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Optimality and Duality for Robust Optimization Problems Involving Intersection of Closed Sets

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Abstract

In this paper, we study a robust optimization problem whose constraints include nonsmooth and nonconvex functions and the intersection of closed sets. Using advanced variational analysis tools, we first provide necessary conditions for the optimality of the robust optimization problem. We then establish sufficient conditions for the optimality of the considered problem under the assumption of generalized convexity. In addition, we present a dual problem to the primal robust optimization problem and examine duality relations.

Keywords Robust nonsmooth optimization · Optimality condition · Mordukhovich/limiting subdifferential · Duality · Constraint · Closed set

1 Introduction

Because of prediction error, fluctuation and disorder, or lack of information, many practical and realistic problems have uncertain data. So *robust optimization* has emerged and become a remarkable and efficient framework for studying mathematical programming problems under data uncertainties; see, e.g., [3, 4].

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Nowadays, robust optimization has been intensively studied in all aspects of theory, method and application [7, 9, 10, 13, 15, 17, 21, 22, 31].

The goal of this paper is to study an *uncertain* optimization problem of the form:

$$\inf_{x\in\mathbb{R}^n}\{f(x,\tau)\mid x\in\bigcap_{j=1}^{m}C_j,\ g_i(x,u_i)\le 0,\ i=1,...,p\},\tag{UP}$$

where x is a decision variable, τ and u_i, $i=1,...,p$, are *uncertain* parameters, which reside in the uncertainty sets T and V_i, respectively, $T\subset\mathbb{R}^k$ and $V_i\subset\mathbb{R}^{n_i}$, $i=1,...,p$, are nonempty compact sets, $C_j\subset\mathbb{R}^n$, $j=1,...,m$, are nonempty closed subsets, and $f:\mathbb{R}^n\times T\to\mathbb{R}$ and $g_i:\mathbb{R}^n\times V_i\to\mathbb{R}$, $i=1,...,p$, are functions.

A *robust* optimization problem associated with (UP) is defined by

$$\inf_{x\in\mathbb{R}^n}\left\{\max_{\tau\in T}f(x,\tau)\mid x\in\bigcap_{j=1}^{m}C_j,\ g_i(x,u_i)\le 0,\ \forall u_i\in V_i,\ i=1,...,p\right\}.\tag{RP}$$

The problem of type (RP) admits a general formulation and so it provides a unified framework for investigating various robust optimization problems. For instance, when $m:=1$, the nonempty subset C_1 is a closed convex set in $\mathbb{R}^n$, f is a continuous convex function, and $g_1,...,g_p$, are continuously differentiable functions, Chieu et al. in [6] examined links among various constraint qualifications including Karush–Kuhn–Tucker conditions for an optimization problem without uncertainties. In the case of $m:=1$ and the constraints related to a convex cone, Ghafari and Mohebi in [12] provided a new characterization of the Robinson constraint qualification, which collapses to the validation of a generalized Slater constraint qualification and a sharpened nondegeneracy condition for a (no uncertainty) nonconvex optimization problem involving nearly convex feasible sets.

Another approach based on a characterization of the normal cone together with the oriented distance function to establish necessary and sufficient optimality conditions for a smooth optimization problem of type (RP) without uncertainties was given by Jalilian and Pirbazari in [14]. When there are no uncertainty and constraint functions g_i, we refer the reader to [1] for a recent result on optimality, which was obtained by using a canonical representation of a closed set via an associated oriented distance function. For a special case of this problem (RP) with $m:=1$, where there is no uncertainty in the objective, C_1 is a closed convex cone of $\mathbb{R}^n$, f and $g_i(\cdot,u_i)$, $u_i\in V_i$, $i=1,...,p$, are convex functions, Lee and Lee in [18] established an optimality theorem for approximate solutions under a new robust characteristic cone constraint qualification. This result was developed by Sun et al. in [30] to a more general class of robust optimization problems in locally convex vector spaces.

In passing, dealing with a robust optimization problem involving many simple geometric constraints C_j's in (RP) is often more preferable than involving a single abstract set due to the technical calculations of related concepts in variational analysis and nonsmooth/nonconvex generalized differentiations. Moreover, general programming problems with finitely many geometric constraints arise frequently in practical applications (see e.g., customer satisfaction modelling within the automotive industry

[14]) and many other popular classes of optimization problems with specific types of constraints [2, 23] can be reformulated and cast into resulting models involving geometric constraints.

To the best of our knowledge, there are not any results related to optimality conditions and duality for the nonsmooth and nonconvex robust optimization problem (RP). This is because there are challenges associated with not only the nonsmooth and nonconvex structures of the related functions and sets but also uncertainty data. In this work, we employ advanced variational analysis tools (see e.g., [24]) and recent advances on nonsmooth robust optimization (see e.g., [7, 8]) to establish necessary conditions for the optimality of the robust optimization problem (RP). We also provide sufficient conditions for the optimality of the considered problem under the assumption of generalized convexity. Moreover, we address a dual problem to the robust optimization problem (RP) and explore duality relations between them.

The organization of the paper is as follows. Section 2 provides some basic concepts and calculus rules from variational analysis needed for proving our main results. In Sect. 3, we establish necessary conditions and sufficient conditions for the optimality of problem (RP). Section 4 is devoted to examining robust duality relations between the problem (RP) and its dual problem. The last section summarizes the obtained results.

2 Preliminaries

Throughout the paper, the inner product and a norm in $\mathbb{R}^n$ are denoted respectively by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, where $n \in \mathbb{N} := \{1, 2, ...\}$. We use the notation $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$ for the nonnegative orthant and nonpositive orthant of $\mathbb{R}^n$, respectively. Let Γ be a nonempty subset of $\mathbb{R}^n$, the interior, the convex hull and the boundary of Γ are denoted respectively by $\operatorname{int} \Gamma$, $\operatorname{co} \Gamma$ and $\operatorname{bd} \Gamma$. The notation $x \xrightarrow{\Gamma} \overline{x}$ means that $x \to \overline{x}$ and $x \in \Gamma$. The *polar cone* of $\Gamma \subset \mathbb{R}^n$ is defined by

$$\Gamma^\circ := \{\vartheta \in \mathbb{R}^n \mid \langle \vartheta, x \rangle \leq 0, \quad \forall x \in \Gamma\}.$$

Let $\mathcal{F} : X \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a multivalued function/set-valued map. $\mathcal{F}$ is *closed* at $\overline{x} \in X$ if for any sequence $\{x_l\} \subset X$, $x_l \to \overline{x}$ and any sequence $\{y_l\} \subset \mathbb{R}^m$, $y_l \in \mathcal{F}(x_l)$, $y_l \to \overline{y}$ as $l \to \infty$, we have $\overline{y} \in \mathcal{F}(\overline{x})$.

Let us recall some concepts and calculus rules from Variational Analysis (see e.g., [24, 27]). Let $\mathcal{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a multivalued function, the *sequential Painlevé-Kuratowski upper/outer limit* of $\mathcal{F}$ at $\overline{x} \in \operatorname{dom} \mathcal{F}$ is given by

$$\operatorname*{Lim\,sup}_{x \xrightarrow{\Gamma} \overline{x}} \mathcal{F}(x) := \left\{ \vartheta \in \mathbb{R}^n \mid \exists \text{ sequences } x_l \xrightarrow{\Gamma} \overline{x} \text{ and } \vartheta_l \to \vartheta \text{ with } \vartheta_l \in \mathcal{F}(x_l) \text{ for all } l \in \mathbb{N} \right\},$$

where $\mathrm{dom}\mathcal{F} := \{x \in \mathbb{R}^n \mid \mathcal{F}(x) \neq \emptyset\}$. The *Fréchet normal cone* (known also as the regular normal cone) $\widehat{N}(\overline{x};\Gamma)$ to Γ at $\overline{x} \in \Gamma$ is defined by

$$\widehat{N}(\overline{x};\Gamma) := \left\{ \vartheta \in \mathbb{R}^n \mid \limsup_{x \xrightarrow{\Gamma} \overline{x}} \frac{\langle \vartheta, x - \overline{x} \rangle}{\|x - \overline{x}\|} \leq 0 \right\}.$$

We put $\widehat{N}(\overline{x};\Gamma) := \emptyset$ for any $\overline{x} \in \mathbb{R}^n \backslash \Gamma$. The *Mordukhovich normal cone* (known also as the limiting normal cone) $N(\overline{x};\Gamma)$ to Γ at $\overline{x} \in \Gamma$ is defined by

$$N(\overline{x};\Gamma) := \operatorname*{Lim\,sup}_{x \xrightarrow{\Gamma} \overline{x}} \widehat{N}(x;\Gamma).$$

If $\overline{x} \in \mathbb{R}^n \backslash \Gamma$, then $N(\overline{x};\Gamma) := \emptyset$. Given a function $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, we denote $\mathrm{epi}\psi := \{(x,\alpha) \in \mathbb{R}^n \times \mathbb{R} \mid \psi(x) \leq \alpha\}$. The *Fréchet subdifferential* and *limiting/Mordukhovich subdifferential* of $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$ at $\overline{x} \in \mathbb{R}^n$ with $|\psi(\overline{x})| < \infty$ are respectively given by

$$\widehat{\partial}\psi(\overline{x}) := \left\{ \vartheta \in \mathbb{R}^n \mid (\vartheta, -1) \in \widehat{N}((\overline{x}, \psi(\overline{x})); \mathrm{epi}\,\psi) \right\},$$
$$\partial\psi(\overline{x}) := \left\{ \vartheta \in \mathbb{R}^n \mid (\vartheta, -1) \in N((\overline{x}, \psi(\overline{x})); \mathrm{epi}\,\psi) \right\}.$$

If $|\psi(\overline{x})| = \infty$, then the above subdifferentials are empty. Given a set $\Gamma \subset \mathbb{R}^n$, we consider an indicator function $\delta(\cdot;\Gamma)$ defined by

$$\delta(\overline{x};\Gamma) := \begin{cases} 0 & \text{if } \overline{x} \in \Gamma, \\ \infty & \text{if } \overline{x} \notin \Gamma. \end{cases} \tag{2.1}$$

By [27, Proposition 1.19], we obtain that

$$N(\overline{x};\Gamma) = \partial\delta(\overline{x};\Gamma) \quad \text{for } \overline{x} \in \Gamma. \tag{2.2}$$

Remark that the above-defined normal cones and subdifferentials reduce to the corresponding concepts of normal cone and subdifferential in convex analysis when Γ is a convex set and ψ is a convex function.

