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On the stochastic inventory problem under order capacity constraints[☆]

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ABSTRACT

We consider the single-item single-stocking location stochastic inventory system under a fixed ordering cost component. A long-standing problem is that of determining the structure of the optimal control policy when this system is subject to order quantity capacity constraints; to date, only partial characterisations of the optimal policy have been discussed. An open question is whether a policy with a single continuous interval over which ordering is prescribed is optimal for this problem. Under the so-called "continuous order property" conjecture, we show that the optimal policy takes the modified multi-(s, S) form. Moreover, we provide a numerical counterexample in which the continuous order property is violated, and hence show that a modified multi-(s, S) policy is not optimal in general. However, in an extensive computational study, we show that instances violating the continuous order property do not surface, and that the plans generated by a modified multi-(s, S) policy can therefore be considered, from a practical standpoint, near-optimal. Finally, we show that a modified (s, S) policy also performs well in this empirical setting.

1. Introduction

This study focuses on one of the fundamental problems in inventory control theory (Arrow, Harris, & Marschak, 1951; Porteus, 2002): the periodic review single-item single-stocking location stochastic inventory system under nonstationary demand, complete backorders, and a fixed ordering cost component. By introducing the concept of K-convexity, Scarf (1960) proved, under mild assumptions, that the optimal control policy takes the well-known (s, S) form: if the inventory level falls below the reorder point s, one should place an order and raise inventory up to level S; otherwise, one should not order. Compared to the case investigated by Scarf, in which the order quantity is unconstrained, the capacitated version of the stochastic inventory problem is inherently harder, both structurally and computationally. This work is concerned with this variant of the problem. We assume the capacity is fixed and known, as opposed to uncertain (e.g. Ciarallo, Akella, & Morton, 1994).

If the fixed ordering cost is absent, but ordering capacity constraints are enforced, a so-called *modified base stock policy* is optimal for both the finite and infinite horizon cases (Federgruen & Zipkin, 1986a; 1986b). While in a classical base stock policy one simply orders up to S, in a modified base stock policy, when the inventory level falls below S, one should order up to S, or as close to S as possible, given the ordering capacity. The classical base stock policy is thus "modified" to embed order saturation.

In the presence of a positive fixed ordering cost, Wijngaard (1972) was the first to investigate the influence of capacity constraints on the structure of the optimal control policy. In analogy to the aforementioned modified base stock policy, Wijngaard conjectured that an optimal strategy may feature a so-called *modified (s, S) structure*: if the inventory level is greater or equal to s, do not order; otherwise, order up to S, or as close to S as possible, given the ordering capacity. Unfortunately, both Wijngaard (1972) and Shaoxiang & Lambrecht (1996) provided counterexamples that ruled out the optimality of a modified (s, S) policy. However, Shaoxiang & Lambrecht (1996) proved that, under stationary demand and a finite horizon, the optimal policy features a so-called $X-Y$ band structure: when initial inventory level is below X, it is optimal to order at full capacity; when initial inventory level

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is above Y, it is optimal to not order. Gallego & Scheller-Wolf (2000) introduced CK-convexity, a generalisation of Scarf's K-convexity; by leveraging this property, they extended the analysis in Shaoxiang & Lambrecht (1996) and further characterized the optimal policy by identifying four regions: in two of these regions the optimal policy is completely specified, while it is only partially specified in the other two regions. Chan & Song (2003) discussed further properties of the optimal order policy when the inventory level falls within Shaoxiang & Lambrecht's $X-Y$ band, and devised an efficient algorithm to compute optimal policy parameters. Shaoxiang (2004) extended the analysis in Shaoxiang & Lambrecht (1996) and proved that the optimal policy continues to exhibit the $X-Y$ band structure under infinite horizon; moreover, Shaoxiang proved that the $X-Y$ band width is no more than the capacity. Gallego & Toktay (2004) investigated the case in which the fixed ordering cost is large relative to the variable cost of a full order; this assumption allowed them to restrict their analysis to full-capacity orders; under this setting they showed that the optimal policy is a threshold policy: if the inventory level falls below the threshold s, issue a full-capacity order; otherwise, do not order. Finally, Shi, Zhang, Chao, & Levi (2014) developed an approximation algorithm with worst-case performance guarantee.

As mentioned in Shi et al. (2014), when order quantity capacity constraints are enforced, *only some partial characterization of the structure of the optimal control policy is available* in the literature. To the best of our knowledge, the problem of determining the structure of the optimal policy of the capacitated stochastic inventory problem remains open. A long-standing open question in the literature, originally posed by Gallego & Scheller-Wolf (2000), is whether a policy with a single continuous interval over which ordering is prescribed is optimal for this problem. This is the so-called "continuous order property" conjecture, which was later also investigated by Chan & Song (2003). To the best of our knowledge, to date this conjecture has never been confirmed or disproved. This gap motivates the present study.

We make the following contributions to the literature on stochastic inventory control.

- In light of the results presented in Shaoxiang (2004), we show how to simplify the optimal policy structure presented by Gallego & Scheller-Wolf (2000). Moreover, we extend the discussion in Gallego & Scheller-Wolf (2000) and provide a full characterisation of the optimal policy for instances for which the continuous order property holds. In particular, we show that the optimal policy takes the *modified multi-*(s,S) *form*.
- We provide a numerical counterexample in which the continuous order property is violated. This closes a fundamental and long standing question in the literature: a policy with a single continuous interval over which ordering is prescribed is not optimal in general. Since generating similar counterexamples is far from trivial, in our Appendix we illustrate the analytical insights we relied upon to generate such instances.
- In an extensive computational study comprising 9720 instances constructed by using realistic demand patterns and cost configurations investigated in the literature, we show that no violation of the continuous order property is found. From a practical standpoint, a modified multi-(s,S) ordering policy can therefore be considered near-optimal for the problem under scrutiny. Moreover, we empirically find that the number of reorder-point/order-up-to-level pairs that this policy features in each period is always less or equal to 6 in our test bed. Finally, we show that a well-known heuristic policy, the modified (s,S) policy (Wijngaard, 1972), also performs well in this empirical setting.

The rest of this paper is organised as follows. In Section 2, we introduce the well-known stochastic inventory problem as originally discussed in Scarf (1960). In Section 3, we extend the problem description to accommodate order quantity capacity constraints. In Section 4 we summarise known properties of the optimal policy from the literature. In Section 5 we introduce the so-called "continuous order property," which has been previously conjectured in the literature, and illustrate the structure that the optimal policy would take if this property were to hold. In Section 6 we present a numerical counterexample in which the continuous order property is violated. In Section 7 we illustrate results of our extensive computational study aimed at showing that no violation of the continuous order property occurs, that a modified multi-(s,S) ordering policy is near-optimal from a practical standpoint, and that a modified (s,S) ordering policy also performs well in this empirical setting. In Section 8 we draw conclusions.

2. Preliminaries on the (s,S) policy

The rest of this work is concerned with a single-item single-stocking point inventory control problem. A finite planning horizon of n discrete time periods, which are labelled in reverse order for convenience, is assumed. Period demands are stochastic, d_t in period t, with known probability density and cumulative distribution functions f_t and F_t, respectively. The cost components that are taken into account include: the ordering cost $c(x)$ for placing an order for x units; the inventory holding cost h for any excess unit of stock carried over to next period; and the shortage cost p that is incurred for each unit of unmet demand in any given period. Unmet demand is backordered. Without loss of generality, it is assumed that there is no lead-time and deliveries are instantaneous.[1]

Let x represent the pre-order inventory level, and $\widehat{C}_n(x)$ denote the minimum expected total cost achieved by employing an optimal replenishment policy over the planning horizon $n,\dots,1$; then one can write

$$\widehat{C}_n(x) \triangleq \min_{x\le y}\left\{c(y-x)+L_n(y)+\int_0^\infty \widehat{C}_{n-1}(y-\xi)f_n(\xi)\mathrm{d}\xi\right\},$$

where $\widehat{C}_0 \triangleq 0$ and $L_n(y) \triangleq \int_0^y h(y-\xi)f_n(\xi)\mathrm{d}\xi + \int_y^\infty p(\xi-y)f_n(\xi)\,\mathrm{d}\xi$.

Following Scarf (1960), we assume that the ordering cost takes the form

$$c(x) \triangleq \begin{cases} 0 & x=0,\\ K+vx & x>0.\end{cases}$$

For convex $L_n(y)$, Scarf (1960) proved that the optimal policy takes the (s,S) form, and thus features two policy control parameters: s and S. In the (s,S) policy, an order of size $S-x$ is placed if and only if the pre-order inventory level is $x<s$.

More specifically, Scarf (1960) introduced the concept of K-convexity (Definition 1).[2]

Definition 1 K-convexity**.** Let $K\ge 0$, $g(x)$ is K-convex if for all x, $a>0$, and $b>0$,

$$\big(K+g(x+a)-g(x)\big)/a \ge \big(g(x)-g(x-b)\big)/b.$$

By leveraging this concept, Scarf proved that $\widehat{G}_n(y)$ is K-convex, where

$$\widehat{G}_n(y) \triangleq vy + L_n(y) + \int_0^\infty \widehat{C}_{n-1}(y-\xi)f_n(\xi)\mathrm{d}\xi.$$

[1] Under complete backordering, it is sufficient to replace the inventory level with the inventory position as the state variable, and modify the demand distribution to account for the presence of positive lead-time (Scarf, 1960, p. 201).

[2] A geometrical interpretation of K-convexity can be found in Porteus (2002, p. 106–107).

This observation implies that the (s, S) policy is optimal, and the policy parameters s and S satisfy $\widehat{G}_n(s) = \widehat{G}_n(S) + K$. Note that when the order quantity is not subject to capacity constraints, S coincides with the global minimizer of $\widehat{G}_n(y)$. In what follows, we will see that this may not be the case when a capacity constraint is enforced on the order quantity.

3. Capacitated ordering

The stochastic inventory problem investigated in Scarf (1960) assumes that order quantity Q in each period can be as large as needed. In practice, one may want to impose the restriction that $0 \le Q \le B$, where B is a positive value denoting the maximum order quantity in each period.

