x)^{-b}\,.$$

Then, for all $1\le\gamma<1/\alpha$ there exist $C,\widetilde{C}>0$ and $\bar{\gamma}$ with $\gamma<\bar{\gamma}<1/\alpha$ such that for all $s\in\mathbb{R}$ we have that for all $h\in\mathsf{V}_{\alpha,\beta,\gamma}$ and for all $n\in\mathbb{N}$,

$$\|\widehat{\psi}^n_{is}h\|_{\alpha,\beta,\gamma}\le\widetilde{C}\left(\kappa^n\|h\|_{\alpha,\beta,\gamma}+C^n\|h\|_{\bar{\gamma}}\right). \tag{5.10}$$

[1] In fact, they are α-Hölder continuous. See Appendix B.

Remark 5.7. In the linear expanding case, i.e., $\eta_+ = \eta_- > 1$, the condition $\kappa < 1$ reduces to $\beta > \alpha$. Also, the constant C is independent of $\bar\gamma$.

Remark 5.8. Restricting $\bar\gamma$ to $(\gamma, 1/\alpha)$ ensures that $h \in \mathsf{V}_{\alpha,\beta,\gamma}$ implies $h \in L^{\bar\gamma}$. To see this, observe that $|R_\alpha h| \lesssim \mathbf{1}$ which yields that $|h|^{\bar\gamma} \lesssim R_{-\alpha\bar\gamma}\mathbf{1}$, and since $\bar\gamma\alpha < 1$, $R_{-\alpha\bar\gamma}\mathbf{1}$ is integrable.

Proof of Lemma 5.6. Let $s \in \mathbb{R}$ and $h \in \mathsf{V}_{\alpha,\beta,\gamma}$ be $\mathbb{R}-$valued. We estimate $|\widehat{\psi}_{is}h|_{\alpha,\beta}$:

$$\begin{aligned}
\operatorname{osc}\left(R_\alpha(\widehat{\psi}_{is}h), B_\varepsilon(x)\right) &= \operatorname{osc}\left(R_\alpha \sum_{j=0}^{k-1}\left(\frac{e^{is\chi}\cdot h}{|\psi'|}\right)\circ\psi_{j+1}^{-1}\mathbf{1}_{\psi[c_j,c_{j+1}]}\,, B_\varepsilon(x)\right)\\
&\le \sum_{j=0}^{k-1}\operatorname{osc}\left(R_\alpha\left(\frac{e^{is\chi}\cdot h}{|\psi'|}\right)\circ\psi_{j+1}^{-1}\,, B_\varepsilon(x)\right)\\
&\le \sum_{j=0}^{k-1}\operatorname{osc}\left(\frac{R_\alpha\mathbf{1}\circ\psi_{j+1}}{R_\alpha\mathbf{1}}\cdot R_\alpha\frac{e^{is\chi}\cdot h}{|\psi'|}, \psi_{j+1}^{-1}B_\varepsilon(x)\cap[c_j,c_{j+1}]\right)\\
&\le \sum_{j=0}^{k-1}\operatorname{osc}\left(\frac{R_\alpha\mathbf{1}\circ\psi_{j+1}}{R_\alpha\mathbf{1}}\cdot R_\alpha\frac{e^{is\chi}\cdot h}{|\psi'|}, B_{\varepsilon/\eta_-}(\psi_{j+1}^{-1}x)\cap[c_j,c_{j+1}]\right)\\
&= \sum_{j=0}^{k-1}\operatorname{osc}\left(\bar R_{j+1}\cdot\frac{e^{is\chi}}{|\psi'|}\cdot R_\alpha h\,, D_{j+1}(x,\varepsilon/\eta_-)\right),
\end{aligned}$$

where $D_{j+1}(x,\varepsilon) := B_\varepsilon(\psi_{j+1}^{-1}x)\cap[c_j,c_{j+1}]$. So, by [38, Prop. 3.2 (iii)] there exists $c > 0$ such that

$$\begin{aligned}
\operatorname{osc}\left(R_\alpha(\widehat{\psi}_{is}h), B_\varepsilon(x)\right) &\le \sum_{j=0}^{k-1}\operatorname{osc}\left(R_\alpha h\,, D_{j+1}(x,\varepsilon/\eta_-)\right)\sup_{D_{j+1}(x,\varepsilon/\eta_-)}\left|\bar R_{j+1}\cdot\frac{e^{is\chi}}{|\psi'|}\right|\\
&\quad+\sum_{j=0}^{k-1}\operatorname{osc}\left(\left|\bar R_{j+1}\cdot\frac{e^{is\chi}}{|\psi'|}\right|, D_{j+1}(x,\varepsilon/\eta_-)\right)\inf_{D_{j+1}(x,\varepsilon/\eta_-)}|R_\alpha h|\,.\\
&\le \left(1+c(\varepsilon\eta_-^{-1})^\vartheta\right)\sum_{j=0}^{k-1}\frac{\operatorname{osc}\left(R_\alpha h\,, B_{\varepsilon/\eta_-}(\psi_{j+1}^{-1}x)\right)}{|\psi'|(\psi_{j+1}^{-1}x)}\sup_{D_{j+1}(x,\varepsilon/\eta_-)}\left|\bar R_{j+1}\right|\\
&\quad+\sum_{j=0}^{k-1}\operatorname{osc}\left(\left|\bar R_{j+1}\cdot\frac{e^{is\chi}}{|\psi'|}\right|, D_{j+1}(x,\varepsilon/\eta_-)\right)|R_\alpha h(\psi_{j+1}^{-1}x)|\,.
\end{aligned}$$

The last inequality follows from the fact that ψ^{-1} is C^1 and its derivative is uniformly $\vartheta-$Hölder.

Hence, using the upper bound (5.7), and then using the definition of the transfer operator $\widehat{\psi}$, we have

$$\begin{aligned}
\operatorname{osc}\left(R_\alpha(\widehat{\psi}_{is}h), B_\varepsilon(x)\right) &\le \left(1+c(\varepsilon\eta_-^{-1})^\vartheta\right)\eta_+^\alpha\widehat{\psi}\big(\operatorname{osc}(R_\alpha h, B_{\varepsilon/\eta_-}(\,\cdot\,))\big)(x)\\
&\quad+\sum_{j=0}^{k-1}|R_\alpha h(\psi_{j+1}^{-1}x)|\operatorname{osc}\left(\left|\bar R_{j+1}\cdot\frac{e^{is\chi}}{|\psi'|}\right|, D_{j+1}(x,\varepsilon/\eta_-)\right)\,.
\end{aligned}\tag{5.11}$$

Taking the integral over the first term in (5.11) and multiplying by $\varepsilon^{-\beta}$ we obtain

$$\varepsilon^{-\beta}\int\left(1+c(\varepsilon\eta_-^{-1})^\vartheta\right)\eta_+^\alpha\widehat{\psi}\big(\operatorname{osc}(R_\alpha h, B_{\varepsilon/\eta_-}(\,\cdot\,))\big)(x)\,\mathrm{d}\lambda_I(x)$$

$$\le \varepsilon^{-\beta}\left(1+c(\varepsilon\eta_-^{-1})^{\vartheta}\right)\eta_+^{\alpha}\int \widehat{\psi}\big(\operatorname{osc}(R_\alpha h, B_{\varepsilon/\eta_-}(\,\cdot\,))\big)(x)\,\mathrm{d}\lambda_I(x)$$
$$= \varepsilon^{-\beta}\left(1+c(\varepsilon\eta_-^{-1})^{\vartheta}\right)\eta_+^{\alpha}\int \operatorname{osc}(R_\alpha h, B_{\varepsilon/\eta_-}(\,\cdot\,))(x)\,\mathrm{d}\lambda_I(x)$$
$$\le \left(1+c(\varepsilon\eta_-^{-1})^{\vartheta}\right)\eta_+^{\alpha}\eta_-^{-\beta}|h|_{\alpha,\beta}$$
$$\le \left(1+c(\varepsilon_0\eta_-^{-1})^{\vartheta}\right)\kappa\|h\|_{\alpha,\beta,\gamma}, \tag{5.12}$$

for all $\gamma\ge 1$. Next, we analyze the second term in (5.11). Again, by [38, Prop. 3.2 (iii)] we have

$$\operatorname{osc}\left(\left|\bar{R}_{j+1}\cdot\frac{e^{is\chi}}{|\psi'|}\right|, D_{j+1}(x,\varepsilon/\eta_-)\right)$$
$$\le \operatorname{osc}\left(\frac{1}{|\psi'|}, B_{\varepsilon/\eta_-}(\psi_{j+1}^{-1}x)\right)\left(\operatorname*{ess\,sup}_{B_{\varepsilon/\eta_-}(\psi_{j+1}^{-1}x)}|\Re\bar{R}_{j+1}e^{is\chi}| + \operatorname*{ess\,sup}_{B_{\varepsilon/\eta_-}(\psi_{j+1}^{-1}x)}|\Im\bar{R}_{j+1}e^{is\chi}|\right)$$
$$+\operatorname{osc}\left(\bar{R}_{j+1}e^{is\chi}, D_{j+1}(x,\varepsilon/\eta_-)\right)\inf_{D_{j+1}(x,\varepsilon/\eta_-)}\frac{1}{|\psi'|}$$
$$\le c(\varepsilon\eta_-^{-1})^{\vartheta}\eta_+^{\alpha}\frac{1}{|\psi'|(\psi_{j+1}^{-1}x)} + (1+c(\varepsilon\eta_-^{-1})^{\vartheta})\frac{\operatorname{osc}\left(\bar{R}_{j+1}e^{is\chi}, D_{j+1}(x,\varepsilon/\eta_-)\right)}{|\psi'|(\psi_{j+1}^{-1}x)}. \tag{5.13}$$

Note that

$$\varepsilon^{-\beta}c(\varepsilon\eta_-^{-1})^{\eta}\eta_+^{\alpha}\int\sum_{j=0}^{k-1}\frac{|R_\alpha h(\psi_{j+1}^{-1}x)|}{|\psi'|(\psi_{j+1}^{-1}x)}\,\mathrm{d}\lambda_I(x) = \varepsilon^{-\beta}c(\varepsilon\eta_-^{-1})^{\vartheta}\eta_+^{\alpha}\int\widehat{\psi}(|R_\alpha h|)\,\mathrm{d}\lambda_I(x)$$
$$= \varepsilon^{-\beta}c(\varepsilon\eta_-^{-1})^{\vartheta}\eta_+^{\alpha}\int|R_\alpha h|\,\mathrm{d}\lambda_I(x)$$
$$\le K_1\varepsilon^{\vartheta-\beta}\|R_\alpha\mathbf{1}\|_{\gamma_1}\|h\|_{\bar{\gamma}} \tag{5.14}$$

where $\gamma_1^{-1}+\bar{\gamma}^{-1}=1$, $K_1 := c\eta_-^{-\vartheta}\eta_+^{\alpha}\|R_\alpha\mathbf{1}\|_{\bar{\gamma}}$ and $\beta<\vartheta$. So, the contribution from the first summand of (5.13) to (5.11) is under control.

