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#### Abstract

By a delicate analysis for the Stein's equation associated to the  $\alpha$ -stable law approximation with  $\alpha \in (0, 2)$ , we prove a quantitative stable central limit theorem in Wasserstein type distance, which generalizes the results in the series of work [9, 10, 21] from the univariate case to the multiple variate case. From an explicit computation for Pareto's distribution, we see that the rate of our approximation is sharp. The analysis of the Stein's equation is new and has independent interest.

**Keywords:** multivariate  $\alpha$ -stable approximation, Stein's method, generalized central limit theorem, rate of convergence, Wasserstein(-type) distance, fractional Laplacian

#### 1 Introduction

This paper is concerned with the multivariate stable approximation by Stein's method. A probability measure  $\pi$  on  $\mathbb{R}^d$  with  $d \geq 2$  is strictly stable if, for any a > 0, there is b > 0 such that  $\widehat{\pi}(\lambda)^a = \widehat{\pi}(b\lambda), \lambda \in \mathbb{R}^d$ , where  $\widehat{\pi}$  is the characteristic function of  $\pi$ . For strictly stable measures, a and b have to satisfy the relation  $b = a^{1/\alpha}$  where  $\alpha \in (0,2)$  is the stability parameter. A strictly  $\alpha$ -stable measure  $\pi$  is characterised by a finite non-zero spectral measure  $\nu$  on the sphere  $\mathbb{S}^{d-1}$  and, in and only in the case  $\alpha = 1$ , a vector  $\gamma \in \mathbb{R}^d$ , see [18, Remark 14.6]. Our working assumption in the case  $\alpha = 1$  is that  $\gamma = 0$  and the center of mass of  $\nu$  vanishes, namely,  $\int_{\mathbb{S}^{d-1}} \theta \nu(d\theta) = 0$ . The first condition is artificial and the second is equivalent to strict stability of  $\pi$  in the case  $\alpha = 1$ . Under the condition  $\gamma = 0$ , we have a unified representation for any  $\alpha \in (0, 2)$ ,

$$\widehat{\pi}(\lambda) = \exp\left[-\int_{\mathbb{S}^{d-1}} \nu(d\theta) \int_0^\infty (e^{i\langle \lambda, r\theta \rangle} - 1 - i\langle \lambda, r\theta \rangle k_\alpha(r)) \frac{dr}{r^{1+\alpha}}\right], \quad \lambda \in \mathbb{R}^d,$$
(1)

where  $k_{\alpha}(r) = \mathbf{1}_{\alpha=1,r\in(0,1]} + \mathbf{1}_{\alpha\in(1,2)}$ . The family of strictly stable laws is therefore as rich as the family of finite measures on  $\mathbb{S}^{d-1}$ . From now on, let  $\psi$  denote the integral in the exponent of (1) and call it the *characteristic exponent*.

The spectral measure  $\nu$  plays a crucial role in the study of multivariate stable laws. The distributional properties of  $\pi$  change dramatically from one type of  $\nu$  to another. For instance, if  $\nu$  is the uniform probability measure on  $\mathbb{S}^{d-1}$ , then  $\psi(\lambda) = \sigma |\lambda|^{\alpha}$  with  $\sigma > 0$  so that  $\pi$  is rotationally invariant. Hereafter, |a| denotes the Euclidean norm of  $a \in \mathbb{R}^d$ . Another example is when  $\nu = \sum_{i=1}^d \delta_{e_i} + \delta_{-e_i}$  where  $\delta$  denotes the Dirac mass at some point and  $\{e_i, 1 \leq i \leq d\}$  is the canonical basis of  $\mathbb{R}^d$ , then  $\psi(\lambda) = \sum_{i=1}^d \sigma_i |\lambda_i|^{\alpha}$  for some  $\sigma_i > 0$  so that the marginal distributions of  $\pi$  are independent one-dimensional symmetric stable laws. A third type of example is when  $\nu$  is a fractal measure on  $\mathbb{S}^{d-1}$ , then  $\pi$  can be wildly anisotropic with correlated marginals. In this paper, we shall consider not only each of the aforementioned types of  $\nu$ , but also mixtures of these types.

To assess convergence rates of a sequence towards a multivariate stable law, we use Stein's method – a vast range of ideas and tools that allow one to study the proximity between a probability measure and a target distribution. The scope of the method has been considerably extended since Stein [19] proposed his elegant approach for normal approximation. In particular, Barbour

[4] devised the generator approach which is applicable to target distributions that can be realized as the stationary distribution of a "nice" Markov process. Barbour's approach is the one adopted in this paper and it takes the following steps. First, one constructs a Markov process  $(X_t)_{t\geq 0}$  with infinitesimal generator  $\mathcal{A}$  that converges in distribution to  $\pi$  as  $t\to\infty$  for any initial condition  $X_0=x\in\mathbb{R}^d$ . Second, one considers Stein's equation (or Poisson equation in the PDE literature)

$$\mathcal{A}f(x) = h(x) - \pi(h) \tag{2}$$

with  $h \in L^1(\pi)$  and  $\pi(h) := \int h(x)\pi(dx)$ . By exploiting properties of the transition semigroup  $(Q_t)_{t\geq 0}$  determined by  $\mathcal{A}$ , in particular  $Q_0h = h$ ,  $Q_\infty h = \int h(x)\pi(dx)$  and the relation  $\frac{d}{dt}Q_t = \mathcal{A}Q_t$ , one argues that

$$f_h(x) := -\int_0^\infty Q_t (h(x) - \pi(h)) dt \tag{3}$$

is in the domain of  $\mathcal{A}$  and solves (2). Third, one uses the integral form (3) and properties of  $(Q_t)_{t\geq 0}$  to derive regularity estimates for the solution (3). To see why these steps lead to an upper bound for the distance between an arbitrary distribution and  $\pi$ , let Z denote a strictly stable random vector with distribution  $\pi$ , for any  $\mathbb{R}^d$ -valued random vector F, one has

$$\mathbb{E}[h(F)] - \mathbb{E}[h(Z)] = \mathbb{E}[\mathcal{A}f_h(F)].$$

Ranging h in a class of functions that is large enough to guarantee convergence in distribution, and using the regularity estimates of (3) obtained in the third step, together with the explicit form of A, one would obtain an upper bound for a certain distance between F and Z.

Carrying out rigorously each of the aforementioned steps and claims in the context of stable approximation is a non-trivial task. In dimension one, Xu [21] considered the case of symmetric  $\alpha$ -stable law with  $\alpha>1$ . The approach of [21] was then generalized in [9] to asymmetric  $\alpha$ -stable law with  $\alpha>1$ , and in [2] to a class of infinitely divisible distributions with finite first moment. Later, Chen et al. [10] considered non-integrable  $\alpha$ -stable approximation ( $\alpha\leq 1$ ). In higher dimensions, Arras and Houdré [3, Theorem 4.2] executed the aforementioned second step (construction of the solution to Stein's equation) for a class of self-decomposable distributions which includes multivariate stable laws. However, regularity estimates of the solution are obtained only when test functions have their 0-th, first and second partial derivatives bounded by 1. Therefore, the results in [3] cannot be used to derive bounds for multivariate stable approximation in Wasserstein(-type) distance that we address in this paper.

The main contribution of this paper is the regularity estimates for the solution to Stein's equation in the context of multivariate stable approximation and *Lipschitz(-type)* test functions, which in turn allows to obtain Wasserstein bounds. Such bounds were not available in previous work. Our

approach relies on delicate density estimates of multivariate strictly stable laws. Recent advances on heat kernel estimates of anisotropic non-local operators e.g. [8, 11, 12, 20] allow us to handle a diversified class of spectral measures. Since real life high dimensional data are often anisotropic (see [14, 15, 17] and the references therein), the rich class of spectral measures that we consider would widen the applicability of our results. In terms of application, we provide the rate of convergence for the classical multivariate stable limit theorem.

The rest of this paper is organized as follows. After introducing the Markov process converging to  $\pi$ , we construct a solution to Stein's equation (Proposition 1), present the regularity estimates for the solution (Theorem 3) and obtain Wasserstein bounds for multivariate stable approximation (Theorem 5). Theorem 3 is proved in Section 3 and Theorem 5 is proved in Section 4. Example is given in Section 5.

# 2 Preliminaries and statement of the main results

#### 2.1 Ornstein-Uhlenbeck type processes

The Markov process we construct in the first step of Barbour's program is the so-called Ornstein-Uhlenbeck type process which is a simple stochastic differential equation (SDE) driven by stable Lévy processes. We refer the reader to Applebaum [1] for background on stochastic calculus of Lévy processes, and Sato [18] for general facts about Lévy processes.

Let  $(Z_t)_{t\geq 0}$  be a strictly stable Lévy process, a process with independent and stationary increments having marginal  $Z_1$  distributed as  $\pi$ , given by (1). Consider the SDE

$$\begin{cases} X_t = -\frac{1}{\alpha} \int_0^t X_s ds + Z_t \\ X_0 = x \end{cases} , \tag{4}$$

Such an equation can be solved explicitly

$$X_t^x = xe^{-\frac{t}{\alpha}} + \int_0^t e^{-\frac{t-s}{\alpha}} dZ_s, \tag{5}$$

see [18, p.105], and provides an interpolation between any Dirac mass and  $\pi$ . This follows from the fact that  $(X_t^x)_{t\geq 0}$  is a scaled and time-changed Lévy process, i.e.

$$X_t^x \stackrel{d}{=} xe^{-\frac{t}{\alpha}} + e^{-\frac{t}{\alpha}} Z_{e^t - 1} \stackrel{d}{=} xe^{-\frac{t}{\alpha}} + Z_{1 - e^{-t}}, \tag{6}$$

see [10, Section 2.3], where  $\stackrel{d}{=}$  denotes equality in distribution. For the second equality we have used the self-similarity of the process  $(Z_t)_{t\geq 0}$ , namely  $Z_{ct} \stackrel{d}{=}$ 

 $c^{1/\alpha}Z_t$  in distribution for any c, t > 0. One sees that as  $t \to \infty$ ,  $X_t^x$  converges in distribution to  $Z_1 \sim \pi$ . For another proof of the latter fact, one may check the condition of a general result [18, Th. 17.5] for self-decomposable distributions.

An application of Itô's formula for semimartingales with jumps to  $(X_t^x)_{t\geq 0}$  shows that (see [1, Chapter 6] for details) the generator of X is

$$\mathcal{A}^{\alpha,\nu}f(x) := \mathcal{L}^{\alpha,\nu}f(x) - \frac{1}{\alpha}\langle x, \nabla f(x) \rangle, \tag{7}$$

where recalling the definition of  $k_{\alpha}(r)$  (1),

$$\mathcal{L}^{\alpha,\nu}f(x) = d_{\alpha} \int_{\mathbb{S}^{d-1}} \nu(d\theta) \int_{0}^{\infty} (f(x+r\theta) - f(x) - k_{\alpha}(r)\langle r\theta, \nabla f(x) \rangle) \frac{dr}{r^{1+\alpha}}.$$
(8)

Here  $d_{\alpha}^{-1} = \int_0^{\infty} (1 - \cos y) y^{-1-\alpha} dy = \alpha^{-1} \Gamma(1-\alpha) \cos \frac{\pi \alpha}{2}$ ,  $\alpha \in (0,2) \setminus \{1\}$  with  $d_1 = \lim_{\alpha \to 1} d_{\alpha} = 2/\pi$ , and  $\nu$  is normalized to have total mass 1.

#### 2.2 Solving Stein's equation

Now one can write down Stein's equation associated with the multivariate stable distribution  $\pi$  as follows

$$\mathcal{A}^{\alpha,\nu}f(x) = h(x) - \pi(h),\tag{9}$$

where  $h \in L^1(\pi)$ . In view of obtaining bounds in Wasserstein distance, we consider h belonging to the space  $\operatorname{Lip}_1$  of Lipschitz continuous functions with Lipschitz constant at most 1. It is standard that  $\operatorname{Lip}_1 \subset L^1(\pi)$  if  $\alpha > 1$ , while  $\operatorname{Lip}_1 \cap L^1(\pi)$  is a strict subset of  $\operatorname{Lip}_1$  if  $\alpha \leq 1$ . Whenever  $\alpha \leq 1$ , we let  $0 < \beta < \alpha$  and consider  $h \in \mathcal{H}_\beta := \operatorname{Lip}_1 \cap \operatorname{H\"ol}(\beta, 1)$  where  $\operatorname{H\"ol}(\beta, 1)$  is the class of H\"older continuous functions of order  $\beta$  and H\"older constant at most 1. Namely  $h \in \mathcal{H}_\beta$  means

$$|h(x) - h(y)| \le |x - y| \wedge |x - y|^{\beta}.$$

The Lipschitz continuity imposes smoothness and the Hölder condition imposes global growth rate of the function h which is crucial to make sense of (9) in the case  $\alpha \leq 1$  thanks to the simple inclusion  $\text{H\"ol}(\beta,1) \subset L^1(\pi)$ .

We construct a solution to Stein's equation by using the process  $(X_t^x)_{t\geq 0}$ , as described in the introduction. Denote by  $p(t,x) := p_t(x)$  the density of the driving process  $(Z_t)_{t\geq 0}$  in (4). Write  $p(x) := p_1(x)$ . By (6), one sees that

$$q(t, x, y) = p_{1-e^{-t}}(y - e^{-t/\alpha}x) = s(t)^{-d/\alpha}p(s(t)^{-1/\alpha}(y - e^{-t/\alpha}x)),$$
 (10)

where  $y \mapsto q(t, x, y)$  is the density of  $X_t^x$ ,  $s(t) = 1 - e^{-t}$  and we used the self-similarity of  $(Z_t)_{t \geq 0}$  in the second equality.

**Proposition 1** (Solution to Stein's equation) Suppose  $h \in \text{Lip}_1$  if  $\alpha > 1$  and  $h \in \mathcal{H}_{\beta}$  for some  $\beta < \alpha$  if  $\alpha \leq 1$ . Set

$$f(x) := -\int_{0}^{\infty} \mathbb{E} \left[ h \left( X_{t}^{x} \right) - \pi(h) \right] dt,$$

$$= -\int_{0}^{\infty} \int p_{1-e^{-t}} (y - e^{-\frac{t}{\alpha}} x) (h(y) - \pi(h)) dy dt$$

$$= -\int_{0}^{\infty} \int_{\mathbb{R}^{d}} p(y) \left[ h ((1 - e^{-t})^{1/\alpha} y + e^{-t/\alpha} x) - h(y) \right] dy dt. \tag{11}$$

Then  $\mathcal{A}^{\alpha,\nu}f$  is well-defined and f solves Stein's equation (9).

The proof of this Proposition somewhat standard in view of recent developments [9, 10, 21] on stable approximation with Lipschitz(-type) test functions, we give a proof in the Appendix for the sake of completeness.