We generalise $\widehat{C}_n(x)$ and $\widehat{G}_n(x)$ to reflect capacity restrictions

$$C_n(x) \triangleq \min_{x \le y \le x+B} \left\{ c(y-x) + L_n(y) + \int_0^\infty C_{n-1}(y-\xi) f_n(\xi) d\xi \right\}; \quad (1)$$

$$G_n(y) \triangleq vy + L_n(y) + \int_0^\infty C_{n-1}(y-\xi) f_n(\xi) d\xi. \quad (2)$$

Finally, we present a useful result that will be used in the coming sections.

Definition 2. A function $f : \mathbb{R} \to \mathbb{R}$ is coercive if $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = \infty$.

Lemma 1. *$G_n(x)$ is coercive.*

Proof. The limiting behaviour of $G_n(x)$ can be characterized as $\lim_{x\to\infty} G'_n(x) = nh$ and $\lim_{x\to-\infty} G'_n(x) = -np$, and from the fundamental theorem of calculus it follows $\lim_{x\to\infty} G_n(x) = \lim_{x\to-\infty} G_n(x) = \infty$. □

4. Review of known properties of the optimal policy

We next introduce[3] "(K, B)-convexity 1" (KBC1) for a function g (Gallego & Scheller-Wolf, 2000).

Definition 3. Let $K \ge 0$, $B \ge 0$, g is KBC1 if it satisfies

$$\big(K + g(x+a) - g(x)\big)/a \ge \big(g(y) - g(y-b)\big)/b$$

for $0 < a \le B$, $0 < b \le B$, and $y \le x$.

Example 1. Consider a planning horizon of $n = 4$ periods, and a demand d_t distributed in each period $t = 1, \dots, n$ according to a Poisson law with rate $\lambda_t \in \{20, 40, 60, 40\}$. Other problem parameters are $K = 100$, $h = 1$ and $p = 10$; to better conceptualise the example we let $v = 0$. In Fig. 1 we plot $G_n(y)$ and illustrate the concept of KBC1 for the case in which $B = 65$.

Lemma 2. *If G_n (resp. C_n) is KBC1 and it is optimal to place an order at x_0, then $G_n(y)$ (resp. $C_n(y)$) is nonincreasing for $y \le x_0$.*

Proof. Since G_n is KBC1, if it is optimal to place an order at x_0, say an order of a units, then $0 \ge (K + G_n(x_0 + a) - G_n(x_0))/a$, and G_n is nonincreasing for $y \le x_0$, since $0 \ge (K + G_n(x_0 + a) - G_n(x_0))/a \ge (G_n(y) - G_n(y-b))/b$, for $y \le x_0$ and $0 < b \le B$. The proof for C_n is identical. □

Lemma 3. *If G_n is KBC1, there exists a pair of values S_m and s_m such that $s_m \triangleq \sup\{x | C_n(x) = G_n(x) - vx\}$ is the maximum inventory level at which it is optimal to place an order, and $S_m \triangleq s_m + a$, where $0 < a \le B$ is the order quantity at s_m.*

[3] This was originally called strong CK-convexity in Gallego & Scheller-Wolf (2000); however, in line with Scarf (1960), in the present work we used C to denote the cost function, and B for the ordering capacity, hence the concept has been renamed (K, B)-convexity.

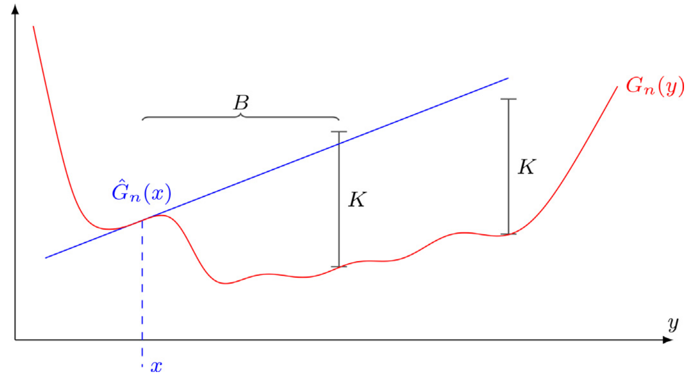

Fig. 1. KBC1 in the context of Example 1, when $B = 65$. For the sake of illustration, $\hat{g}(y) \triangleq (g(y) - g(y-b))/b$, $x = y$, and b is a small positive number, obtaining $K + G_n(x+a) - G_n(x) - a\hat{G}_n(x) \ge 0$ for $0 < a \le B$.

Proof. Let x_0 be any point at which it is optimal to order, say, a units, $0 < a \le B$. $G_n(y)$ is a nonincreasing function for $y \le x_0$ (Lemma 2). This result implies that there must exist an upper bound on inventory level beyond which no ordering is optimal. Otherwise $G_n(y)$ would be a nonincreasing function for all y, which contradicts Lemma 1. □

We next introduce "(K, B)-convexity 2" (KBC2) for a function g (Shaoxiang, 2004).

Definition 4. Let $K \ge 0$, $B \ge 0$, g is KBC2 if it satisfies

$$\big(K + g(x+a) - g(x)\big)/a \ge \big(K + g(y) - g(y-B)\big)/B$$

for $0 < a \le B$ and $y \le x$.

Example 2. In Fig. 2 we plot $C_n(y)$ and illustrate the concept of KBC2 for our numerical example.

Definition 5. g is (K, B)-convex if it satisfies KBC1 and KBC2.

Theorem 1. *$G_n(x)$ and $C_n(x)$ are (K, B)-convex.*

Proof. Proofs are available in Gallego & Scheller-Wolf (2000), Shaoxiang (2004). □

Originally in Shaoxiang & Lambrecht (1996), and then by introducing the concept of KBC2 in Shaoxiang (2004), Shaoxiang & Lambrecht established existence of a level $Y \triangleq s_m$ beyond which it is not optimal to order, and of another level $X \triangleq Y - B$ below which it is optimal to order up to capacity. The optimal policy therefore features a so-called "$X - Y$ band" structure.

Lemma 4. *If G_n is KBC2, it is optimal to order up to capacity at any $y \le s_m - B$.*

Proof. Let x_0 be any point at which it is optimal to order something. By KBC2,

$$0 > \big(K + G_n(x_0 + a) - G_n(x_0)\big)/a \ge \big(K + G_n(y) - G_n(y-B)\big)/B,$$

for all $y \le x_0$. Thus, $0 > K + G_n(y) - G_n(y-B)$, because G_n is nonincreasing for $y \le x_0$. Hence, it is optimal to order up to capacity at any $y \le x_0 - B$. □

Gallego & Scheller-Wolf (2000) further characterised the structure of the optimal policy within Shaoxiang & Lambrecht's $X - Y$

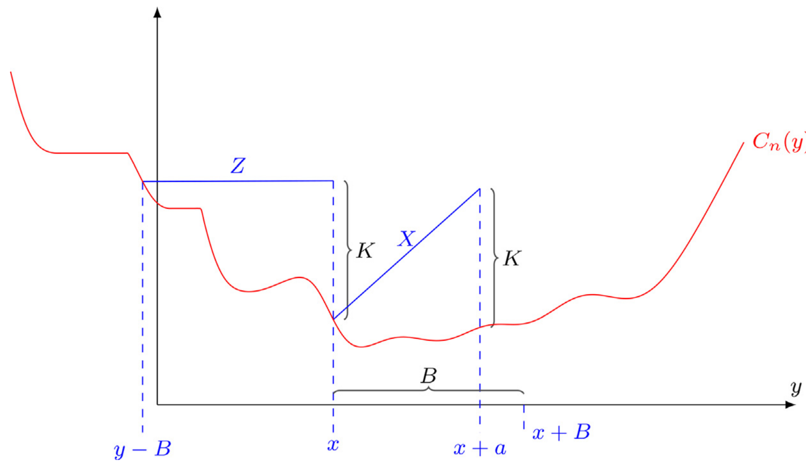

Fig. 2. KBC2 in the context of Example 1, when $B = 65$. For the sake of illustration, we set $x = y$. Intuitively, for all $a \leq B$, the slope of segment X is greater or equal to the slope of segment Z.

band. In particular, they showed that

$$C_n(x) = \begin{cases} G_n^B(x) & x < \min\{s' - B, s\} \\ \alpha \min\{-vx + G_n(x), G_n^B(x)\} + (1-\alpha)G_n^S(x) & \min\{s' - B, s\} \leq x < \max\{s' - B, s\} \\ \min\{-vx + G_n(x), G_n^S(x)\} & \max\{s' - B, s\} \leq x \leq s' \\ -vx + G_n(x) & x > s' \end{cases} \quad (3)$$

where

$$G_n^B(x) \triangleq K - vx + G_n(x + B)$$

$$G_n^S(x) \triangleq K - vx + \min_{x \leq y \leq x+B} G_n(y)$$

$$s \triangleq \inf\left\{x | K + \min_{x \leq y \leq x+B} G_n(y) - G_n(x) \geq 0\right\}$$

$$s' \triangleq \max\left\{x \leq S_m | K + \min_{x \leq y \leq x+B} G_n(y) - G_n(x) \leq 0\right\}$$

and α is an indicator variable that takes value 1 if $s' - s > B$, and 0 otherwise.

Lemma 5. $s' - B < s \leq s'$

Proof. Observe that $s_m = s'$, thus $s \leq s'$; by Lemma 4, it is optimal to order up to capacity at any $x \leq s_m - B$; hence $C_n(x) = G_n^B(x)$ for $x < s' - B$, and $C_n(x) = G_n^S(x)$ at $x = s' - B$; therefore $s > s' - B$. □

By leveraging Lemma 5, it is possible to further simplify Gallego & Scheller-Wolf's structure of the optimal policy as follows. To the best of our knowledge, this simplified policy structure has not been previously discussed in the literature.

Lemma 6.

$$C_n(x) = \begin{cases} G_n^B(x) & x < s_m - B \\ G_n^S(x) & s_m - B \leq x < s \\ \min\{-vx + G_n(x), G_n^S(x)\} & s \leq x \leq s_m \\ -vx + G_n(x) & x > s_m \end{cases} \quad (4)$$

Proof. Observe that $s_m = s'$; because of Lemma 5, it is clear that $s_m - s \leq B$ and $\alpha = 0$. □

5. The modified multi-(s, S) policy

We next introduce the continuous order property, and characterise the structure of the optimal policy for instances for which this property holds.

Definition 6 (Continuous Order Property)**.** Let x_0 be an inventory level at which it is optimal to place an order, C_n is said to have the continuous order property if it is optimal to place an order at y, for all $y < x_0$.