To estimate the contribution from second summand of (5.13) to (5.11) we note that for all j and for all $A\subset[c_j,c_{j+1}]$, we have

$$\operatorname{osc}\left(\bar{R}_{j+1}e^{is\chi}, A\right) = \operatorname{osc}\left(\sum_{j=0}^{k-1}\bar{R}_{j+1}e^{is\chi}\mathbf{1}_{[c_j,c_{j+1})}, A\right),$$

and therefore, we can bound this contribution by

$$(1+c(\varepsilon\eta_-^{-1})^{\vartheta})\sum_{j=0}^{k-1}\frac{|R_\alpha h(\psi_{j+1}^{-1}x)|}{|\psi'|(\psi_{j+1}^{-1}x)}\operatorname{osc}\left(F\,,\,B_{\varepsilon/\eta_-}(\psi_{j+1}^{-1}x)\right)$$
$$= (1+c(\varepsilon\eta_-^{-1})^{\vartheta})\widehat{\psi}\left(|R_\alpha h|\operatorname{osc}\left(F\,,\,B_{\varepsilon/\eta_-}(\,\cdot\,)\right)\right), \tag{5.15}$$

where

$$F(x) = e^{is\chi(x)}\sum_{j=0}^{k-1}\bar{R}_{j+1}(x)\mathbf{1}_{[c_j,c_{j+1})}(x) = e^{is\chi(x)}\sum_{j=0}^{k-1}\frac{R_\alpha\mathbf{1}\circ\psi_{j+1}(x)}{R_\alpha\mathbf{1}(x)}\mathbf{1}_{[c_j,c_{j+1})}(x).$$

This is bounded by

$$(1+c(\varepsilon\eta_-^{-1})^{\vartheta})\int\widehat{\psi}\left(|R_\alpha h|\operatorname{osc}\left(F\,,\,B_{\varepsilon/\eta_-}(\,\cdot\,)\right)\right)(x)\,\mathrm{d}\lambda_I(x)$$

$$= (1 + c(\varepsilon\eta_-^{-1})^{\vartheta}) \int |R_\alpha h|(x) \operatorname{osc}\left(F\,,\, B_{\varepsilon/\eta_-}(x)\right) \,\mathrm{d}\lambda_I(x)$$

$$= (1 + c(\varepsilon\eta_-^{-1})^{\vartheta}) \int |h(x)| \cdot \left(R_\alpha \operatorname{osc}\left(F\,,\, B_{\varepsilon/\eta_-}(x)\right)\right) \mathrm{d}\lambda_I(x)\,. \tag{5.16}$$

To estimate the integral we split it as follows.

$$\int |h(x)| \cdot \left(R_\alpha \operatorname{osc}\left(F\,,\, B_{\varepsilon/\eta_-}(x)\right)\right) \mathrm{d}\lambda_I(x)$$
$$= \left(\sum_{j=1}^{k-1} \int_{c_j-\varepsilon^\iota-\varepsilon}^{c_j+\varepsilon^\iota+\varepsilon} + \sum_{j=1}^{k} \int_{c_{j-1}+\varepsilon^\iota+\varepsilon}^{c_j-\varepsilon-\varepsilon^\iota} + \int_0^{\varepsilon+\varepsilon^\iota} + \int_{1-\varepsilon-\varepsilon^\iota}^{1}\right) |h(x)| \cdot \left(R_\alpha \operatorname{osc}\left(F\,,\, B_{\varepsilon/\eta_-}(x)\right)\right) \mathrm{d}\lambda_I(x)$$

where we choose for ι any number fulfilling

$$\frac{\beta}{1-\alpha} < \iota < \frac{1-\beta}{1-\alpha}. \tag{5.17}$$

Because $\beta < 1/2$ such a choice is possible. Note that for $j = 1, \dots k-1$, $x \in [c_j - \varepsilon^\iota - \varepsilon, c_j + \varepsilon^\iota + \varepsilon]$,

$$\operatorname{osc}\left(F\,,\, B_{\varepsilon/\eta_-}(x)\right) \le 2(\sup \bar{R}_j + \sup \bar{R}_{j+1}) \le 4K$$

and for $x \in [0, \varepsilon + \varepsilon^\iota) \cup (1 - \varepsilon - \varepsilon^\iota, 1]$,

$$\operatorname{osc}\left(F\,,\, B_{\varepsilon/\eta_-}(x)\right) \le 2(\sup \bar{R}_0 + \sup \bar{R}_k) \le 4K$$

where $K := \sup_j \sup R_{j+1} < \infty$. So,

$$\sum_{j=0}^{k} \int_{(c_j-\varepsilon^\iota-\varepsilon)\vee 0}^{(c_j+\varepsilon^\iota+\varepsilon)\wedge 1} |h(x)| \cdot \left(R_\alpha \operatorname{osc}\left(F\,,\, B_{\varepsilon/\eta_-}(x)\right)\right) \mathrm{d}\lambda_I(x)$$
$$\le \|h\|_{\bar\gamma} \sum_{j=0}^{k} \left(\int_{(c_j-\varepsilon^\iota-\varepsilon)\vee 0}^{(c_j+\varepsilon^\iota+\varepsilon)\wedge 1} \left(R_\alpha \operatorname{osc}\left(F\,,\, B_{\varepsilon/\eta_-}(x)\right)\right)^{\gamma_1} \mathrm{d}\lambda_I(x)\right)^{1/\gamma_1}$$
$$\le K_\alpha \varepsilon^{\iota/\gamma_1} \|h\|_{\bar\gamma} \tag{5.18}$$

where $\gamma_1^{-1} + \bar\gamma^{-1} = 1$ and $K_\alpha = 4^{\iota/\gamma_1} 2^{-2\alpha} K$. Here, we choose $\bar\gamma$ such that

$$\max\left\{\gamma,\, \frac{\iota}{\iota-\beta},\, \frac{1}{1-b\beta+\alpha}\,,\, \frac{1}{1-b(\beta-\alpha)}\right\} < \bar\gamma < \frac{1}{\alpha}. \tag{5.19}$$

We will see later in the proof why this restrictions on $\bar\gamma$ are needed.

Now, we show that such a choice is possible. Since we were assuming that $\iota > \beta/(1-\alpha)$, we have $\iota/(\iota-\beta) < 1/\alpha$. We note that when $b \le 1$, $1 - b\beta + \alpha \ge 1 - b(\beta - \alpha)$, and it is enough to see whether $\alpha < 1 - b(\beta - \alpha)$. In fact, this is true because $b(\beta - \alpha) < \beta - \alpha < 1 - \alpha$. On the contrary, when $b > 1$, we have $1 - b\beta + \alpha < 1 - b(\beta - \alpha)$, and it is enough to see whether $\alpha < 1 - b\beta + \alpha$. This is true because $\beta < 1/b$.

To estimate the remaining terms we note, using (5.7) and [38, Prop. 3.2(iii)], that for all $j = 0, \dots, k-1$, for all $x \in [c_j + \varepsilon^\iota + \varepsilon, c_{j+1} - \varepsilon^\iota - \varepsilon]$,

$$\operatorname{osc}\left(F\,,\, B_{\varepsilon/\eta_-}(x)\right)$$
$$= \operatorname{osc}\left(e^{is\chi}\bar{R}_{j+1}\,,\, B_{\varepsilon/\eta_-}(x)\right)$$
$$\le \sup_{B_\varepsilon(x)} (\Re|e^{is\chi}| + \Im|e^{is\chi}|) \operatorname{osc}\left(\bar{R}_{j+1}\,,\, B_{\varepsilon/\eta_-}(x)\right) + \operatorname{osc}\left(e^{is\chi}\,,\, B_{\varepsilon/\eta_-}(x)\right) \inf_{B_\varepsilon(x)} \bar{R}_{j+1}$$
$$\le 2 \sup_{B_\varepsilon(x)} |\bar{R}'_{j+1}| \,\frac{\varepsilon}{\eta_-} + \operatorname{osc}\left(e^{is\chi}\,,\, B_{\varepsilon/\eta_-}(x)\right) \eta_+^\alpha,$$

and thus,

$$\begin{aligned}
&\sum_{j=0}^{k-1}\int_{c_j+\varepsilon^\iota+\varepsilon}^{c_{j+1}-\varepsilon^\iota-\varepsilon} |h(x)|\cdot\left(R_\alpha \operatorname{osc}\left(F\,,\,B_{\varepsilon/\eta_-}(x)\right)\right)\,\mathrm{d}\lambda_I(x)\\
&\quad\le \frac{2\varepsilon}{\eta_-}\left\||h|\sum_{j=0}^{k-1}\mathbf{1}_{[c_j+\varepsilon^\iota+\varepsilon,c_{j+1}-\varepsilon^\iota-\varepsilon]}\right\|_1 \sup_{x\in[c_j+\varepsilon^\iota+\varepsilon,c_{j+1}-\varepsilon^\iota-\varepsilon]} R_\alpha \sup_{B_\varepsilon(x)}|\bar{R}'_{j+1}|\\
&\qquad + \eta_+^\alpha\left\|\sum_{j=0}^{k-1}\mathbf{1}_{[c_j+\varepsilon^\iota+\varepsilon,c_{j+1}-\varepsilon-\varepsilon^\iota]}\cdot|h|\left(R_\alpha \operatorname{osc}\left(e^{is\chi}\,,\,B_{\varepsilon/\eta_-}(\cdot)\right)\right)\right\|_1\\
&\quad\le \frac{2\varepsilon}{\eta_-}\|h\|_1 \sup_{x\in[c_j+\varepsilon^\iota+\varepsilon,c_{j+1}-\varepsilon^\iota-\varepsilon]} R_\alpha \sup_{B_\varepsilon(x)}|\bar{R}'_{j+1}| + \eta_+^\alpha\|h\|_{\bar\gamma}\left\|R_\alpha \operatorname{osc}\left(e^{is\chi}\,,\,B_{\varepsilon/\eta_-}(\cdot)\right)\right\|_{\bar\gamma}.
\end{aligned}$$