When ψ is locally Lipschitz at $\overline{x} \in \mathbb{R}^n$, i.e., there exist a neighborhood U of $\overline{x}$ and a real number $\mathcal{L} > 0$ such that

$$|\psi(x_1) - \psi(x_2)| \leq \mathcal{L}\|x_1 - x_2\| \;\; \forall x_1,\, x_2 \in U,$$

we assert by [24, Corollary 1.81] that $\|\vartheta\| \leq \mathcal{L}$ for any $\vartheta \in \partial\psi(\overline{x})$. Moreover, if $\overline{x}$ is a *local minimizer* for ψ, then we get by the *nonsmooth version of Fermat's rule* (see [24, Proposition 1.114]) that

$$0 \in \partial\psi(\overline{x}). \tag{2.3}$$

Lemma 2.1 ([24, Theorem 3.36]) *Let the functions $\psi_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, ..., p$, $p \geq 2$, be lower semicontinuous around $\overline{x} \in \mathbb{R}^n$, and let all but one of these be locally Lipschitz at $\overline{x}$. Then one has*

$$\partial(\psi_1 + \cdots + \psi_p)(\overline{x}) \subset \partial\psi_1(\overline{x}) + \cdots + \partial\psi_p(\overline{x}). \tag{2.4}$$

In the rest of this section, we recall a calculus rule for calculating the limiting subdifferential of maximum of finitely many functions.

Lemma 2.2 ([24, Theorem 3.36 and Theorem 3.46]) *Let the functions $\psi_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, ..., p$, $p \geq 2$, be locally Lipschitz at $\overline{x} \in \mathbb{R}^n$ and denote the maximum function by $\psi(x) := \max\limits_{i=1,...,p} \psi_i(x)$ for $x \in \mathbb{R}^n$. Then*

$$\begin{aligned}\partial\psi(\overline{x}) \;\; \subset \bigcup \Big\{ &\sum_{i=1}^{p} \mu_i \partial\psi_i(\overline{x}) \;\mid\; \sum_{i=1}^{p} \mu_i = 1,\;\; \mu_i \geq 0, \\ &\mu_i(\psi_i(\overline{x}) - \psi(\overline{x})) = 0, i = 1, \ldots, p \Big\}.\end{aligned} \tag{2.5}$$

3 Robust Optimality Conditions

In this section, we first present necessary conditions for the optimality of the robust problem (RP). We then establish sufficient conditions by employing the generalized convexity for a set of finitely many real-valued functions.

In what follows, we assume that the objective function f together with the constraint functions $g_1, ..., g_p$ of the problem (RP) satisfy the following assumptions:

(A1) Given a fixed $\overline{x} \in \mathbb{R}^n$, there exist neighborhoods U_i, $i = 0, ..., p$, of $\overline{x}$ such that the functions $\tau \in T \mapsto f(x, \tau)$, $x \in U_0$, and $u_i \in V_i \mapsto g_i(x, u_i)$, $x \in U_i, i = 1, \ldots, p$ are upper semicontinuous and the functions f and g_i are partially uniformly Lipschitz of ranks $\mathcal{L}_0 > 0$ and $\mathcal{L}_i > 0$ on U_0 and U_i, respectively, i.e.,

$$\begin{aligned}&|f(x_1, \tau) - f(x_2, \tau)| \leq \mathcal{L}_0 \,\|x_1 - x_2\| \quad \forall x_1,\; x_2 \in U_0,\; \forall \tau \in T, \\ &|g_i(x_1, u_i) - g_i(x_2, u_i)| \leq \mathcal{L}_i \,\|x_1 - x_2\| \quad \forall x_1,\; x_2 \in U_i,\; \forall u_i \in V_i,\; i = 1, ..., p.\end{aligned}$$

(A2) For the above $\overline{x} \in \mathbb{R}^n$, the multivalued function $(x, \tau) \in U_0 \times T \rightrightarrows \partial_x f(x, \tau) \subset \mathbb{R}^n$ is closed at $(\overline{x}, \overline{\tau})$ for each $\overline{\tau} \in T(\overline{x})$, and the multivalued functions $(x, u_i) \in U_i \times V_i \rightrightarrows \partial_x g_i(x, u_i) \subset \mathbb{R}^n, i = 1, ..., p$ are closed at $(\overline{x}, \overline{u}_i)$ for each $\overline{u}_i \in V_i(\overline{x})$, where the symbol ∂_x stands for the limiting subdifferential operation with respect to the first variable x and

$$T(\overline{x}) := \{\tau \in T \;\mid\; f(\overline{x}, \tau) = \mathcal{F}(\overline{x})\},\; V_i(\overline{x}) := \{u_i \in V_i \;\mid\; g_i(\overline{x}, u_i) = \mathcal{G}_i(\overline{x})\} \tag{3.1}$$

with

$$\mathcal{F}(\overline{x}) := \max_{\tau \in T} f(\overline{x}, \tau),\ \mathcal{G}_i(\overline{x}) := \max_{u_i \in V_i} g_i(\overline{x}, u_i),\ i = 1, ..., p. \tag{3.2}$$

It is worth mentioning that the above assumptions are commonly found in the study of robust optimization problems or in the nonsmooth analysis such as calculating the nonsmooth subdifferentials/subgradients of max or supremum functions over an infinite set. More precisely, the hypothesis (A1) ensures that the functions $\mathcal{F}$ and $\mathcal{G}_i,\ i = 1, ..., p$, are well-defined, and furthermore, it entails that the functions $\mathcal{F}$ and $\mathcal{G}_i$ are locally Lipschitz of ranks $\mathcal{L}_0$ and $\mathcal{L}_i,\ i = 1, ..., p$, respectively. The hypothesis (A2) can be viewed as a relaxation of subdifferentials for the class of convex functions, and this assumption is automatically satisfied in *smooth settings* as their gradients are continuous (see Corollary 3.1 below). In fact, (A2) holds for a broader class of regular *nonsmooth functions* including subsmooth, and continuously prox-regularity functions whenever (A1) holds. We refer the interested readers to [7, 8] and the references therein for a detailed review.

Let us introduce the following constraint qualification (CQ), which will be necessary to derive the *Karush–Kuhn–Tucker* (KKT) condition for the robust problem (RP).

Definition 3.1 For the problem (RP), let $\overline{x} \in S := \{x \in \mathbb{R}^n \mid x \in \bigcap_{j=1}^{m} C_j,\ g_i(x, u_i) \leq 0,\ \forall u_i \in V_i,\ i = 1, ..., p\}$ and denote

$$\Lambda(\overline{x}) := \{(\lambda_1, ..., \lambda_p) \in \mathbb{R}^p_+ \mid \sum_{i=1}^{p} \lambda_i = 1,\ \lambda_i \max_{u_i \in V_i} g_i(\overline{x}, u_i) = 0,\ i = 1, ..., p\}.$$

We say that the *constraint qualification* (CQ) is satisfied at $\overline{x}$ if there does not exist $(\mu_1, ..., \mu_p) \in \Lambda(\overline{x})$ such that

$$0 \in \sum_{i=1}^{p} \mu_i \mathrm{co}\{\partial_x g_i(\overline{x}, u_i) \mid u_i \in V_i(\overline{x})\} + \sum_{j=1}^{m} N(\overline{x}; C_j),$$

where $V_i(\overline{x})$ is defined by (3.1).

Observe that the concept of (CQ) in Definition 3.1 reduces to the *(extended) Mangasarian-Fromovitz constraint qualification* in the case of smooth setting with $C_j = \mathbb{R}^n,\ j = 1, ..., m$ (see, e.g., [5, 24, 27] for more details).

We are now ready to present necessary optimality conditions for the robust problem (RP) in terms of the limiting/Mordukhovich subdifferentials and normal cones.