Remark 1(i) Note that the last two identities follow from (10) and a change of variables. We end this subsection by verifying that (11) is well-defined. When  $\alpha \in (1, 2)$ , this is obvious. When  $\alpha \in (0, 1]$ , since  $h \in \mathcal{H}_{\beta}$ , we have

$$\begin{aligned} & \left| h((1 - e^{-t})^{1/\alpha} y + e^{-t/\alpha} x) - h(y) \right| \\ & \leq e^{-t/\alpha} |x| + e^{-t\beta/\alpha} |x|^{\beta} + |y(1 - (1 - e^{-t})^{1/\alpha})| \wedge |y(1 - (1 - e^{-t})^{1/\alpha})|^{\beta}, \end{aligned}$$

which is integrable with respect to  $1_{t>0}dt \otimes p(y)dy$ , as desired.

(ii) Throughout the paper, we often use the facts that  $\mathbb{E}|Z_1|^{\beta} < \infty$  for any  $\beta \in (0, \alpha)$ , where  $Z_t$  is the strictly  $\alpha$ -stable Lévy process. For the convenience of readers, we obtain the moment estimate of  $Z_1$  in Lemma 17 in Appendix A.

# 2.3 Probability metric

In the case  $\alpha > 1$ , we shall consider multivariate stable approximation in the classical Wasserstein distance given by

$$d_W(\mu_1, \mu_2) = \sup_{h \in \text{Lip}_1} |\mu_1(h) - \mu_2(h)|.$$

In the case  $\alpha \leq 1$ , we shall consider Wasserstein-type distance given by

$$d_{W_{\beta}}(\mu_1, \mu_2) = \sup_{h \in \mathcal{H}_{\beta}} |\mu_1(h) - \mu_2(h)|, \quad 0 < \beta < \alpha.$$

## 2.4 Spectral measures

Obtaining sharp density estimates for general multivariate stable law is very sensitive to the form of the spectral measure. The seminal work of Watanabe

[20] seems to be the first which identifies the best, worse and a range of different rates of decay of stable densities in relation to the spectral measures.

For our purpose, we need not only density estimates but also gradient and fractional derivative (the operator  $\mathcal{L}^{\alpha,\nu}$ ) estimates of stable densities. Our setting is that of Bogdan *et al.* [8]. In fact, [8] addressed Lévy-type operators with a jump kernel equivalent to the Lévy measure of a general multivariate stable law. The condition on the spectral measure therein is that the corresponding Lévy measure is a  $\gamma$ -measure. Let  $\mu$  denote the Lévy measure of a multivariate strictly stable distribution, i.e., for measurable  $A \subset \mathbb{R}^d$ ,

$$\mu(A) := \int_{\mathbb{S}^{d-1}} \nu(d\theta) \int_0^\infty \mathbf{1}_A(r\theta) \frac{dr}{r^{1+\alpha}}.$$

**Definition 1** We say  $\mu$  is a  $\gamma$ -measure for some  $\gamma \geq 0$  if there exists a finite c > 0 such that for all  $x \in \mathbb{S}^{d-1}$  and 0 < r < 1/2, one has

$$\mu(B(x,r)) \le cr^{\gamma}$$
.

Here B(x,r) is the Euclidean ball with center x and radius r > 0.

It is easy to see that the Lévy measure of stable laws are  $\gamma$ -measures with  $\gamma \in [1, d]$ . The following examples show that this is a rather general setting.

Example 1 Suppose that  $\nu$  is absolutely continuous with respect to uniform measure on  $\mathbb{S}^{d-1}$  with density  $\kappa$  bounded from above and below. Then for measurable  $A \subset \mathbb{R}^d$ ,

$$\mu(A) = \int_A \frac{\kappa(x/|x|)dx}{|x|^{d+\alpha}}.$$

This is the setting (restricting to the Lévy case) of Chen and Zhang [11, 12] where the densities, their gradient and fractional derivatives are estimated. One readily checks that  $\mu$  is a d-measure.

Example 2 Suppose that  $\nu = \sum_{i=1}^k a_i \delta_{x_i}$  where  $\delta$  is a Dirac mass and  $\{x_1, ..., x_k\} \subset \mathbb{S}^{d-1}$ . Then  $\mu$  is a d-measure. In the particular case where  $x_i$ 's are the canonical basis and k = d, we have a multivariate stable law with independent marginals. The desired estimates follows from their counterparts in dimension 1.

Example 3 Let  $1 \le \gamma < d$ . Suppose that  $\nu$  is supported on  $E \subset \mathbb{S}^{d-1}$  which is Ahlfors regular of order  $\gamma - 1$ , namely, there exists c such that

$$\mathcal{H}^{\gamma-1}(B(x,r)\cap E) \le cr^{\gamma-1}, \quad x \in E,$$

where  $\mathcal{H}^s$  is the s-dimensional Hausdorff measure. Then  $\nu$  is  $\gamma$ -measure whenever it is absolutely continuous with respect to  $(\gamma - 1)$ -dimensional Hausdorff measure with bounded density. In [8] only density and gradient estimates were obtained. Computing fractional derivatives of stable densities requires a little more work, which we do in this paper following the arguments of [11, 12]. It was observed in [20] that, subject to further lower bounds on the  $\nu$ -measure of balls, sharp two-sided

estimates can be obtained for stable densities. Since we are only concerned with upper bounds, we do not need to impose these additional assumptions. For aspects of fractal measures, we refer to [16].

Example 4 Any linear combination of  $\nu$  in the previous three examples is again a  $\gamma$ -measure with an appropriate  $\gamma$ .

We gather estimates about stable densities which are useful for our purpose.

**Lemma 2** Let p(x) be the density of a multivariate strictly  $\alpha$ -stable law with characteristic exponent (1) for all  $\alpha \in (0,2)$  and  $\int_{\mathbb{S}^{d-1}} \theta \nu(d\theta) = 0$  when  $\alpha = 1$ . Suppose that the Lévy measure  $\mu$  is a  $\gamma$ -measure with  $\gamma \in [1,d]$  satisfying  $\gamma > d - \alpha$ . Suppose also that  $\nu$  is symmetric i.e.  $\nu(B) = \nu(-B)$  for any  $B \subset \mathbb{S}^{d-1}$ . Then, there exists a finite constant  $C = C_{\alpha,d,\nu} > 0$  such that for all  $x,y \in \mathbb{R}^d$ 

$$|p(x)| \leqslant \frac{C}{(1+|x|)^{\alpha+\gamma}},\tag{12}$$

$$|\nabla p(x)| \leqslant \frac{C}{(1+|x|)^{\alpha+\gamma}},\tag{13}$$

$$\|\nabla^2 p(x)\|_{\text{op}} \leqslant \frac{C}{(1+|x|)^{\alpha+\gamma}},\tag{14}$$

and

$$|p(x) - p(y)| \le C(|x - y| \land 1) \left( \frac{1}{(1 + |x|)^{\alpha + \gamma}} + \frac{1}{(1 + |y|)^{\alpha + \gamma}} \right),$$
 (15)

where  $\nabla^2$  is the Hessian and  $\|\cdot\|_{op}$  denotes the operator norm.

- Remark 2 a) In the setting of Example 1, Chen and Zhang [12] obtained these estimates with  $\gamma = d$  without the symmetry assumption on the spectral measure, extending their earlier work [11]. It is an open problem to obtain these general estimates in the setting of [8] without the symmetry condition.
- b) Regularity estimates of solutions to Stein's equation and Wasserstein bounds in the sequel rely on this Lemma. In view of Item a), all the upcoming results extend to non-symmetric  $\nu$  in the setting of Example 1.
- c) Equations (12)-(14) were proved in [8, Lemma 2.4]. Equation (15) follows from (13) by the mean value theorem.
- d) In the above Lemma, if  $\nu(d\theta) = \frac{1}{V(\mathbb{S}^{d-1})}$  (the rotationally invariant  $\alpha$ -stable Lévy process), where  $V(\mathbb{S}^{d-1})$  is the surface area of  $\mathbb{S}^{d-1}$  and  $V(\mathbb{S}^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ , then we can obtain that

$$p(x) \leqslant \frac{\max\left\{2^{-d+1}\pi^{-\frac{d}{2}}\frac{\Gamma(d/\alpha)}{\alpha\Gamma(d/2)}, \frac{\alpha 2^{\alpha-1}\sin\frac{\alpha\pi}{2}\Gamma((d+\alpha)/2)\Gamma(\alpha/2)}{\pi^{d/2+1}}\right\}}{(1+|x|)^{\alpha+d}},$$

that is, the dependence of the constant C on the dimension in (12) is exponential and similar conclusions can be made about the gradient (13) and Hessian matrix (14). The specific details will be given in Lemma 18 in Appendix A. For the general case, the dependence of the constants on the dimension may need to analysis some infinite series, which is beyond the scope of this paper, we omit here.

#### 2.5 Main results

**Theorem 3** (Regularity estimates for the solution) Let f be given by (11) for  $h \in \text{Lip}_1$  in the case  $\alpha > 1$  and  $h \in \mathcal{H}_{\beta}$  for some  $\beta < \alpha$  in the case  $\alpha \leq 1$ . Suppose that the assumption of Lemma 2 holds.

i) If  $\alpha \in (1,2)$ . Then there exists a finite constant C > 0 depending on  $\alpha, d, \nu$ ,

$$\|\nabla f\|_{\infty} \leqslant \alpha,\tag{16}$$

$$\sup_{x \in \mathbb{R}^d} \|\nabla^2 f(x)\|_{\text{op}} \leqslant C,\tag{17}$$

where  $||f||_{\infty} = \sup_{x \in \mathbb{R}^d} |f(x)|$  is the  $L^{\infty}$  norm. Further, for all  $x, y \in \mathbb{R}^d$ 

$$\left| \mathcal{L}^{\alpha,\nu} f(x) - \mathcal{L}^{\alpha,\nu} f(y) \right| \leqslant \frac{2d_{\alpha}C}{\alpha(2-\alpha)(\alpha-1)} |x-y|^{2-\alpha}. \tag{18}$$

ii) If  $\alpha = 1$ , then there exists finite C > 0 depending on  $\beta, d, \nu$  such that

$$\|\nabla f\|_{\infty} \leqslant 1,\tag{19}$$

$$\|\mathcal{L}^{1,\nu}f\|_{\infty} \leqslant C \tag{20}$$

and for any  $x, y \in \mathbb{R}^d$  with |x - y| < 1 and  $w \in \mathbb{R}^d$ ,

$$|\nabla f(x) - \nabla f(y)| \leqslant C(1 - \log|x - y|)|x - y|. \tag{21}$$

$$|f(x+w) - f(x)| \leqslant C|w| \wedge |w|^{\beta}, \tag{22}$$

$$|\langle x, \nabla f(x) \rangle| \leqslant C(1+|x|^{\beta}). \tag{23}$$

$$|\mathcal{L}^{1,\nu}f(x) - \mathcal{L}^{1,\nu}f(y)| \le C|x - y|(1 - \log|x - y|).$$
 (24)

iii) If  $\alpha \in (0,1)$ , then there exists finite C>0 depending on  $\alpha,\beta,d,\nu$  such that

$$\|\nabla f\|_{\infty} \leqslant \alpha,\tag{25}$$

$$\|\mathcal{L}^{\alpha,\nu}f\|_{\infty} \leqslant C,\tag{26}$$

$$|\nabla f(x) - \nabla f(y)| \leqslant C|x - y|^{\alpha} \tag{27}$$

and for any  $x, y \in \mathbb{R}^d$ , one has

$$|f(x+y) - f(x)| \leqslant C|y| \wedge |y|^{\beta}, \tag{28}$$

$$|\langle x, \nabla f(x) \rangle| \leqslant C(1+|x|^{\beta}). \tag{29}$$

Further, for any  $\eta \in (0,1)$ , there exists finite C > 0 depending on  $\alpha, \beta, d, \nu, \eta$ such that for any  $x, y \in \mathbb{R}^d$ ,

$$|\mathcal{L}^{\alpha,\nu}f(x) - \mathcal{L}^{\alpha,\nu}f(y)| \leqslant C|x - y|^{\eta}. \tag{30}$$

In addition, we denote  $\mathcal{F}_{\beta}$  is the class of functions  $h: \mathbb{R}^d \to (\mathbb{R}, d_{\beta})$  such that  $|\nabla h(x)| \leq \frac{1}{1+|x|^{1-\beta}}$ . Then, we have the following proposition.

**Proposition 4** Let  $\alpha \in (0, \frac{1}{2}]$  and f be given by (11) for  $h \in \mathcal{H}_{\beta} \cap \mathcal{F}_{\beta}$  with  $\beta \in (0, \alpha)$ . Suppose that the assumption of Lemma 2 holds and denote  $\tilde{\beta} := \max\{\beta, d - \gamma\} \in$  $(0,\alpha)$ . Then, there exists a finite constant C>0 depending on  $\alpha,\beta,d,\nu,\gamma$  such that for any  $x \in \mathbb{R}^d$ ,

$$|\nabla f(x)| \leqslant C\left(1 \wedge |x|^{\tilde{\beta}-1}\right).$$

The second result is concerned with Wasserstein(-type) bounds for limit theorems with multivariate stable limit.

Let  $n \in \mathbb{N}$  and let  $\zeta_{n,1}, \zeta_{n,2}, \cdots, \zeta_{n,n}$  be a sequence of independent random vectors satisfying  $\mathbb{E}|\zeta_{n,i}|^{\beta} < \infty$  for any  $\beta \in (0,\alpha)$  and  $i = 1, \dots, n$ .

$$S_{n} = \begin{cases} \zeta_{n,1} - \mathbb{E}\zeta_{n,1} + \dots + \zeta_{n,n} - \mathbb{E}\zeta_{n,n}, & \alpha \in (1,2), \\ \zeta_{n,1} - \mathbb{E}\zeta_{n,1}\mathbf{1}_{(0,1]}(|\zeta_{n,1}|) + \dots + \zeta_{n,n} - \mathbb{E}\zeta_{n,n}\mathbf{1}_{(0,1]}(|\zeta_{n,n}|), & \alpha = 1, \\ \zeta_{n,1} + \zeta_{n,2} + \dots + \zeta_{n,n}, & \alpha \in (0,1) \end{cases}$$

and

$$S_n(i) = S_n - \zeta_{n,i}, \qquad 1 \leqslant i \leqslant n.$$

Denote  $l_n = \frac{\alpha}{d_{\alpha}} n$  and set  $\eta_{n,i} = l_n^{1/\alpha} \zeta_{n,i}$ , then we have the following Theorem.