Now, in order to estimate the first summand taking the maximum over j of the supremum in (5.6) above with $\delta=\varepsilon^\iota$ yields that the outer supremum above is bounded by $C\varepsilon^{\iota(\alpha-1)}$ for some constant $C>0$. For the second summand, from (5.19), we have that $\bar\gamma^{-1}<1-b\beta+\alpha$ which implies that $b\beta<1-\bar\gamma^{-1}+\alpha=\bar\gamma^{-1}+\alpha$, and hence, when $b>1$, we have the condition (5.3). Also from (5.19), $\bar\gamma^{-1}<1-b(\beta-\alpha)$ which implies that $b\beta<\bar\gamma^{-1}+b\alpha$, and hence, when $b\le 1$, we have (5.3). Therefore, we can apply Lemma 5.3 with α, β, b, $\bar\gamma$, ε/η_- to conclude

$$\left\|R_\alpha \operatorname{osc}\left(e^{is\chi}\,,\,B_{\varepsilon/\eta_-}(\cdot)\right)\right\|_{\bar\gamma}\le C_{\varepsilon_0}\varepsilon^\beta\eta_-^{-\beta}$$

where C_{ε_0} is independent of $\bar\gamma$. Therefore, for all $s\neq 0$,

$$\begin{aligned}
&\sum_{j=0}^{k-1}\int_{c_j+\varepsilon^\iota+\varepsilon}^{c_{j+1}-\varepsilon^\iota-\varepsilon} |h(x)|\cdot\left(R_\alpha \operatorname{osc}\left(F\,,\,B_{\varepsilon/\eta_-}(x)\right)\right)\,\mathrm{d}\lambda_I(x)\\
&\quad\le \bar{C}_{\varepsilon_0}\varepsilon^{\min\{1-\iota(1-\alpha),\beta\}}\|h\|_{\bar\gamma}\,. \qquad (5.20)
\end{aligned}$$

Finally, combining (5.18) and (5.20), we estimate (5.16) multiplied by $\varepsilon^{-\beta}$ by

$$\begin{aligned}
&\varepsilon^{-\beta}(1+c(\varepsilon\eta_-^{-1})^\vartheta)\int\widehat{\psi}\left(|R_\alpha h|\operatorname{osc}\left(F\,,\,B_{\varepsilon/\eta_-}(\,\cdot\,)\right)\right)(x)\,\mathrm{d}\lambda_I(x)\\
&\quad\le \varepsilon^{(\iota/\bar\gamma\wedge(1-\iota(1-\alpha))\wedge\beta)-\beta}C_{\varepsilon_0}\|h\|_{\bar\gamma}\le C_{\varepsilon_0}\|h\|_{\bar\gamma}. \qquad (5.21)
\end{aligned}$$

To justify the last inequality, we analyse the exponent of ε. By (5.19) and the relation $\gamma_1^{-1}+\bar\gamma^{-1}=1$ we have $\iota/\gamma_1>\iota(1-\bar\gamma^{-1})>\iota(1-(\iota-\beta)/\iota))=\beta$. Furthermore, the second inequality of (5.17) implies that $1-\iota(1-\alpha)>\beta$.

Combining (5.11), (5.12), (5.14) and (5.21), we have

$$\begin{aligned}
|\widehat{\psi}_{is}h|_{\alpha,\beta}&=\sup_{\varepsilon\in(0,\varepsilon_0)}\int\frac{\operatorname{osc}\left(R_\alpha(\widehat{\psi}_{is}h),B_\varepsilon(x)\right)}{\varepsilon^\beta}\,\mathrm{d}\lambda_I(x)\\
&\le\left(1+c(\varepsilon_0\eta_-^{-1})^\vartheta\right)\kappa\|h\|_{\alpha,\beta,\gamma}+C_{\varepsilon_0}\|h\|_{\bar\gamma},
\end{aligned}$$

for all $\gamma\ge 1$. Therefore, for all $\bar\gamma$ chosen appropriately

$$\begin{aligned}
\|\widehat{\psi}_{is}h\|_{\alpha,\beta,\gamma}&=|\widehat{\psi}_{is}h|_{\alpha,\beta}+\|\widehat{\psi}_{is}h\|_\gamma\\
&\le\left(1+c(\varepsilon_0\eta_-^{-1})^\vartheta\right)\kappa\|h\|_{\alpha,\beta,\gamma}+C_{\varepsilon_0}\|h\|_{\bar\gamma}+C_\gamma\|h\|_\gamma\\
&\le\bar\kappa\|h\|_{\alpha,\beta,\gamma}\bar{C}\|h\|_{\bar\gamma}
\end{aligned}$$

where $\bar{\kappa} = \left(1 + c(\varepsilon_0 \eta_-^{-1})^\eta\right) \kappa < 1$ for sufficiently small ε_0, and $\bar{C} = C_{\varepsilon_0} + C_\gamma$, where C_γ is given in Lemma 5.1.

Iterating, we obtain the following DFLY inequality: for all $h \in \mathsf{V}_{\alpha,\beta,\gamma}$

$$\begin{aligned}\sup_s \|\widehat{\psi}_{is}^n h\|_{\alpha,\beta,\gamma} &\le \kappa \|\widehat{\psi}_{is}^{n-1} h\|_{\alpha,\beta,\gamma} + \bar{C}\|\widehat{\psi}_{is}^{n-1} h\|_{\bar{\gamma}} \\ &\le \kappa^2 \|\widehat{\psi}_{is}^{n-2} h\|_{\alpha,\beta,\gamma} + \kappa\bar{C}\|\widehat{\psi}_{is}^{n-2} h\|_{\bar{\gamma}} + \bar{C}\bar{C}^{n-1}\|h\|_{\bar{\gamma}} \\ &\le \kappa^n \|h\|_{\alpha,\beta,\gamma} + \bar{C}\|h\|_{\bar{\gamma}} \sum_{j=0}^{n-1} \kappa^j \bar{C}^{n-1-j} \\ &\le \kappa^n \|h\|_{\alpha,\beta,\gamma} + C\bar{C}^{n+1}\|h\|_{\bar{\gamma}}\end{aligned}$$

for some $C > 0$.

In the proof above, we assumed that h is $\mathbb{R}-$valued. When $h = h_1 + ih_2$ where h_j, $j = 1, 2$ are $\mathbb{R}-$valued, using linearity of the operator

$$\|\widehat{\psi}_{is}^n h\|_{\alpha,\beta,\gamma} \le \|\widehat{\psi}_{is}^n h_1\|_{\alpha,\beta,\gamma} + \|\widehat{\psi}_{is}^n h_2\|_{\alpha,\beta,\gamma}\,,$$

and also, $\|h_j\|_{\alpha,\beta,\gamma} \le \|h\|_{\alpha,\beta,\gamma}$ and $\|h_j\|_{\bar{\gamma}} \le \|h\|_{\bar{\gamma}}$ for all $j = 1, 2$. So, applying DLFY inequality proven above in the $\mathbb{R}-$valued case to h_1 and h_2, we conclude that DFLY in the general case of h holds up to a constant multiple. □

6. Proofs of the Main Theorems

Finally, we give the proofs of our main theorems. We start with the theorems from Section 2.4.

6.1. Proofs of limit theorems for expanding interval maps.

Proof of Theorem 2.3. From (2.7) we obtain that there exist α, β fulfilling

$$a < \alpha < \beta \cdot \min\left\{1, \frac{\log \eta_-}{\log \eta_+}\right\} < \min\left\{\vartheta, \frac{1}{b}, \frac{1}{2}\right\} \cdot \min\left\{1, \frac{\log \eta_-}{\log \eta_+}\right\}. \tag{6.1}$$

Furthermore, since $\alpha > a$, the inequality $\beta < 1/b$ which we can deduce immediately from (6.1) that $\frac{1}{b} < \frac{1}{b-a}$. So, by Lemma 4.8, we obtain $|\chi|_{\alpha,\beta} < \infty$ and also $\chi \in \mathsf{V}_{\alpha,\beta,2} \hookrightarrow L^2$. Furthermore, from the second inequality of (6.1) we obtain $\eta_+^\alpha / \eta_-^\beta < 1$.

Since ψ is a piecewise C^2 uniformly expanding and a covering map of the interval, it has a unique absolutely continuous invariant mixing probability (acip) with a bounded invariant density; see [31]. Let's call this acip π. Then $L^2 \hookrightarrow L^2(\pi)$ because

$$\int |h|^2 \,\mathrm{d}\pi = \int |h|^2 \frac{\mathrm{d}\pi}{\mathrm{d}\lambda_I}\,\mathrm{d}\lambda_I \le \left\|\frac{\mathrm{d}\pi}{\mathrm{d}\lambda_I}\right\|_\infty \int |h|^2 \,\mathrm{d}\lambda_I\,.$$

We claim that $\widehat{\psi}$ has a spectral gap in $\mathsf{V}_{\alpha,\beta,\gamma}$ with $\gamma = 2$. In Appendix A.2, we show that $\mathsf{V}_{\alpha,\beta,2}$ is continuously embedded in $L^{\bar{\gamma}}$ where $\bar{\gamma} \in (2, 1/\alpha)$ and that the unit ball of $\mathsf{V}_{\alpha,\beta,2}$ is relatively compact in $L^{\bar{\gamma}}$. A suitable $\bar{\gamma}$ exists by the condition $\alpha < 1/2$ from (6.1). So, the claim follows from [5, Lemma B.15] due to the DFLY inequality (5.10) with $s = 0$ and Remark A.9.

Now, the CLT (in the stationary case) follows directly from Proposition 3.1 applied to $\chi - \pi(\chi)$. That is, from (3.1) we have

$$\mathbb{P}_\pi\left(\frac{S_n(\chi) - n\,\pi(\chi)}{\sigma\sqrt{n}} \le x\right) - \mathfrak{N}(x) = o(1), \quad \text{as} \quad n \to \infty$$

with $\sigma^2 > 0$ because χ is not a coboundary. □

Next, we will continue with the proof of Theorem 2.6 as the proof of Theorem 2.4 will need similar methods to those of Theorem 2.6.

Proof of Theorem 2.6. (2.10) implies that there exist α, β such that $\alpha > a$ and

$$3\bar{\alpha} := 3\min\{2\alpha, \max\{\alpha, \alpha + b - 2\}\} < \beta \cdot \min\left\{1, \frac{\log \eta_+}{\log \eta_-}\right\} < \min\left\{\vartheta, \frac{1}{b}, \frac{1}{2}\right\} \cdot \min\left\{1, \frac{\log \eta_+}{\log \eta_-}\right\}.$$

Since either $b < a + 1$ or $1/b < (1 + \alpha - a)/(b - a)$, we obtain by Lemma 4.8 that $|\chi|_{\alpha,\beta} < \infty$ and additionally we obtain by the last inequality that

(*a*) $0 < 3\bar{\alpha} < \beta < \min\{\vartheta, 1/b, 1/2\}$,
(*b*) $\eta_+^{3\bar{\alpha}} < \eta_-^{\beta}$.