Theorem 3.1 *Let the assumptions (A1) and (A2) hold for an optimal solution $\overline{x}$ of problem (RP). Assume that the equation $\nu_1 + \cdots + \nu_m = 0$, where $\nu_j \in N(\overline{x}; C_j),\ j = 1, ..., m$, has only the trivial solution $\nu_j = 0,\ j = 1, ..., m$. Then, there exists*

$(\mu_0, \mu_1, ..., \mu_p) \in \mathbb{R}_+^{p+1}$ *with* $\sum_{i=0}^{p} \mu_i = 1$ *such that*

$$0 \in \mu_0 \,\text{co}\{\partial_x f(\overline{x}, \tau) \mid \tau \in T(\overline{x})\} + \sum_{i=1}^{p} \mu_i \text{co}\{\partial_x g_i(\overline{x}, u_i) \mid u_i \in V_i(\overline{x})\} + \sum_{j=1}^{m} N(\overline{x}; C_j), \tag{3.3}$$

$$\mu_i \max_{u_i \in V_i} g_i(\overline{x}, u_i) = 0, \; i = 1, ..., p. \tag{3.4}$$

If assume additionally that (CQ) *holds at* $\overline{x}$*, then* μ_0 *in the relation of* (3.3) *is a positive number.*

Proof Assume that $\overline{x}$ is an optimal solution of problem (RP), we define the corresponding function $\psi : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\psi(x) := \max\{\mathcal{F}(x) - \mathcal{F}(\overline{x}), \; \mathcal{G}_i(x), \; i = 1, ..., p\}, \; x \in \mathbb{R}^n,$$

where $\mathcal{F}$ and $\mathcal{G}_i$, $i = 1, ..., p$, are defined by (3.2).

We claim that

$$\psi(x) \geq 0 = \psi(\overline{x}) \text{ for all } x \in \bigcap_{j=1}^{m} C_j. \tag{3.5}$$

Indeed, by $\mathcal{G}_i(\overline{x}) \leq 0, \; i = 1, ..., p$, it holds that $\psi(\overline{x}) = 0$. Now, take any $x \in \bigcap_{j=1}^{m} C_j$. It is easy to see that $\mathcal{F}(x) \geq \mathcal{F}(\overline{x})$ if x is a feasible point of problem (RP), which entails that $\psi(x) \geq 0$. Otherwise, it is true that $\max_{i=1,...,p} \mathcal{G}_i(x) > 0$ and so $\psi(x) > 0$.

By (3.5), we see that $\overline{x}$ is a minimizer of the following optimization problem

$$\min_{x \in \mathbb{R}^n} \left\{ \psi(x) + \delta(x; \bigcap_{j=1}^{m} C_j) \right\}, \tag{3.6}$$

where δ is the indicator function defined as in (2.1). Applying the nonsmooth version of Fermat's rule in (2.3) to the problem (3.6) gives us

$$0 \in \partial\big(\psi + \delta(\cdot; \bigcap_{j=1}^{m} C_j)\big)(\overline{x}).$$

Note that, under (A1), the function ψ is locally Lipschitz around $\overline{x}$ and that, due to the closedness of the sets $C_1, ..., C_m$, the indicator function δ is lower semicontinuous around this point. Therefore, invoking the sum rule for the limiting subdifferential in Lemma 2.1 and the formula (2.2), we arrive at

$$0 \in \partial\psi(\overline{x}) + N(\overline{x}; \bigcap_{j=1}^{m} C_j). \tag{3.7}$$

Moreover, since the system $\nu_1 + \cdots + \nu_m = 0$, where $\nu_j \in N(\overline{x}; C_j),\ j = 1, ..., m$, has only one solution, $\nu_j = 0,\ j = 1, ..., m$, we apply the formula of normal cones to finite set intersections (cf. [27, Corollary 2.17]) to arrive at

$$N(\overline{x}; \bigcap_{j=1}^{m} C_j) \subset N(\overline{x}; C_1) + \cdots + N(\overline{x}; C_m).$$

This, together with the calculus rule in Lemma 2.2 and the inclusion (3.7), yields

$$\begin{aligned} 0 \in \bigcup \Big\{ & \alpha\partial\mathcal{F}(\overline{x}) + \sum_{i=1}^{p} \alpha_i \partial\mathcal{G}_i(\overline{x}) \mid \alpha \geq 0,\ \alpha_i \geq 0, \\ & \alpha + \sum_{i=1}^{p} \alpha_i = 1,\ \alpha_i \mathcal{G}_i(\overline{x}) = 0,\ i = 1, ..., p \Big\} + \sum_{j=1}^{m} N(\overline{x}; C_j). \end{aligned} \tag{3.8}$$

Under the assumptions of (A1) and (A2), we argue similarly as in the proof of [7, Theorem 3.3] to arrive at

$$\begin{aligned} &\partial\mathcal{F}(\overline{x}) \subset \mathrm{co}\{\partial_x f(\overline{x}, \tau) \mid \tau \in T(\overline{x})\}, \\ &\partial\mathcal{G}_i(\overline{x}) \subset \mathrm{co}\{\partial_x g_i(\overline{x}, u_i) \mid u_i \in V_i(\overline{x})\},\ i = 1, ..., p, \end{aligned} \tag{3.9}$$

where $T(\overline{x})$ and $V_i(\overline{x})$ are given as in (3.1).

Next, combining (3.8) and (3.9) shows that there exists $(\mu_0, ..., \mu_p) \in \mathbb{R}_+^{p+1}$ such that $\sum_{i=0}^{p} \mu_i = 1$ and

$$\begin{aligned} 0 \in\ & \mu_0 \mathrm{co}\{\partial_x f(\overline{x}, \tau) \mid \tau \in T(\overline{x})\} \\ & + \sum_{i=1}^{p} \mu_i \mathrm{co}\{\partial_x g_i(\overline{x}, u_i) \mid u_i \in V_i(\overline{x})\} + \sum_{j=1}^{m} N(\overline{x}; C_j), \\ & \mu_i \max_{u_i \in V_i} g_i(\overline{x}, u_i) = 0,\ i = 1, ..., p. \end{aligned}$$

So, (3.3) and (3.4) have been justified.

Now, let the (CQ) be satisfied at $\overline{x}$. We obtain from (3.3) and (3.4) that $\mu_0 > 0$, which completes the proof. □

The following corollary gives necessary optimality conditions for the robust problem (RP) under the smoothness of related functions and the convexity of uncertainty sets. In this case, both the hypotheses (A1) and (A2) are automatically satisfied. In what follows, we use $\nabla_x h(\overline{x}, \overline{\tau})$ to denote the derivative of a differentiable function h with respect to the first variable at a given point $(\overline{x}, \overline{\tau})$.

Corollary 3.1 *Let $\overline{x}$ be an optimal solution of problem* (RP), *where T and V_i, $i = 1, ..., p$, are convex sets. Let U_i, $i = 0, ..., p$, be neighborhoods of $\overline{x}$ such that for each $x \in U_0$ and $x \in U_i$, i=1,...,p $f(x, \cdot)$, $g_i(x, \cdot)$ are concave functions on T and V_i, respectively. Assume that f and g_i, $i = 1, ..., p$, are strictly differentiable with respect to the first variable on $U_0 \times T$ and $U_i \times V_i$, respectively. Assume further that maps $(x, \tau) \mapsto \nabla_x f(x, \tau)$ and $(x, u_i) \mapsto \nabla_x g_i(x, u_i)$ are continuous on $U_0 \times T$ and $U_i \times V_i$, respectively. If the system $\nu_1 + \cdots + \nu_m = 0$, where $\nu_j \in N(\overline{x}; C_j)$, $j = 1, ..., m$, has only one solution $\nu_j = 0$, $j = 1, ..., m$, then there exist $\overline{\tau} \in T(\overline{x})$, $\overline{u}_i \in V_i(\overline{x})$, $i = 1, ..., p$ and $(\mu_0, \mu_1, ..., \mu_p) \in \mathbb{R}_+^{p+1}$ with $\sum_{i=0}^{p} \mu_i = 1$ such that*

$$0 \in \mu_0 \nabla_x f(\overline{x}, \overline{\tau}) + \sum_{i=1}^{p} \mu_i \nabla_x g_i(\overline{x}, \overline{u}_i) + \sum_{j=1}^{m} N(\overline{x}; C_j), \tag{3.10}$$

$$\mu_i g_i(\overline{x}, \overline{u}_i) = 0, \; i = 1, ..., p. \tag{3.11}$$

Moreover, we have $\mu_0 > 0$ if the condition (CQ) *is satisfied at $\overline{x}$.*

Proof Observe first that the hypothesis (A2) is automatically satistied at $\overline{x}$ for our setting as the maps $(x, \tau) \mapsto \nabla_x f(x, \tau)$ and $(x, u_i) \mapsto \nabla_x g_i(x, u_i)$, $i = 1, ..., p$ are continuous on $U_0 \times T$ and $U_i \times V_i$, respectively. To verify the hypothesis (A1), we only justify for the function f as the similarities go for the functions g_i, $i = 1, ..., p$. To see this, we first claim that for each $\epsilon > 0$, there exists neighborhood U_ϵ of $\overline{x}$ satisfying

$$\|\nabla_x f(x, \tau) - \nabla_x f(y, \tau)\| \leq \epsilon, \quad \forall x, \; y \in U_\epsilon, \forall \tau \in T, \tag{3.12}$$

where U_ϵ can be chosen such that $U_\epsilon \subset U_0$ and it is a convex set. To prove (3.12), suppose on the contrary that there exist $\overline{\epsilon} > 0$ and a sequence $\{(x_k, y_k, \tau_k)\}$ in $U_0 \times U_0 \times T$ such that $(x_k, y_k) \to (\overline{x}, \overline{x})$ as $k \to \infty$ and

$$\|\nabla_x f(x_k, \tau_k) - \nabla_x f(y_k, \tau_k)\| > \overline{\epsilon} \quad \forall k \in \mathbb{N}. \tag{3.13}$$

Because of the compactness of T and passing to a subsequence if necessary, we may assume that $\{\tau_k\}$ converges to some $\overline{\tau} \in T$. Besides, due to the continuity of the map $(x, \tau) \mapsto \nabla_x f(x, \tau)$ on $U_0 \times T$, it follows that

$$\nabla_x f(x_k, \tau_k) \to \nabla_x f(\overline{x}, \overline{\tau}) \text{ and } \nabla_x f(y_k, \tau_k) \to \nabla_x f(\overline{x}, \overline{\tau}) \text{ as } k \to \infty,$$

which contradicts (3.13).