Theorem 5 (Wassertein bounds) Suppose that the assumption of Lemma 2 holds. Let  $n \in \mathbb{N}$  and  $\zeta_{n,i}$ ,  $\eta_{n,i}$ ,  $i = 1, \dots, n$  are defined as above. Denote the density

function of  $\eta_{n,i}$  by  $p_{\eta_{n,i}}(r)dr\nu(d\theta)$  and when  $\alpha \in (0,1]$ , further assume that  $p_{\eta_{n,i}}(r)$  is non-increasing,

(1) When  $\alpha \in (1,2)$ , there exists a finite constant C>0 depending on  $\alpha,d,\nu$  such that

$$d_{W}\left(\mathcal{L}(S_{n}), \pi\right)$$

$$\leq C \sum_{i=1}^{n} \left\{ n^{-\frac{2}{\alpha}} \mathbb{E} |\eta_{n,i}|^{2-\alpha} + n^{-\frac{2}{\alpha}} \left( \mathbb{E} |\eta_{n,i}| \right)^{2} \right.$$

$$\left. + n^{-\frac{2}{\alpha}} \int_{0}^{l_{n}^{\frac{1}{\alpha}}} r^{2} \left| \frac{\alpha}{r^{\alpha+1}} - p_{\eta_{n,i}}(r) \left| dr + n^{-\frac{1}{\alpha}} \int_{l_{n}^{\frac{1}{\alpha}}}^{\infty} \left| \frac{\alpha}{r^{\alpha}} - r p_{\eta_{n,i}}(r) dr \right| \right\}.$$

(2) When  $\alpha = 1$ , there exists a finite constant C > 0 depending on  $\beta, d, \nu$  such that

$$\begin{split} &d_{W_{\beta}}\left(\mathcal{L}(S_{n}),\mu\right) \\ \leqslant &C\sum_{i=1}^{n} \Big\{n^{-2} \int_{0}^{l_{n}} r \left(1 - \log(l_{n}^{-1}r)\right) p_{\eta_{n,i}}(r) dr + n^{-\beta} \int_{l_{n}}^{\infty} r^{\beta} \left|d\left[\frac{1}{r} - r p_{\eta_{n,i}}(r)\right]\right| \\ &+ n^{-1} \int_{l_{n}}^{\infty} p_{\eta_{n,i}}(r) dr + n^{-2} \int_{0}^{l_{n}} r^{2} \left(1 - \log(l_{n}^{-1}r)\right) \left|\frac{1}{r^{2}} dr - p_{\eta_{n,i}}(r) dr\right| \Big\}. \end{split}$$

(3) When  $\alpha \in (\frac{1}{2}, 1)$ , there exists a finite constant C > 0 depending on  $\alpha, \beta, d, \nu$  such that

$$\begin{split} & d_{W_{\beta}}\left(\mathcal{L}(S_{n}), \mu\right) \\ \leqslant & C \sum_{i=1}^{n} \left\{ n^{-\frac{3\alpha+1}{2\alpha}} \int_{0}^{l^{\frac{1}{\alpha}}} r^{\frac{\alpha+1}{2}} p_{\eta_{n,i}}(r) dr + n^{-\frac{\beta}{\alpha}} \int_{l^{\frac{1}{\alpha}}_{n}}^{\infty} r^{\beta} \left| d[\frac{\alpha}{r^{\alpha}} - r p_{\eta_{n,i}}(r)] \right| \right. \\ & + n^{-1} \int_{l^{\frac{1}{\alpha}}_{n}}^{\infty} p_{\eta_{n,i}}(r) dr + n^{-\frac{1+\alpha}{\alpha}} \int_{0}^{l^{\frac{1}{\alpha}}_{n}} r^{\alpha+1} \left| \frac{\alpha}{r^{\alpha+1}} - p_{\eta_{n,i}}(r) \right| dr + \mathcal{R}_{n,\alpha,i} \right\}, \end{split}$$

where

$$\mathcal{R}_{n,\alpha,i} = n^{-\frac{1}{\alpha}} \Big| \int_{\mathbb{S}^{d-1}} \theta \nu(d\theta) \Big| \Big| \frac{\alpha}{1-\alpha} l_n^{\frac{1-\alpha}{\alpha}} - \int_0^{l_n^{\frac{1}{\alpha}}} r p_{\eta_{n,i}}(r) dr \Big|.$$

(4) When  $\alpha \in (0, \frac{1}{2}]$ , there exists a finite constant C > 0 depending on  $\alpha, \beta, d, \nu$  such that

$$\sup_{h \in \mathcal{H}_{\beta} \cap \mathcal{F}_{\beta}} \left| \mathbb{E}h(S_n) - \mu(h) \right|$$

$$\leq C \sum_{i=1}^{n} \left\{ n^{-\frac{3\alpha+1}{2\alpha}} \int_{0}^{l_{n}^{\frac{1}{\alpha}}} r^{\frac{\alpha+1}{2}} p_{\eta_{n,i}}(r) dr + n^{-\frac{\beta}{\alpha}} \int_{l_{n}^{\frac{1}{\alpha}}}^{\infty} t^{\beta} \left| d[\frac{\alpha}{t^{\alpha}} - t p_{\eta_{n,i}}(t)] \right| \right. \\ \left. + n^{-1} \int_{l_{n}^{\frac{1}{\alpha}}}^{\infty} p_{\eta_{n,i}}(r) dr + n^{-\frac{1+\alpha}{\alpha}} \int_{0}^{l_{n}^{\frac{1}{\alpha}}} r^{\alpha+1} \left| \frac{\alpha}{r^{\alpha+1}} - p_{\eta_{n,i}}(r) \right| dr + \mathcal{R}_{n,\alpha,i} \right\}.$$

Remark 3 In order to use the Leave-one-out method to prove the above theorem, when  $\alpha \in (0, \frac{1}{2}]$ , we need better decaying property with respect to the solution of the Stein's equation (9), hence we consider the case  $h \in \mathcal{H}_{\beta} \cap \mathcal{F}_{\beta}$ . An example  $\nu$ -Paretian distribution for this theorem will be given in Section 5.

## 3 Proof of Theorem 3 and Proposition 4

#### 3.1 Proof of Theorem 3

Now we prove all the claims of Theorem 3.

Proof of (16), (19) and (25) For any  $\alpha \in (0,2)$ , denote  $s=(1-\mathrm{e}^{-t})$  and  $z=y-\mathrm{e}^{-\frac{t}{\alpha}}x$ , it is easy to check

$$\nabla_x p(s,z) = -e^{-\frac{t}{\alpha}} \nabla_z p(s,z), \qquad \nabla_y p(s,z) = \nabla_z p(s,z).$$

Notice that  $\|\nabla h\|_{\infty} \leq 1$ , from which it is readily checked that one can differentiate under the integral sign in (11). Hence,

$$\nabla f(x) = -\int_{0}^{\infty} \int_{\mathbb{R}^{d}} \nabla_{x} p(s, z) (h(y) - \mu(h)) dy dt$$

$$= \int_{0}^{\infty} \int_{\mathbb{R}^{d}} e^{-\frac{t}{\alpha}} \nabla_{z} p(s, z) (h(y) - \mu(h)) dy dt$$

$$= \int_{0}^{\infty} \int_{\mathbb{R}^{d}} e^{-\frac{t}{\alpha}} \nabla_{y} p(s, z) (h(y) - \mu(h)) dy dt$$

$$= -\int_{0}^{\infty} \int_{\mathbb{R}^{d}} e^{-\frac{t}{\alpha}} p(s, z) \nabla h(y) dy dt.$$
(31)

Therefore,

$$\begin{split} \|\nabla f\|_{\infty} &\leqslant \|\nabla h\|_{\infty} \int_{0}^{\infty} \mathrm{e}^{-\frac{t}{\alpha}} \int_{\mathbb{R}^{d}} p(s,z) \mathrm{d}y \mathrm{d}t \\ &= \|\nabla h\|_{\infty} \int_{0}^{\infty} \mathrm{e}^{-\frac{t}{\alpha}} \int_{\mathbb{R}^{d}} p(s,z) \mathrm{d}z \mathrm{d}t = \alpha \|\nabla h\|_{\infty}. \end{split}$$

Proof of (17) From (31) we see that

$$\|\nabla^2 f(x)\|_{\text{op}} \leqslant \int_0^\infty \int_{\mathbb{R}^d} e^{-\frac{2t}{\alpha}} |\nabla_z p(s, z)| |\nabla h(y)| dy dt.$$

Thanks to the scaling property  $p(s,z) = s^{-d/\alpha} p(s^{-1/\alpha}z)$ , we have

$$\|\nabla^2 f(x)\|_{\text{op}} \leqslant \|\nabla h\|_{\infty} \int_0^{\infty} e^{-\frac{2t}{\alpha}} \int_{\mathbb{R}^d} s^{-\frac{d+1}{\alpha}} |\nabla p(s^{-\frac{1}{\alpha}}z)| dy dt$$

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$$= \|\nabla h\|_{\infty} \int_{0}^{\infty} s^{-1/\alpha} e^{-\frac{2t}{\alpha}} \int_{\mathbb{R}^{d}} |\nabla p(u)| du dt,$$

where the equality is by taking  $u = s^{-1/\alpha}z$ . Then, by Lemma 2 (13), there exists a finite C > 0 such that

$$\|\nabla^2 f(x)\|_{\text{op}} \leqslant C \|\nabla h\|_{\infty} \int_0^{\infty} s^{-1/\alpha} e^{-\frac{2t}{\alpha}} dt = C \|\nabla h\|_{\infty} B\left(\frac{\alpha - 1}{\alpha}, \frac{2}{\alpha}\right),$$

where B(a, b) is the Beta function.

Proof of (18) Before proving (18), we give another representation of the operator  $\mathcal{L}^{\alpha,\nu}$ . Fix  $\alpha \in (1,2)$ . Let  $f \in C^2(\mathbb{R}^d)$  be such that  $\|\nabla^2 f\|_{\infty} + \|\nabla f\|_{\infty} < \infty$ . We have

$$\mathcal{L}^{\alpha,\nu}f(x) = \frac{d_{\alpha}}{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{\langle \theta, \nabla f(x+u\theta) \rangle - \langle \theta, \nabla f(x) \rangle}{u^{\alpha}} du \nu(d\theta), \quad x \in \mathbb{R}^{d}.$$
 (32)

Indeed, one can write

$$\mathcal{L}^{\alpha,\nu}f(x) = d_{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \int_{0}^{r} \frac{\langle \theta, \nabla f(x+u\theta) \rangle - \langle \theta, \nabla f(x) \rangle}{r^{1+\alpha}} du dr \nu(d\theta)$$

$$= d_{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \int_{u}^{\infty} \frac{\langle \theta, \nabla f(x+u\theta) \rangle - \langle \theta, \nabla f(x) \rangle}{r^{1+\alpha}} dr du \nu(d\theta)$$

$$= \frac{d_{\alpha}}{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{\langle \theta, \nabla f(x+u\theta) \rangle - \langle \theta, \nabla f(x) \rangle}{u^{\alpha}} du \nu(d\theta),$$

implying (32). Using (32), we can write

$$\begin{split} &\frac{\alpha}{d_{\alpha}} \left| \mathcal{L}^{\alpha,\nu} f(x) - \mathcal{L}^{\alpha,\nu} f(y) \right| \\ = & \left| \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{\langle \theta, \nabla f(x+u\theta) \rangle - \langle \theta, \nabla f(x) \rangle - \langle \theta, \nabla f(y+u\theta) \rangle + \langle \theta, \nabla f(y) \rangle}{u^{\alpha}} du \nu(d\theta) \right| \\ \leqslant & \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{\left| \langle \theta, \nabla f(x+u\theta) - \nabla f(x) - \nabla f(y+u\theta) + \nabla f(y) \rangle \right|}{u^{\alpha}} du \nu(d\theta) \\ \leqslant & 2 \|\nabla^{2} f\| |x-y| \int_{\mathbb{S}^{d-1}} \int_{|x-y|}^{\infty} \frac{1}{u^{\alpha}} du \nu(d\theta) + 2 \|\nabla^{2} f\| \int_{\mathbb{S}^{d-1}} \int_{0}^{|x-y|} \frac{1}{u^{\alpha-1}} du \nu(d\theta) \\ = & \frac{2 \|\nabla^{2} f\|_{\infty}}{(2-\alpha)(\alpha-1)} |x-y|^{2-\alpha}, \end{split}$$

ending the proof.

Proof of (21) and (27) Let  $\alpha \in (0,1]$ . Differentiating under the integral, we have

$$\nabla f(x) = -\int_0^\infty \int_{\mathbb{R}^d} e^{-t/\alpha} p(u) \nabla h(s(t)^{1/\alpha} u + e^{-t/\alpha} x) du dt.$$
 (33)

Choose  $B = |x - y|^{\alpha}$ . Applying successively (33), a change of variables, and Lemma 2, we get that

$$\begin{split} &|\nabla f(x) - \nabla f(y)|\\ & \leq \int_0^\infty e^{-t/\alpha} \int_{\mathbb{R}^d} |p(u-s(t)^{-1/\alpha} e^{-t/\alpha} x) - p(u-s(t)^{-1/\alpha} e^{-t/\alpha} y)||\nabla h(us(t)^{1/\alpha})| du dt\\ & \leq C \|\nabla h\|_\infty \int_0^\infty e^{-t/\alpha} \big( (s(t)^{-1/\alpha} e^{-t/\alpha} |x-y|) \wedge 1 \big) dt \end{split}$$

$$\begin{split} &\leqslant C\|\nabla h\|_{\infty}\int_{0}^{\infty}e^{-t/\alpha}\big(t^{-1/\alpha}|x-y|)\wedge 1\big)dt\\ &\leqslant C\Big(\int_{0}^{B}e^{-t/\alpha}dt+\int_{B}^{\infty}e^{-t/\alpha}t^{-1/\alpha}dt|x-y|\Big)\leqslant C\big(B+\int_{B}^{\infty}t^{-1/\alpha}e^{-t/\alpha}dt|x-y|\big), \end{split}$$

where in the third inequality, we use the fact that  $s(t)^{-1/\alpha}e^{-t/\alpha}=(e^t-1)^{-1/\alpha}\leqslant t^{-1/\alpha}$ . If  $\alpha=1$ , then for B=|x-y|<1,

$$\begin{aligned} |\nabla f(x) - \nabla f(y)| &\leqslant C \Big( B + \Big( \int_B^1 t^{-1} dt + \int_1^\infty e^{-t} dt \Big) |x - y| \Big) \\ &\leqslant C (1 - \log|x - z|) |x - z|. \end{aligned}$$

If  $\alpha \in (0,1)$ , then for  $B = |x-y|^{\alpha}$ 

$$|\nabla f(x) - \nabla f(y)| \le C(B + \int_{B}^{\infty} t^{-1/\alpha} dt |x - z|) \le C|x - y|^{\alpha}.$$

Proof of (22) and (28) For  $\alpha \in (0,1]$ , one has by (11)

$$f(x+w) - f(x)$$

$$= -\int_{0}^{\infty} \int_{\mathbb{R}^{d}} p(z) (h(s(t)^{-1/\alpha}z + e^{-t/\alpha}(x+w)) - h(s(t)^{-1/\alpha}z + e^{-t/\alpha}x)) dz dt.$$