Hence, under our assumptions, we have the following:

(1) The second inequality in (2.6) and $|\chi|_{\alpha,\beta} < \infty$ imply that $|e^{is\chi}|_{0,\beta} < \infty$ for all $s > 0$ (see Lemma 4.4). So, due to Corollary 5.2 (1), we have $\widehat{\psi}_{is} \in \mathcal{L}(\mathsf{V}_{\tilde{\alpha},\beta,\tilde{\gamma}})$ for all $0 < \tilde{\alpha} < \beta$ and $\tilde{\gamma} \geq 1$.
(2) Since $|\chi|_{\alpha,\beta} < \infty$, from Lemma 4.6, for all $\alpha^* > 0$ close to 0,
$$\lim_{s\to 0} |1 - e^{is\chi}|_{\alpha^*,\beta} = 0\,.$$
Along with Corollary 5.2 (2), this yields that for all $0 \leq \alpha_0 < \beta$, $\gamma_0 \geq 1$,
$$s \mapsto \widehat{\psi}_{is} \in \mathcal{L}(\mathsf{V}_{\alpha_0,\beta,\gamma_0}, \mathsf{V}_{\alpha_1,\beta,\gamma_1})$$
is continuous for $\alpha_1 = \alpha^* + \alpha_0$ and $1 \leq \gamma_1 \leq \gamma_0$.
(3) From the second inequality in (2.6) and $|\chi|_{\alpha,\beta} < \infty$, for all $\alpha^{**} > \min\{2\alpha, \max\{\alpha + b - 2, \alpha\}\}$,
$$\lim_{s\to 0} \left| \frac{e^{is\chi} - 1 - is\chi}{s} \right|_{\alpha^{**},\beta} = 0$$
due to Lemma 4.7. Then, we have that for all $0 \leq \alpha_1 < \beta$, and $\gamma_1 \geq 1$,
$$s \mapsto \widehat{\psi}_{is} \in \mathcal{L}(\mathsf{V}_{\alpha_1,\beta,\gamma_1}, \mathsf{V}_{\alpha_2,\beta,\gamma_2})$$
is continuously differentiable, for all $\alpha_2 = \alpha^* + \max\{\alpha^*, \alpha^{**}\} + \alpha_1$ and $1 \leq \gamma_2 \leq (\gamma_1^{-1} + \gamma^{-1})^{-1}$ due to Corollary 5.2 (2) and (3).

Next, we define the following chain of spaces in order to invoke Proposition 3.3 with $r = 1$:

$$\mathsf{V}_{\alpha_0,\beta,\gamma_0} \hookrightarrow \mathsf{V}_{\alpha_1,\beta,\gamma_1} \hookrightarrow \mathsf{V}_{\alpha_2,\beta,\gamma_2} \hookrightarrow \mathsf{V}_{\alpha_3,\beta,\gamma_3} \hookrightarrow \mathsf{V}_{\alpha_4,\beta,\gamma_4} \hookrightarrow \mathsf{V}_{\alpha_5,\beta,\gamma_5} \hookrightarrow \mathsf{V}_{\alpha_6,\beta,\gamma_6} \hookrightarrow \mathsf{V}_{\alpha_7,\beta,\gamma_7},$$

where $\alpha_0 = 0$, $\alpha_{2j} - \alpha_{2j-1} \geq \min\{2\alpha, \max\{\alpha + b - 2, \alpha\}\}$, for $j = 1, 2, 3$, $\alpha_{2j+1} > \alpha_{2j}$ for $j = 0, 1, 2, 3$, and $\alpha_7 < \beta$. By (*a*) such a choice is possible. Furthermore, we assume that the γ_js are chosen such that $\gamma_0 = M \gg 1$ sufficiently large, $\gamma_{2j+1} = \gamma_{2j}$ and $\gamma_{2j} < (\alpha^{-1} + \gamma_{2j-1}^{-1})^{-1}$.

Now, to prove the theorem, we verify the conditions in Proposition 3.3 for the above sequence of Banach spaces. We notice that if for some function φ it holds that $|\varphi|_{\alpha,\beta} < \infty$, then $\|\varphi\|_{\alpha,\beta,\gamma} < \infty$ as long as $\gamma < 1/\alpha$. We next verify that it is possible to construct valid spaces with the above choice of parameters. First, we notice that by (*a*) it is possible to construct $\alpha_0 \leq \ldots \leq \alpha_7$ with the above properties that $\alpha_7 < \beta$ and thus $\alpha_j < \beta$ for all j. Furthermore, by (*a*) we have $\alpha < 1/3$. Thus, it is possible that $1 \leq \gamma_{2j} \leq (\gamma^{-1} + \gamma_{2j-1}^{-1})$ holds together with $1/\gamma_j > \alpha_j$. Moreover, under (*b*) we have that $\eta_+^{\alpha_j}/\eta_-^{\beta} < 1$ holds for all j.

With that it becomes immediate from applying the conditions of this theorem on the parameters in the Banach spaces and from the calculations in (1)–(3) applied to all indices j that conditions (I)-(III) of Proposition 3.3 are satisfied.

For each j, we apply Lemma 5.6 with $\gamma = \gamma_j$ and we choose $\bar{\gamma} = \bar{\gamma}_j$ as in the proof of the lemma. In Appendix A.2, we show that $\mathsf{V}_{\alpha_j,\beta,\gamma_j}$ is continuously embedded in $L^{\bar{\gamma}_j}$ and that the unit ball of $\mathsf{V}_{\alpha_j,\beta,\gamma_j}$ is relatively compact in $L^{\bar\gamma_j}$. Also, we recall from Lemma 5.1 that for all $h \in L^{\bar\gamma_j}$, $\|\widehat{\psi}_{is}(h)\|_{\bar\gamma_j} \le C_{\bar\gamma_j}\|h\|_{\bar\gamma_j}$ where $C_{\bar\gamma_j} > 1$. Therefore, $\|\widehat{\psi}^n_{is}(h)\|_{\bar\gamma_j} \le C_{\bar\gamma_j}\|\widehat{\psi}^{n-1}_{is}(h)\| \le C^n_{\bar\gamma_j}\|h\|_{\bar\gamma_j}$ which gives us $\|\widehat{\psi}^n_{is}\|_{L^{\bar\gamma_j}\to L^{\bar\gamma_j}} \le C^n_{\bar\gamma_j}$. Choose $\kappa = \max_{0\le j\le 7} \eta_+^{\alpha_j}\eta_-^{-\beta} < 1$. Also, by our previous constructions, we have that $\gamma_j < 1/\alpha_j$ for all j. So, due to Lemma 5.6, we have the DFLY inequality: for all $h \in \mathsf{V}_{\alpha_j,\beta,\gamma_j}$

$$\|\widehat{\psi}^n_{is}h\|_{\alpha_j,\beta,\gamma_j} \le \widetilde{C}\left(\kappa^n\|h\|_{\alpha_j,\beta,\gamma_j} + C^n\|h\|_{\bar\gamma_j}\right)$$

for some $\gamma_j < \bar\gamma_j < 1/\alpha_j$ and C uniform in j and s. Therefore, we have the first conclusion, equation (8), of [26, Theorem 1] uniformly over all spaces. That is, there exist v and w such that

$$\sup_{z\in D_\kappa} \|(z\mathsf{Id} - \widehat{\psi}_{is})^{-1}h\|_{\mathsf{V}_{\alpha_j,\beta,\gamma_j}\to\mathsf{V}_{\alpha_j,\beta,\gamma_j}} \le v\|h\|_{\alpha_j,\beta,\gamma_j} + w\|h\|_{\bar\gamma_j}$$

for all space pairs $\mathsf{V}_{\alpha_j,\beta,\gamma_j} \hookrightarrow L^{\bar\gamma_j}$ and $s \in \mathbb{R}$. This gives (IV) of Proposition 3.3.

The conditions (V)–(VII) of Proposition 3.3 are equivalent to Assumption (B) in [11, Section I.1.2] for a single dynamical system, i.e., when Assumptions (0) and (A)(1) in [11, Section I.1.2] are trivially true. Moreover, as discussed in [11], [11, Lemma 4.5] implies Assumption (B). Therefore, we verify the conditions (with a slight modification) in [11, Lemma 4.5] to establish (V)–(VII):

- We have assumed that χ is non-arithmetic.
- Due to Remark A.9 and the DFLY inequality (5.10), we can apply [5, Lemma B.15] to conclude that for all s the essential spectral radius of $\widehat{\psi}_{is}$ on $\mathsf{V}_{\alpha_j,\beta,\gamma_j}$ is at most κ. This is precisely the conclusion of [11, Proposition 4.3].
- We know that $\mathsf{V}_{\alpha_j,\beta,\gamma_j} \hookrightarrow L^1$ for all j, and that $\|\widehat{\psi}_{is}h\|_1 \le \|\widehat{\psi}h\|_1 \le \|h\|_1$ for all $h \in L^1$. So, the spectral radius of $\widehat{\psi}_{is}$ on L^1, and hence, on $\mathsf{V}_{\alpha_j,\beta,\gamma_j}$ for all j, is at most 1.
- Since ψ is a uniformly expanding, piecewise C^2 and a full branch map with finitely many branches, ψ is exact (cf. [17, Theorem 3]) and $\psi^{-1}x$ is finite for all x.
- The Assumption (A)(1) in [11] is trivially true because there is only a single dynamical system in Figure 2 of [11].

Hence, (V) and (VI) are true due to the first part of [11, Lemma 4.5]. To establish (VII), we need a slight modification of the second part of [11, Lemma 4.5]. First, we note that $\chi \in \mathsf{V}_{\alpha,\beta,\gamma} \hookrightarrow L^2$, for $\gamma \ge 3$, and $\widehat{\psi}$ has a spectral gap on $\mathsf{V}_{\alpha,\beta,\gamma}$. So, we can repeat the argument in the first part of the proof of [11, Lemma 4.5] to conclude that $\sum_{k=0}^{n-1} \bar\chi\circ\psi^k$ is L^2-bounded. So, it has an L^2-weakly convergent subsequence. This establishes (VII).

Finally, the non-arithmeticity of χ implies that χ is not cohomologous to a constant, and hence, we have (VIII) of Proposition 3.3. ☐

Proof of Theorem 2.4. To prove this theorem we use Proposition 3.2. By Theorem 2.3 we immediately obtain (V) of Proposition 3.2.

Next, we define the following chain of spaces.

$$\mathsf{V}_{\alpha_0,\beta,M} \hookrightarrow \mathsf{V}_{\alpha_1,\beta,M} \hookrightarrow L^p \hookrightarrow L^1(\pi)$$

with $p \le M$ where the choices correspond to $0 \le \alpha_0 < \alpha_1 < \beta$ and $\gamma_0 = \gamma_1 = M \ge 1$ in the proof of Theorem 2.6. Then, the conditions (I)–(IV) and (VI) of Proposition 3.2 follow as in the proof of Theorem 2.6 due to Corollary 5.2 (2) and [11, Lemma 4.5]. ☐

Proofs of the results in Example 2.8. We first note that

$$|\chi'(x)| \lesssim x^{-c}(1-x)^{-c}$$

and

$$|\chi'(x)| = \left| -cx^{-c-1}\sin\left(\frac{1}{x}\right) - x^{-c-2}\cos\left(\frac{1}{x}\right)\right| \lesssim x^{-c-2}(1-x)^{-c-2}.$$

So, we obtain $a = c$ and $b = c+2$ in the notation of Theorems 2.3, 2.4 and 2.6. In order to prove (1) we note that (2.7) then simplifies to

$$c < \min\left\{\vartheta, \frac{1}{2+c}\right\}\min\left\{1, \frac{\log\eta_-}{\log\eta_+}\right\}.$$

So, on the one hand, we have the requirement $c < \vartheta\tilde{\eta}$ and on the other hand, we have the condition $c < \tilde{\eta}/(c+2)$ which, given that we assume $c \geq 0$, is equivalent to $c < \sqrt{1+\tilde{\eta}} - 1$ giving (1). Furthermore, in the doubling map case we have $\vartheta = 2$ and $\tilde{\eta} = 1$ implying (1a).