Let (3.12) hold for a given $\epsilon > 0$. For any $x,\ y \in U_\epsilon$ with $x \neq y$ and for any $\tau \in T$, by the mean value theorem (cf. [11, Theorem 2.4, p. 75]), we find $z \in (x, y) \subset U_\epsilon$ satisfying the condition

$$f(x, \tau) - f(y, \tau) = \langle \nabla_x f(z, \tau), x - y\rangle,$$

where $(x, y) := \mathrm{co}\{x, y\} \setminus \{x, y\}$. This, together with (3.12), gives us

$$\begin{aligned} |\langle \nabla_x f(y, \tau), x - y\rangle - f(x, \tau) + f(y, \tau)| &= |\langle \nabla_x f(y, \tau) - \nabla_x f(z, \tau), x - y\rangle| \\ &\leq \|\nabla_x f(y, \tau) - \nabla_x f(z, \tau)\| \cdot \|x - y\| \\ &\leq \epsilon\|x - y\|. \end{aligned} \tag{3.14}$$

Therefore, it follows that

$$|f(x, \tau) - f(y, \tau)| \leq \big(\|\nabla_x f(y, \tau)\| + \epsilon\big)\|x - y\|.$$

Moreover, by (3.12), $\|\nabla_x f(y, \tau)\| \leq \|\nabla_x f(\overline{x}, \tau)\| + \epsilon$, we arrive at

$$|f(x, \tau) - f(y, \tau)| \leq \big(\|\nabla_x f(\overline{x}, \tau) + 2\epsilon\|\big)\|x - y\|. \tag{3.15}$$

We conclude that (3.15) holds for every $x,\ y \in U_\epsilon$ and $\tau \in T$ as the case of $x = y$ also holds trivially.

By the continuity of function $\tau \mapsto \|\nabla_x f(\overline{x}, \tau)\|$ on the compact set T, it admits the maximum value over T, and so we can take $\mathcal{L}_0 \geq \max\limits_{\tau\in T}\big\{\|\nabla_x f(\overline{x}, \tau)\| + 2\epsilon\big\}$. Consequently, the hypothesis (A1) is satisfied.

In this setting, we can verify that (see e.g., similar arguments as in [7, Corollary 3.4]), the sets $\{\nabla_x f(\overline{x}, \tau) \mid \tau \in T(\overline{x})\}$ and $\{\nabla_x g_i(\overline{x}, u_i) \mid u_i \in V_i(\overline{x})\}$, $i = 1, ..., p$, are convex. Applying Theorem 3.1, we find $(\mu_0, \mu_1, ..., \mu_p) \in \mathbb{R}^{p+1}_+$ with $\sum\limits_{i=0}^{p} \mu_i = 1$ such that

$$0 \in \mu_0 \{\nabla_x f(\overline{x}, \tau) \,|\, \tau \in T(\overline{x})\} + \sum_{i=1}^{p} \mu_i\{\nabla_x g_i(\overline{x}, u_i) \mid u_i \in V_i(\overline{x})\} + \sum_{j=1}^{m} N(\overline{x}; C_j),$$
$$\mu_i \max_{u_i\in V_i} g_i(\overline{x}, u_i) = 0,\ i = 1, ..., p,$$

which imply the assertions in (3.10) and (3.11), and thus completes the proof. □

Remark 3.1 By considering $m := 1$, Corollary 3.1 can be regarded as a generalization version of [19, Theorem 2.3], which was obtained by another approach.

Let us now illustrate the necessary optimality conditions given in Theorem 3.1.

Example 3.1 Let $f : \mathbb{R}^2 \times T \to \mathbb{R}$ and $g_i : \mathbb{R}^2 \times V_i \to \mathbb{R}$, $i = 1, 2, 3$ be given respectively by

$$\begin{aligned}
&f(x,\tau) := |x_1 + \frac{4}{5}| - |x_2 + 1| - \tau^2,\\
&g_1(x,u_1) := -|x_1 + \frac{4}{5}| + |x_2 - \frac{3}{5}| + 2u_1^2 - u_1 - 3,\\
&g_2(x,u_2) := 2|x_1 + 1| + |x_2 + 1| - u_2,\\
&g_3(x,u_3) := x_1 \sin u_3 + x_2 \cos u_3 - 1,
\end{aligned}$$

where $x := (x_1, x_2) \in \mathbb{R}^2$, $\tau \in T := [-2, -1] \cup [1, 2]$, $u_1 \in V_1 := [0, 1]$, $u_2 \in V_2 := [2, 5]$ and $u_3 \in V_3 := [-\frac{\pi}{2}, \pi]$. We consider the robust optimization problem (RP) with geometric constraints C_j, $j = 1, 2$, given by $C_1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 2\}$ and $C_2 := [-2, 0] \times [-1, \frac{3}{5}]$, which is the following problem

$$\inf_{x\in\mathbb{R}^2} \left\{ \max_{\tau\in T} f(x,\tau) \mid x \in C_1 \cap C_2,\ g_i(x,u_i) \leq 0,\ \forall u_i \in V_i,\ i = 1,2,3 \right\}. \tag{EP1}$$

In this setting, for each $x := (x_1, x_2) \in \mathbb{R}^2$, we see that

$$\begin{aligned}
&\mathcal{F}(x) := \max_{\tau\in T} f(x,\tau) = |x_1 + \frac{4}{5}| - |x_2 + 1| - 1,\\
&\mathcal{G}_1(x) := \max_{u_1\in V_1} g_1(x,u_1) = -|x_1 + \frac{4}{5}| + |x_2 - \frac{3}{5}| - 2,\\
&\mathcal{G}_2(x) := \max_{u_2\in V_2} g_2(x,u_2) = 2|x_1 + 1| + |x_2 + 1| - 2,
\end{aligned}$$

and (cf. [7, Example 3.6]) that

$$\mathcal{G}_3(x) := \max_{u_3\in V_3} g_3(x,u_3) = \begin{cases} \max\{-x_1, -x_2\} - 1 & \text{if } x \in (-\infty, 0) \times (-\infty, 0),\\ \|x\| - 1 & \text{otherwise.} \end{cases}$$

Denote $S := \{x := (x_1, x_2) \in \mathbb{R}^2 \mid x \in C_1 \cap C_2,\ \mathcal{G}_i(x) \leq 0,\ i = 1, 2, 3\}$. Then, S is the feasible set of problem (EP1) and is depicted in the gray shade of Fig. 1.

Letting $\bar{x} := \left(-\frac{4}{5}, \frac{3}{5}\right) \in S$, we can verify that the assumptions (A1) and (A2) are satisfied at $\bar{x}$. Moreover, we can also check that $\bar{x}$ is an optimal solution of problem (EP1). By direct calculation, we obtain that

$$\begin{aligned}
&N(\bar{x}; C_1) = \{(0,0)\},\ \ N(\bar{x}; C_2) = \{(0,b) \in \mathbb{R}^2 \mid b \geq 0\},\\
&T(\bar{x}) = \{-1, 1\},\ V_1(\bar{x}) = \{1\},\ V_2(\bar{x}) = \{2\},\\
&V_3(\bar{x}) = \{\bar{u}_3 \in [-\frac{\pi}{2}, \pi] \mid -\frac{4}{5}\sin\bar{u}_3 + \frac{3}{5}\cos\bar{u}_3 = 1\}.\\
&\mathcal{G}_1(\bar{x}) = -2,\ \mathcal{G}_2(\bar{x}) = 0,\ \mathcal{G}_3(\bar{x}) = 0.
\end{aligned}$$