Thus, for  $h \in \mathcal{H}_{\beta}$  with  $\beta \in (0, \alpha)$ ,

$$|f(x+w)-f(x)| \leq \int_0^\infty \int_{\mathbb{R}^d} p(z) e^{-\beta t/\alpha} \, dz \, dt (|w|^\beta \wedge |w|) = \frac{\alpha}{\beta} (|w|^\beta \wedge |w|).$$

Proof of (20) and (26) We first prove (26). By (28), for  $\alpha \in (0,1)$ ,

$$|\mathcal{L}^{\alpha,\nu}f(x)| \le d_{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{|f(x+r\theta) - f(x)|}{r^{\alpha+1}} dr \nu(d\theta)$$

$$\le \frac{\alpha d_{\alpha}}{\beta} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{r^{\beta} \wedge r}{r^{\alpha+1}} dr \nu(d\theta) \le C.$$

Now it remains to bound  $\|\mathcal{L}^{1,\nu}f\|_{\infty}$ . By (21), for  $|w| \leq 1$ , one has

$$|f(x+w) - f(x) - \langle \nabla f(x), w \rangle| = \left| \int_0^1 \langle \nabla f(x+uw) - \nabla f(x), w \rangle du \right|$$

$$\leq |w| \int_0^1 |\nabla f(x+uw) - \nabla f(x)| du$$

$$\leq C|w|^2 \int_0^1 u(1 + \log \frac{1}{|u|}) du$$

$$\leq C|w|^2 (1 + \log \frac{1}{|u|}).$$

It follows from (22) that

$$|\mathcal{L}^{1,\nu}f(x)|\leqslant C\Big(\int_{\mathbb{S}^{d-1}}\int_0^1(1+\log\frac{1}{r})dr\nu(d\theta)+\int_{\mathbb{S}^{d-1}}\int_1^\infty r^{\beta-2}dr\nu(d\theta)\Big)\leqslant C.$$

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Proof of (23) and (29) For  $\alpha \in (0,1]$ , one has by (9) that

$$|\langle x, \nabla f(x)| = \alpha | \mathcal{L}^{\alpha, \nu} f(x) - [h(x) - h(0)] + [\pi(h) - h(0)] |.$$

Thus, by (20) and (26), we have

$$|\langle x, \nabla f(x) \rangle| \le C[1 + |x| \wedge |x|^{\beta}] \le C(1 + |x|^{\beta}).$$

The proof is complete.

It remains to prove (24) and (30). We need two lemmas.

**Lemma 6** Let  $\alpha \in (0,1]$  and  $h \in \mathcal{H}_{\beta}$  with  $\beta \in (0,\alpha)$ .

1) If  $\alpha = 1$ , then for any a > 0,

$$\left| \int_{\mathbb{R}^d} \mathcal{L}^{1,\nu} p(y) h(ay) \, dy \right| \le C(a^{\beta} + a).$$

2) If  $\alpha \in (0,1)$ , then for any a > 0,

$$\left| \int_{\mathbb{R}^d} \mathcal{L}^{\alpha,\nu} p(y) h(ay) \, dy \right| \le Ca^{\alpha}.$$

*Proof* 1) Notice that for any  $y \in \mathbb{R}^d$ ,

$$\mathcal{L}^{1,\nu}p(y) = \int_{\mathbb{S}^{d-1}} \int_{1}^{\infty} \frac{p(y+r\theta) - p(y)}{r^2} dr \nu(d\theta) + \int_{\mathbb{S}^{d-1}} \int_{0}^{1} \frac{p(y+r\theta) - p(y) - \langle r\theta, \nabla p(y) \rangle}{r^2} dr \nu(d\theta).$$

By Fubuni's theorem,

$$\begin{split} & \Big| \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \int_1^\infty \frac{p(y+r\theta)-p(y)}{r^2} dr \nu(d\theta) h(ay) dy \Big| \\ = & \Big| \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \int_1^\infty \frac{p(y)}{r^2} \Big( h(ay-ar\theta) - h(ay) \Big) dr \nu(d\theta) dy \Big| \\ \leqslant & a^\beta \int_{\mathbb{S}^{d-1}} \int_1^\infty \int_{\mathbb{R}^d} \frac{p(y)r^\beta}{r^2} dy dr \nu(d\theta) \leqslant Ca^\beta. \end{split}$$

Applying Fubini's theorem, integration by parts and the estimate of  $\nabla p(x)$  (Lemma 2), we get

$$\begin{split} & \Big| \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \int_0^1 \frac{p(y+r\theta) - p(y) - \langle r\theta, \nabla p(y) \rangle}{r^2} dr \nu(d\theta) h(ay) dy \Big| \\ = & \Big| \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \int_0^1 \int_0^1 \frac{\langle \nabla p(y+ur\theta) - \nabla p(y), r\theta \rangle}{r^2} du dr \nu(d\theta) h(ay) dy \Big| \\ \leqslant & a \|\nabla h\|_{\infty} \Big| \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \int_0^1 \int_0^1 \frac{|p(y+ur\theta) - p(y)|}{r} du dr \nu(d\theta) dy \Big| \\ \leqslant & Ca \|\nabla h\|_{\infty} \int_{\mathbb{S}^{d-1}} \int_0^1 \int_0^1 \frac{ur}{r} du dr \nu(d\theta) \leqslant Ca. \end{split}$$

2) We have by Fubini's theorem that, for any a > 0,

$$\left| \int_{\mathbb{R}^{d}} (\mathcal{L}^{\alpha,\nu} p)(y) h(ay) \, dy \right|$$

$$= d_{\alpha} \left| \int_{\mathbb{R}^{d}} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{p(y+r\theta) - p(y)}{r^{\alpha+1}} dr \nu(d\theta) h(ay) dy \right|$$

$$= d_{\alpha} \left| \int_{\mathbb{R}^{d}} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{p(y)}{r^{\alpha+1}} (h(ay-ar\theta) - h(ay)) dr \nu(d\theta) dy \right|$$

$$= d_{\alpha} a^{\alpha} \left| \int_{\mathbb{R}^{d}} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{p(y)}{u^{\alpha+1}} (h(ay-u\theta) - h(ay)) du \nu(d\theta) dy \right|$$

$$\leq d_{\alpha} a^{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \int_{\mathbb{R}^{d}} \frac{p(y)(u \wedge u^{\beta})}{u^{\alpha+1}} dy du \nu(d\theta) \leq Ca^{\alpha}.$$

Thus, the assertion is proved.

**Lemma 7** Let  $\alpha \in (0,1]$ . Then

$$\int_{\mathbb{R}^d} |\mathcal{L}^{\alpha,\nu} p(z)| dz \leqslant C.$$

*Proof* If  $\alpha = 1$ , then by Lemma 2, we get that, for any  $|u| \leq 1$ ,

$$|\nabla^2 p(z+u)| \le \frac{C_d}{(1+|z+u|)^{\gamma+1}} \le \frac{C}{(1+|z|)^{\gamma+1}},$$

where in the last inequality, we use the fact that  $2(1+|z+u|) \ge 2+|z|-|u| \ge 1+|z|$ . It follows that, for any  $|w| \le 1$ ,

$$|p(z+w)-p(z)-w\cdot\nabla p(z)|\leqslant rac{C}{(1+|z|)^{\gamma+1}}|w|^2.$$

Thus, we have that

$$\begin{split} \int_{\mathbb{R}^d} |\mathcal{L}^{1,\nu} p(z)| dz &\leqslant d_1 \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \int_1^\infty \frac{p(z+r\theta)+p(z)}{r^2} dr \nu(d\theta) dz \\ &+ d_1 \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \int_0^1 \frac{|p(z+r\theta)-p(z)-\langle r\theta, \nabla p(z)\rangle|}{r^2} dr \nu(d\theta) dz \\ &\leqslant 2d_1 + d_1 \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \int_0^1 \frac{C}{(1+|z|)^{\gamma+1}} dr \nu(d\theta) dz \leqslant C. \end{split}$$

If  $\alpha \in (0,1)$ , then by Lemma 2,

$$\int_{\mathbb{R}^d} |\mathcal{L}^{\alpha,\nu} p(z)| dz \leqslant d_{\alpha} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \int_0^{\infty} \frac{|p(z+r\theta) - p(z)|}{r^{\alpha+1}} dr \nu(d\theta) dz$$

$$\leqslant C \int_{\mathbb{S}^{d-1}} \int_0^{\infty} \frac{r \wedge 1}{r^{\alpha+1}} dr \nu(d\theta) \leqslant C.$$

We are ready to complete the proof of Theorem 3.

Proof of (24) and (30) Set  $s(t) = 1 - e^{-t}$  and  $\tilde{h} = h - \mathbb{E}[h(Z)]$ . We claim that

$$\mathcal{L}^{\alpha,\nu}f(x) = -\int_0^\infty \int_{\mathbb{R}^d} \mathcal{L}^{\alpha,\nu}q(t,\cdot,y)(x)\widetilde{h}(y) \,dy \,dt$$
$$= -\int_0^\infty s(t)^{-1}e^{-t} \,dt \int_{\mathbb{R}^d} (\mathcal{L}^{\alpha,\nu}p)(z)\widetilde{h}\left(s(t)^{1/\alpha}z + e^{-t/\alpha}x\right) \,dz. \tag{34}$$

The second equality follows from (10). To see that the first one holds, note that  $h \in \mathcal{H}_{\beta}$  so that Fubini's theorem implies

$$\mathcal{L}^{\alpha,\nu}f(x) = -\int_0^\infty \mathcal{L}^{\alpha,\nu}\left(\int_{\mathbb{R}^d} q(t,\cdot,y)\widetilde{h}(y)dy\right)(x)dt.$$

For each fixed t > 0, applying Lemma 2 justifies a further use of Fubini's theorem, we are led to

$$\mathcal{L}^{\alpha,\nu}\left(\int_{\mathbb{R}^d}q(t,\cdot,y)\widetilde{h}(y)dy\right)(x)=\int_{\mathbb{R}^d}\mathcal{L}^{\alpha,\nu}q(t,\cdot,y)(x)\widetilde{h}(y)dy,$$

and the claim follows. Now let  $\alpha = 1$  and let  $x, y \in \mathbb{R}^d$  be such that  $|x - y| \le 1$ . By Lemma 7, we get that

$$\left| \int_{\mathbb{R}^d} (\mathcal{L}^{1,\nu} p)(z) (h(s(t)^{1/\alpha} z + e^{-t/\alpha} x) - h(s(t)^{1/\alpha} z + e^{-t/\alpha} y)) dz \right|$$

$$\leq e^{-t} |x - y| \int_{\mathbb{R}^d} \left| (\mathcal{L}^{1,\nu} p)(z) \right| dz.$$

By Lemma 6, we get that, for t < 1,

$$\left| \int_{\mathbb{R}^d} (\mathcal{L}^{1,\nu} p)(z) h\left(s(t)^{1/\alpha} z + e^{-t/\alpha} x\right) dz - \int_{\mathbb{R}^d} (\mathcal{L}^{1,\nu} p)(z) h\left(s(t)^{1/\alpha} z + e^{-t/\alpha} y\right) dz \right|$$

$$\leq C(s(t) + s(t)^{\beta}) \leq Cs(t)^{\beta}.$$
(35)

Let  $B = |x - y|^{1/\beta}$ . Then by (34), we get that

$$\begin{aligned} |\mathcal{L}^{1,\nu}f(x) - \mathcal{L}^{1,\nu}f(y)| &\leq C \Big( \int_0^B s(t)^{-1} s(t)^\beta e^{-t} \, dt + \int_B^\infty s(t)^{-1} e^{-2t} \, dt |x - y| \Big) \\ &\leq C \Big( B^\beta + \int_B^\infty t^{-1} e^{-t} \, dt |x - y| \Big) \\ &= C \Big( B^\beta + \left( \int_B^1 t^{-1} \, dt + \int_1^\infty e^{-t} \, dt \right) |x - y| \Big) \\ &\leq C |x - y| \left( 1 - \log |x - y| \right), \end{aligned}$$

If  $\alpha \in (0,1)$ , by Lemma 7, we get that

$$\left| \int_{\mathbb{R}^d} (\mathcal{L}^{\alpha,\nu} p)(z) (\widetilde{h}(s(t)^{1/\alpha} z + e^{-t/\alpha} x) - \widetilde{h}(s(t)^{1/\alpha} z + e^{-t/\alpha} y)) dz \right|$$

$$\leq e^{-t/\alpha} |x - y| \int_{\mathbb{R}^d} \left| (\mathcal{L}^{\alpha,\nu} p)(z) \right| dz \leq C|x - y|.$$

By Lemma 6 applied to  $\widetilde{h}(\cdot + e^{-t/\alpha}x)$ ,  $\widetilde{h}(\cdot + e^{-t/\alpha}y) \in \mathcal{H}_{\beta}$ , we get that,

$$\begin{split} &\left| \int_{\mathbb{R}^d} (\mathcal{L}^{\alpha,\nu} p)(z) \widetilde{h} \left( (s(t)^{1/\alpha} z + e^{-t/\alpha} x \right) dz - \int_{\mathbb{R}^d} (\mathcal{L}^{\alpha,\nu} p)(z) \widetilde{h} \left( (s(t)^{1/\alpha} z + e^{-t/\alpha} y \right) dz \right| \\ &\leq C s(t). \end{split}$$

Thus, we get that, for any  $\eta \in [0, 1]$ ,

$$\left| \int_{\mathbb{R}^d} (\mathcal{L}^{\alpha,\nu} p)(z) \widetilde{h} \left( s(t)^{1/\alpha} z + e^{-t/\alpha} x \right) dz - \int_{\mathbb{R}^d} (\mathcal{L}^{\alpha,\nu} p)(z) \widetilde{h} \left( s(t)^{1/\alpha} z + e^{-t/\alpha} y \right) dz \right|$$

$$\leq C \left( s(t) \wedge |x - y| \right) \leq C s(t) (1 \wedge s(t)^{-1} |x - y|)^{\eta} \leqslant C s(t)^{1-\eta} |x - y|^{\eta}.$$
(36)

Then, by (34) and (36), we get that

$$\begin{aligned} |\mathcal{L}^{\alpha,\nu}f(x) - \mathcal{L}^{\alpha,\nu}f(y)| &\leq C \int_0^\infty s(t)^{-1}s(t)^{1-\eta}e^{-t} \, dt |x-y|^{\eta} \\ &\leq C \int_0^\infty s(t)^{-\eta}e^{-t} \, dt |x-y|^{\eta} \\ &\leq C \int_0^\infty t^{-\eta}e^{-(1-\eta)t} \, dt |x-y|^{\eta} \leqslant C|x-y|^{\eta}, \end{aligned}$$

completing the proof.

# 3.2 Proof of Proposition 4

When |x| < 1, the conclusion is obvious.