Next, we notice that (2.10) in our case simplifies to

$$3c < \min\left\{\vartheta, \frac{1}{2+c}\right\}\min\left\{1, \frac{\log\eta_+}{\log\eta_-}\right\}.$$

With a similar calculation as above applying Theorem 2.6 gives (2) and as above we get (2a). ☐

6.2. **Proofs of limit theorems for the Boolean-type transformation.** Now we give the proofs from Section 2.5. We start with the following technical lemmas:

Lemma 6.1. *For all $r \in \mathbb{N}$, the rth asymptotic moments of both $S_n(\chi)$ and $\widetilde{S}_n(h)$ are equal.*

Proof. It is enough to show that $\mathbb{E}_\mu(\widetilde{S}_n^r(h)) = \mathbb{E}_{\lambda_I}(S_n^r(\chi))$ for all r. In fact, due to (2.13)

$$\begin{aligned}\mathbb{E}_\mu(h\circ\phi^{j_1}\, h\circ\phi^{j_2}\,\cdots\, h\circ\phi^{j_k}) &= \mathbb{E}_{\lambda_I}(h\circ\xi\circ\psi^{j_1}\, h\circ\xi\circ\psi^{j_2}\,\cdots\, h\circ\xi\circ\psi^{j_k})\\ &= \mathbb{E}_{\lambda_I}(\chi\circ\psi^{j_1}\,\chi\circ\psi^{j_2}\,\cdots\,\chi\circ\psi^{j_k})\end{aligned}$$

for all $j_1, \ldots, j_k \in \mathbb{N}_0$ such that $j_1 + \cdots + j_k = r$. ☐

Lemma 6.2. *Let $h : \mathbb{R} \to \mathbb{R}$ be such that the left and right derivatives exist and there exist $u, v \geq 0$ fulfilling*

$$h(x) \lesssim |x|^u \qquad \text{and} \qquad \max\left\{\left|h'(x-)\right|, \left|h'(x+)\right|\right\} \lesssim |x|^v,$$

and let $\chi : I \to \mathbb{R}$ be given by $\chi = h\circ\xi$ with $\xi(x) := \cot(\pi x)$, then we have

$$|\chi(x)| \lesssim x^{-u}(1-x)^{-u}$$

and

$$\max\{|\chi'(x+)|, |\chi'(x-)|\} \lesssim x^{-b}(1-x)^{-b}, \quad b = 2+v.$$

Further, if

$$\begin{gathered}\alpha > u\,,\\ \beta < (1+\alpha-u)/(2+v-u) \quad \text{or} \quad 1+v<u\,, \quad \text{and}\\ 1 \leq \gamma < 1/u\,,\end{gathered} \tag{6.2}$$

then $\|\chi\|_{\alpha,\beta,\gamma} < \infty$. In particular, if $u < 1/(2+v-u)$, then there exist $0 < \alpha < \beta < 1$ such that $|\chi|_{\alpha,\beta} < \infty$.

Proof. We will apply Lemma 4.8. First, we note that

$$\lim_{x\to 0}\xi(x)x = 1/\pi \quad \text{and} \quad \lim_{x\to 1}\xi(x)(1-x) = 1/\pi\,.$$

This and (2.14) imply

$$|\chi(x)| \lesssim x^{-u}(1-x)^{-u}, \tag{6.3}$$

and in particular, $\chi \in L^\gamma$ with $1 \leq \gamma < 1/u$.

For simplicity, we assume χ is differentiable. Otherwise, at a point where χ is not differentiable, both one-sided derivatives will exist and the following estimates do hold for them.

Note that we have $|h'(\xi(x))| \lesssim x^{-v}(1-x)^{-v}$. Using the chain rule $|\chi'(x)| = |h'(\xi(x))||\xi'(x)|$. Since $\xi'(x) = -\pi/\sin^2(\pi x)$, we have that

$$|\chi'(x)| \lesssim x^{-2-v}(1-x)^{-2-v}. \tag{6.4}$$

So, we have $|\chi'(x)| \lesssim x^{-b}(1-x)^{-b}$ with $b = 2+v > 2$. The lemma then follows immediately by applying Lemma 4.8. □

With this we are able to prove the results from Section 2.5

Proof of Proposition 2.9. To prove the statement it is enough to prove its counterpart for $S_n(\chi,\psi)$ where $\chi = h\circ\xi$ and ψ is the doubling map.

From Lemma 6.2, we have

$$|\chi(x)| \lesssim x^{-u}(1-x)^{-u} \quad \text{and} \quad \max\{|\chi'(x+)|, |\chi'(x-)|\} \lesssim x^{-b}(1-x)^{-b}, \quad b = 2+v\,.$$

Now, we invoke Theorem 2.3 with ψ, $\eta_+ = \eta_- = 2$ and $\log\eta_-/\log\eta_+ = 1$. Since ψ is linear, $\vartheta = 1$. Hence, (2.7) simplifies to $u < 1/(2+v)$. Also, the assumption that h is not an $L^2(\mu)$ coboundary implies that χ is not an $L^2(\lambda)$ coboundary.

Therefore, χ and ψ satisfy the conditions of Theorem 2.3, and hence satisfy the CLT given by (2.8) with

$$\sigma^2 = \mathbb{E}_\lambda(\chi^2) + 2\sum_{k=1}^{\infty}\mathbb{E}_\lambda(\chi\cdot\chi\circ\psi^k) \in (0,\infty)\,.$$

From Lemma 6.1, $\widetilde{\sigma}^2 = \sigma^2$ and $\mathbb{E}_\mu(h) = \mathbb{E}_{\lambda_I}(\chi)$. As a direct consequence of (2.13), we obtain the required CLT given by (2.15). □

We next prove the MLCLT for a class of observables in $\mathfrak{F}$.

Proof of Proposition 2.10. Our assumption allows us to apply Theorem 2.4 to the Birkhoff sum $S_n(\chi) = \sum_{k=0}^{n-1}\chi\circ\psi^k$ with $\chi = h\circ\xi$ and ψ the doubling map and conclude

$$\sup_{\ell\in\mathbb{R}}\left|\sigma\sqrt{2\pi n}\,\mathbb{E}_{\xi_*\mathfrak{m}}(U\circ\xi\circ\psi^n\,V(S_n(\overline{\chi})-\ell)\,W\circ\xi) - e^{-\frac{\ell^2}{2n\sigma^2}}\,\mathbb{E}_\pi(U\circ\xi)\,\mathbb{E}_{\xi_*\mathfrak{m}}(W\circ\xi)\int V(x)\,\mathrm{d}x\right| = o(1)\,.$$

From Lemma 6.1 and the fact that ξ is a conjugacy, we have

$$\sup_{\ell\in\mathbb{R}}\left|\widetilde{\sigma}\sqrt{2\pi n}\,\mathbb{E}_{\mathfrak{m}}(U\circ\phi^n\,V(\widetilde{S}_n(\overline{h})-\ell)\,W) - e^{-\frac{\ell^2}{2n\widetilde{\sigma}^2}}\,\mathbb{E}_\mu(U)\,\mathbb{E}_{\mathfrak{m}}(W)\int V(x)\,\mathrm{d}x\right| = o(1)\,.$$

This is because the two LHSs are exactly the same. □

Now, we prove that corollaries that show the validity of the CLT and MLCLT for the real part, imaginary part and the absolute value of the Riemann zeta function when sampled over the trajectories of ϕ.

Proof of Corollary 2.11. To apply Proposition 2.9, we have to show the existence of u, v as in (2.14). It is well-known that for any $s\in(0,1)$, for any $\delta > 0$,

$$\max\left\{|\zeta|(s+ix), |\zeta'|(s+ix)\right\} \lesssim |x|^{(1-s)/2+\delta}\,; \tag{6.5}$$

see, for example, [42].

So, we pick $u = v = (1-s)/2+\delta$ and this is possible when $((1-s)/2+\delta)\,((1-s)/2+\delta+2) < 1$ and such $\delta > 0$ exists iff $(1-s)(5-s) < 4$ iff $s\in(3-2\sqrt{2},1)$. So, for such choices of s we can apply Proposition 2.9 and obtain the CLT provided that h is not $\phi-$cohomologous to a constant. The MLCLT follows from Proposition 2.10 analogously, when ϕ is non-arithmetic. □

Proof of Corollary 2.13. To apply Proposition 2.9, we have to show the existence of u, v as in (2.14). We assume $a \geq 1$, set $\tilde{h}(x) = h(x)^{1/a}$. Note that $h'(x) = a\tilde{h}(x)^{a-1}\tilde{h}'(x)$. Since we restrict ourselves to the critical line, $s = 1/2$, $|\tilde{h}(x)| \lesssim |x|^{13/84+\delta}$ and $|\tilde{h}'(x)| \lesssim |x|^{13/84+\delta}$ for all $\delta > 0$, due to (6.5). So, we can take $u = 13a/84 + \delta$ and $v = 13(a-1)/84 + 13/84 + \delta = 13a/84 + \delta$, and the condition in Proposition 2.9 for u, v reduces to $(13a/84)(13a/84 + 2) < 1$. This is equivalent to $1 \leq a < 84/13(\sqrt{2}-1)$. So, for such choices of a, we can apply Proposition 2.9 and obtain the CLT provided that h is not $\phi-$cohomologous to a constant. The MLCLT follows from Proposition 2.10 analogously, when ϕ is non-arithmetic. □

Finally, we look at the proof for the First Order Edgeworth Expansion for observables over the Boolean-type transformation.

Proof of Proposition 2.15. We follow the proof of Proposition 2.9 and invoke Theorem 2.6.