Lemma 7. *If C_n has the continuous order property, $\{x | C_n(x) - (G_n(x) - vx) < 0\}$ is a convex set.*

Proof. If C_n has the continuous order property, in Gallego & Scheller-Wolf's policy $s = s'$; hence for all $x \leq s'$ it is optimal to order, that is $C_n(x) - (G_n(x) - vx) \leq 0$, and for all $x > s'$ it is optimal to not order, that is $C_n(x) - (G_n(x) - vx) > 0$; hence $\{x | C_n(x) - (G_n(x) - vx) < 0\}$ is a convex set. □

In Fig. 3 we illustrate Lemma 7 for our numerical example, which incidentally satisfies the continuous order property.

Consider C_n as defined in Eq. (1), let this function be (K, B)-convex, and assume that the continuous order property holds. When inventory falls below the reorder threshold s_m, defined in Lemma 3, the optimal policy takes the following form: at the beginning of each period, let x be the initial inventory, the order quantity Q is computed as

$$Q = \begin{cases} \min\{S_k - x, B\} & s_{k-1} < x \leq s_k, \\ 0 & x > s_m; \end{cases} \quad (5)$$

where $k = 1, \ldots, m$ and $s_0 = -\infty$. In essence, the policy features m reorder thresholds $s_1 < s_2 < \ldots < s_m$ and associated order-up-to-levels $S_1 < S_2 < \ldots < S_m$; at the beginning of each period, if inventory drops between reorder threshold s_k and reorder threshold s_{k-1}, it is optimal to order a quantity $Q = \min\{S_i - x, B\}$. For convenience, we denote the case $Q = B$ as *saturated ordering*, and the case $0 < Q < B$ as *unsaturated ordering*. We shall name this control rule *modified multi-*(s, S) *policy*, or (s_k, S_k) policy in short. This policy structure was also described in Gallego & Scheller-Wolf (2000, p. 612); however, Gallego & Scheller-Wolf did not establish a relation between the continuous order property and the optimality of this policy.

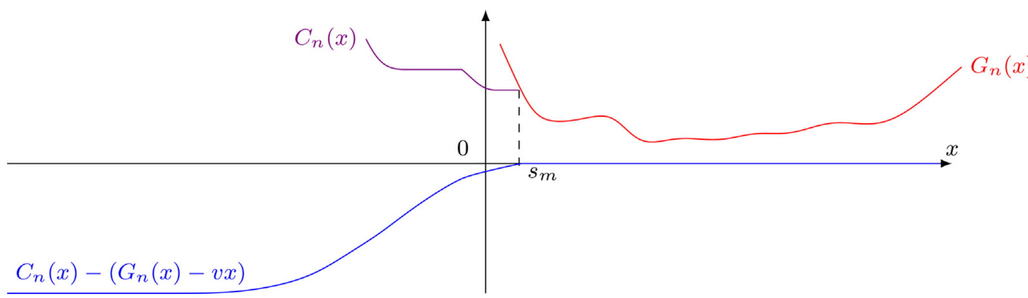

Fig. 3. Lemma 7 in the context of Example 1, when $B = 65$.

Lemma 8. *Consider S_m and s_m as defined in Lemma 3, and let $S^* \triangleq \arg\min_y G_n(y)$,*

(a) $S_m \leq S^*$;
(b) $G_n(S_m) \leq G_n(x)$ *for* $x < S_m$;
(c) $G_n(s_m) > G_n(x)$ *for* $s_m < x \leq S_m$.

Proof. (a) If $s_m \geq S^* - B$, then we must necessarily order up to S^* as no point dominates a global minimum. If $s_m < S^* - B$, then we do not have sufficient capacity to reach S^*, hence the optimum order quantity will be a value $a \leq B$; and from s_m we will order up to a point $S_m \triangleq s_m + a \leq S^*$. (b) Assume, ex absurdo, $G_n(S_m) > G_n(S)$ for some S such that $s_m < S < S_m$; then from s_m it would not be optimal to order up to S_m, which contradicts Lemma 3. (c) Assume, ex absurdo, $G_n(s_m) \leq G_n(s)$ for some s such that $s_m < s \leq S_m$; then from s it would be optimal to order up to S_m, this contradicts the fact that s_m is the maximum inventory level at which it is optimal to place an order (Lemma 3). □

Observe that S_m is not necessarily a minimizer of G_n; this is further illustrated in Appendix A.

By building upon (K, B)-convexity of $G_n(x)$ and $C_n(x)$, and upon the assumption that the continuous order property in Definition 6 holds, we next establish existence of reorder thresholds $s_1 < s_2 < \ldots < s_m$ and associated order-up-to-levels $S_1 < S_2 < \ldots < S_m$ that can be used to control the system according to the optimal ordering policy in Eq. (5).

Definition 7. A function $g : \mathcal{D} \to \mathbb{R}$ defined on a convex subset $\mathcal{D} \in \mathbb{R}$ is *quasiconvex* if, for all $x, y \in \mathcal{D}$ and $\lambda \in [0, 1]$,

$$g\big(\lambda x + (1 - \lambda)y\big) \leq \max\big\{g(x), g(y)\big\}.$$

Definition 8. The *quasiconvex envelope* (QCE) $\tilde{g}$ of a function g on a convex subset $\mathcal{D} \in \mathbb{R}$ is defined as

$$\sup\big\{\tilde{g}(x) | \tilde{g} : \mathbb{R} \to \mathbb{R} \text{ quasiconvex}, \tilde{g}(x) \leq g(x)\ \forall x \in \mathcal{D}\big\}.$$

Lemma 9. *The QCE of G_n on interval (s_m, S_m) is nonincreasing.*

Proof. From Lemma 8b and Lemma 8c, it follows $G_n(s_m) > G_n(x) \geq G_n(S_m)$ for $s_m < x < S_m$. Hence, the QCE of G_n on interval (s_m, S_m) is a nonincreasing function. □

Definition 9. Consider a function $g : \mathbb{R} \to \mathbb{R}$, a point x in the domain of g is a *strict local minimum from the right* if there exists $\delta > 0$ such that $g(y) > g(x)$ for all $y \in (x, x + \delta]$.

Definition 10. Let $[a, b]$, $a \leq b$, in the domain of a function g be a compact interval such that b is a strict local minimum from the right, $g(x) = g(b)$ for all $x \in [a, b]$, and $g(a) = \tilde{g}(a)$; $[a, b]$ *nontrivially belongs to the QCE* $\tilde{g}$ of g, if there exists $\delta > 0$ such that $g(y) > g(x)$ and $g(y) = \tilde{g}(y)$ for all $y \in (a - \delta, a]$; $[a, b]$ *trivially belongs to the QCE* $\tilde{g}$ of g, if there is no $\delta > 0$ such that $g(y) = \tilde{g}(y)$ for all $y \in (a - \delta, a)$.

The concepts introduced in Definition 10 are illustrated in Fig. 4.

Assume G_n is (K, B)-convex; this function must be increasing over some intervals in (s_m, ∞), otherwise $G_n(y)$ would be a nonincreasing function for all y, which contradicts Lemma 1. Let $\widehat{\mathcal{S}}$ be the set of all points a such that interval $[a, b] \in (s_m, S_m)$ *nontrivially belongs to the QCE* of G_n.

Lemma 10. *Let x_0 be any point at which it is optimal to place an order; then either it is the case that $\arg\min_{y \in (x_0, x_0 + B]} G_n(y) = x_0 + B$, or that $\arg\min_{y \in (x_0, x_0 + B]} G_n(y) = \widehat{S}_k$, for some $\widehat{S}_k \in \widehat{\mathcal{S}}$.*

Proof. Assume that at x_0 it is optimal to place an order. Then either the lowest cost will be attained by ordering up to $x_0 + B$, or by ordering up to some local minimum $S \in (x_0, x_0 + B)$. Consider this latter case. We first show that S must belong to the QCE of G_n on (s_m, S_m). Assume, ex absurdo, that S does not belong to the QCE of G_n on (s_m, S_m); since the QCE of G_n is nonincreasing on (s_m, S_m) (Lemma 9), there must exist some other local minimum $\widehat{S}$, such that $s_m < \widehat{S} < S$ and $G_n(\widehat{S}) < G_n(S)$, which contradicts the fact that at x_0 it is optimal to order up to S. Finally, assume interval $[S, b]$, for some $b \geq S$, trivially belongs to the QCE of G_n on (s_m, S_m), this means there must exist some other local minimum $\widehat{S}$, such that $s_m < \widehat{S} < S$ and $G_n(\widehat{S}) = G_n(S)$; hence ordering up to S is no better than ordering up to $\widehat{S}$. □

Lemma 10 is further illustrated in a numerical example presented in Appendix B.

In what follows, we shall assume that $\widehat{\mathcal{S}} \triangleq \{\widehat{S}_1, \widehat{S}_2, \ldots, \widehat{S}_{w-1}\} \subseteq \mathcal{S}$ is an ordered set, so that $s_m < \widehat{S}_1 < \widehat{S}_2 < \ldots < \widehat{S}_{w-1} < S_m$, and $|\widehat{\mathcal{S}}| \geq 0$.

Lemma 11. $G_n(s_m) > G_n(\widehat{S}_1) > G_n(\widehat{S}_2) > \ldots > G_n(\widehat{S}_{w-1}) > G_n(S_m)$.

Proof. Immediately follows from the definition of $\widehat{\mathcal{S}}$ and from Lemma 9. □

Corollary 1. *$\widehat{\mathcal{S}}$ is empty if G_n is quasiconvex on (s_m, S_m).*

Proof. If G_n quasiconvex on (s_m, S_m), from Lemma 9 it follows that G_n is nonincreasing, and hence it does not admit any strict local minima from the right in this interval. □

For the sake of convenience let $\widehat{S}_w \triangleq S_m$.