When  $|x| \ge 1$ , notice that

$$\nabla f(x) = -\int_0^\infty \int_{\mathbb{R}^d} e^{-t/\alpha} p(y) \nabla h \left( s(t)^{1/\alpha} y + e^{-t/\alpha} x \right) dy dt,$$

as  $|y| \ge 2s(t)^{-1/\alpha}e^{-t/\alpha}|x|$ , that is,  $s(t)^{1/\alpha}|y| \ge 2e^{-t/\alpha}|x|$ , we have

$$\begin{split} &e^{\frac{\beta-1}{\alpha}t}|x|^{1-\beta}|\nabla h\big(s(t)^{1/\alpha}y + e^{-t/\alpha}x\big)| \\ \leqslant &\frac{e^{\frac{\beta-1}{\alpha}t}|x|^{1-\beta}}{1+\left|s(t)^{1/\alpha}y + e^{-t/\alpha}x\right|^{1-\beta}} \\ \leqslant &\frac{e^{\frac{\beta-1}{\alpha}t}|x|^{1-\beta}}{1+\left(s(t)^{1/\alpha}|y| - e^{-t/\alpha}|x|\right)^{1-\beta}} \leqslant \frac{e^{\frac{\beta-1}{\alpha}t}|x|^{1-\beta}}{1+e^{\frac{\beta-1}{\alpha}t}|x|^{1-\beta}} \leqslant 1, \end{split}$$

which implies

$$|\nabla h(s(t)^{1/\alpha}y + e^{-t/\alpha}x)| \le e^{\frac{1-\beta}{\alpha}t}|x|^{\beta-1}.$$

These imply

$$\begin{split} &\left| \int_0^\infty \int_{|y|\geqslant 2s(t)^{-1/\alpha}e^{-t/\alpha}|x|} e^{-\frac{t}{\alpha}} p(y) \nabla h \big( s(t)^{1/\alpha} y + e^{-t/\alpha} x \big) dy dt \right| \\ \leqslant & \int_0^\infty \int_{|y|\geqslant 2s(t)^{-1/\alpha}e^{-t/\alpha}|x|} e^{-\frac{t}{\alpha}} p(y) e^{\frac{1-\beta}{\alpha}t} |x|^{\beta-1} dy dt \\ \leqslant & |x|^{\beta-1} \int_0^\infty \int_{\mathbb{R}^d} e^{-\frac{\beta}{\alpha}t} p(y) dy dt = \frac{\alpha}{\beta} |x|^{\beta-1}. \end{split}$$

As  $|y| \leqslant \frac{1}{2}s(t)^{-1/\alpha}e^{-t/\alpha}|x|$ , that is,  $s(t)^{1/\alpha}|y| \leqslant \frac{1}{2}e^{-t/\alpha}|x|$ , we have

$$\begin{split} e^{\frac{\beta-1}{\alpha}t}|x|^{1-\beta}|\nabla h\big(s(t)^{1/\alpha}y + e^{-t/\alpha}x\big)| \leqslant & \frac{e^{\frac{\beta-1}{\alpha}t}|x|^{1-\beta}}{1 + \big(e^{-t/\alpha}|x| - s(t)^{1/\alpha}|y|\big)^{1-\beta}} \\ \leqslant & \frac{e^{\frac{\beta-1}{\alpha}t}|x|^{1-\beta}}{1 + 2^{\beta-1}e^{\frac{\beta-1}{\alpha}t}|x|^{1-\beta}} \leqslant 2, \end{split}$$

which implies

$$|\nabla h(s(t)^{1/\alpha}y + e^{-t/\alpha}x)| \le 2e^{\frac{1-\beta}{\alpha}t}|x|^{\beta-1}$$

These imply

$$\left| \int_0^\infty \int_{|y| \leqslant \frac{1}{2} s(t)^{-1/\alpha} e^{-t/\alpha} |x|} e^{-\frac{t}{\alpha}} p(y) \nabla h \big( s(t)^{1/\alpha} y + e^{-t/\alpha} x \big) dy dt \right| \leqslant \frac{2\alpha}{\beta} |x|^{\beta - 1}.$$

As  $\frac{1}{2}s(t)^{-1/\alpha}e^{-t/\alpha}|x| \le |y| \le 2s(t)^{-1/\alpha}e^{-t/\alpha}|x|$ , (12) implies

$$\begin{split} & \left| \int_{0}^{\infty} \int_{\frac{1}{2}s(t)^{-1/\alpha} e^{-t/\alpha} |x| \leq |y| \leq 2s(t)^{-1/\alpha} e^{-t/\alpha} |x|} e^{-\frac{t}{\alpha}} p(y) \nabla h \left( s(t)^{1/\alpha} y + e^{-t/\alpha} x \right) dy dt \right| \\ & \leq C \left[ \int_{0}^{\ln(1+|x|^{\alpha})} e^{-\frac{t}{\alpha}} \int_{\frac{1}{2}e^{-t/\alpha} |x| \leq s(t)^{1/\alpha} |y| \leq 2e^{-t/\alpha} |x|} \frac{|y|^{-(\alpha+\gamma)}}{\left| s(t)^{1/\alpha} y + e^{-t/\alpha} x \right|^{1-\beta}} dy dt \right. \\ & + \int_{\ln(1+|x|^{\alpha})}^{\infty} e^{-\frac{t}{\alpha}} \int_{\frac{1}{2}s(t)^{-1/\alpha} e^{-t/\alpha} |x| \leq |y| \leq 2s(t)^{-1/\alpha} e^{-t/\alpha} |x|} dy dt \right]. \end{split}$$

For the first term, we have

$$\int_{0}^{\ln(1+|x|^{\alpha})} e^{-\frac{t}{\alpha}} \int_{\frac{1}{2}e^{-t/\alpha}|x| \leqslant s(t)^{1/\alpha}|y| \leqslant 2e^{-t/\alpha}|x|} \frac{|y|^{-(\alpha+\gamma)}}{|s(t)^{1/\alpha}y + e^{-t/\alpha}x|^{1-\beta}} dy dt$$

$$\leqslant \int_{0}^{\ln(1+|x|^{\alpha})} e^{-\frac{t}{\alpha}} \int_{\mathbb{S}^{d-1}} \int_{\frac{1}{2}s(t)^{-1/\alpha}e^{-t/\alpha}|x|}^{2s(t)^{-1/\alpha}e^{-t/\alpha}|x|} \frac{r^{-(\alpha+1+\gamma-d)}}{|s(t)^{1/\alpha}r\theta - e^{-t/\alpha}|x||^{1-\beta}} dr d\theta dt$$

$$\leqslant C \int_{0}^{\ln(1+|x|^{\alpha})} e^{t} s(t)^{\frac{\alpha+1}{\alpha}} \int_{\frac{1}{2}s(t)^{-1/\alpha}e^{-t/\alpha}|x|}^{s(t)^{-1/\alpha}e^{-t/\alpha}|x|} \frac{|x|^{-(\alpha+\gamma+1-d)}}{(e^{-t/\alpha}|x| - s(t)^{1/\alpha}r)^{1-\beta}} dr dt$$

$$+ C \int_{0}^{\ln(1+|x|^{\alpha})} e^{t} s(t)^{\frac{\alpha+1}{\alpha}} \int_{s(t)^{-1/\alpha}e^{-t/\alpha}|x|}^{2s(t)^{-1/\alpha}e^{-t/\alpha}|x|} \frac{|x|^{-(\alpha+\gamma+1-d)}}{(s(t)^{1/\alpha}r - e^{-t/\alpha}|x|)^{1-\beta}} dr dt.$$

Then, one can write

$$\int_{0}^{\ln(1+|x|^{\alpha})} e^{t} s(t)^{\frac{\alpha+1}{\alpha}} \int_{\frac{1}{2}s(t)^{-1/\alpha}e^{-t/\alpha}|x|}^{s(t)^{-1/\alpha}e^{-t/\alpha}|x|} \frac{|x|^{-(\alpha+\gamma+1-d)}}{(e^{-t/\alpha}|x| - s(t)^{1/\alpha}r)^{1-\beta}} dr dt$$

$$= \frac{1}{|x|^{\alpha+\gamma+1-d}} \int_{0}^{\ln(1+|x|^{\alpha})} e^{t} s(t) \int_{0}^{\frac{1}{2}e^{-t/\alpha}|x|} \frac{1}{r^{1-\beta}} dr dt$$

$$\leq \frac{C}{|x|^{\alpha+\gamma+1-d-\beta}} \int_{0}^{\ln(1+|x|^{\alpha})} e^{\frac{\alpha-\beta}{\alpha}t} dt$$

$$\leq \frac{C}{|x|^{\alpha+\gamma+1-d-\beta}} e^{\frac{\alpha-\beta}{\alpha}\ln(1+|x|^{\alpha})} \leq \frac{C}{|x|^{\gamma+1-d}},$$

whereas

$$\int_{0}^{\ln(1+|x|^{\alpha})} e^{t} s(t)^{\frac{\alpha+1}{\alpha}} \int_{s(t)^{-1/\alpha} e^{-t/\alpha}|x|}^{2s(t)^{-1/\alpha} e^{-t/\alpha}|x|} \frac{|x|^{-(\alpha+\gamma+1-d)}}{\left(s(t)^{1/\alpha} r - e^{-t/\alpha}|x|\right)^{1-\beta}} dr dt \\ \leqslant \frac{C}{|x|^{\gamma+1-d}}.$$

These imply

$$\int_{0}^{\ln(1+|x|^{\alpha})} e^{-\frac{t}{\alpha}} \int_{\frac{1}{2}e^{-t/\alpha}|x| \leqslant s(t)^{1/\alpha}|y| \leqslant 2e^{-t/\alpha}|x|} \frac{|y|^{-(\alpha+\gamma)}}{\left|s(t)^{1/\alpha}y + e^{-t/\alpha}x\right|^{1-\beta}} dydt \leqslant \frac{C}{|x|^{\gamma+1-d}}.$$

For the second term,

$$\int_{\ln(1+|x|^{\alpha})}^{\infty} e^{-\frac{t}{\alpha}} \int_{\frac{1}{2}s(t)^{-1/\alpha}e^{-t/\alpha}|x| \le |y| \le 2s(t)^{-1/\alpha}e^{-t/\alpha}|x|} dydt$$

$$\leq C|x| \int_{\ln(1+|x|^{\alpha})}^{\infty} e^{-\frac{t}{\alpha}} (e^{t} - 1)^{-1/\alpha} dt$$

$$\leq \frac{C|x|}{|x|} \int_{\ln(1+|x|^{\alpha})}^{\infty} e^{-\frac{t}{\alpha}} dt \le Ce^{-\frac{1}{\alpha}\ln(1+|x|^{\alpha})} \le C|x|^{-1}.$$

The proof is complete.

# 4 Proof of Theorem 5

# 4.1 Alternate expressions for $\mathcal{L}^{\alpha,\nu}$

The following lemma gathers useful alternate expressions for the operator  $\mathcal{L}^{\alpha,\nu}$ .

**Lemma 8** Let  $\alpha \in (0,2)$  and  $f \in C^2(\mathbb{R}^d)$ . We have, for all  $x \in \mathbb{R}^d$  and a > 0, a.) When  $\alpha \in (1,2)$ ,

$$(\mathcal{L}^{\alpha,\nu}f)(x) = \frac{d_{\alpha}}{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{\langle \theta, \nabla f(x+u\theta) \rangle - \langle \theta, \nabla f(x) \rangle}{u^{\alpha}} du \nu(d\theta)$$
$$= \frac{d_{\alpha}a^{1-\alpha}}{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{\langle u\theta, \nabla f(x+u\theta) \rangle - \langle u\theta, \nabla f(x) \rangle}{u^{\alpha+1}} du \nu(d\theta)$$

provided that  $\|\nabla f\|_{\infty} < \infty$  and  $\sup_{x \in \mathbb{R}^d} \|\nabla^2 f(x)\|_{\text{op}} < \infty$ . b.) When  $\alpha = 1$ ,

$$(\mathcal{L}^{1,\nu}f)(x) = d_1 \int_{\mathbb{S}^{d-1}} \int_0^\infty \frac{\langle \theta, \nabla f(x+u\theta) \rangle - \langle \theta, \nabla f(x) \mathbf{1}_{(0,1]}(u) \rangle}{u} du \nu(d\theta)$$
$$= d_1 \int_{\mathbb{S}^{d-1}} \int_0^\infty \frac{\langle u\theta, \nabla f(x+au\theta) \rangle - \langle u\theta, \nabla f(x) \mathbf{1}_{(0,1]}(u) \rangle}{u^2} du \nu(d\theta)$$

provided that  $\int_{\mathbb{S}^{d-1}} \int_0^\infty \frac{|f(x+u\theta)-f(x)-\langle u\theta,\nabla f(x)\mathbf{1}_{(0,1]}(u)\rangle|}{u^2} du\nu(d\theta) < \infty \text{ and } \int_{\mathbb{S}^{d-1}} \int_0^\infty \frac{|\langle \theta,\nabla f(x+u\theta)\rangle-\langle \theta,\nabla f(x)\mathbf{1}_{(0,1]}(u)\rangle|}{u} du\nu(d\theta) < \infty.$  c.) When  $\alpha \in (0,1)$ ,

$$(\mathcal{L}^{\alpha,\nu}f)(x) = \frac{d_{\alpha}}{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{\langle \theta, \nabla f(x+u\theta) \rangle}{u^{\alpha}} du \nu(d\theta)$$
$$= \frac{d_{\alpha}a^{1-\alpha}}{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{\langle u\theta, \nabla f(x+au\theta) \rangle}{u^{\alpha+1}} du \nu(d\theta).$$

provided  $\int_{\mathbb{S}^{d-1}} \int_0^\infty \frac{|f(x+u\theta)-f(x)|}{u^{\alpha+1}} du\nu(d\theta) < \infty$ ,  $\int_{\mathbb{S}^{d-1}} \int_0^\infty \frac{|\langle \theta, \nabla f(x+u\theta) \rangle|}{u^{\alpha}} du\nu(d\theta) < \infty$ .