Consider $S_n(\chi, \psi)$ where $\chi = \xi \circ h$ and ψ is the doubling map. Remember that from Lemma 6.2, we have

$$|\chi(x)| \lesssim x^{-u}(1-x)^{-u} \quad \text{and} \quad \max\{|\chi'(x+)|, |\chi'(x-)|\} \lesssim x^{-b}(1-x)^{-b}, \quad b = 2 + v\,.$$

Next, to apply Theorem 2.6 we observe that $\eta_+ = \eta_- = 2$ and $\log \eta_- / \log \eta^+ = 1$ and since ψ is linear $\vartheta = 1$. Hence, (2.10) simplifies to (2.17). Also, the assumption that h is not an $L^2(\mu)$ coboundary implies that χ is not an $L^2(\lambda)$ coboundary. □

Appendix A. The Banach Spaces $\mathsf{V}_{\alpha,\beta,\gamma}$

The spaces $\mathsf{V}_{\alpha,\beta}$ with their particular norm considered in [40] are not complete, and thus, are not Banach spaces. However, with the norm we introduce here, we can construct a family of Banach spaces $\mathsf{V}_{\alpha,\beta,\gamma}$, $\alpha \in (0,1), \beta \in (0,1]$ and $\gamma \geq 1$, and use it to correct the proofs in [40], and even generalize the results appearing there.

First, we show that $\|\cdot\|_{\alpha,\beta,\gamma}$ is indeed a norm.

Lemma A.1. *For all $\alpha \in (0,1)$, $\beta \in (0,1]$ and $\gamma \geq 1$, we have that $\|\cdot\|_{\alpha,\beta,\gamma}$ is a norm.*

Proof. We have for $f, g \in \mathsf{V}_{\alpha,\beta}$ that

$$\begin{aligned}
|f+g|_{\alpha,\beta} &= \sup_{\varepsilon \in (0,\varepsilon_0]} \int \frac{\operatorname{osc}\left(R_\alpha\left(f+g\right), B_\varepsilon(x)\right)}{\varepsilon^\beta} \mathrm{d}\lambda_I(x) \\
&= \sup_{\varepsilon \in (0,\varepsilon_0]} \int \frac{\operatorname{osc}\left(R_\alpha f + R_\alpha g, B_\varepsilon(x)\right)}{\varepsilon^\beta} \mathrm{d}\lambda_I(x) \\
&\leq \sup_{\varepsilon \in (0,\varepsilon_0]} \int \frac{\operatorname{osc}\left(R_\alpha f, B_\varepsilon(x)\right)}{\varepsilon^\beta} \mathrm{d}\lambda_I(x) + \sup_{\varepsilon \in (0,\varepsilon_0]} \int \frac{\operatorname{osc}\left(R_\alpha g, B_\varepsilon(x)\right)}{\varepsilon^\beta} \mathrm{d}\lambda_I(x) \\
&= |f|_{\alpha,\beta} + |g|_{\alpha,\beta}
\end{aligned}$$

and thus

$$\|f+g\|_{\alpha,\beta,\gamma} = \|f+g\|_\gamma + |f+g|_{\alpha,\beta} \leq \|f\|_\gamma + \|g\|_\gamma + |f|_{\alpha,\beta} + |g|_{\alpha,\beta} = \|f\|_{\alpha,\beta,\gamma} + \|g\|_{\alpha,\beta,\gamma}\,.$$

It is obviously true that $\|af\|_{\alpha,\beta,\gamma} = a\,\|f\|_{\alpha,\beta,\gamma}$, for any positive a. Since $\|\cdot\|_\gamma$ is already a norm and $|f|_{\alpha,\beta} = 0$ if $f = 0$ almost surely, we know that $\|f\|_{\alpha,\beta,\gamma} = 0$ if and only if $f = 0$ almost surely. □

A.1. **Completeness.** Here we verify that $\mathsf{V}_{\alpha,\beta,\gamma}$ are, in fact, Banach spaces.

Lemma A.2. *For $\alpha \in (0,1), \beta \in (0,1]$ and $\gamma \geq 1$, $\mathsf{V}_{\alpha,\beta,\gamma}$ is complete.*

Proof. Let (f_n) be a Cauchy sequence with respect to $\|\cdot\|_{\alpha,\beta,\gamma}$. Then, in particular (f_n) is also a Cauchy sequence with respect to $\|\cdot\|_\gamma$, we set f as its limit. Also, there exists a subsequence, say (f_{n_r}), that converges to f pointwise almost everywhere.

Since (f_n) is a Cauchy sequence with respect to $\|\cdot\|_{\alpha,\beta,\gamma}$, for each $\delta > 0$ we can choose $L > 0$ such that $\|f_k - f_\ell\|_{\alpha,\beta,\gamma} < \delta$ for all $k, \ell > L$. Let $\delta > 0$ and choose k, ℓ sufficiently large so that $n_k, n_\ell > L$. Then,

$$\|f_{n_k} - f_{n_\ell}\|_{\alpha,\beta,\gamma} = \|f_{n_k} - f_{n_\ell}\|_\gamma + \sup_{\varepsilon\in(0,\varepsilon_0]} \frac{\int \operatorname{osc}\left(R_\alpha\left(f_{n_k} - f_{n_\ell}\right), B_\varepsilon(x)\right)\, \mathrm{d}\lambda_I\left(x\right)}{\varepsilon^\beta} < \delta\,.$$

Then, by Fatou's Lemma, $\|f_{n_k} - f\|_\gamma \leq \liminf_{\ell\to\infty} \|f_{n_k} - f_{n_\ell}\|_\gamma$ and

$$\begin{aligned}
\frac{\int \operatorname{osc}\left(R_\alpha\left(f_{n_k} - f\right), B_\varepsilon(x)\right)\, \mathrm{d}\lambda_I\left(x\right)}{\varepsilon^\beta} &\leq \frac{\int \liminf_{\ell\to\infty} \operatorname{osc}\left(R_\alpha\left(f_{n_k} - f_{n_\ell}\right), B_\varepsilon(x)\right)\, \mathrm{d}\lambda_I\left(x\right)}{\varepsilon^\beta}\\
&\leq \liminf_{\ell\to\infty} \frac{\int \operatorname{osc}\left(R_\alpha\left(f_{n_k} - f_{n_\ell}\right), B_\varepsilon(x)\right)\, \mathrm{d}\lambda_I\left(x\right)}{\varepsilon^\beta}\\
&\leq \liminf_{\ell\to\infty} \sup_{\varepsilon\in(0,\varepsilon_0]} \frac{\int \operatorname{osc}\left(R_\alpha\left(f_{n_k} - f_{n_\ell}\right), B_\varepsilon(x)\right)\, \mathrm{d}\lambda_I\left(x\right)}{\varepsilon^\beta}\,.
\end{aligned}$$

As a result, for all k sufficiently large so that $n_k > L$,

$$\begin{aligned}
\|f_{n_k} - f\|_{\alpha,\beta,\gamma} &\leq \liminf_{\ell\to\infty} \|f_{n_k} - f_{n_\ell}\|_\gamma + \liminf_{\ell\to\infty} \sup_{\varepsilon\in(0,\varepsilon_0]} \frac{\int \operatorname{osc}\left(R_\alpha\left(f_{n_k} - f_{n_\ell}\right), B_\varepsilon(x)\right)\, \mathrm{d}\lambda_I\left(x\right)}{\varepsilon^\beta}\\
&\leq \liminf_{\ell\to\infty} \left(\|f_{n_k} - f_{n_\ell}\|_\gamma + \sup_{\varepsilon\in(0,\varepsilon_0]} \frac{\int \operatorname{osc}\left(R_\alpha\left(f_{n_k} - f_{n_\ell}\right), B_\varepsilon(x)\right)\, \mathrm{d}\lambda_I\left(x\right)}{\varepsilon^\beta}\right) \leq \delta.
\end{aligned}$$

Now, choose r sufficiently large so that $n_r > L$ and $k > L$. Then,

$$\|f_k - f\|_{\alpha,\beta,\gamma} \leq \|f_k - f_{n_r}\|_{\alpha,\beta,\gamma} + \|f_{n_r} - f\|_{\alpha,\beta,\gamma} < 2\delta\,.$$

Thus, $f \in \mathsf{V}_{\alpha,\beta,\gamma}$ and (f_n) converges to f with respect to $\|\cdot\|_{\alpha,\beta,\gamma}$ giving completeness. ☐

Now, we discuss properties of $\mathsf{V}_{\alpha,\beta,\gamma}$ that are relevant for the application of Proposition 3.3 to our setting. First, we prove that constant functions belong to the spaces we consider.

Lemma A.3. *For $\alpha \in (0,1)$, $\beta \in (0,1]$ and $\gamma \geq 1$, the constant function, $\mathbf{1} \in \mathsf{V}_{\alpha,\beta,\gamma}$.*

Proof. Since $\|\mathbf{1}\|_\gamma = 1$, we only have to show that $|\mathbf{1}|_{\alpha,\beta} < \infty$. Observe that $R_\alpha\mathbf{1}$ is bounded by $2^{-2\alpha}$, symmetric about $x = 1/2$ and strictly increasing on $[0, 1/2]$ with a strictly decreasing derivative. Hence, for any $0 < \varepsilon \leq \varepsilon_0 < 1/4$,

$$\begin{aligned}
\int \operatorname{osc}(R_\alpha\mathbf{1}, B_\varepsilon(x))\mathrm{d}\lambda_I(x) &\leq \int_{2\varepsilon}^{1-2\varepsilon} \operatorname{osc}(R_\alpha\mathbf{1}, B_\varepsilon(x))\,\mathrm{d}\lambda_I(x) + 2^{-2\alpha}\left(\int_0^{2\varepsilon} \mathrm{d}\lambda_I(x) + \int_{1-2\varepsilon}^1 \mathrm{d}\lambda_I(x)\right)\\
&\leq 4\varepsilon \int_{2\varepsilon}^{1/2} \max_{B_\varepsilon(x)} |(R_\alpha\mathbf{1})'|\,\mathrm{d}\lambda_I(x) + 2^{2-2\alpha}\varepsilon\\
&= 4\varepsilon \int_{2\varepsilon}^{1/2} (R_\alpha\mathbf{1})'(x-\varepsilon)\,\mathrm{d}\lambda_I(x) + 2^{2-2\alpha}\varepsilon\\
&= 4\varepsilon\left(R_\alpha\mathbf{1}\big(1/2-\varepsilon\big) - R_\alpha\mathbf{1}(\varepsilon)\right) + 2^{2-2\alpha}\varepsilon \leq 2^{3-2\alpha}\varepsilon.
\end{aligned}$$

This implies that $|\mathbf{1}|_{\alpha,\beta} \leq 2^{3-2\alpha}\varepsilon_0^{1-\beta}$. ☐

Next, we state two lemmas about the inclusion properties of $\mathsf{V}_{\alpha,\beta,\gamma}$.