Lemma 12. *For each $\widehat{S}_k \in \widehat{\mathcal{S}}$ there exists a nonempty set $\{b | \widehat{S}_k < b < \widehat{S}_{k+1}, G_n(b) \geq G_n(\widehat{S}_k)\}$.*

Proof. Consider s_m and S_m as defined in Lemma 3. From Lemma 11, $G_n(\widehat{S}_k) > G_n(\widehat{S}_{k+1})$, for $s_m < \widehat{S}_k < \widehat{S}_{k+1} < S_m$. The result in this lemma follows from the extreme value theorem, since G_n must attain a local maximum at $x^* \in (\widehat{S}_k, \widehat{S}_{k+1})$, such that

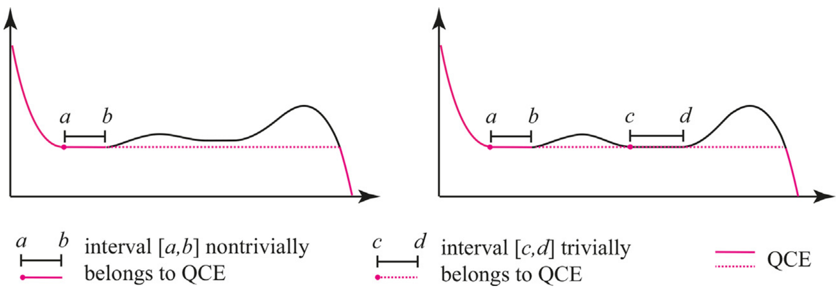

Fig. 4. Graphical illustration of the concepts introduced in Definition 10; note that intervals $[a, b]$ and $[c, d]$ can be degenerate, and reduce to a single point.

$G_n(x^*) > G_n(\widehat{S}_k) > G_n(\widehat{S}_{k+1})$. Note that there cannot be a point $S \in \widehat{\mathcal{S}}$, such that $\widehat{S}_k < S < \widehat{S}_{k+1}$. $\square$

Definition 11. For $k = 1, \ldots, w-1$, $b_k \triangleq \max\{b | \widehat{S}_k < b < \widehat{S}_{k+1}, G_n(b) \geq G_n(\widehat{S}_k)\}$, and $s_k \triangleq b_k - B$; finally, for the sake of convenience, we define $s_0 \triangleq -\infty$.

Lemma 13. $s_{k-1} < \widehat{S}_k - B < s_k$

Proof. This follows from Definition 11. $\square$

Lemma 14. $C_n(x) = -vx + \min\{G_n(x), \min_{x \leq y \leq x+B} G_n(y) + K\}$ *takes the general form*

$$C_n(x) = \begin{cases} K - vx + G_n(x+B) & s_{k-1} < x \leq \widehat{S}_k - B & k = 1, \ldots, w-1 \\ K - vx + G_n(\widehat{S}_k) & \widehat{S}_k - B < x \leq s_k & k = 1, \ldots, w-1 \\ K - vx + G_n(x+B) & s_{w-1} < x \leq S_m - B & \\ K - vx + G_n(S_m) & S_m - B < x \leq s_m & \\ -vx + G_n(x) & x > s_m. & \end{cases}$$

Proof. If at x it is optimal to order $a \triangleq S - x$ units, where $a > 0$, then $C_n(x) = K - vx + G_n(S)$. We consider each interval for x in order.

$x > s_m$: this case follows from Lemma 3, since s_m denotes an inventory level beyond which no ordering is optimal. Conversely, because of the continuous order property, for $x \leq s_m$ it is always optimal to order;

$S_m - B < x \leq s_m$: in this interval, $\arg\min_{y \in (x, x+B]} G_n(y) = S_m$, this follows from the definition of S_m (Lemma 3) and from the fact that G_n is nonincreasing in $(-\infty, s_m]$ (Lemma 2);

$s_{w-1} < x \leq S_m - B$: in this interval, from Definition 11 it follows that $\arg\min_{y \in (x, x+B]} G_n(y) = x + B$, since $G_n(\widehat{S}_k) > G_n(S_m)$, for all $k = 1, \ldots, w-1$; $\widehat{S}_k - B < x \leq s_k$, for all $k = 1, \ldots, w-1$: in this interval, from Definition 11 and from Lemma 13, it follows that $\arg\min_{y \in (x, x+B]} G_n(y) = \widehat{S}_k$;

$s_{k-1} < x \leq \widehat{S}_k - B$, for all $k = 1, \ldots, w-1$: in this interval, from Definition 11 and from Lemma 13, it follows that $\arg\min_{y \in (x, x+B]} G_n(y) = x + B$, since $G_n(\widehat{S}_k) > G_n(\widehat{S}_{k+1})$; finally, note that if $s_0 < x \leq \widehat{S}_1 - B$, then $\arg\min_{y \in (x, x+B]} G_n(y) = x + B$, since G_n is nonincreasing in $(-\infty, \widehat{S}_1]$: in fact, G_n is nonincreasing in $(-\infty, s_m]$ (Lemma 2), $\widehat{\mathcal{S}}$ is an ordered set, hence by definition there exists no point $s_m < S < \widehat{S}_1$ that is a strict local minimum from the right, $G_n(s_m) > G_n(\widehat{S}_1)$ (Lemma 11), and thus G_n is nonincreasing in $(s_m, \widehat{S}_1]$. $\square$

Definition 12. $S_k \triangleq \widehat{S}_k$, for all $k = 1, \ldots, w-1$; and, for convenience, let $m \triangleq w$.

By applying Definition 12, we can rewrite, for $k = 1, \ldots, m$,

$$C_n(x) = \begin{cases} K - vx + G_n(x+B) & s_{k-1} < x \leq S_k - B & \text{(saturated ordering)} \\ K - vx + G_n(S_k) & S_k - B < x \leq s_k & \text{(unsaturated ordering)} \\ -vx + G_n(x) & x > s_m & \text{(no order)} \end{cases} \quad (6)$$

where $S_1, \ldots, S_m$ are the order-up-to-levels and $s_1, \ldots, s_m$ the reorder points of the (s_k, S_k) policy.

Corollary 2. *If the continuous order property holds, the (s_k, S_k) policy generalises the $X - Y$ band discussed in Shaoxiang (2004).*

Proof. In Shaoxiang, $Y \triangleq s_m$ and $X \triangleq Y - B$, where X denotes an inventory level below which it is optimal to order up to capacity; hence, their $X - Y$ band has size B. According to Lemma 14, it is optimal to order up to capacity for all $x \leq S_1 - B$. According to Lemma 8c, $s_m < S_1$, and thus $s_m - B < S_1 - B$. By letting $\bar{X} \triangleq S_1 - B$, we obtain a tighter band $\bar{X} - Y$. $\square$

Corollary 3. *If the continuous order property holds, the (s_k, S_k) policy generalises the policy discussed in Gallego & Scheller-Wolf (2000).*

Proof. Gallego & Scheller-Wolf's optimal policy structure features two thresholds: s and s', where $-\infty \leq s \leq s' \leq S^*$, and $S^* = \arg\min_y G_n(y)$. Clearly, s' is the same threshold we denoted as s_m, and under the assumption that the continuous order property holds, it follows that $s = s'$. Gallego & Scheller-Wolf's optimal policy therefore reduces to

$$C_n(x) = \begin{cases} K - vx + G_n(x+B) & x \leq s_m - B & \text{(saturated)} \\ K + \min_{x \leq y \leq x+B}\{G_n(y) - vx\} & s_m - B < x \leq s_m & \text{(unsaturated or saturated)} \\ -vx + G_n(x) & x > s_m & \text{(no order),} \end{cases}$$

which is equivalent to Shaoxiang's X-Y band. $\square$

Corollary 4. *If the continuous order property holds, the (s_k, S_k) policy generalises the (s, S) policy discussed in Scarf (1960).*

Proof. When $B = \infty$, $S_m - B = -\infty$, and from Lemma 14 it is clear that the optimal policy must feature a single reorder threshold s and order-up-to-level S. $\square$

In Fig. 5 we illustrate $G_n(y)$ for different ordering capacities ($B \in \{35, 65, 71, \infty\}$) imposed for the problem in Example 1.

The optimal (s_k, S_k) ordering policy under ordering capacity constraints for our numerical example is shown in Table 1, and in Fig. 6 for the case in which $B = 65$.

In Appendix C we characterise the structure of the optimal policy for the open numerical example in Shaoxiang & Lambrecht (1996, p. 1015), for which the continuous order property holds.

Table 1
Optimal (s_k, S_k) ordering policy under ordering capacity constraints ($B \in \{35, 65, 71, \infty\}$) for our numerical example; in all cases the continuous order property holds.

Period	B							
	35		65		71		∞	
	s_k	S_k	s_k	S_k	s_k	S_k	s_k	S_k
1	39	68	-11	31	-16	27	15	67
	46	81	14	70	7	71		
					13	84		
2	64	99	-5	51	27	76	28	49
			28	82	34	105		
			35	100				
3	61	96	18	71	12	71	55	109
			55	109	55	109		
4	28	49	28	49	28	49	28	49

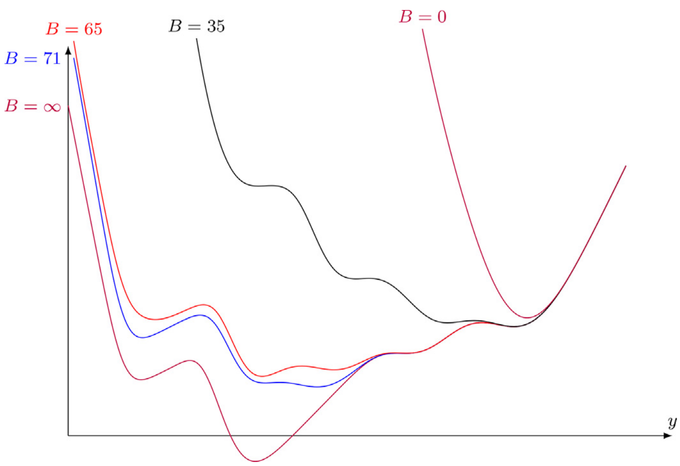

Fig. 5. Numerical example illustrating $G_n(y)$ for different ordering capacities.

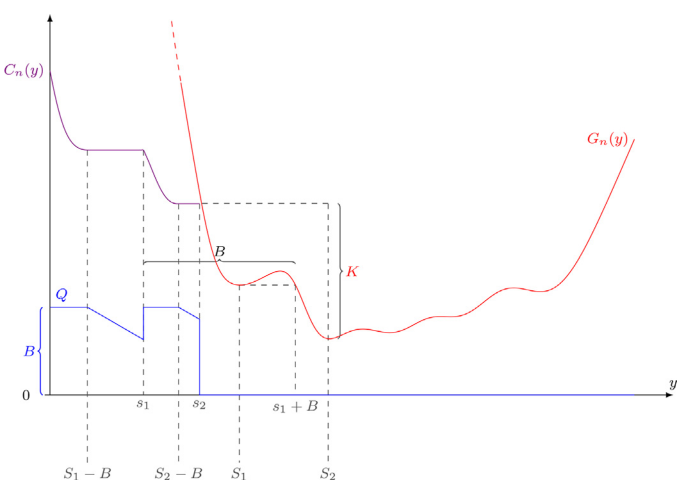

Fig. 6. Optimal ordering policy in period 1 when $B = 65$; note that $G_n(y)$ and Q are not plotted according to the same vertical scale.