Proof Note that the conditions on f ensure that all the integrals are well defined and we can use Fubini's theorem in the following proof.

$$(\mathcal{L}^{\alpha,\nu}f)(x) = d_{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \left( f(x+r\theta) - f(x) - k_{\alpha}(r) \langle r\theta, \nabla f(x) \rangle \right) \frac{dr}{r^{1+\alpha}} \nu(d\theta)$$

$$= d_{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \int_{0}^{r} \left( \langle \theta, \nabla f(x+u\theta) \rangle - k_{\alpha}(r) \langle \theta, \nabla f(x) \rangle \right) du \frac{dr}{r^{1+\alpha}} \nu(d\theta)$$

$$= d_{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \int_{u}^{\infty} \left( \langle \theta, \nabla f(x+u\theta) \rangle - k_{\alpha}(r) \langle \theta, \nabla f(x) \rangle \right) \frac{dr}{r^{1+\alpha}} du \nu(d\theta),$$

since  $k_{\alpha}(r) = \mathbf{1}_{\alpha=1, r \in (0,1)} + \mathbf{1}_{\alpha \in (1,2)}$  and  $\int_{\mathbb{S}^{d-1}} \theta \nu(d\theta) = 0$  when  $\alpha = 1$ , we further have

$$(\mathcal{L}^{\alpha,\nu}f)(x) = \frac{d_{\alpha}}{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{\langle \theta, \nabla f(x+u\theta) - k_{\alpha}(u) \langle \theta, \nabla f(x) \rangle}{u^{\alpha}} du \nu(d\theta)$$
$$= \frac{d_{\alpha}a^{1-\alpha}}{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{\langle u\theta, \nabla f(x+au\theta) - k_{\alpha}(u) \langle u\theta, \nabla f(x) \rangle}{u^{\alpha+1}} du \nu(d\theta).$$

Now we check that the solution f to Stein's equation satisfies the integrability condition of the previous proposition.

**Lemma 9** Let  $\alpha \in (\frac{1}{2}, 1]$  and  $h \in \mathcal{H}_{\beta}$  with  $\beta \in (0, \alpha)$ . Let f be defined as (11). If  $\alpha = 1$  and  $\int_{\mathbb{S}^{d-1}} \theta \nu(d\theta) = 0$ , then for any  $x \in \mathbb{R}^d$ ,

$$\int_{\mathbb{S}^{d-1}} \int_0^\infty \frac{|\langle \theta, \nabla f(x+r\theta) \rangle - \langle \theta, \nabla f(x) \mathbf{1}_{(0,1]}(r) \rangle|}{r} \mathrm{d}r \mathrm{d}\theta \leqslant C(1+\ln|x|).$$

If  $\alpha \in (0,1)$ , then for any  $x \in \mathbb{R}^d$ ,

$$\int_{\mathbb{S}^{d-1}} \int_0^\infty \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r^{\alpha}} dr d\theta \leqslant C(1+|x|^{1-\alpha}).$$

*Proof* When  $\alpha = 1$  and |x| < 1, we have

$$\begin{split} &\int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle - \langle \theta, \nabla f(x) \mathbf{1}_{(0,1]}(r) \rangle|}{r} \mathrm{d}r \mathrm{d}\theta \\ &= \int_{\mathbb{S}^{d-1}} \int_{0}^{1} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle - \langle \theta, \nabla f(x) \rangle|}{r} \mathrm{d}r \mathrm{d}\theta + \int_{\mathbb{S}^{d-1}} \int_{1}^{\infty} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r} \mathrm{d}r \mathrm{d}\theta. \end{split}$$

By (21), we have

$$\int_{\mathbb{S}^{d-1}} \int_0^1 \frac{|\langle \theta, \nabla f(x+r\theta) \rangle - \langle \theta, \nabla f(x) \rangle|}{r} \mathrm{d}r \mathrm{d}\theta \leqslant C \int_0^1 (1 - \ln r) \mathrm{d}r \mathrm{d}\theta \leqslant C,$$

and by (19) and (23), we have

$$\int_{\mathbb{S}^{d-1}} \int_{1}^{\infty} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r} dr d\theta$$

$$= \int_{\mathbb{S}^{d-1}} \int_{1}^{\infty} \frac{|\langle r\theta, \nabla f(x+r\theta) \rangle|}{r^{2}} dr d\theta$$

$$\leq \int_{\mathbb{S}^{d-1}} \int_{1}^{\infty} \frac{|\langle (x+r\theta), \nabla f(x+r\theta) \rangle| + |\langle x, \nabla f(x+r\theta) \rangle|}{r^{2}} dr d\theta$$

$$\leq C \int_{\mathbb{S}^{d-1}} \int_{1}^{\infty} \frac{1 + |x+r\theta|^{\beta} + |x|}{r^{2}} dr d\theta \leq C \int_{1}^{\infty} \frac{1 + |x| + r^{\beta}}{r^{2}} \leq C(1 + |x|). \quad (37)$$

When  $\alpha = 1$  and  $|x| \ge 1$ , we have

$$\begin{split} &\int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle - \langle \theta, \nabla f(x) \mathbf{1}_{(0,1]}(r) \rangle|}{r} \mathrm{d}r \mathrm{d}\theta \\ &= \int_{\mathbb{S}^{d-1}} \int_{0}^{|x|} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle - \langle \theta, \nabla f(x) \rangle|}{r} \mathrm{d}r \mathrm{d}\theta + \int_{\mathbb{S}^{d-1}} \int_{|x|}^{\infty} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r} \mathrm{d}r \mathrm{d}\theta. \end{split}$$

By (21) and (19), we have

$$\begin{split} &\int_{\mathbb{S}^{d-1}} \int_{0}^{|x|} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle - \langle \theta, \nabla f(x) \mathbf{1}_{(0,1]}(r) \rangle|}{r} \mathrm{d}r \mathrm{d}\theta \\ &= \int_{\mathbb{S}^{d-1}} \int_{0}^{1} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle - \langle \theta, \nabla f(x) \rangle|}{r} \mathrm{d}r \mathrm{d}\theta + \int_{\mathbb{S}^{d-1}} \int_{1}^{|x|} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r} \mathrm{d}r \mathrm{d}\theta \\ &\leq C(1+\ln|x|), \end{split}$$

and by (19) and (23), we have

$$\int_{\mathbb{S}^{d-1}} \int_{|x|}^{\infty} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r} \mathrm{d}r \mathrm{d}\theta$$

$$\begin{split} &= \int_{\mathbb{S}^{d-1}} \int_{|x|}^{\infty} \frac{|\langle r\theta, \nabla f(x+r\theta) \rangle|}{r^2} \mathrm{d}r \mathrm{d}\theta \\ &\leq \int_{\mathbb{S}^{d-1}} \int_{|x|}^{\infty} \frac{|\langle (x+r\theta), \nabla f(x+r\theta) \rangle| + |\langle x, \nabla f(x+r\theta) \rangle|}{r^2} \mathrm{d}r \mathrm{d}\theta \\ &\leq C \int_{\mathbb{S}^{d-1}} \int_{|x|}^{\infty} \frac{1+|x+r\theta|^{\beta}+|x|}{r^2} \mathrm{d}r \mathrm{d}\theta \\ &\leq C \int_{|x|}^{\infty} \frac{1+|x|+r^{\beta}}{r^2} \leq C(1+|x|)|x|^{-1}+|x|^{\beta-1} \leq C. \end{split}$$

When  $\alpha \in (0,1)$  and |x| < 1, we have

$$\int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r^{\alpha}} dr d\theta$$

$$= \int_{\mathbb{S}^{d-1}} \int_{0}^{1} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r^{\alpha}} dr d\theta + \int_{\mathbb{S}^{d-1}} \int_{1}^{\infty} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r^{\alpha}} dr d\theta.$$

By (25), we have

$$\int_{\mathbb{S}^{d-1}} \int_0^1 \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r^\alpha} \mathrm{d}r \mathrm{d}\theta \leqslant C \int_0^1 \frac{1}{r^\alpha} \mathrm{d}r \mathrm{d}\theta \leqslant C,$$

and by (25) and (29), we have

$$\int_{\mathbb{S}^{d-1}} \int_{1}^{\infty} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r^{\alpha}} dr d\theta$$

$$= \int_{\mathbb{S}^{d-1}} \int_{1}^{\infty} \frac{|\langle r\theta, \nabla f(x+r\theta) \rangle|}{r^{\alpha+1}} dr d\theta$$

$$\leq \int_{\mathbb{S}^{d-1}} \int_{1}^{\infty} \frac{|\langle (x+r\theta), \nabla f(x+r\theta) \rangle| + |\langle x, \nabla f(x+r\theta) \rangle|}{r^{\alpha+1}} dr d\theta$$

$$\leq C \int_{\mathbb{S}^{d-1}} \int_{1}^{\infty} \frac{1 + |x+r\theta|^{\beta} + |x|}{r^{\alpha+1}} dr d\theta \leq C \int_{1}^{\infty} \frac{1 + |x| + r^{\beta}}{r^{\alpha+1}} \leq C(1+|x|). \quad (38)$$

When  $\alpha \in (0,1)$  and  $|x| \ge 1$ , we have

$$\begin{split} &\int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r^{\alpha}} \mathrm{d}r \mathrm{d}\theta \\ &= \int_{\mathbb{S}^{d-1}} \int_{0}^{|x|} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r^{\alpha}} \mathrm{d}r \mathrm{d}\theta + \int_{\mathbb{S}^{d-1}} \int_{|x|}^{\infty} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r^{\alpha}} \mathrm{d}r \mathrm{d}\theta. \end{split}$$

By (25), we have

$$\int_{\mathbb{S}^{d-1}} \int_0^{|x|} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r^{\alpha}} \mathrm{d}r \mathrm{d}\theta \leqslant C \int_0^{|x|} \frac{1}{r^{\alpha}} \mathrm{d}r \mathrm{d}\theta \leqslant C |x|^{1-\alpha},$$

and by (25) and (29), we have

$$\begin{split} &\int_{\mathbb{S}^{d-1}} \int_{|x|}^{\infty} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r^{\alpha}} \mathrm{d}r \mathrm{d}\theta \\ &= \int_{\mathbb{S}^{d-1}} \int_{|x|}^{\infty} \frac{|\langle r\theta, \nabla f(x+r\theta)|}{r^{\alpha+1}} \mathrm{d}r \mathrm{d}\theta \\ &\leq \int_{\mathbb{S}^{d-1}} \int_{|x|}^{\infty} \frac{|\langle (x+r\theta), \nabla f(x+r\theta) \rangle| + |\langle x, \nabla f(x+r\theta) \rangle|}{r^{\alpha+1}} \mathrm{d}r \mathrm{d}\theta \end{split}$$

$$\leq C \int_{\mathbb{S}^{d-1}} \int_{|x|}^{\infty} \frac{1 + |x + r\theta|^{\beta} + |x|}{r^{\alpha+1}} dr d\theta 
\leq C \int_{|x|}^{\infty} \frac{1 + |x| + r^{\beta}}{r^{\alpha+1}} \leq C(1 + |x|) |x|^{-\alpha} + |x|^{\beta-\alpha} \leq C(1 + |x|^{1-\alpha}).$$

The proof is complete.

**Lemma 10** Let  $\alpha \in (0, \frac{1}{2}]$  and  $h \in \mathcal{H}_{\beta} \cap \mathcal{F}_{\beta}$  with  $\beta \in (0, \alpha)$ . Let f be defined as (11) and denote  $\tilde{\beta} := \max\{\beta, d - \gamma\} \in (0, \alpha)$ . Then, for any  $x \in \mathbb{R}^d$ ,

$$\int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r^{\alpha}} dr d\theta \leqslant C(1+|x|^{\tilde{\beta}-\alpha}).$$

*Proof* When |x| < 1, the proof is similar to the proof of Lemma 9. When  $|x| \ge 1$ , Proposition 4 implies that

$$\begin{split} &\int_{\mathbb{S}^{d-1}} \int_0^\infty \frac{|\langle \theta, \nabla f(x+r\theta) \rangle|}{r^\alpha} \mathrm{d}r \mathrm{d}\theta \\ &\leqslant C \int_{\mathbb{S}^{d-1}} \int_0^\infty \frac{|x+r\theta|^{\tilde{\beta}-1}}{r^\alpha} \mathrm{d}r \mathrm{d}\theta \\ &\leqslant C \int_0^\infty \frac{||x|-r|^{\tilde{\beta}-1}}{r^\alpha} \mathrm{d}r = C \left[ \int_0^{|x|} \frac{(|x|-r)^{\tilde{\beta}-1}}{r^\alpha} \mathrm{d}r + \int_{|x|}^\infty \frac{(r-|x|)^{\tilde{\beta}-1}}{r^\alpha} \mathrm{d}r \right]. \end{split}$$

One can write

$$\int_{0}^{|x|} \frac{(|x|-r)^{\tilde{\beta}-1}}{r^{\alpha}} dr = \int_{0}^{\frac{|x|}{2}} \frac{(|x|-r)^{\tilde{\beta}-1}}{r^{\alpha}} dr + \int_{\frac{|x|}{2}}^{|x|} \frac{(|x|-r)^{\tilde{\beta}-1}}{r^{\alpha}} dr 
\leq 2^{1-\tilde{\beta}} |x|^{\tilde{\beta}-1} \int_{0}^{\frac{|x|}{2}} \frac{1}{r^{\alpha}} dr + 2^{\alpha} |x|^{-\alpha} \int_{\frac{|x|}{2}}^{|x|} (|x|-r)^{\tilde{\beta}-1} dr 
\leq C|x|^{\tilde{\beta}-\alpha},$$

whereas

$$\int_{|x|}^{\infty} \frac{(r-|x|)^{\tilde{\beta}-1}}{r^{\alpha}} dr = \frac{1}{\tilde{\beta}} \int_{|x|}^{\infty} \frac{1}{r^{\alpha}} d(r-|x|)^{\tilde{\beta}} = \frac{\alpha}{\tilde{\beta}} \int_{|x|}^{\infty} \frac{(r-|x|)^{\tilde{\beta}}}{r^{\alpha+1}} dr \\ \leqslant \frac{\alpha}{\tilde{\beta}(\alpha-\beta)} |x|^{\tilde{\beta}-\alpha}.$$

The proof is complete.

## 4.2 Taylor-like expansion

In order to prove the main Theorem, we shall make use of the following lemmas.