Lemma A.4. *For $\beta \in (0,1]$ and $\gamma \geq 1$,*

$$\mathsf{V}_{0,\beta,\gamma} \hookrightarrow \mathsf{V}_{0,\beta,1} \hookrightarrow L^\infty .$$

Proof. This follows from [38, Proposition 3.4] applied to the real and imaginary parts of functions in $\mathsf{V}_{0,\beta,1}$ and the fact that $L^\gamma \hookrightarrow L^1$. □

Remark A.5. Note that, if $f \in \mathsf{V}_{\alpha,\beta,\gamma}$, then $R_\alpha f \in \mathsf{V}_{0,\beta,\gamma}$. So, $\operatorname{ess\,sup} R_\alpha f < \infty$. This fact will be useful in proofs.

Lemma A.6. *Suppose $0 < \alpha_1 \leq \alpha_2 < 1$, $0 < \beta_2 \leq \beta_1 \leq 1$ and $1 \leq \gamma_2 \leq \gamma_1$. Then*

$$\mathsf{V}_{\alpha_1,\beta_1,\gamma_1} \hookrightarrow \mathsf{V}_{\alpha_2,\beta_2,\gamma_2} \hookrightarrow L^1 .$$

Proof. Since $\|f\|_{\gamma_2} \leq \|f\|_{\gamma_1}$, it is enough to show that $|f|_{\alpha_2,\beta_2} \lesssim \|f\|_{\alpha_1,\beta_1,\gamma_1}$. By applying [38, Proposition 3.2 (iii)] to the real and imaginary parts of f, we have,

$$\begin{aligned}\operatorname{osc}(R_{\alpha_2} f, B_\varepsilon(x)) &= \operatorname{osc}(R_{\alpha_2-\alpha_1}\mathbf{1} \cdot R_{\alpha_1} f, B_\varepsilon(x)) \\ &\leq \operatorname{ess\,sup}|R_{\alpha_1} f| \cdot \operatorname{osc}(R_{\alpha_2-\alpha_1}\mathbf{1}, B_\varepsilon(x)) + \operatorname{osc}(R_{\alpha_1} f, B_\varepsilon(x)) \cdot \sup_{B_\varepsilon(x)} R_{\alpha_2-\alpha_1}\mathbf{1} ,\end{aligned}$$

and due to Lemma A.4,

$$\operatorname{ess\,sup}|R_{\alpha_1} f| \lesssim |R_{\alpha_1} f|_{0,\beta_1} + \|R_{\alpha_1} f\|_1 \leq |f|_{\alpha_1,\beta_1} + \|R_{\alpha_1}\mathbf{1}\|_{\bar\gamma}\|f\|_{\gamma_1} \lesssim \|f\|_{\alpha_1,\beta_1,\gamma_1}$$

with $\bar\gamma = (1-\gamma_1^{-1})^{-1}$. Therefore,

$$\begin{aligned}&\varepsilon^{-\beta_2}\operatorname{osc}(R_{\alpha_2} f, B_\varepsilon(x)) \\ &\quad\lesssim \varepsilon^{-\beta_1}\operatorname{osc}(R_{\alpha_2-\alpha_1}\mathbf{1}, B_\varepsilon(x))\|f\|_{\alpha_1,\beta_1,\gamma_1} + \sup_{B_\varepsilon(x)} R_{\alpha_2-\alpha_1}\mathbf{1} \cdot \varepsilon^{-\beta_1}\operatorname{osc}(R_{\alpha_1} f, B_\varepsilon(x)) .\end{aligned}$$

Integrating and taking the supremum over ε,

$$|f|_{\alpha_2,\beta_2} \lesssim \|f\|_{\alpha_1,\beta_1,\gamma_1},$$

and the inclusion follows. □

A.2. **Continuous inclusion and relative compactness.** To apply Hennion-Nassbaum theory, see [26, 5], we have to show that our *weak* spaces, L^p, are continuously embedded in *strong* spaces, $\mathsf{V}_{\alpha,\beta,\gamma}$, and that the closed bounded sets in strong spaces are compact with respect to weak norms.

Lemma A.7. *Let $\alpha \in (0,1), \beta \in (0,1]$ and $\gamma \geq 1$. Then for all $\bar\gamma$ such that $\gamma < \bar\gamma < 1/\alpha$, $L^{\bar\gamma}$ is continuously embedded in $\mathsf{V}_{\alpha,\beta,\gamma}$.*

Proof. Due to Remark 5.8 and the assumption $\bar\gamma < 1/\alpha$, if $h \in \mathsf{V}_{\alpha,\beta,\gamma}$, then $h \in L^{\bar\gamma}$. So, $\mathsf{V}_{\alpha,\beta,\gamma} \subseteq L^{\bar\gamma}$. To show that this inclusion is continuous we need to show that if $f_n \to 0$ in $\mathsf{V}_{\alpha,\beta,\gamma}$, then $f_n \to 0$ in $L^{\bar\gamma}$. Let $\|f_n\|_{\alpha,\beta,\gamma} \to 0$. Then, $|R_\alpha f_n| \in \mathsf{V}_{0,\beta,1}$ and $\|R_\alpha f_n\|_{0,\beta,1} \to 0$. However, $\mathsf{V}_{0,\beta,1} \hookrightarrow L^\infty$. So, $\|R_\alpha f_n\|_\infty \to 0$. Therefore, $\|f_n^{\bar\gamma}\|_1 \leq \|R_{-\alpha\bar\gamma}\mathbf{1}\|_1\|R_\alpha f_n\|_\infty^{\bar\gamma} \to 0$ proving the claim. □

Lemma A.8. *Let α, β, γ and $\bar\gamma$ be as in the previous lemma. Then, the closed unit ball of $\mathsf{V}_{\alpha,\beta,\gamma}$ is compact in $L^{\bar\gamma}$.*

Proof. Let $\{f_n\}$ be such that $\|f_n\|_{\alpha,\beta,\gamma} \leq 1$. It is enough to show that there is $f \in \mathsf{V}_{\alpha,\beta,\gamma}$ such that $\|f\|_{\alpha,\beta,\gamma} \leq 1$ and $\{f_n\}$ converges to f in $L^{\bar\gamma}$ over a subsequence. To do this, we recall from [25, Theorem 1.13] that closed subsets of $\mathsf{V}_{0,\beta,\gamma}$ are compact in L^γ. Since $\{R_\alpha f_n\} \subset \mathsf{V}_{0,\beta,\gamma}$ is a bounded sequence, it has an L^γ convergent subsequence, and in turn, it has a pointwise almost everywhere convergence subsequence. Let's call this subsequence $\{R_\alpha f_{n_k}\}$ and its point-wise limit f.

We claim $f_{n_k} \to R_{-\alpha} f$ in $L^{\bar\gamma}$. Observe that $f_{n_k} \to R_{-\alpha} f$ point-wise almost everywhere, and since $\mathsf{V}_{0,\beta,\gamma} \hookrightarrow L^\infty$, $|f_{n_k}| \le |R_{-\alpha}\mathbf{1}||R_\alpha f_{n_k}| \le C|R_{-\alpha}\mathbf{1}| \in L^{\bar\gamma}$. So, $f_{n_k} \to R_{-\alpha} f$ in $L^{\bar\gamma}$ if $\alpha\bar\gamma < 1$. Moreover, we claim $\|R_{-\alpha} f\|_{\alpha,\beta,\gamma} \le 1$. To see this, observe that since $L^{\bar\gamma}$ convergence implies L^γ convergence, we apply [25, Lemma 1.12] to conclude that $\liminf_k |f_{n_k}|_{\alpha,\beta} = \liminf_k |R_\alpha f_{n_k}|_{0,\beta} \ge |f|_{0,\beta} = |R_{-\alpha} f|_{\alpha,\beta}$. Since strong convergence implies weak convergence, we have $\liminf_k \|f_{n_k}\|_\gamma \ge \|R_{-\alpha} f\|_\gamma$, and finally,

$$\begin{aligned}\|R_{-\alpha} f\|_{\alpha,\beta,\gamma} = |R_{-\alpha} f|_{\alpha,\beta} + \|R_{-\alpha} f\|_\gamma &\le \liminf_k |f_{n_k}|_{\alpha,\beta} + \liminf_k \|f_{n_k}\|_\gamma \\ &\le \liminf_k \left(|f_{n_k}|_{\alpha,\beta} + \|f_{n_k}\|_\gamma\right) = \liminf_k \|f_{n_k}\|_{\alpha,\beta,\gamma} \le 1\end{aligned}$$

as claimed. ☐

Remark A.9. In particular, the above implies that $\|\cdot\|_{\alpha,\beta,\gamma}$-bounded sequences have $\|\cdot\|_{\bar\gamma}$-Cauchy subsequences.

Appendix B. Hölder Continuity of $\bar R_{j+1}$

Lemma B.1. *For all $j = 0,\dots,k-1$, let $\bar R_{j+1} : [c_j, c_{j+1}] \to \mathbb{R}$ be given by*

$$\bar R_{j+1} = \frac{(R_\alpha \mathbf{1}) \circ \psi_{j+1}}{R_\alpha \mathbf{1}}.$$

Then $\bar R_{j+1}$ is bounded and α-Hölder continuous for all j.

Proof. Our strategy is to prove the following two steps:

(1) There exists $\delta_0 > 0$ such that $\bar R_1'$ is bounded on the interval $[0, c_1 - \delta_0)$, $\bar R_{k+1}'$ is bounded on the interval $(c_k + \delta_0, 1]$ and $\bar R_{j+1}'$, $j = 1,\dots,k-1$ is bounded on the interval $(c_j + \delta_0, c_{j+1} - \delta_0)$.
(2) Since $\bar R_{j+1}(c_j) = \bar R_{j+1}(c_{j+1}) = 0$ for $j = 1,\dots,k-1$, it is enough to show that there exists $C > 0$ such that $\bar R_{j+1}(c_j + \varepsilon) \le C\varepsilon^\alpha$ and $\bar R_{j+1}(c_{j+1} - \varepsilon) \le C\varepsilon^\alpha$, for all $\varepsilon > 0$.

We have

$$\bar R_{j+1}'(x) = \alpha \cdot \frac{\psi_{j+1}'(x)(1 - 2\psi_{j+1}(x))x(1-x) - \psi_{j+1}(x)(1-\psi_{j+1}(x))(1-2x)}{\left(\psi_{j+1}(x)(1-\psi_{j+1}(x))\right)^{1-\alpha} \left(x\,(1-x)\right)^{1+\alpha}}. \tag{B.1}$$

The numerator is bounded, and for $j = 1,\dots,k-2$, the denominator has zeros only at c_j and c_{j+1}. So, we immediately get that $\bar R_{j+1}'$ is bounded on $(c_j + \delta_0, c_{j+1} - \delta_0)$.