6. A counterexample

The continuous order property in Definition 6 has been originally conjectured by Gallego & Scheller-Wolf (2000), and it was later further investigated by Chan & Song (2003). Gallego & Scheller-Wolf (2000) wrote:

> A number of problems still remain. The most vexing is the possibility that under the current structure there could exist a number of intervals [...] where it is optimal to start and stop ordering. An optimal policy with a single continuous interval over which ordering is prescribed, as was found for all of the cases tested [...], is much more analytically appealing. [...] Unfortunately, the proof of this has thus far eluded us. It should be mentioned that it is likewise possible, although we believe it unlikely, that such a structure simply does not exist. To show this requires a problem instance in which the optimal policy has multiple disjoint intervals in which ordering is optimal. Our computational study suggests that this is not the case.

Chan & Song (2003) wrote:

> If our conjecture [the continuous order property] holds, the computational time for obtaining the optimal ordering policy parameters can be further reduced [...]. We can only show that this conjecture holds for a special case where [the capacity] is large enough [...]. It should be an interesting problem for researchers to prove or disprove the conjecture is true for small [capacity].

In the rest of this section, we introduce a numerical instance that violates the continuous order property. To the best of our knowledge, no such instance has ever been discussed in the literature.

Table 2
Probability mass functions of the nonstationary demand d_t considered in Example 3.

d_1	34 (0.018)	159 (0.888)	281 (0.046)	286 (0.048)
d_2	14 (0.028)	223 (0.271)	225 (0.170)	232 (0.531)
d_3	5 (0.041)	64 (0.027)	115 (0.889)	171 (0.043)
d_4	35 (0.069)	48 (0.008)	145 (0.019)	210 (0.904)

Example 3. Consider a planning horizon of $n = 4$ periods and a nonstationary demand d_t distributed in each period t according to the probability mass function shown in Table 2. Other problem parameters are $K = 250$, $B = 41$, $h = 1$ and $p = 26$ and $v = 0$.

In Table 3 we report an extract of the tabulated optimal policy in which the continuous order property is violated (Fig. 7).

Our numerical example confirms that it is possible to construct instances for which it is optimal to start and stop ordering, and that the continuous order property conjectured in Gallego & Scheller-Wolf (2000), Chan & Song (2003) does not hold for the general case of the stochastic inventory problem under order quantity capacity constraints. In Appendix D we discuss the rationale underpinning the generation of our counterexample.

Table 3
An extract of the optimal policy for period $t = 1$ of Example 3, in which the continuous order property is violated.

Starting inventory level	593	594	595	596	597	598	599	600	601
Optimal order quantity	41	40	39	38	37	36	35	34	33
Starting inventory level	602	603	604	605	606	607	608	609	610
Optimal order quantity	0	0	0	0	0	0	0	0	0
Starting inventory level	611	612	613	614	615	616	617	618	619
Optimal order quantity	0	0	0	0	0	41	41	41	0

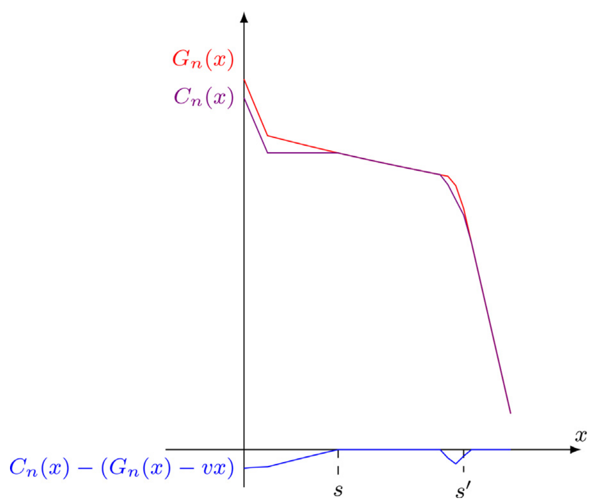

Fig. 7. Lemma 7 does not hold in the context of Example 3: $\{x|C_n(x) - (G_n(x) - vx) < 0\}$ is not a convex set; hence the continuous order property is violated and $s < s'$.

Table 4
Modified (s, S) policy (Wijngaard, 1972) parameters and optimality gaps for Example 1, when $B \in \{35, 65, 71\}$.

Period	B					
	35		65		71	
	s_m	S_m	s_m	S_m	s_m	S_m
1	46	81	14	70	13	84
2	64	99	35	100	34	105
3	61	96	55	109	55	109
4	28	49	28	49	28	49
Optimality gap (%)	0.000		0.123		0.192	

Table 5
Pivot table for our computational study: discrete uniform demand.

		modified (s, S) % optimality gap		modified multi-(s, S) max thresholds	instances
		avg	max		
K	250	0.004	0.122	3	270
	500	0.000	0.050	3	270
	1000	0.000	0.007	3	270
v	2	0.002	0.122	3	270
	5	0.000	0.057	3	270
	10	0.000	0.032	3	270
p	5	0.000	0.057	3	270
	10	0.001	0.122	3	270
	15	0.001	0.115	3	270
B	2.0D	0.000	0.047	2	270
	3.0D	0.001	0.122	3	270
	4.0D	0.000	0.062	3	270
Demand	EMP1	0.002	0.122	3	81
	EMP2	0.003	0.044	3	81
	EMP3	0.003	0.039	3	81
	EMP4	0.003	0.057	3	81
	LC1	0.000	0.000	1	81
	LC2	0.002	0.008	1	81
	RAND	0.006	0.018	2	81
	SIN1	0.000	0.002	3	81
	SIN2	0.000	0.001	1	81
	STA	0.000	0.001	1	81
Overall		0.000	0.122	3	810

7. Computational study

Albeit in the previous section we demonstrated that it is possible to construct instances for which the continuous order property does not hold, we must underscore that these instances are hard to generate, as they do not show up in numerical experiments featuring conventional parameter ranges found in the literature. This is also the reason why the conjecture in Gallego & Scheller-Wolf (2000), Chan & Song (2003) remained open for over twenty years. In this section, we consider an extensive test bed comprising a broad family of demand distributions and problem parameters; our aim is threefold. First, we aim to show empirically that instances that violate the continuous order property do not surface when realistic cost configurations and demand patterns investigated in the literature are considered. In turn, this means that the plans generated by the modified multi-(s, S) ordering policy can therefore be considered, from a practical standpoint, near-optimal. Second, the modified multi-(s, S) ordering policy may feature, in each period, a variable number of thresholds s_k and associated order-up-to-levels S_k. In our computational study, the number of thresholds in a modified multi-(s, S) policy remains less or equal to 6 in each period. Finally, as shown in Table 4, a modified (s, S) policy (Wijngaard, 1972) with parameters (s_m, S_m) appears to perform well in the context of Example 1; in our study we proceed to show that this simple policy, which has been known for decades, also performs well across all instances considered.

7.1. Test bed

In our test bed, the planning horizon comprises $n = 20$ periods. We consider 10 different patterns for the expected value of the demand in each period, as shown in Fig. 8: 2 life cycle patterns (LCY1 and LCY2), 2 sinusoidal patterns (SIN1 and SIN2), 1 stationary pattern (STA), 1 random pattern (RAND), and 4 empirical patterns (EMP1, EMP2, EMP3, EMP4) derived from demand data in Kurawarwala & Matsuo (1996). Further details of expected demand rates in each period are given in Table E.1 in Appendix E.

We consider a broad family of demand distributions commonly used in practice: discrete uniform, geometric, Poisson, normal, lognormal, and gamma. Demands in different periods are assumed to be mutually independent. More specifically, let μ_t denote the mean demand in period t, we investigate a demand that follows a discrete uniform distribution in $[0, 2\mu_t)$; a demand that follows a geometric distribution with mean μ_t; and a demand that follows a Poisson distribution with rate μ_t. Finally, given the coefficient of variation of the demand in each period $c_v = \sigma_t/\mu_t$, where σ_t is the standard deviation of the demand in period t, we consider normal-