•  $\alpha \in (1,2)$  :

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**Lemma 11** Consider  $\alpha \in (1,2)$ . Let X be a d-dimensional random vector with density function  $p_X(r)dr\nu(d\theta)$  and Y be a d-dimensional random vector, which is independent of X. For any a > 0 and f is defined as (11), denote

$$T_1 := \left| \mathbb{E}[\langle X, \nabla f(Y + aX) \rangle - \langle X, \nabla f(Y) \rangle] - \frac{\alpha^2}{d\alpha} a^{\alpha - 1} \mathbb{E}[\mathcal{L}^{\alpha, \nu} f(Y)] \right|,$$

then, we have

$$T_1 \leqslant C \left( a \int_0^{a^{-1}} r^2 \left| \frac{\alpha}{r^{\alpha+1}} - p_X(r) \right| dr + \int_{a^{-1}}^{\infty} \left| \frac{\alpha}{r^{\alpha}} - r p_X(r) dr \right| \right).$$

Proof From Lemma 8, we have

$$\frac{\alpha^2}{d_{\alpha}}a^{\alpha-1}\mathbb{E}[\mathcal{L}^{\alpha,\nu}f(Y)] = \alpha\mathbb{E}\Big[\int_{\mathbb{S}^{d-1}}\int_0^{\infty}\frac{\langle r\theta,\nabla f(Y+ar\theta)\rangle - \langle r\theta,\nabla f(Y)\rangle}{r^{\alpha+1}}dr\nu(d\theta)\Big],$$

and

$$\mathbb{E}[\langle X, \nabla f(Y+aX) \rangle - \langle X, \nabla f(Y) \rangle]$$

$$= \mathbb{E}\Big[\int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \left( \langle r\theta, \nabla f(Y+ar\theta) \rangle - \langle r\theta, \nabla f(Y) \rangle \right) p_{X}(r) dr \nu(d\theta) \Big].$$

These imply

$$T_{1} \leqslant \mathbb{E} \Big| \int_{\mathbb{S}^{d-1}} \int_{0}^{a^{-1}} \langle r\theta, \nabla f(Y + ar\theta) - \nabla f(Y) \rangle \Big[ \frac{\alpha}{r^{\alpha+1}} dr \nu(d\theta) - p_{X}(r) dr \nu(d\theta) \Big] \Big|$$

$$+ \mathbb{E} \Big| \int_{\mathbb{S}^{d-1}} \int_{a^{-1}}^{\infty} \langle r\theta, \nabla f(Y + ar\theta) - \nabla f(Y) \rangle \Big[ \frac{\alpha}{r^{\alpha+1}} dr \nu(d\theta) - p_{X}(r) dr \nu(d\theta) \Big] \Big|$$

$$:= \mathcal{I}_{1} + \mathcal{I}_{2}.$$

$$(39)$$

Then, one can write by (17) that

$$\mathcal{I}_1 \leqslant Ca \int_{\mathbb{S}^{d-1}} \int_0^{a^{-1}} r^2 \left| \frac{\alpha}{r^{\alpha+1}} dr - p_X(r) dr \right| \nu(d\theta) \leqslant Ca \int_0^{a^{-1}} r^2 \left| \frac{\alpha}{r^{\alpha+1}} - p_X(r) \right| dr,$$
 whereas by (16)

$$\mathcal{I}_{2} \leqslant C \int_{\mathbb{S}^{d-1}} \int_{a^{-1}}^{\infty} \left| \frac{\alpha}{r^{\alpha}} - rp_{X}(r)dr \right| \nu(d\theta) \leqslant C \int_{a^{-1}}^{\infty} \left| \frac{\alpha}{r^{\alpha}} - rp_{X}(r)dr \right|,$$

the desired result follows.

•  $\underline{\alpha=1}$ :

**Lemma 12** Consider  $\alpha = 1$  and  $\int_{\mathbb{S}^{d-1}} \theta \nu(d\theta) = 0$ . Let X be a d-dimensional random vector with density function  $p_X(r)dr\nu(d\theta)$  and suppose that  $p_X(r)$  is non-increasing. Let Y be a d-dimensional random vector, which is independent of X. For any a > 0 and f is defined as (11), denote

$$T_2 := \left| \mathbb{E}[\langle X, \nabla f(Y + aX) \rangle - \langle X, \nabla f(Y) \mathbf{1}_{(0,1]}(a|X|) \rangle] - \frac{1}{d_*} \mathbb{E}[\mathcal{L}^{1,\nu} f(Y)] \right|,$$

then, we have

$$T_{2} \leqslant C \left( a \int_{0}^{a^{-1}} r^{2} \left( 1 - \log(ar) \right) \left| \frac{\alpha}{r^{2}} - p_{X}(r) \right| dr + a^{\beta - 1} \int_{a^{-1}}^{\infty} t^{\beta} \left| d \left[ \frac{\alpha}{t} - t p_{X}(t) \right] \right| \right).$$

*Proof* By the same argument as the proof of (39), we have

$$T_{2} \leqslant \mathbb{E} \Big| \int_{\mathbb{S}^{d-1}} \int_{0}^{a^{-1}} \langle r\theta, \nabla f(Y + ar\theta) - \nabla f(Y) \rangle \Big] \Big[ \frac{1}{r^{2}} dr \nu(d\theta) - p_{X}(r) dr \nu(d\theta) \Big] \Big|$$

$$+ \mathbb{E} \Big| \int_{\mathbb{S}^{d-1}} \int_{a^{-1}}^{\infty} \langle r\theta, \nabla f(Y + ar\theta) \rangle \Big[ \frac{1}{r^{2}} dr \nu(d\theta) - p_{X}(r) dr \nu(d\theta) \Big] \Big|$$

$$:= \mathcal{J}_{1} + \mathcal{J}_{2}.$$

$$(40)$$

On the one hand, (21) derives that

$$\mathcal{J}_{1} \leqslant \mathbb{E} \Big[ \int_{\mathbb{S}^{d-1}} \int_{0}^{a^{-1}} |\langle r\theta, \nabla f(Y + ar\theta) - \nabla f(Y) \rangle| \Big| \frac{1}{r^{2}} dr - p_{X}(r) dr \Big| \nu(d\theta) \Big] \\
\leqslant Ca \int_{\mathbb{S}^{d-1}} \int_{0}^{a^{-1}} r^{2} \Big( 1 - \log(ar) \Big) \Big| \frac{1}{r^{2}} - p_{X}(r) \Big| dr \nu(d\theta) \\
\leqslant Ca \int_{0}^{a^{-1}} r^{2} \Big( 1 - \log(ar) \Big) \Big| \frac{1}{r^{2}} - p_{X}(r) \Big| dr.$$

On the other hand, noting that  $p_X(r)$  is non-increasing and  $\int_0^\infty p_X(r)dr < \infty$ , which imply  $\lim_{r\to\infty} rp_X(r) = 0$ . So we have by integration by parts that

$$\begin{split} \mathcal{J}_2 = & \mathbb{E} \Big| \int_{\mathbb{S}^{d-1}} \int_{a^{-1}}^{\infty} \langle \theta, \nabla f(Y + ar\theta) \rangle \Big[ \frac{1}{r} - r p_X(r) \Big] dr \nu(d\theta) \Big| \\ = & \mathbb{E} \Big| \int_{\mathbb{S}^{d-1}} \int_{a^{-1}}^{\infty} \langle \theta, \nabla f(Y + ar\theta) \rangle dr \int_{r}^{\infty} d \Big[ \frac{1}{t} - t p_X(t) \Big] \nu(d\theta) \Big| \\ = & \mathbb{E} \Big| \int_{\mathbb{S}^{d-1}} \int_{a^{-1}}^{\infty} d \Big[ \frac{1}{t} - t p_X(t) \Big] \int_{a^{-1}}^{t} \langle \theta, \nabla f(Y + ar\theta) \rangle dr \nu(d\theta) \Big| \\ = & a^{-1} \mathbb{E} \Big| \int_{\mathbb{S}^{d-1}} \int_{a^{-1}}^{\infty} \Big( f(Y + at\theta) - f(Y + \theta) \Big) d \Big[ \frac{1}{t} - t p_X(t) \Big] \nu(d\theta) \Big|, \end{split}$$

then we have by (22)

$$\mathcal{J}_{2} \leqslant Ca^{\beta-1} \int_{\mathbb{S}^{d-1}} \int_{a^{-1}}^{\infty} t^{\beta} \left| d \left[ \frac{1}{t} - t p_{X}(t) \right] \right| \nu(d\theta) \leqslant Ca^{\beta-1} \int_{a^{-1}}^{\infty} t^{\beta} \left| d \left[ \frac{1}{t} - t p_{X}(t) \right] \right|, \tag{41}$$

the desired conclusion follows.

•  $\alpha \in (0,1)$ : For any  $x \in \mathbb{R}^d$ , we have

$$\int_{\mathbb{S}^{d-1}} \int_0^1 \frac{\langle r\theta, \nabla f(x) \rangle}{r^{\alpha+1}} dr \nu(d\theta) = \frac{1}{1-\alpha} \int_{\mathbb{S}^{d-1}} \langle \theta, \nabla f(x) \rangle \nu(d\theta),$$

which follows that

$$\frac{1}{d_{\alpha}} \mathcal{L}^{\alpha} f(x) - \frac{1}{\alpha(1-\alpha)} \int_{\mathbb{S}^{d-1}} \langle \theta, \nabla f(Y) \rangle \nu(d\theta)$$

$$= \frac{1}{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \frac{\langle r\theta, \nabla f(x+r\theta) \rangle - \langle r\theta, \nabla f(x) \mathbf{1}_{(0,1]}(r) \rangle}{r^{\alpha+1}} dr \nu(d\theta). \tag{42}$$

According to (42), we have the following Taylor-like expansion.

**Lemma 13** Consider  $\alpha \in (0,1)$  and when  $\alpha \in (0,\frac{1}{2}]$ , we further assume  $h \in \mathcal{F}_{\beta}$ . Let X be a d-dimensional random vector with density function  $p_X(r)dr\nu(d\theta)$  and suppose that  $p_X(r)$  is non-increasing. Let Y be a d-dimensional random vector, which is independent of X. For any a > 0 and f is defined as (11), denote

$$T_{3} := \left| \mathbb{E}[\langle X, \nabla f(Y + aX) \rangle - \langle X, \nabla f(Y) \mathbf{1}_{(0,1]}(a|X|) \rangle] - \frac{\alpha^{2}}{d_{\alpha}} a^{\alpha - 1} \mathbb{E}[\mathcal{L}^{\alpha, \nu} f(Y) - \frac{d_{\alpha}}{\alpha(1 - \alpha)} \int_{\mathbb{S}^{d-1}} \langle \theta, \nabla f(Y) \nu(d\theta) \rangle] \right|,$$

then, we have

$$T_3 \leqslant C \Big( a^{\alpha} \int_0^{a^{-1}} r^{\alpha+1} \Big| \frac{\alpha}{r^{\alpha+1}} - p_X(r) \Big| dr + a^{\beta-1} \int_{a^{-1}}^{\infty} t^{\beta} \Big| d \Big[ \frac{\alpha}{t^{\alpha}} - t p_X(t) \Big] \Big| \Big).$$

*Proof* According to (42), by the same argument as the proof of (39), we have

$$T_{3} \leqslant \mathbb{E} \Big| \int_{\mathbb{S}^{d-1}} \int_{0}^{a^{-1}} [\langle r\theta, \nabla f(Y + ar\theta) - \nabla f(Y) \rangle] \Big[ \frac{\alpha}{r^{\alpha+1}} dr \nu(d\theta) - p_{X}(r) dr \nu(d\theta) \Big] \Big|$$

$$+ \mathbb{E} \Big| \int_{\mathbb{S}^{d-1}} \int_{a^{-1}}^{\infty} \langle r\theta, \nabla f(Y + ar\theta) \rangle \Big[ \frac{\alpha}{r^{\alpha+1}} dr \nu(d\theta) - p_{X}(r) dr \nu(d\theta) \Big] \Big|$$

$$:= \mathcal{I} + \mathcal{I} \mathcal{I}.$$

One can write by (27) that

$$\mathcal{I} \leqslant \mathbb{E} \Big[ \int_{\mathbb{S}^{d-1}} \int_{0}^{a^{-1}} |\langle r\theta, \nabla f(Y + ar\theta) \rangle - \langle r\theta, \nabla f(Y) \rangle| \Big| \frac{\alpha}{r^{\alpha+1}} dr - p_X(r) dr \Big| \nu(d\theta) \Big] \\
\leqslant Ca^{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{a^{-1}} r^{\alpha+1} \Big| \frac{\alpha}{r^{\alpha+1}} - p_X(r) \Big| dr \nu(d\theta) \\
= Ca^{\alpha} \int_{0}^{a^{-1}} r^{\alpha+1} \Big| \frac{\alpha}{r^{\alpha+1}} - p_X(r) \Big| dr,$$

whereas by the same argument as the proof of (41),

$$\mathcal{I}\mathcal{I} \leqslant Ca^{\beta-1} \int_{\mathbb{S}^{d-1}} \int_{a^{-1}}^{\infty} t^{\beta} \left| d \left[ \frac{\alpha}{t^{\alpha}} - t p_X(t) \right] \right| \nu(d\theta) = Ca^{\beta-1} \int_{a^{-1}}^{\infty} t^{\beta} \left| d \left[ \frac{\alpha}{t^{\alpha}} - t p_X(t) \right] \right|,$$
 the desired conclusion follows.

#### 4.3 Truncation for random variable X

In the case  $\alpha \in (0,1]$ , the random variable X considered here satisfies  $\mathbb{E}|X|^{\alpha} = \infty$ . Therefore, we need to truncate the random variable X.

**Lemma 14** Consider  $\alpha \in (0,1]$  and when  $\alpha = 1$  we assume  $\int_{\mathbb{S}^{d-1}} \theta \nu(d\theta) = 0$ . Let X be a d-dimensional random vector with density function  $p_X(r)dr\nu(d\theta)$  and f be defined as (11). Then for any 0 < a < 1 and  $z \in \mathbb{R}^d$ , we have 1.) when  $\alpha = 1$ ,

$$\mathbb{E} |\mathcal{L}^{1,\nu} f(z) - \mathcal{L}^{1,\nu} f(z + aX)| \leq C \Big( \int_{a^{-1}}^{\infty} p_X(r) dr + a \int_{0}^{a^{-1}} r (1 - \log(ar)) p_X(r) dr \Big).$$

2.) when  $\alpha \in (0, 1)$ ,

$$\mathbb{E}\left|\mathcal{L}^{\alpha,\nu}f(z) - \mathcal{L}^{\alpha,\nu}f(z+aX)\right| \leqslant C\left(\int_{a^{-1}}^{\infty}p_X(r)dr + a^{\frac{1+\alpha}{2}}\int_{0}^{a^{-1}}r^{\frac{1+\alpha}{2}}p_X(r)dr\right).$$

Proof Observe

$$\begin{split} & \mathbb{E}\Big[ \big| \mathcal{L}^{\alpha,\nu} f(z) - \mathcal{L}^{\alpha,\nu} f(z+aX) \big| \Big] \\ = & \mathbb{E}\Big[ \big| \mathcal{L}^{\alpha,\nu} f(z) - \mathcal{L}^{\alpha,\nu} f(z+aX) \big| \big[ \mathbf{1}_{(a^{-1},\infty)}(|X|) + \mathbf{1}_{((0,a^{-1}))}(|X|) \big] \Big] := \mathrm{I} + \mathrm{II}. \end{split}$$

When  $\alpha = 1$ , one can write by (20)

$$I \leqslant C\mathbb{P}(|X| > a^{-1}) = C \int_{\mathbb{S}^{d-1}} \int_{a^{-1}}^{\infty} p_X(r) dr \nu(d\theta) \leqslant C \int_{a^{-1}}^{\infty} p_X(r) dr,$$

whereas by (24)

$$II \leqslant Ca\mathbb{E}[|X|(1 - \log(a|X|))\mathbf{1}_{(0,a^{-1})}(|X|)] \leqslant Ca\int_0^{a^{-1}} r(1 - \log(ar))p_X(r)dr.$$

When  $\alpha \in (0,1)$ , one can write by (26)

$$\mathbf{I} \leqslant C_{\alpha,\beta} \mathbb{P}(|X| > a^{-1}) = C \int_{\mathbb{S}^{d-1}} \int_{a^{-1}}^{\infty} p_X(r) dr \nu(d\theta) = C \int_{a^{-1}}^{\infty} p_X(r) dr,$$

whereas by (30) with  $\eta = \frac{1+\alpha}{2} \in (\alpha, 1)$ 

$$II \leqslant Ca^{\frac{1+\alpha}{2}} \mathbb{E}\left[|X|^{\frac{1+\alpha}{2}} \mathbf{1}_{(0,a^{-1})}(|X|)\right] = Ca^{\frac{1+\alpha}{2}} \int_0^{a^{-1}} r^{\frac{1+\alpha}{2}} p_X(r) dr,$$

the desired conclusion follows.