We only have to further consider the cases $j = 0$ and $j = k-1$. We have to show that $\bar R_1'(x)$ is bounded in a neighbourhood of 0. Since ψ_1 has a bounded second derivative, we can write $\psi_1(x) = \psi_1(0) + \psi_1'(0)x + \mathcal{O}(x^2) = \psi_1'(0)x + \mathcal{O}(x^2)$. This yields $\left(\psi_1(x)(1-\psi_1(x))\right)^{1-\alpha} \left(x\,(1-x)\right)^{1+\alpha} = \Omega(x^2)$.[2] On the other hand, by simply multiplying out we obtain

$$\psi_1'(x)(1-2\psi_1(x))x(1-x) - \psi_1(x)(1-\psi_1(x))(1-2x) = \mathcal{O}(x^2)$$

implying that $\lim_{x\to 0} \bar R_1(x) < \infty$. The calculation for $\lim_{x\to 1} \bar R_k(x)$ follows analogously.

In order to analyse the behaviour for $x \to c_j$ and $x \to c_{j+1}$ with x starting from $[c_j, c_{j+1}]$ we note that $\bar R_{j+1}'$ can be written as

(B.2)
$$\bar R_{j+1}'(x) = \alpha \cdot \frac{\psi_{j+1}'(x)(1-2\psi_{j+1}(x))}{\left(\psi_{j+1}(x)(1-\psi_{j+1}(x))\right)^{1-\alpha} \left(x\,(1-x)\right)^{\alpha}} - \alpha \cdot \frac{\left(\psi_{j+1}(x)(1-\psi_{j+1}(x))\right)^{\alpha}(1-2x)}{\left(x\,(1-x)\right)^{1+\alpha}}.$$

The minuend tends to ∞ for $x \to c_j$ and to $-\infty$ for $x \to c_{j+1}$ since $\psi_{j+1}(x)$ and $1 - \psi_{j+1}(x)$ tend to zero, respectively, and the numerator remains bounded and is positive near c_j and negative near

[2] $f(x) = \Omega(g(x))$ as $x \to 0$ if $\liminf_{x\to 0} |f(x)|/g(x) > 0$.

c_{j+1}. The subtrahend is bounded on an interval $[\delta_0, 1-\delta_0]$. Thus, $\bar{R}'_{j+1}(x)$ tends to ∞ for $x \to c_j$ and to $-\infty$ for $x \to c_{j+1}$ except if $c_j = 0$ or $c_{j+1} = 1$.

Hence, we can conclude that $|\bar{R}_{j+1}(x) - \bar{R}_{j+1}(y)| \leq \bar{R}_{j+1}(c_j + |x-y|) - \bar{R}_{j+1}(c_j) = \bar{R}_{j+1}(c_j + |x-y|)$ for $x, y \in [c_j, c_j + \delta_0]$ and $\delta_0 > 0$ sufficiently small. Similarly, we have $|\bar{R}_{j+1}(x) - \bar{R}_{j+1}(y)| \leq \bar{R}_{j+1}(c_{j+1} - |x-y|) - \bar{R}_{j+1}(c_{j+1}) = \bar{R}_{j+1}(c_{j+1} - |x-y|)$ for $x, y \in [c_{j+1} - \delta_0, c_{j+1}]$ and $\delta_0 > 0$ sufficiently small. On the other hand, we have

$$\bar{R}_{j+1}(c_j - \varepsilon) = \left(\frac{\psi_{j+1}(c_j - \varepsilon)(1 - \psi_{j+1}(c_j - \varepsilon))}{(c_j - \varepsilon)(1 - c_j + \varepsilon)} \right)^{\alpha}.$$

There exists $C_{j,\delta_0} > 0$ such that

$$\left(\frac{\psi_{j+1}(c_j - \varepsilon)}{(c_j - \varepsilon)(1 - c_j + \varepsilon)} \right)^{\alpha} < C_{j,\delta_0}$$

uniformly for all $\varepsilon \in (0, \delta_0)$ and thus

$$\bar{R}_{j+1}(c_j - \varepsilon) \leq C_{j,\delta_0} \left(\eta_+ \varepsilon\right)^{\alpha}.$$

Similarly, we have

$$\bar{R}_{j+1}(c_{j-1} + \varepsilon) = \left(\frac{\psi_{j+1}(c_{j-1} + \varepsilon)(1 - \psi_{j+1}(c_{j-1} + \varepsilon))}{(c_{j-1} + \varepsilon)(1 - c_{j-1} - \varepsilon)} \right)^{\alpha}$$

and there exists $\bar{C}_{j,\delta_0} > 0$ such that

$$\left(\frac{1 - \psi_{j+1}(c_{j-1} + \varepsilon)}{(c_{j-1} + \varepsilon)(1 - c_{j-1} + \varepsilon)} \right)^{\alpha} < \bar{C}_{j,\delta_0}$$

uniformly for all $\varepsilon \in (0, \delta_0)$ and thus

$$\bar{R}_{j+1}(c_j - \varepsilon) \leq \bar{C}_{j,\delta_0} \left(\eta_+ \varepsilon\right)^{\alpha}.$$

Setting $C = \max_j \max\{C_{\delta_0,j}, \bar{C}_{\delta_0,j}\}$ concludes the proof of the lemma. □

Appendix C. A Key Estimate

In this appendix we will prove the following key lemma:

Lemma C.1. *Define*

$$\Theta_1 := \lim_{s \to 0} \sup_{\varepsilon \leq \varepsilon_0} \frac{1}{\varepsilon^{\beta}} \int_0^{\delta} \operatorname{osc}(R_{\alpha^*} \Re(1 - e^{is\chi})_+, \bar{D}(\delta, \varepsilon, x)) \, \mathrm{d}\lambda_I(x)$$

and

$$\Theta_2 := \lim_{s \to 0} \sup_{\varepsilon \leq \varepsilon_0} \frac{1}{s\varepsilon^{\beta}} \int_0^{\delta} \operatorname{osc}(R_{\alpha^*} \Re(1 - e^{is\chi})_+, \bar{D}(\delta, \varepsilon, x)) \, \mathrm{d}\lambda_I(x)$$

with $\alpha^* \geq 0$, $\delta = \delta(\varepsilon, s) = \varepsilon^{\kappa} s^{\iota}$ where $\iota, \kappa > 0$ and with $\bar{D}$ as in (4.25). Suppose $|\chi|_{\alpha,\beta} < \infty$ with $0 \leq \alpha < \beta \leq 1$.

(1) *If*

$$\iota > 0 \quad \text{and} \quad \kappa \geq \frac{\beta}{\alpha^* + 1}, \tag{C.1}$$

then $\Theta_1 = 0$.

(2) *If*

$$\iota > 1, \quad \text{and} \quad \kappa \geq \frac{\beta}{\alpha^* + 1}, \tag{C.2}$$

then $\Theta_2 = 0$.

Proof. Without loss of generality, we assume that $s > 0$. Note that due to (4.26), we have

$$\begin{aligned}\Theta_1 &\le \lim_{s\to 0}\sup_{\varepsilon\le\varepsilon_0} \frac{\int_0^\delta 2(x+\varepsilon)^{\alpha^*}\,\mathrm{d}\lambda_I(x)}{\varepsilon^\beta}\\ &= \lim_{s\to 0}\sup_{\varepsilon\le\varepsilon_0} \frac{2\left((\delta+\varepsilon)^{\alpha^*+1}-\varepsilon^{\alpha^*+1}\right)}{(\alpha^*+1)\varepsilon^\beta} = \lim_{s\to 0} L(s,\varepsilon_0)\end{aligned}$$

where

$$L(s,\varepsilon_0) := \frac{2}{(\alpha^*+1)}\sup_{\varepsilon\le\varepsilon_0} J(s,\varepsilon)\,,$$

and

$$J(s,\varepsilon) := \frac{(\delta+\varepsilon)^{\alpha^*+1}-\varepsilon^{\alpha^*+1}}{\varepsilon^\beta} = (\varepsilon^\kappa s^\iota+\varepsilon)^{\alpha^*+1}\varepsilon^{-\beta} - \varepsilon^{\alpha^*+1-\beta}\,.$$

First, we note that

$$(\varepsilon^\kappa s^\iota+\varepsilon)^{\alpha^*+1}\varepsilon^{-\beta} = (\varepsilon^{\kappa-\beta/(\alpha^*+1)}s^\iota + \varepsilon^{1-\beta/(\alpha^*+1)})^{\alpha^*+1}\,, \tag{C.3}$$

and hence, for $J(s,\varepsilon)$ to not blow up near $\varepsilon = 0$, we should have (C.1). Due to the first inequality in (C.1) and (C.3), we have for given $s > 0$ that $\sup_{\varepsilon\le\varepsilon_0} J(s,\varepsilon) = J(s,\varepsilon_0)$ and thus $J(0,\varepsilon) := \lim_{s\to 0} J(s,\varepsilon) = 0$ for all ε. Therefore, under the assumption (C.1), $\Theta_1 = 0$ as claimed because

$$\Theta_1 \le \lim_{s\to 0} L(s,\varepsilon_0) = \frac{2}{\alpha^*+1}\lim_{s\to 0} J(s,\varepsilon_0) = 0\,.$$

Now, using (4.26) and l'Hôpital's rule, we obtain

$$\begin{aligned}\Theta_2 &= \lim_{s\to 0}\sup_{\varepsilon\le\varepsilon_0}\frac{\int \operatorname{osc}(R_\alpha\Re(1-e^{is\chi})_+,\bar{D}(\delta,\varepsilon,x))\mathbf{1}_{[0,\delta]}(x)\,\mathrm{d}\lambda_I(x)}{s\varepsilon^\beta}\\ &\le \lim_{s\to 0}\frac{1}{s}\sup_{\varepsilon\le\varepsilon_0}\frac{2\left((\delta+\varepsilon)^{\alpha^*+1}-\varepsilon^{\alpha^*+1}\right)}{(\alpha^*+1)\varepsilon^\beta} = \frac{\mathrm{d}}{\mathrm{d}s}L(s,\varepsilon_0)\Big|_{s=0}.\end{aligned}$$

We note that the last equality follows by the above calculation, namely that $\sup_{\varepsilon\le\varepsilon_0} J(s,\varepsilon) = J(s,\varepsilon_0)$ holds because of $1+\alpha^* > \beta$ and the additional conditions $\iota > 0$ and $\kappa \ge \beta/(\alpha^*+1)$.

Next, taking the derivative of L wrt s, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}L(s,\varepsilon_0) = \frac{2}{\alpha^*+1}\frac{\mathrm{d}}{\mathrm{d}s}(\varepsilon_0^\kappa s^\iota+\varepsilon_0)^{\alpha^*+1}\varepsilon_0^{-\beta} = 2\iota(\varepsilon_0^\kappa s^\iota+\varepsilon_0)^{\alpha^*}\varepsilon_0^{-\beta}\varepsilon_0^\kappa s^{\iota-1}.$$

Note that for $\Theta_2 = 0$ we should have $\frac{\mathrm{d}}{\mathrm{d}s}L(s,\varepsilon_0)\big|_{s=0} = 0$ and this is true, if

$$\iota > 1\,.$$

Therefore under (C.2), we have that $\Theta_2 = 0$ as claimed. □

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