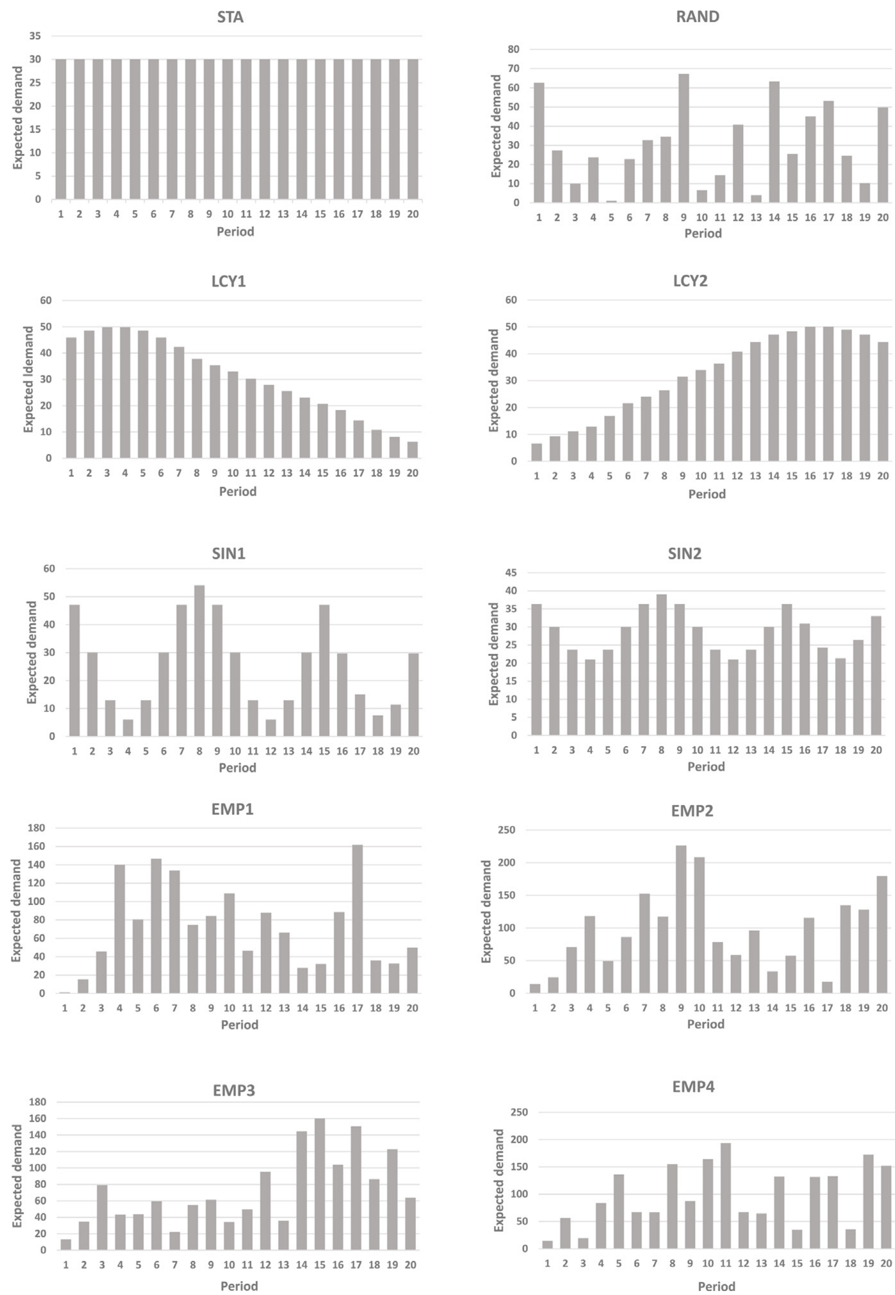

Fig. 8. Demand patterns in our computational study.

, lognormal-, and gamma-distributed demands with mean μ_t and standard deviation σ_t.

Fixed ordering cost K takes values in $\{250, 500, 1000\}$; inventory holding cost h is 1; unit variable ordering cost v takes values in $\{2, 5, 10\}$; unit penalty cost p ranges in $\{5, 10, 15\}$. For the case of normal, lognormal, and gamma distributed demand, the coefficient of variation takes values in $\{0.1, 0.2, 0.3\}$. Let D denote the average demand rate over the whole n periods horizon for a given demand pattern; the maximum order quantity B takes values in $\{\text{round}(2D), \text{round}(3D), \text{round}(4D)\}$, where the round operator rounds the value to the nearest integer.

Since we adopt a full factorial design, we consider 810 instances for discrete uniform, geometric, and Poisson distributed demand, respectively; and 2430 instances for normal, lognormal, and gamma distributed demands, respectively, since in these latter cases we must also consider the three levels of the coefficient of variation. In total, our computational study comprises 9720 instances. Our experimental design is similar to that investigated in a number of existing studies (see e.g. Dural-Selcuk, Rossi, Kilic, & Tarim, 2020; Xiang, Rossi, Martin-Barragan, & Tarim, 2018).

7.2. Results

We run experiments on an Intel(R) Xeon(R) @ 3.5GHz with 16Gb of RAM. The library used in our experiments is `jsdp` (Rossi, 2022b).[4] SDP state space boundaries are fixed — inventory may range in $(-10000, 10000)$ — and in all cases we adopt a unit dis-

[4] The Java code is available at (Rossi, 2018); a self-contained Python code is also available at (Rossi, 2022a).

Table 6
Pivot table for our computational study: geometric demand.

		modified (s, S)		modified multi-(s, S)	
		% optimality gap		max thresholds	instances
		avg	max		
K	250	0.274	0.610	3	270
	500	0.238	0.530	3	270
	1000	0.200	0.427	2	270
v	2	0.284	0.610	3	270
	5	0.235	0.478	3	270
	10	0.193	0.383	3	270
p	5	0.174	0.368	2	270
	10	0.242	0.510	3	270
	15	0.296	0.610	2	270
B	2.0D	0.279	0.610	2	270
	3.0D	0.234	0.495	2	270
	4.0D	0.199	0.416	3	270
Demand	EMP1	0.279	0.596	2	81
	EMP2	0.280	0.602	3	81
	EMP3	0.240	0.488	2	81
	EMP4	0.272	0.610	2	81
	LC1	0.241	0.573	1	81
	LC2	0.205	0.435	1	81
	RAND	0.239	0.501	2	81
	SIN1	0.224	0.490	2	81
	SIN2	0.199	0.451	1	81
	STA	0.196	0.452	1	81
Overall		0.237	0.610	3	810

Table 7
Pivot table for our computational study: Poisson demand.

		modified (s, S)		modified multi-(s, S)	
		% optimality gap		max thresholds	instances
		avg	max		
K	250	0.125	1.918	5	270
	500	0.130	1.583	5	270
	1000	0.029	0.424	5	270
v	2	0.146	1.918	5	270
	5	0.086	0.972	5	270
	10	0.052	0.650	5	270
p	5	0.070	1.048	4	270
	10	0.100	1.623	5	270
	15	0.114	1.918	5	270
B	2.0D	0.103	1.918	4	270
	3.0D	0.100	1.623	4	270
	4.0D	0.081	1.583	5	270
Demand	EMP1	0.204	1.623	4	81
	EMP2	0.176	1.918	4	81
	EMP3	0.181	1.479	5	81
	EMP4	0.248	1.583	5	81
	LC1	0.018	0.154	4	81
	LC2	0.027	0.160	4	81
	RAND	0.043	0.429	4	81
	SIN1	0.016	0.088	3	81
	SIN2	0.017	0.106	4	81
	STA	0.016	0.093	5	81
Overall		0.095	1.918	5	810

cretization, therefore running time for each instance is constant; a continuity correction is introduced for continuous distributions. Monte Carlo simulation runs are determined by targeting an estimation error of 0.01% for the mean estimated at 95% confidence level; we adopt a common random number strategy (Kahn & Marshall, 1953) across all instances.

In Tables 5–10 we present the results of our study for each of the demand distributions under scrutiny. For all instances investigated, a modified multi-(s, S) policy is optimal. Moreover, the maximum number of thresholds observed in any given period never exceeds 6 over the whole test bed. We also report the average and maximum % optimality gap of a modified (s, S) policy with parameters (s_m, S_m) extracted from the SDP tables. This policy is found to be near optimal in our study, since its average % optimality gap is consistently negligible, while the maximum % optimality gap observed never exceeds 2%.

Table 8
Pivot table for our computational study: normal demand.

		modified (s, S)		modified multi-(s, S)	
		% optimality gap		max thresholds	instances
		avg	max		
K	250	0.092	2.006	5	810
	500	0.056	1.435	6	810
	1000	0.015	0.565	6	810
v	2	0.083	2.006	6	810
	5	0.050	0.971	6	810
	10	0.029	0.476	6	810
p	5	0.034	0.893	5	810
	10	0.060	1.597	6	810
	15	0.068	2.006	6	810
B	2.0D	0.050	2.006	5	810
	3.0D	0.068	1.597	5	810
	4.0D	0.045	1.435	6	810
Demand	EMP1	0.104	1.597	4	243
	EMP2	0.088	2.006	4	243
	EMP3	0.092	1.435	6	243
	EMP4	0.120	1.392	5	243
	LC1	0.016	0.357	5	243
	LC2	0.019	0.910	6	243
	RAND	0.040	1.347	5	243
	SIN1	0.024	0.437	5	243
	SIN2	0.024	0.742	5	243
	STA	0.015	0.327	5	243
c_v	0.1	0.111	2.006	6	810
	0.2	0.047	1.237	5	810
	0.3	0.006	0.565	4	810
Overall		0.054	2.006	6	2430

Table 9
Pivot table for our computational study: lognormal demand.

		modified (s, S)		modified multi-(s, S)	
		% optimality gap		max thresholds	instances
		avg	max		
K	250	0.108	1.891	5	810
	500	0.072	1.424	6	810
	1000	0.033	0.579	6	810
v	2	0.099	1.891	6	810
	5	0.067	0.931	6	810
	10	0.047	0.729	6	810
p	5	0.050	0.893	5	810
	10	0.076	1.578	6	810
	15	0.086	1.891	6	810
B	2.0D	0.066	1.891	5	810
	3.0D	0.086	1.578	5	810
	4.0D	0.061	1.424	6	810
Demand	EMP1	0.130	1.578	4	243
	EMP2	0.100	1.891	4	243
	EMP3	0.103	1.424	6	243
	EMP4	0.139	1.285	5	243
	LC1	0.033	0.286	5	243
	LC2	0.035	0.887	6	243
	RAND	0.056	1.262	5	243
	SIN1	0.042	0.600	5	243
	SIN2	0.041	0.695	5	243
	STA	0.030	0.311	5	243
c_v	0.1	0.110	1.891	6	810
	0.2	0.055	0.923	5	810
	0.3	0.048	0.579	4	810
Overall		0.071	1.891	6	2430

Table 10
Pivot table for our computational study: gamma demand.

		modified (s, S) % optimality gap		modified multi-(s, S) max thresholds	instances
		avg	max		
K	250	0.101	1.930	5	810
	500	0.068	1.424	6	810
	1000	0.029	0.570	6	810
v	2	0.094	1.930	6	810
	5	0.062	0.923	6	810
	10	0.043	0.731	6	810
p	5	0.046	0.894	5	810
	10	0.071	1.585	6	810
	15	0.081	1.930	6	810
B	2.0D	0.062	1.930	5	810
	3.0D	0.080	1.585	5	810
	4.0D	0.057	1.424	6	810
Demand	EMP1	0.125	1.585	4	243
	EMP2	0.095	1.930	4	243
	EMP3	0.100	1.424	6	243
	EMP4	0.130	1.312	5	243
	LC1	0.029	0.308	5	243
	LC2	0.032	0.894	6	243
	RAND	0.052	1.290	5	243
	SIN1	0.036	0.421	5	243
	SIN2	0.037	0.708	5	243
	STA	0.027	0.317	5	243
c_v	0.1	0.109	1.930	6	810
	0.2	0.050	1.013	5	810
	0.3	0.040	0.570	4	810
Overall		0.066	1.930	6	2430

8. Conclusions

The periodic review single-item single-stocking location stochastic inventory system under nonstationary demand, complete backorders, a fixed ordering cost component, and order quantity capacity constraints is one of the fundamental problems in inventory management.