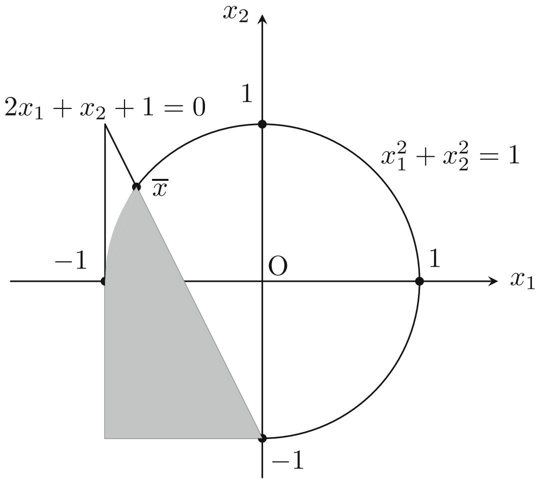

Fig. 1 The feasible set S of problem (EP1) is shaded in gray

Then, for each $\tau \in T(\overline{x})$, $u_1 \in V_1(\overline{x})$, $u_2 \in V_2(\overline{x})$ and $u_3 \in V_3(\overline{x})$, we have

$$\begin{aligned}
&\partial_x f(\overline{x}, \tau) = [-1, 1] \times \{-1\} \Rightarrow \text{co}\, \partial_x f(\overline{x}, \tau) = [-1, 1] \times \{-1\},\\
&\partial_x g_1(\overline{x}, u_1) = \{-1, 1\} \times [-1, 1] \Rightarrow \text{co}\, \{\partial_x g_1(\overline{x}, u_1)\} = [-1, 1] \times [-1, 1],\\
&\partial_x g_2(\overline{x}, u_2) = \{(2, 1)\} \Rightarrow \text{co}\, \{\partial_x g_2(\overline{x}, u_2)\} = \{(2, 1)\},\\
&\partial_x g_3(\overline{x}, u_3) = \{(\sin u_3, \cos u_3)\} \Rightarrow \text{co}\, \{\partial_x g_3(\overline{x}, u_3)\} = \{(\sin u_3, \cos u_3)\}.
\end{aligned}$$

Taking $\mu_0 = \dfrac{3}{4}$, $\mu_2 = \dfrac{1}{4}$ and $\mu_1 = \mu_3 = 0$, we see that

$$\begin{aligned}
&0 \in \mu_0\, \text{co}\{\partial_x f(\overline{x}, \tau) \mid \tau \in T(\overline{x})\}\\
&\quad + \sum_{i=1}^{3} \mu_i \text{co}\{\partial_x g_i(\overline{x}, u_i) \mid u_i \in V_i(\overline{x})\} + \sum_{j=1}^{2} N(\overline{x}; C_j),\\
&\mu_i \max_{u_i \in V_i} g_i(\overline{x}, u_i) = 0, \ i = 1, 2, 3,
\end{aligned}$$

which show that the necessary optimality conditions in Theorem 3.1 hold for the problem (EP1).

Let us state a robust Karush–Kuhn–Tucker (KKT) condition for the problem (RP).

Definition 3.2 Let $\overline{x}$ be a feasible point of problem (RP). We say that the *robust (KKT) condition* is satisfied at $\overline{x}$ if there exists $(\mu_1, ..., \mu_p) \in \mathbb{R}^p_+$ such that

$$0 \in \text{co}\{\partial_x f(\overline{x}, \tau) \mid \tau \in T(\overline{x})\} + \sum_{i=1}^{p} \mu_i \text{co}\{\partial_x g_i(\overline{x}, u_i) \mid u_i \in V_i(\overline{x})\} + \sum_{j=1}^{m} N(\overline{x}; C_j),$$

$$\mu_i \max_{u_i \in V_i} g_i(\overline{x}, u_i) = 0, \ i = 1, ..., p.$$

Remark 3.2 With the assumptions as in Theorem 3.1, it can be seen that if the condition (CQ) satisfies at an optimal solution $\overline{x}$ of problem (RP), then the robust (KKT) condition holds at $\overline{x}$. However, if the robust (KKT) condition is satisfied at a feasible point $\overline{x}$ of problem (RP), we may not conclude that $\overline{x}$ is an optimal solution of the problem as the following example shows.

Example 3.2 Let $f : \mathbb{R}^2 \times T \to \mathbb{R}$ and $g : \mathbb{R}^2 \times V_1 \to \mathbb{R}$ be defined by

$$f(x, \tau) := x_1^3 + \tau x_2^2 - 3x_1x_2, \ g(x, u) := u(x_1^2 + x_2^2),$$

where $x := (x_1, x_2) \in \mathbb{R}^2$, $\tau \in T := [\frac{2}{3}, 1]$ and $u \in V_1 := [-3, 0]$. Consider the following problem

$$\inf_{x \in \mathbb{R}^2} \left\{ \max_{\tau \in T} f(x, \tau) \ | \ x \in C_1, \ g(x, u) \le 0, \ \forall u \in V_1 \right\} \tag{EP2}$$

with the geometric constraint $C_1 := \{x \in \mathbb{R}^2 \ | \ x_1^2 + x_2^2 \le 1\}$.

Let $\overline{x} = (0, 0)$ be a feasible point of problem (EP2). By direct computation, we obtain

$$\begin{aligned} &\mathcal{F}(x) := \max_{\tau \in T} f(x, \tau) = x_1^3 + x_2^2 - 3x_1x_2, \ \mathcal{G}(x) := \max_{u \in V_1} g(x, u) = 0, \ x \in \mathbb{R}^2, \\ &\partial_x f(\overline{x}, \tau) = \{(0, 0)\}, \ \partial_x g(\overline{x}, u) = \{(0, 0)\} \text{ and } N(\overline{x}; C_1) = \{(0, 0)\}. \end{aligned}$$

Observe that the robust (KKT) condition satisfies at $\overline{x}$. However, $\overline{x}$ is not an optimal solution of (EP2) as $\mathcal{F}(\overline{x}) = 0 > -1 = \mathcal{F}(\widehat{x})$, where $\widehat{x} = (-1, 0)$.

Inspired by [7], we define the following generalized convexity. For the convenience in the sequel, we employ the notations $\mathcal{F}(x) := \max_{\tau \in T} f(x, \tau)$, and $\mathcal{G}_i(x) := \max_{u_i \in V_i} g_i(x, u_i), \ i = 1, ..., p$, for $x \in \mathbb{R}^n$.

Definition 3.3 The combination $(\mathcal{F}, \mathcal{G}_1, ..., \mathcal{G}_p)$ is called *generalized convex* at $\overline{x} \in \bigcap_{j=1}^m C_j$, if for any $x \in \bigcap_{j=1}^m C_j$, there exists $w \in \left(\sum_{j=1}^m N(\overline{x}; C_j)\right)^\circ$ such that

$$\begin{aligned} &f(x, \tau) - f(\overline{x}, \tau) \ge \langle x^*, w \rangle \quad \forall x^* \in \partial_x f(\overline{x}, \tau), \ \forall \tau \in T(\overline{x}), \\ &g_i(x, u_i) - g_i(\overline{x}, u_i) \ge \langle x_i^*, w \rangle \quad \forall x_i^* \in \partial_x g_i(\overline{x}, u_i), \ \forall u_i \in V_i(\overline{x}), \ i = 1, ..., p, \end{aligned}$$

where $T(\overline{x})$ and $V_i(\overline{x}), \ i = 1, ..., p$, are defined as in (3.1).

We can verify that if the functions $f(\cdot, \tau), \ \tau \in T$ and $g_i(\cdot, u_i), \ u_i \in V_i, \ i = 1, ..., p$, are convex, then the inequalities in Definition 3.3 are automatically satisfied at any $\overline{x} \in \bigcap_{j=1}^m C_j$ by letting $w := x - \overline{x}$ for each $x \in \bigcap_{j=1}^m C_j$. However, the reverse is not true in general as the following example shows.

Example 3.3 Let $f : \mathbb{R} \times T \to \mathbb{R}$ and $g : \mathbb{R} \times V_1 \to \mathbb{R}$ be given by

$$f(x, \tau) := x^2 + \tau,\ x \in \mathbb{R},\ \tau \in T,$$
$$g(x, u) := \begin{cases} ux^4 & \text{if } u \in V_1 \setminus \{0\} \text{ and } x \in \mathbb{R}, \\ \dfrac{x}{2} & \text{if } u = 0 \text{ and } x \geq 0, \\ 2x & \text{if } u = 0 \text{ and } x < 0, \end{cases}$$

where $T := [0, 1]$ and $V_1 := [0, 2]$.

Denote $\mathcal{F}(x) := \max_{\tau \in T} f(x, \tau)$ and $\mathcal{G}(x) := \max_{u \in V_1} g(x, u)$ for $x \in \mathbb{R}$. We consider $C_1 := [-2, 0]$ and $\overline{x} = 0 \in C_1$. It holds that

$$T(\overline{x}) = \{1\},\ V_1(\overline{x}) := [0, 2],\ \partial_x f(\overline{x}, 1) = \{0\},$$
$$\partial_x g(\overline{x}, u) = \{0\} \text{ for } u \in V_1 \setminus \{0\},\ \partial_x g(\overline{x}, 0) = \{\frac{1}{2}, 2\} \text{ and } N(\overline{x}; C_1)^\circ = (-\infty, 0].$$

In this setting, we can verify that the generalized convexity of $(\mathcal{F}, \mathcal{G})$ is satisfied at $\overline{x}$, while $g_1(\cdot, 0)$ is not a convex function.

The next theorem supplies sufficient conditions for the optimality of problem (RP).