A long standing open question in the literature is whether a policy with a single continuous interval over which ordering is prescribed is optimal for this problem. The so-called “continuous order property” conjecture was originally posited by Gallego & Scheller-Wolf (2000), and later also investigated by Chan & Song (2003). To the best of our knowledge, to date this conjecture has never been confirmed or disproved.

In this work, we provided a numerical counterexample that violates the continuous order property. This closes a fundamental and long standing problem in the literature: a policy with a single continuous interval over which ordering is prescribed is not optimal.

Gallego & Scheller-Wolf (2000) provided a partial characterisation of the optimal policy to the problem. In light of the results presented in Shaoxiang (2004), we showed how to simplify the optimal policy structure presented by Gallego & Scheller-Wolf (2000). Gallego & Scheller-Wolf (2000) also briefly sketched the form that an optimal policy would take under moderate values of K. We formalised this discussion and provided a full characterisation of the optimal policy for instances for which the continuous order property holds. In particular, we showed that under this assumption the optimal policy takes the *modified multi-(s, S) form*.

By leveraging an extensive computational study, we showed that instances violating the continuous order property do not surface when realistic cost configurations and demand patterns investigated in the literature are considered. The modified multi-(s, S) ordering policy can therefore be considered near-optimal for the problem under scrutiny. Moreover, we observed that the number of thresholds in a modified multi-(s, S) policy remains less or equal to 6 in each period. Finally, we showed that a well-known heuristic policy, the modified (s, S) policy (Wijngaard, 1972), also performs well across all instances considered.

Since a policy with a single continuous interval over which ordering is prescribed is not optimal in general, future works may focus on establishing what restrictions (if any) to the problem statement, e.g. nature of the demand distribution, may ensure that such a policy is optimal.

Appendix A. Possible scenarios one may observe when inventory hits level s_m

There are two possible cases one may encounter when inventory hits reorder threshold s_m: either we order less than B, or we order the maximum allowed quantity B. We next illustrate these two possible cases via Example 1.

Case 1: The first case ($B = 65$) is shown in Fig. 6. In this case there are $m = 2$ local minima up to (and including) the global minimizer S_m. Let y denote the initial inventory and apply Eq. (5). Since $s_2 + B \geq S_2$, if $s_1 < y < s_2$ we order $x = \min\{S_2 - y, B\}$; if $y < s_1$ we order $x = \min\{S_1 - y, B\}$. Finally, if $y \geq s_2$, we do not order.

Case 2: The second case ($B = 71$) is shown in Fig. A.1.

In this case there are $m = 3$ local minima up to (and including) the global minimizer S^*. Let y denote the initial inventory and apply Eq. (5). Since capacity B is insufficient to reach the global minimizer S^*, if $s_2 < y < s_3$ we order $x = S_3 - s_3 = B$; if $s_1 < y < s_2$ we order $x = \min\{S_2 - y, B\}$; and if $y < s_1$, we order $x = \min\{S_1 - y, B\}$. Finally, if $y \geq s_3$, we do not order.

These two cases exhaust all possible scenarios one may observe when inventory hits level s_m.

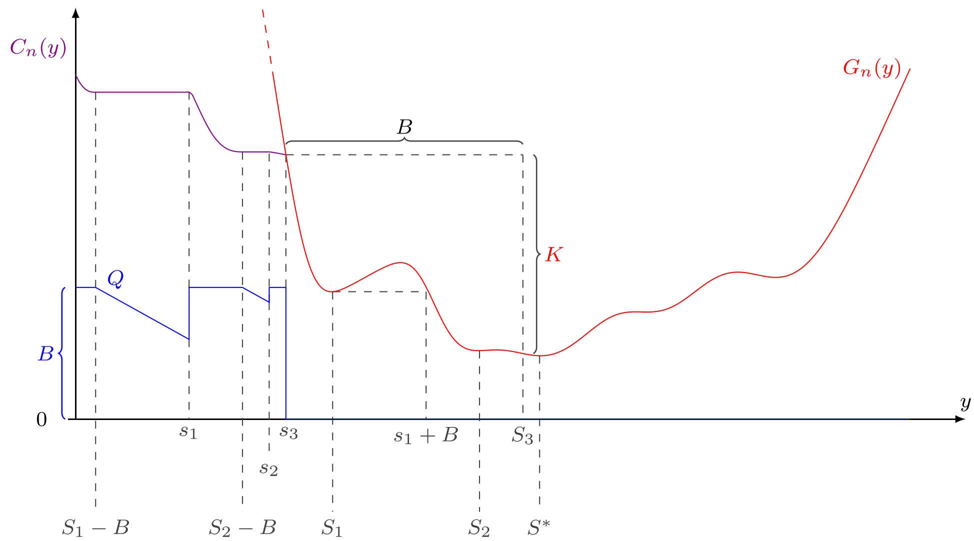

Fig. A.1. Optimal ordering policy in period 1 when $B = 71$; note that $G_n(y)$ and Q are not plotted according to the same vertical scale. If $y > s_3$, it is not convenient to order.

Appendix B. Numerical example illustrating Lemma 10

Example 4. Consider a planning horizon of $n = 12$ periods; a demand d_t distributed in each period $t = 1, \dots, n$ according to a Poisson law with rate $\lambda_t \in \{151, 152, 58, 78, 134, 13, 22, 161, 43, 55, 110, 37\}$; $K = 494$, $v = 0$, $h = 1$, $p = 15$, and $B = 128$.

We focus on period 6, and in Fig. B.1 we plot $G_6(y)$ for an initial inventory $y \in (60, 145)$. It is clear that at any point x_0 in which it is optimal to place an order, if we have sufficient capacity to order beyond b_1, we should do so; however, if we do not have sufficient capacity, then we would never order up to S, as this point is clearly dominated by $\widehat{S}$. Observe that while $\widehat{S}$ belongs to the QCE of G_6 — illustrated as a dashed line where it departs from G_6 — S does not.

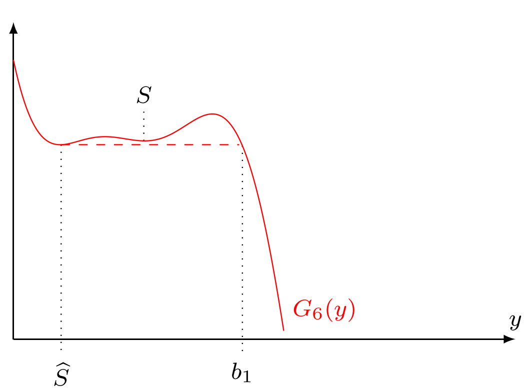

Fig. B.1. Example 4, plot of function $G_6(y)$ for an initial inventory $y \in (60, 145)$; the QCE of G_6, when it departs from G_6, is illustrated as a dashed line. Observe that $\widehat{S}$ belongs to the QCE of G_6, while S does not.

Table C.1
Optimal policy as illustrated in Shaoxiang (2004, p. 417).

Starting inventory level	-3	-2	-1	0	1	2	3	4	5	6	7
Optimal order quantity	9	8	7	9	8	7	9	8	7	0	0

Appendix C. Example from Shaoxiang & Lambrecht (1996)

We hereby illustrate that an (s_k, S_k) ordering policy is optimal for the numerical example originally presented in Shaoxiang & Lambrecht (1996, p. 1015) and also investigated in Shaoxiang (2004) under an infinite horizon.

Example 5. Consider a planning horizon of $n = 20$ periods and a stationary demand d distributed in each period according to the following probability mass function: $\Pr\{d = 6\} = 0.95$ and $\Pr\{d = 7\} = 0.05$. Other problem parameters are $K = 22$, $B = 9$, $h = 1$ and $p = 10$ and $v = 1$; note that, if the planning horizon is sufficiently long, v can be safely ignored. The discount factor is $\alpha = 0.9$.

In Table C.1 we report the tabulated optimal policy as illustrated in Shaoxiang (2004, p. 417).

In Fig. C.1 we plot $G_n(y)$ for an initial inventory $y \in (-5, 50)$ and $n = 20$. The optimal (s_k, S_k) policy is shown in Table C.2; this is equivalent to the policy illustrated in Shaoxiang & Lambrecht (1996, p. 1015) and to the stationary policy tabulated in Shaoxiang (2004, p. 417).

Table C.2
Optimal (s_k, S_k) policy for a generic period t of the example in (Shaoxiang & Lambrecht, 1996).

s_k	S_k
-1	6
2	9
5	12

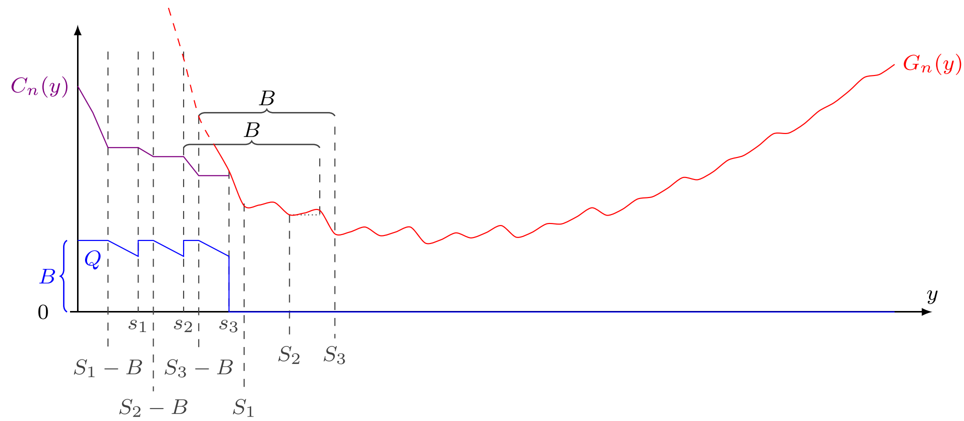

Fig. C.1. Optimal ordering policy for the stationary example in Shaoxiang & Lambrecht (1996); note that $G_n(y)$ and Q are not plotted according to the same vertical scale.

Appendix D. Generating counterexamples to the continuous order property

Generating counterexamples to the continuous order property is not trivial. We believe this is the reason why the continuous order property originally conjectured by Gallego & Scheller-Wolf (2000) has not been so far confirmed or disproved. In this section, we outline the reasoning we followed to generate our counterexample. Our analysis was inspired by the work of Gallego & Toktay (2004).

Lemma 15. *Let f be convex, and S be a minimizer of f, then*

$$g(x) \triangleq \min_{y\in[x,x+B]} f(y) - f(x) = \begin{cases} 0 & S \le x \\ f(S) - f(x) & S - B \le x \le S \\ f(x+B) - f(x) & x \le S - B \end{cases}$$

is nondecreasing.

Proof. Following (Karush, 1959), $g(x)$ is constant for $S \le x$; it is nondecreasing for $S - B \le x \le S$, since $f(S)$ is constant, and f is nonincreasing in this region; finally, it is nondecreasing for $x \le S - B$, since f is convex and hence $f(x+B) - f(x)$ is nondecreasing for all x. □

Consider G_n and C_n as defined in Eqs. (1) and (2), respectively, and let these functions be (K, B)-convex. To show that the continuous order property holds, one must show that $\{x | C_n(x) - (G_n(x) - vx) < 0\}$ is the convex set $(-\infty, s_m)$.

Recall that

$$C_n(x) = \min \left\{ \begin{array}{l} L_n(x) + \int_0^\infty C_{n-1}(x-\xi) f_n(\xi) \mathrm{d}\xi, \\ \min_{x<y\le x+B} \{K + v(y-x) + L_n(y) \\ + \int_0^\infty C_{n-1}(y-\xi) f_n(\xi) \mathrm{d}\xi\} \end{array} \right\},$$

$$G_n(x) = vx + L_n(x) + \int_0^\infty C_{n-1}(x-\xi) f_n(\xi) \mathrm{d}\xi,$$

$$C_n(x) = -vx + \min \left\{ G_n(x), K + \min_{x\le y\le x+B} G_n(y) \right\}.$$

To prove that $\{x | C_n(x) - (G_n(x) - vx) < 0\}$ is a convex set, it is sufficient to show that the function

$$V_n(x) \triangleq C_n(x) - (G_n(x) - vx)$$

is nondecreasing in x for each n. Let $[x]^- \triangleq \min\{0, x\}$, and note that

$$V_n(x) = \left[K + \min_{x\le y\le x+B} G_n(y) - G_n(x) \right]^-.$$

One may want to try and show by induction that $V_n(x)$ is nondecreasing in x for each n. Let $C_0 \triangleq 0$, then

$$V_1(x) = \left[K + \min_{x\le y\le x+B} \{v(y-x) + L_1(y)\} - L_1(x) \right]^-;$$

since the unit cost v is linear, and L_1 is convex, from Lemma 15 it follows that $V_1(x)$ is nondecreasing. Given this base case, we may then assume that $V_n(x)$ is nondecreasing in x, and try to show that $V_{n+1}(x)$ is nondecreasing in x.

First, observe that

$$V_{n+1}(x) = \Bigg[K + \min_{x\le y\le x+B} \left(vy + L_{n+1}(y) + \int_0^\infty C_n(y-\xi) f_{n+1}(\xi) \mathrm{d}\xi \right) - \left(vx + L_{n+1}(x) + \int_0^\infty C_n(x-\xi) f_{n+1}(\xi) \mathrm{d}\xi \right) \Bigg]^-.$$

To investigate whether $V_{n+1}(x)$ is nondecreasing, we shall analyse

$$\begin{aligned} K + \min_{x\le y\le x+B} & v(y-x) + C_n(y) - C_n(x) \\ &= \min_{x\le y\le x+B} \left\{ K + G_n(y) - G_n(x) - V_n(x) + V_n(y) \right\}, \end{aligned}$$

since $C_n(x) = V_n(x) + G_n(x) - vx$. Consider s_m as defined in Lemma 3, and recall this value denotes an inventory level beyond which no ordering is optimal. There are three intervals we need to analyse: $x \le s_m - B$, $s_m - B < x \le s_m$, and $x > s_m$. Observe that, from the definition of s_m in Lemma 3, if $x = s_m$, then $K + \min_{x\le y\le x+B} G_n(y) - G_n(x) \le 0$; moreover, by induction hypothesis $V_n(x)$ is assumed nondecreasing, hence $V_n(x) = K + \min_{x\le y\le x+B} G_n(y) - G_n(x)$ for $x \le s_m$.

Let $x \le s_m - B$; in this interval $V_n(x) = K + \min_{x\le y\le x+B} G_n(y) - G_n(x)$, thus

$$\begin{aligned} &\min_{x\le y\le x+B} \left\{ K + G_n(y) - G_n(x) - V_n(x) + V_n(y) \right\} \\ &= \min_{x\le y\le x+B} \left\{ K - (\min_{x\le z\le x+B} G_n(z)) + (\min_{y\le w\le y+B} G_n(w)) \right\} \\ &= \min_{x\le y\le x+B} \left\{ K - G_n(x+B) + \min_{y\le w\le y+B} G_n(w) \right\} \\ &= K + \min_{x\le y\le x+2B} G_n(y) - G_n(x+B), \\ &= K + \min_{x+B\le y\le x+2B} G_n(y) - G_n(x+B), \end{aligned} \tag{D.1}$$

because $G_n(x)$ is assumed (K, B)-convex and, by Lemma 2, it is nonincreasing for $x \le s_m$, therefore it is also nonincreasing in $(x, x+B)$, since $x \le s_m - B$.

Let $s_m - B < x \le s_m$, in this interval $V_n(x) = K + \min_{x\le z\le x+B} G_n(z) - G_n(x)$, thus

$$\begin{aligned} &\min_{x\le y\le x+B} \left\{ K + G_n(y) - G_n(x) - V_n(x) + V_n(y) \right\} \\ &= \min_{x\le y\le x+B} \left\{ G_n(y) + V_n(y) \right\} - \min_{x\le z\le x+B} G_n(z) \end{aligned}$$

Table E.1
Expected demand values for demand patterns in our test bed.

Pattern	Expected demand values																			
STA	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30
LC1	46	49	50	50	49	46	42	38	35	33	30	28	26	23	21	18	14	11	8	6
LC2	7	9	11	13	17	22	24	26	32	34	36	41	44	47	48	50	50	49	47	44
SIN1	47	30	13	6	13	30	47	54	47	30	13	6	13	30	47	30	15	8	11	30
SIN2	36	30	24	21	24	30	36	39	36	30	24	21	24	30	36	31	24	21	26	33
RAND	63	27	10	24	1	23	33	35	67	7	14	41	4	63	26	45	53	25	10	50
EMP1	5	15	46	140	80	147	134	74	84	109	47	88	66	28	32	89	162	36	32	50
EMP2	14	24	71	118	49	86	152	117	226	208	78	59	96	33	57	116	18	135	128	180
EMP3	13	35	79	43	44	59	22	55	61	34	50	95	36	145	160	104	151	86	123	64
EMP4	15	56	19	84	136	67	67	155	87	164	194	67	65	132	35	131	133	36	173	152
Period	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

$$= \min_{x\le y\le x+B}\left\{C_n(y)+\nu y\right\} - \min_{x\le z\le x+B} G_n(z)$$
$$= \min_{s_m<y\le x+B}\left\{C_n(y)+\nu y\right\} - \min_{s_m<z\le x+B} G_n(z) = 0, \quad \text{(D.2)}$$

because $G_n(x)$ and $C_n(x)$ are assumed (K,B)-convex and, by Lemma 2, they are nonincreasing for $x\le s_m$; and since no ordering is optimal beyond s_m, then $\min_{s_m<y\le x+B} C_n(y)+\nu y = \min_{s_m<z\le x+B} G_n(z)$.

Let $x>s_m$, in this interval $K+\min_{x\le y\le x+B} G_n(y)-G_n(x)>0$, hence $V_n(x)=0$, $V_n(y)=0$, and

$$\min_{x\le y\le x+B}\left\{K+G_n(y)-G_n(x)-V_n(x)+V_n(y)\right\}$$
$$= K+\min_{x\le y\le x+B} G_n(y)-G_n(x)>0. \quad \text{(D.3)}$$

Equipped with Eqs. (D.1), (D.2), and (D.3) for the intervals we considered, it is immediate to see that

$$\left[K+\min_{x\le y\le x+B}\nu(y-x)+C_n(y)-C_n(x)\right]^- = \begin{cases} V_n(x+B) & x\le s_m-B \\ 0 & s_m-B<x\le s_m \\ 0 & x>s_m \end{cases}$$

is nondecreasing. However, it is not possible to determine if $[K+\min_{x\le y\le x+B}\nu(y-x)+\int_0^\infty (C_n(y-\xi)-C_n(x-\xi))f_{n+1}(\xi)\mathrm{d}\xi]^-$ is nondecreasing; and reintroducing term $\min_{x\le y\le x+B} L_{n+1}(y)-L_{n+1}(x)$ only worsens the matter. But because of the behavior of $\left[K+\min_{x\le y\le x+B}\nu(y-x)+C_n(y)-C_n(x)\right]^-$ in intervals $s_m-B<x\le s_m$ and $x\le s_m-B$, one may observe that a $V_{n+1}(x)$ function featuring some decreasing regions may be produced by the convolution $\int_0^\infty (C_n(y-\xi)-C_n(x-\xi))f_{n+1}(\xi)\mathrm{d}\xi$, provided demand is sufficiently "lumpy." In other words, the instance must feature demand whose probability mass function features some values larger than B possessing non negligible probability mass. A demand that is so structured may ensure that the convolution "bends" sufficiently $V_{n+1}(x)$ beyond s_m so that it turns negative.

On the basis of this observation, we have generated several random instances as follows. The fixed ordering cost is a randomly generated value uniformly distributed between 1 and 500; holding cost is 1; penalty cost is a randomly generated value uniformly distributed between 1 and 30; the ordering capacity is a randomly generated value uniformly distributed between 20 and 200; demand distribution in each period is obtained as follows: the probability mass function comprises only four values in the support, one of these values must fall below the given order capacity, the other three values must fall above, and be smaller or equal to 300; probability masses are then allocated uniformly to each of these values. The Java code to generate instances that violate the continuous order property is available on http://gwr3n.github.io/jsdp/.[5]

[5] File https://github.com/gwr3n/jsdp/blob/master/jsdp/src/main/java/jsdp/app/standalone/stochastic/capacitated/CapacitatedStochasticLotSizingFast.java

Appendix E. Expected demand values in our test bed

Expected demand values for demand patterns in our test bed are shown in Table E.1.

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