Theorem 3.2 *Suppose that the robust (KKT) condition holds at a feasible point $\overline{x}$ of problem (RP). If $(\mathcal{F}, \mathcal{G}_1, ..., \mathcal{G}_p)$ is generalized convex at $\overline{x}$, then $\overline{x}$ is an optimal solution of problem (RP).*

Proof As $\overline{x}$ satisfies the robust (KKT) condition, we can find $(\mu_1, ..., \mu_p) \in \mathbb{R}^p_+$ and

$$x^* \in \text{co}\{\partial_x f(\overline{x}, \tau) \mid \tau \in T(\overline{x})\},\ x_i^* \in \text{co}\{\partial_x g_i(\overline{x}, u_i) \mid u_i \in V_i(\overline{x})\},\ i = 1, ..., p \tag{3.16}$$

such that

$$-\left(x^* + \sum_{i=1}^{p} \mu_i x_i^*\right) \in \sum_{j=1}^{m} N(\overline{x}; C_j), \tag{3.17}$$
$$\mu_i \max_{u_i \in V_i} g_i(\overline{x}, u_i) = 0,\ i = 1, ..., p. \tag{3.18}$$

By (3.16), there exist $\lambda_l \geq 0$, $x_l^* \in \partial_x f(\overline{x}, \tau_l)$, $\tau_l \in T(\overline{x})$, $l = 1, ..., l_\tau$, $l_\tau \in \mathbb{N}$, $\mu_{ik} \geq 0$, $x_{ik}^* \in \partial_x g_i(\overline{x}, u_{ik})$, $u_{ik} \in V_i(\overline{x})$, $k = 1, ..., k_i$, $k_i \in \mathbb{N}$ such that $\sum_{k=1}^{k_i} \mu_{ik} = 1$, $\sum_{l=1}^{l_\tau} \lambda_l = 1$ and

$$x^* = \sum_{l=1}^{l_\tau} \lambda_l x_l^*,\ x_i^* = \sum_{k=1}^{k_i} \mu_{ik} x_{ik}^*,\ i = 1, ..., p.$$

Therefore, we get from (3.17) that

$$-\left(\sum_{l=1}^{l_\tau}\lambda_l x_l^* + \sum_{i=1}^{p}\mu_i \sum_{k=1}^{k_i}\mu_{ik}x_{ik}^*\right) \in \sum_{j=1}^{m} N(\overline{x}; C_j). \tag{3.19}$$

Assume on the contrary that $\overline{x}$ is not an optimal solution. Then, there exists a feasible point $\widehat{x}$ of problem (RP) satisfying

$$\mathcal{F}(\widehat{x}) < \mathcal{F}(\overline{x}). \tag{3.20}$$

where $\mathcal{F}(x) := \max_{\tau\in T} f(x,\tau)$ for each $x \in \mathbb{R}^n$.

By the generalized convexity of $(\mathcal{F}, \mathcal{G}_1, ..., \mathcal{G}_p)$ at $\overline{x}$, for the above $\widehat{x}$, one can find $w \in \big(\sum_{j=1}^{m} N(\overline{x}; C_j)\big)^\circ$ such that

$$\begin{aligned} 0 &\leq \sum_{l=1}^{l_\tau}\lambda_l\langle x_l^*, w\rangle + \sum_{i=1}^{p}\mu_i\Big(\sum_{k=1}^{k_i}\mu_{ik}\langle x_{ik}^*, w\rangle\Big) \\ &\leq \sum_{l=1}^{l_\tau}\lambda_l(f(\widehat{x},\tau_l) - f(\overline{x},\tau_l)) + \sum_{i=1}^{p}\mu_i\Big(\sum_{k=1}^{k_i}\mu_{ik}(g_i(\widehat{x},u_{ik}) - g_i(\overline{x},u_{ik}))\Big), \end{aligned}$$

where the first inequality holds due to (3.19) and the definition of polar cone. This gives us

$$\begin{aligned} &\sum_{l=1}^{l_\tau}\lambda_l f(\overline{x},\tau_l) + \sum_{i=1}^{p}\mu_i\left(\sum_{k=1}^{k_i}\mu_{ik}g_i(\overline{x},u_{ik})\right) \\ &\quad\leq \sum_{l=1}^{l_\tau}\lambda_l f(\widehat{x},\tau_l) + \sum_{i=1}^{p}\mu_i\left(\sum_{k=1}^{k_i}\mu_{ik}g_i(\widehat{x},u_{ik})\right). \end{aligned} \tag{3.21}$$

It is worth noting that from the definition of max functions $\mathcal{F}(x)$ and $\mathcal{G}_i(x)$, $i = 1, ..., p$, we arrive at $f(\widehat{x},\tau_l) \leq \mathcal{F}(\widehat{x})$, $g_i(\widehat{x},u_{ik}) \leq \mathcal{G}_i(\widehat{x})$, $f(\overline{x},\tau_l) = \mathcal{F}(\overline{x})$, $g_i(\overline{x},u_{ik}) = \mathcal{G}_i(\overline{x})$ for $l = 1, ..., l_\tau$, $l_\tau \in \mathbb{N}$, $\tau_l \in T(\overline{x})$, $k = 1, ..., k_i$, $k_i \in \mathbb{N}$, $u_{ik} \in V_i(\overline{x})$. So, it follows by (3.21) that

$$\mathcal{F}(\overline{x}) + \sum_{i=1}^{p}\sum_{k=1}^{k_i}\mu_{ik}\mu_i\mathcal{G}_i(\overline{x}) \leq \mathcal{F}(\widehat{x}) + \sum_{i=1}^{p}\sum_{k=1}^{k_i}\mu_{ik}\mu_i\mathcal{G}_i(\widehat{x}). \tag{3.22}$$

Note that $\mathcal{G}_i(\widehat{x}) \leq 0$ for $i = 1, ..., p$ and by (3.18) that $\mu_i \mathcal{G}_i(\overline{x}) = 0$. Therefore, the inequality (3.22) shows that

$$\mathcal{F}(\overline{x}) \leq \mathcal{F}(\widehat{x}),$$

which entails a contradiction to (3.20), and so the proof is complete. □

Remark 3.3 Theorem 3.2 develops [19, Theorem 2.4] with $m = 1$. In the case, where there are no geometric constraints, we refer the interested reader to [17, Proposition 2.1] for a necessary and sufficient condition for robust optimal solutions of a robust *convex* optimization problem.

The next example shows how one can utilize Theorem 3.2 to verify an optimal solution.

Example 3.4 Let $f : \mathbb{R}^2 \times T \to \mathbb{R}$ and $g : \mathbb{R}^2 \times V_1 \to \mathbb{R}$ be defined by

$$f(x, \tau) := |x_1| + x_2^2 + \tau_1\tau_2 \text{ and } g(x, u) := \begin{cases} \dfrac{x_1}{2} + \dfrac{x_2}{2} + u & \text{if } x_1 + x_2 > 0, \\ x_1 + x_2 + u & \text{if } x_1 + x_2 \leq 0, \end{cases}$$

where $x := (x_1, x_2) \in \mathbb{R}^2$, $\tau := (\tau_1, \tau_2) \in T := \{(\tau_1, \tau_2) \in \mathbb{R}^2 \mid \tau_1 \geq 0,\ \tau_2 \geq 0,\ \tau_1^2 + \tau_2^2 \leq 1\}$ and $u \in V_1 := [-2, 0]$. Consider a robust optimization problem:

$$\inf_{x \in \mathbb{R}^2} \Big\{ \max_{\tau \in T} f(x, \tau) \mid x \in C_1 \cap C_2,\ g(x, u) \leq 0 \Big\}, \tag{EP3}$$

where the geometric constraints C_1 and C_2 are described by

$$C_1 := \mathbb{R}^2_- \text{ and } C_2 := \{x \in \mathbb{R}^2 \mid x_1x_2 \leq 1\}.$$

It is clear that $\overline{x} := (0, 0)$ is a feasible point of problem (EP3). Denote $\mathcal{F}(x) := \max\limits_{\tau \in T} f(x, \tau)$ and $\mathcal{G}(x) := \max\limits_{u \in V_1} g(x, u)$ for $x \in \mathbb{R}^2$. By direct calculation, one has

$$\begin{aligned} &T(\overline{x}) = \Big\{ \Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big) \Big\},\ V_1(\overline{x}) = \{0\}, \\ &\partial_x f(\overline{x}, \tau) = [-1, 1] \times \{0\} \text{ for } \tau \in T(\overline{x}) \\ &\quad \text{and } \partial_x g(\overline{x}, u) = \Big\{ \Big(\frac{1}{2}, \frac{1}{2}\Big),\ (1, 1) \Big\} \text{ for } u \in V_1(\overline{x}). \end{aligned}$$

Then, we can verify that the robust (KKT) condition of problem (EP3) is satisfied at $\overline{x}$.

To verify the generalized convexity of $(\mathcal{F}, \mathcal{G})$ at $\overline{x}$, take arbitrarily $x := (x_1, x_2) \in C_1 \cap C_2$. Then, by taking $w := (0, 2x_1 + 2x_2) \in (N(\overline{x}; C_1) + N(\overline{x}; C_2))^\circ = \mathbb{R}^2_-$, we have

$$f(x, \tau) - f(\overline{x}, \tau) \geq \langle u^*, w \rangle \quad \forall u^* \in \partial_x f(\overline{x}, \tau),\ \forall \tau \in T(\overline{x}),$$

$$g(x,u) - g(\overline{x},u) \geq \langle v^*, w\rangle \quad \forall v^* \in \partial_x g(\overline{x},u),\ \forall u \in V_1(\overline{x}),$$

which show that $(\mathcal{F}, \mathcal{G})$ is generalized convex at $\overline{x}$. Now, applying Theorem 3.2, we claim that $\overline{x}$ is an optimal solution of the considered problem.

4 Duality in Robust Optimization

In this section, we propose a dual problem to the uncertain optimization problem (UP) and examine some robust duality relations for the pair of primal and dual problems. For the sake of convenience, we recall here the notations

$$\begin{aligned}
&\mathcal{F}(y) := \max_{\tau\in T} f(y,\tau),\ \ \mathcal{G}_i(y) := \max_{u_i\in V_i} g_i(y,u_i),\ i = 1, ..., p,\\
&T(y) := \{\tau \in T \mid f(y,\tau) = \mathcal{F}(y)\} \quad \text{and}\\
&\ \ V_i(y) := \{u_i \in V_i \mid g_i(y,u_i) = \mathcal{G}_i(y)\},\ i = 1, ..., p,
\end{aligned}$$

where $y \in \mathbb{R}^n$.

For the uncertain optimization problem (UP), we address an *uncertain dual* optimization problem as follows:

$$\max\{f(y,\tau) \mid (y,\mu) \in S_D\}, \tag{DU}$$

where $\tau \in T$ and the feasible set S_D is defined by

$$\begin{aligned}
S_D := \Big\{&(y,\mu) \in (\bigcap_{j=1}^{m} C_j) \times \mathbb{R}^p_+ \mid 0 \in \qquad \mathrm{co}\{\partial_x f(y,\tau) \mid \tau \in T(y)\}\\
&+ \sum_{i=1}^{p} \mu_i \mathrm{co}\{\partial_x g_i(y,u_i) \mid u_i \in V_i(y)\}\\
&+ \sum_{j=1}^{m} N(y; C_j),\ \sum_{i=1}^{p} \mu_i \mathcal{G}_i(y) \geq 0\Big\}.
\end{aligned}$$

We investigate the problem (DU) by analyzing its *robust* (worst-case) counterpart:

$$\max\{\max_{\tau\in T} f(y,\tau) \mid (y,\mu) \in S_D\}. \tag{DR}$$

Note that a point $(\overline{y}, \overline{\mu}) \in S_D$ is a solution of (DR) if $\mathcal{F}(y) \leq \mathcal{F}(\overline{y})$ for every $(y,\mu) \in S_D$.

The first theorem in this section provides a weak duality relation between (RP) and (DR).

Theorem 4.1 (Weak duality) *Let x be a feasible point of the problem* (RP) *and let* (y,μ) *be a feasible point of the problem* (DR). *If $(\mathcal{F}, \mathcal{G}_1, ..., \mathcal{G}_p)$ is generalized convex at y,* *then we have*

$$\mathcal{F}(x) \geq \mathcal{F}(y).$$

Proof Let $(y, \mu) \in S_D$. This means that $\mu := (\mu_1, ..., \mu_p) \in \mathbb{R}^p_+$ and there exist

$$x^* \in \text{co}\{\partial_x f(y, \tau) \,|\, \tau \in T(y)\}, \;\; x_i^* \in \text{co}\{\partial_x g_i(y, u_i) \;|\; u_i \in V_i(y)\}, \; i = 1, ..., p \tag{4.1}$$

such that

$$-\left(x^* + \sum_{i=1}^{p} \mu_i x_i^*\right) \in \sum_{j=1}^{m} N(y; C_j), \tag{4.2}$$

$$\sum_{i=1}^{p} \mu_i \mathcal{G}_i(y) \geq 0. \tag{4.3}$$

Assume that the family $(\mathcal{F}, \mathcal{G}_1, ..., \mathcal{G}_p)$ is generalized convex at y. Then, for $x \in \bigcap_{j=1}^{m} C_j$ above, we find $w \in \big(\sum_{j=1}^{m} N(y; C_j)\big)^\circ$ such that

$$\begin{aligned} &f(x, \tau) - f(y, \tau) \geq \langle u^*, w\rangle \quad \forall u^* \in \partial_x f(y, \tau), \; \forall \tau \in T(y), \\ &g_i(x, u_i) - g_i(y, u_i) \geq \langle v^*, w\rangle \quad \forall v^* \in \partial_x g_i(y, u_i), \; \forall u_i \in V_i(y), \; \forall i = 1, ..., p. \end{aligned} \tag{4.4}$$

We assert by (4.1) that there exist $\lambda_l \geq 0$, $y_l^* \in \partial_x f(y, \tau_l)$, $\tau_l \in T(y)$, $l = 1, ..., l_\tau$, $l_\tau \in \mathbb{N}$, such that $\sum_{l=1}^{l_\tau} \lambda_l = 1$ and

$$x^* = \sum_{l=1}^{l_\tau} \lambda_l y_l^*.$$

Therefore, from (4.4), we have

$$\langle x^*, w\rangle = \sum_{l=1}^{l_\tau} \lambda_l \langle y_l^*, w\rangle \leq \sum_{l=1}^{l_\tau} \lambda_l (f(x, \tau_l) - f(y, \tau_l)).$$

Since $f(x, \tau_l) \leq \mathcal{F}(x)$ and $f(y, \tau_l) = \mathcal{F}(y)$ for all $\tau_l \in T(y)$, $l = 1, ..., l_\tau$, we arrive at

$$\langle x^*, w\rangle \leq \sum_{l=1}^{l_\tau} \lambda_l (\mathcal{F}(x) - \mathcal{F}(y)) = \mathcal{F}(x) - \mathcal{F}(y). \tag{4.5}$$

Similarly, we can verify that for each $i = 1, ..., p$, there exist $\mu_{ik} \geq 0$, $k = 1, ..., k_i$, $k_i \in \mathbb{N}$, such that $\sum_{k=1}^{k_i} \mu_{ik} = 1$, and then

$$\langle x_i^*, w\rangle \leq \sum_{k=1}^{k_i} \mu_{ik}(\mathcal{G}_i(x) - \mathcal{G}_i(y)) = \mathcal{G}_i(x) - \mathcal{G}_i(y). \tag{4.6}$$

Combining now (4.2) with (4.5) and (4.6) gives us

$$0 \leq \langle x^*, w\rangle + \sum_{i=1}^{p} \mu_i \langle x_i^*, w\rangle \leq \mathcal{F}(x) - \mathcal{F}(y) + \sum_{i=1}^{p} \mu_i(\mathcal{G}_i(x) - \mathcal{G}_i(y)).$$

This shows that

$$\mathcal{F}(y) + \sum_{i=1}^{p} \mu_i \mathcal{G}_i(y) \leq \mathcal{F}(x) + \sum_{i=1}^{p} \mu_i \mathcal{G}_i(x). \tag{4.7}$$

Furthermore, $\sum_{i=1}^{p} \mu_i \mathcal{G}_i(y) \geq 0$ due to (4.3) and $\mathcal{G}_i(x) \leq 0$ for all $i = 1, ..., p$, because x is a feasible point of problem (RP). So, we get by (4.7) that

$$\mathcal{F}(y) \leq \mathcal{F}(x),$$

which completes the proof of the theorem. □

The following example emphasizes the importance of the generalized convexity imposed in Theorem 4.1.

Example 4.1 Let $f : \mathbb{R}^2 \times T \to \mathbb{R}$ and $g : \mathbb{R}^2 \times V_1 \to \mathbb{R}$ be defined respectively by

$$f(x, \tau) := x_1^3 + x_2^3 - 3x_1x_2 + \tau \text{ and } g(x, u_1) := u_1(x_1^2 + x_2^2),$$

where $x := (x_1, x_2) \in \mathbb{R}^2$, $\tau \in T := [-2, 0]$ and $u_1 \in V_1 := [-1, 0]$. Consider a robust optimization problem:

$$\inf_{x\in\mathbb{R}^2} \Big\{\max_{\tau\in T} f(x, \tau) \mid x \in C_1,\ g(x, u_1) \leq 0\Big\}, \tag{EP4}$$

where the geometric constraint C_1 is given by $C_1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$. Then, the dual problem in terms of (DR) for (EP4) is defined by

$$\max \{\max_{\tau\in T} f(y, \tau) \mid (y, \mu) \in S_D\}. \tag{ED4}$$

Let $\overline{x} = \left(\frac{1}{2}, \frac{1}{2}\right)$, $\overline{\mu} = 1$ and $\overline{y} = (0, 0)$ and denote $\mathcal{F}(x) := \max_{\tau \in T} f(x, \tau)$ and $\mathcal{G}(x) := \max_{u_1 \in V_1} g(x, u_1)$ for $x \in \mathbb{R}^2$. A direct calculation shows that

$$\begin{aligned}
&N(\overline{y}; C_1)^\circ = \mathbb{R}^2,\ T(\overline{y}) = \{0\},\\
&V_1(\overline{y}) = [-1, 0],\ \partial_x f(\overline{y}, \tau) = \{(0, 0)\},\ \partial_x g(\overline{y}, \tau) = \{(0, 0)\},
\end{aligned}$$

and so we can check that $\overline{x}$ is feasible for (EP4) and $(\overline{x}, \overline{\mu})$ is feasible for (ED4). However, it holds that

$$\mathcal{F}(\overline{x}) = -\frac{1}{2} < 0 = \mathcal{F}(\overline{y}),$$

which shows that Theorem 4.1 is not true for this setting. The reason is that the generalized convexity of $(\mathcal{F}, \mathcal{G})$ at $\overline{y}$ is violated.

In the following theorem, we establish strong duality and converse duality relations between (RP) and (DR).

Theorem 4.2 (Strong and converse duality) *Consider the robust optimization problem* (RP) *and its dual problem* (DR).

(i) *Let assumptions (A1) and (A2) hold at an optimal solution $\overline{x}$ of problem* (RP). *Asssume that the condition* (CQ) *is satisfied at $\overline{x}$ and that the equation $\nu_1 + \cdots + \nu_m = 0$, where $\nu_j \in N(\overline{x}; C_j)$, $j = 1, ..., m$, has only the trivial solution $\nu_1 = \cdots = \nu_m = 0$. If $(\mathcal{F}, \mathcal{G}_1, ..., \mathcal{G}_p)$ is generalized convex at any $y \in \bigcap_{j=1}^{m} C_j$, then there exists $\overline{\mu} \in \mathbb{R}^p_+$ such that $(\overline{x}, \overline{\mu})$ is a solution of problem* (DR).

(ii) *Let $(\overline{x}, \overline{\mu})$ be a feasible point of problem* (DR). *If $\overline{x}$ is a feasible point of problem* (RP) *and $(\mathcal{F}, \mathcal{G}_1, ..., \mathcal{G}_p)$ is generalized convex at $\overline{x}$, then $\overline{x}$ is an optimal solution of* (RP).

Proof (i) As $\overline{x}$ is an optimal solution of problem (RP), it follows that $\overline{x} \in \bigcap_{j=1}^{m} C_j$. According to Theorem 3.1, we can find $(\mu_0, \mu_1, ..., \mu_p) \in \mathbb{R}^{p+1}_+$ with $\mu_0 > 0$ and $\sum_{i=0}^{p} \mu_i = 1$ such that

$$\begin{aligned}
0 \in\ &\mu_0 \mathrm{co}\{\partial_x f(\overline{x}, \tau) \mid \tau \in T(\overline{x})\} + \sum_{i=1}^{p} \mu_i \mathrm{co}\{\partial_x g_i(\overline{x}, u_i) \mid u_i \in V_i(\overline{x})\}\\
&+ \sum_{j=1}^{m} N(\overline{x}; C_j),
\end{aligned}$$

$$\mu_i \max_{u_i \in V_i} g_i(\overline{x}, u_i) = 0, \ i = 1, ..., p.$$

By dividing the above relationships by μ_0, and then setting $\overline{\mu}_i := \dfrac{\mu_i}{\mu_0}$, $i = 1, ..., p$, we obtain that $\overline{\mu} := (\overline{\mu}_1, ..., \overline{\mu}_p) \in \mathbb{R}^p_+$ and $(\overline{x}, \overline{\mu}) \in S_D$.

Now, let $(\mathcal{F}, \mathcal{G}_1, ..., \mathcal{G}_p)$ be generalized convex at any $y \in \bigcap_{j=1}^m C_j$. For any $(y, \mu) \in S_D$, we invoke Theorem 4.1 to conclude that

$$\mathcal{F}(\overline{x}) \geq \mathcal{F}(y).$$

This means that $(\overline{x}, \overline{\mu})$ is a solution of problem (DR).

(ii) As $(\overline{x}, \overline{\mu}) \in S_D$, we have $\overline{\mu} := (\overline{\mu}_1, ..., \overline{\mu}_p) \in \mathbb{R}^p_+$ and

$$0 \in \mathrm{co}\{\partial_x f(\overline{x}, \tau) \mid \tau \in T(\overline{x})\} + \sum_{i=1}^p \overline{\mu}_i \mathrm{co}\{\partial_x g_i(\overline{x}, u_i) \ \mid \ u_i \in V_i(\overline{x})\} + \sum_{j=1}^m N(\overline{x}; C_j), \tag{4.8}$$

$$\sum_{j=1}^p \overline{\mu}_i \max_{u_i \in V_i} g_i(\overline{x}, u_i) \geq 0, \ i = 1, ..., p. \tag{4.9}$$

Let $\overline{x}$ be a feasible point of problem (RP). Then, it entails that $\overline{\mu}_i \max_{u_i \in V_i} g_i(\overline{x}, u_i) \leq 0$ for all $i = 1, ..., p$. This, together with (4.9), ensures that

$$\overline{\mu}_i \max_{u_i \in V_i} g_i(\overline{x}, u_i) = 0, \ i = 1, ..., p.$$

Namely, the robust (KKT) condition of problem (RP) holds at $\overline{x}$. To complete the proof, it suffices to invoke Theorem 3.2. □

We finish this section by giving an example that shows how one can calculate an optimal solution of a robust optimization problem through its dual counterpart.

Example 4.2 Let $f : \mathbb{R}^2 \times T \to \mathbb{R}$ and $g_i : \mathbb{R}^2 \times V_i \to \mathbb{R}$, $i = 1, 2$ be defined by

$$f(x, \tau) := |x_1| + |x_2| + 1 - \tau,$$
$$g_1(x, u_1) := -x_1^2 + 2|x_2| + u_1, \ g_2(x, u_2) := u_2|x_1| - 2u_2,$$

where $x := (x_1, x_2) \in \mathbb{R}^2$, $\tau \in T := [1, 2]$, $u_1 \in V_1 := [-5, -3]$ and $u_2 \in V_2 := [0, 2]$.

Consider a robust optimization problem:

$$\inf_{x \in \mathbb{R}^2} \Big\{ \max_{\tau \in T} f(x, \tau) \ \mid \ x \in C_1 \cap C_2, \ g_i(x, u_i) \leq 0, \ i = 1, 2 \Big\}, \tag{EP5}$$

where the geometric constraints C_1 and C_2 are given respectively by

$$C_1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0\} \text{ and } C_2 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0,\ x_1^2 + x_2^2 \leq 1\}.$$

Then, the dual problem in terms of (DR) for (EP5) is defined by

$$\max \{\max_{\tau \in T} f(y, \tau) \mid (y, \mu) \in S_D\}, \tag{ED5}$$

where the feasible set S_D is given by

$$\begin{aligned} S_D := \Big\{(y, \mu) \in (C_1 \cap C_2) \times \mathbb{R}^2_+ \mid 0 \in \quad & \operatorname{co}\{\partial_x f(y, \tau) \mid \tau \in T(y)\} \\ + \sum_{i=1}^{2} \mu_i \operatorname{co}\{\partial_x g_i(y, u_i) \mid u_i \in V_i(y)\} & \\ + \sum_{j=1}^{2} N(y; C_j),\ \sum_{i=1}^{2} \mu_i \mathcal{G}_i(y) \geq 0\Big\} & \end{aligned}$$

with $\mathcal{G}_i(x) := \max_{u_i \in V_i} g_i(x, u_i),\ i = 1, 2$ for $x \in \mathbb{R}^2$.

Denoting $\overline{x} := (0, 0)$, we see that $\overline{x}$ is a feasible point of problem (EP5). By direct calculation, we have

$$\begin{gathered} T(\overline{x}) = \{1\},\ V_1(\overline{x}) = \{-3\},\ V_2(\overline{x}) = \{0\}, \\ \partial_x f(\overline{x}, 1) = [-1, 1] \times [-1, 1],\ \partial_x g_1(\overline{x}, -3) = \{0\} \times [-2, 2],\ \partial_x g_2(\overline{x}, 0) = \{(0, 0)\}, \\ N(\overline{x}; C_1) = \{(x_1, 0) \in \mathbb{R}^2 \mid x_1 \in \mathbb{R}\} \text{ and } N(\overline{x}; C_2) = \{(0, x_2) \in \mathbb{R}^2 \mid x_2 \leq 0\}. \end{gathered}$$

Taking $\overline{\mu} := (1, 1)$, we see that $(\overline{x}, \overline{\mu}) \in S_D$. Moreover, we can verify that $(\mathcal{F}, \mathcal{G}_1, \mathcal{G}_2)$ is generalized convex at $\overline{x}$, where $\mathcal{F}(x) := \max_{\tau \in T} f(x, \tau)$ for $x \in \mathbb{R}^2$. Employing now Theorem 4.2(ii), we assert that $\bar{x}$ is an optimal solution of problem (EP5).

5 Conclusions

This paper studied a robust optimization problem, where the related data are nonsmooth and nonconvex functions, and moreover the constraint set involves an intersection of finitely many closed sets. More concretely, we presented necessary conditions for the optimality of the underlying problem. Under the additional assumption of generalized convexity and the robust (KKT) condition satisfying at a given feasible point, we provided sufficient conditions that allow the referenced feasible point to be optimal. We also established a dual problem to the robust optimization problem and proposed their duality relationships.

It would be interesting to see how we can develop numerical schemes based on the obtained optimality conditions to find optimal solutions or optimal value for the considered robust optimization problem. When the intersection of finitely geometric constraints involves data uncertainties, the investigation of the corresponding robust optimization problem might become intractable and the current approach would not be applicable as we encounter an uncertainty setting, where the robust optimization counterpart inherits an infinite number of geometric constraint sets. Some possible approaches such as tangential extremal principles for a countable set system in [25, 26] could be developed to study this type of uncertain/robust optimization problems. Moreover, some appropriate applications to examine other general classes of robust optimization problems are well worth a further study.

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Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

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