## 4.4 the proof of Theorem 5

Now, we are ready to use the Leave-one-out method to prove our second main result.

Proof of Theorem 5 By Eq. (9), we have

$$\alpha \left| \mathbb{E}[h(S_n)] - \pi(h) \right| = \left| \mathbb{E}[\alpha \mathcal{L}^{\alpha,\nu} f(S_n) - \langle S_n, \nabla f(S_n) \rangle] \right| \leqslant \mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3,$$

where

$$\mathcal{N}_1 = \frac{\alpha}{n} \sum_{i=1}^n \left| \mathbb{E} \left[ (\mathcal{L}^{\alpha, \nu} f)(S_n(i)) - \mathbb{E} \left[ (\mathcal{L}^{\alpha, \nu} f)(S_n) \right] \right|.$$

If  $\alpha \in (1, 2)$ ,

$$\mathcal{N}_{2} = l_{n}^{-\frac{1}{\alpha}} \sum_{i=1}^{n} \left| \mathbb{E} \left[ \langle \eta_{n,i}, \nabla f(S_{n}(i) + l_{n}^{-\frac{1}{\alpha}} \eta_{n,i}) \rangle \right] - \mathbb{E} \left[ \langle \eta_{n,i}, \nabla f(S_{n}(i)) \rangle \right] - \frac{\alpha^{2}}{d_{\alpha}} l_{n}^{\frac{1-\alpha}{\alpha}} \mathbb{E} \left[ (\mathcal{L}^{\alpha,\nu} f)(S_{n}(i)) \right] \right|$$

$$\mathcal{N}_3 = l_n^{-\frac{1}{\alpha}} \sum_{i=1}^n \left| \mathbb{E} \left[ \eta_{n,i} \right] \right| \left| \mathbb{E} \left[ \nabla f(S_n(i)) - \nabla f \left( S_n(i) + l_n^{-\frac{1}{\alpha}} \eta_{n,i} \right) \right] \right|;$$

If 
$$\alpha = 1$$

$$\mathcal{N}_{2} = l_{n}^{-1} \sum_{i=1}^{n} \left| \mathbb{E} \left[ \langle \eta_{n,i}, \nabla f(S_{n}(i) + l_{n}^{-1} \eta_{n,i}) \rangle \right] - \mathbb{E} \left[ \langle \eta_{n,i}, \nabla f(S_{n}(i)) \mathbf{1}_{(0,l_{n}]}(|\eta_{n,i}|) \rangle \right] - \frac{1}{d_{1}} \mathbb{E} \left[ (\mathcal{L}^{1,\nu} f)(S_{n}(i)) \right] \right|,$$

$$\mathcal{N}_{3} = l_{n}^{-1} \sum_{i=1}^{n} \left| \mathbb{E} \left[ \eta_{n,i} \mathbf{1}_{(0,l_{n}]}(|\eta_{n,i}|) \right] \right| \left| \mathbb{E} \left[ \nabla f(S_{n}(i)) - \nabla f \left( S_{n}(i) + l_{n}^{-1} \eta_{n,i} \right) \right] \right|;$$

If  $\alpha \in (0,1)$ ,

$$\mathcal{N}_{2} = l_{n}^{-\frac{1}{\alpha}} \sum_{i=1}^{n} \left| \mathbb{E} \left[ \langle \eta_{n,i}, \nabla f(S_{n}(i) + l_{n}^{-\frac{1}{\alpha}} \eta_{n,i}) \rangle \right] - \mathbb{E} \left[ \langle \eta_{n,i}, \nabla f(S_{n}(i)) \mathbf{1}_{(0,l_{n}^{\frac{1}{\alpha}})} (|\eta_{n,i}|) \rangle \right] - \frac{\alpha^{2}}{d_{\alpha}} l_{n}^{\frac{1-\alpha}{\alpha}} \mathbb{E} \left[ (\mathcal{L}^{\alpha,\nu} f)(S_{n}(i)) - \frac{d_{\alpha}}{\alpha(1-\alpha)} \int_{\mathbb{S}^{d-1}} \langle \theta, \nabla f(S_{n}(i)) \rangle \nu(d\theta) \right] \right|,$$

$$\mathcal{N}_{3} = l_{n}^{-\frac{1}{\alpha}} \sum_{i=1}^{n} \left| \frac{\alpha}{1-\alpha} l_{n}^{\frac{1-\alpha}{\alpha}} \mathbb{E} \left[ \int_{\mathbb{S}^{d-1}} \langle \theta, \nabla f(S_{n}(i)) \rangle \nu(d\theta) \right] - \mathbb{E} \left[ \langle \eta_{n,i}, \nabla f(S_{n}(i)) \mathbf{1}_{(0,l_{n}^{\frac{1}{\alpha}})} (|\eta_{n,i}|) \rangle \right] \right|.$$

1) When  $\alpha \in (1,2)$ , we have by (18)

$$\mathcal{N}_1 \leqslant C l_n^{-\frac{2}{\alpha}} \sum_{i=1}^n \mathbb{E} |\eta_{n,i}|^{2-\alpha}$$

and Lemma 11 implies that

$$\mathcal{N}_2 \leqslant C \sum_{i=1}^n \left( l_n^{-\frac{2}{\alpha}} \int_0^{l_n^{\frac{1}{\alpha}}} r^2 \left| \frac{\alpha}{r^{\alpha+1}} - p_{\eta_{n,i}}(r) \right| dr + l_n^{-\frac{1}{\alpha}} \int_{l_n^{\frac{1}{\alpha}}}^{\infty} \left| \frac{\alpha}{r^{\alpha}} - r p_{\eta_{n,i}}(r) dr \right| \right).$$

For the third term, one can derive from (17) that

$$\mathcal{N}_3 \leqslant C l_n^{-\frac{2}{\alpha}} \sum_{i=1}^n (\mathbb{E}|\eta_{n,i}|)^2$$
.

2) When  $\alpha=1$  and  $\int_{\mathbb{S}^{d-1}}\theta\nu(d\theta)=0$ , we have by Lemma 14

$$\mathcal{N}_1 \leqslant \frac{C}{n} \sum_{i=1}^n \left( l_n^{-1} \int_0^{l_n} r \left( 1 - \log(l_n^{-1} r) \right) p_{\eta_{n,i}}(r) dr + \int_{l_n}^{\infty} p_{\eta_{n,i}}(r) dr \right).$$

By Lemma 12, we have

$$\mathcal{N}_{2} \leqslant C l_{n}^{-1} \sum_{i=1}^{n} \left( l_{n}^{-1} \int_{0}^{l_{n}} r^{2} \left( 1 - \log(l_{n}^{-1}r) \right) \left| \frac{1}{r^{2}} - p_{\eta_{n,i}}(r) \right| dr + l_{n}^{1-\beta} \int_{l_{n}}^{\infty} t^{\beta} \left| d \left[ \frac{1}{t} - t p_{\eta_{n,i}}(t) \right] \right| \right).$$

In addition, noticing that  $\int_{\mathbb{S}^{d-1}} \theta \nu(d\theta) = 0$ , we have  $\mathcal{N}_3 = 0$ . 3) When  $\alpha \in (0,1)$ , we have by Lemma 14,

$$\mathcal{N}_1 \leqslant C \frac{\alpha}{n} \sum_{i=1}^n \left( l_n^{-\frac{\alpha+1}{2\alpha}} \int_0^{l_n^{\frac{1}{\alpha}}} r^{\frac{\alpha+1}{2}} p_{\eta_{n,i}}(r) dr + \int_{l_n^{\frac{1}{\alpha}}}^{\infty} p_{\eta_{n,i}}(r) dr \right).$$

By Lemma 13, we have

$$\mathcal{N}_{2} \leqslant C l_{n}^{-\frac{1}{\alpha}} \sum_{i=1}^{n} \left( l_{n}^{-1} \int_{0}^{l_{n}^{\frac{1}{\alpha}}} r^{\alpha+1} \left| \frac{\alpha}{r^{\alpha+1}} - p_{\eta_{n,i}}(r) \right| dr + l_{n}^{\frac{1-\beta}{\alpha}} \int_{l_{n}^{\frac{1}{\alpha}}}^{\infty} t^{\beta} \left| d \left[ \frac{\alpha}{t^{\alpha}} - t p_{\eta_{n,i}}(t) \right] \right| \right).$$

In addition, we have

$$\mathbb{E}\left[\left\langle \eta_{n,i}, \nabla f(S_n(i)) \mathbf{1}_{\left(0, l_n^{\frac{1}{\alpha}}\right]}(|\eta_{n,i}|) \right\rangle\right] = \int_{\mathbb{S}^{d-1}} \int_0^{l_n^{\frac{1}{\alpha}}} \left\langle r\theta, \nabla f(S_n(i)) \right\rangle p_{\eta_{n,i}}(r) dr \nu(d\theta),$$

which implies that

$$\mathcal{N}_{3} \leqslant l_{n}^{-\frac{1}{\alpha}} \sum_{i=1}^{n} \left| \mathbb{E} \left[ \int_{\mathbb{S}^{d-1}} \langle \theta, \nabla f(S_{n}(i)) \rangle \nu(d\theta) \right] \right| \left| \frac{\alpha}{1-\alpha} l_{n}^{\frac{1-\alpha}{\alpha}} - \int_{0}^{\frac{1}{\alpha}} r p_{\eta_{n,i}}(r) dr \right|$$
$$\leqslant \alpha l_{n}^{-\frac{1}{\alpha}} \sum_{i=1}^{n} \left| \int_{\mathbb{S}^{d-1}} \theta \nu(d\theta) \right| \left| \frac{\alpha}{1-\alpha} l_{n}^{\frac{1-\alpha}{\alpha}} - \int_{0}^{l_{n}^{\frac{1}{\alpha}}} r p_{\eta_{n,i}}(r) dr \right|.$$

Combining all of above, the desired conclusion follows.

# 5 Example: $\nu$ -Paretian distribution

In [13], Davydov and Nagaev considered the Pareto distribution  $\xi$ , that is, the density of the random variable  $\xi$  is

$$p_{\xi}(u) = \begin{cases} \alpha u^{-1-\alpha} & if \quad u \geqslant 1, \\ 0 & if \quad u < 1. \end{cases}$$

$$(43)$$

It is convenient to adhere the following definition.

**Definition 2** We call a distribution  $\nu$ -Paretian if it corresponds to a random vector  $\tau$  admitting the representation  $\xi \varepsilon$ , where  $\xi$  and  $\varepsilon$  are independent,  $\xi$  has the density (43) while  $\varepsilon$  is a random unit vector satisfying

$$P(\varepsilon \in E) = \nu(E),\tag{44}$$

where  $E \in \mathcal{B}_{\mathbb{S}^{d-1}}$ , the Borel sets of  $\mathbb{S}^{d-1}$ .

In [13], the authors assumed that  $\nu$  is symmetric and

$$m_{\nu} = \min_{e \in \mathbb{S}^{d-1}} \Sigma_{\alpha}(e, \nu) > 0,$$

where  $\Sigma_{\alpha}(e,\nu) = \int_{\mathbb{S}^{d-1}} |\langle e,\theta \rangle|^{\alpha} \nu(\mathrm{d}\theta)$ . That means the v-Paretian distribution is strictly d-dimensional. Consider a sequence of i.i.d. random vectors such that

$$\tau_i = ^d \tau, \qquad i = 1, 2, \cdots.$$

Set

$$T_n = n^{-1/\alpha} \sum_{i=1}^n \tau_i. \tag{45}$$

Then, [13] gave the following approximation of multidimensional stable law:

**Theorem 15** [13, Theorem 3.2] Let  $T_n$  be defined by (45). If the underlying distribution is  $\nu$ -Paretian then as  $n \to \infty$ 

$$\sup_{A \in \mathcal{B}(\mathbb{R}^d)} |\mathbb{P}(T \in A) - \mathbb{P}(T_n \in A)| = \mathbf{O}(n^{-\delta}),$$

where  $\delta = \frac{\min(\alpha, 2-\alpha)}{d+\alpha}$  and T has the stable distribution determined by the characteristic function  $Ee^{i\langle\lambda,T\rangle} = \exp(-\frac{\alpha}{d_{\alpha}}|\lambda|^{\alpha}\Sigma_{\alpha}(e_{\lambda}, \nu)), \ \lambda \in \mathbb{R}^{d}, d \geqslant 1.$ 

According to Theorem 5, here we can consider the more general  $\nu$  (see the assumptions in Lemma 2) and obtain a better convergence rate in Wasserstein(-type) distance.

**Theorem 16** Keep the same assumptions as in Lemma 2. Set

$$\zeta_{n,i} = \left(\frac{\alpha}{d_{\alpha}}\right)^{-\frac{1}{\alpha}} \frac{\tau_i}{n_{\alpha}^{\frac{1}{\alpha}}}$$

and

$$S_{n} = \begin{cases} \zeta_{n,1} - \mathbb{E}\zeta_{n,1} + \dots + \zeta_{n,n} - \mathbb{E}\zeta_{n,n}, & \alpha \in (1,2), \\ \zeta_{n,1} - \mathbb{E}\zeta_{n,1}\mathbf{1}_{(0,1]}(|\zeta_{n,1}|) + \dots + \zeta_{n,n} - \mathbb{E}\zeta_{n,n}\mathbf{1}_{(0,1]}(|\zeta_{n,n}|), & \alpha = 1, \\ \zeta_{n,1} + \zeta_{n,2} + \dots + \zeta_{n,n}, & \alpha \in (0,1). \end{cases}$$

Then,

$$d_W(\mathcal{L}(S_n), \pi) \leqslant Cn^{\frac{\alpha-2}{\alpha}}, \quad \alpha \in (1, 2),$$

and for any  $\beta \in (0, \alpha)$ ,

$$d_{W_{\beta}}\left(\mathcal{L}(S_n), \mu\right) \leqslant C \begin{cases} n^{-1} (\log n)^2, & \alpha = 1, \\ n^{-1} + \left| \int_{\mathbb{S}^{d-1}} \theta \nu(d\theta) \right| n^{\frac{\alpha - 1}{\alpha}}, & \alpha \in (\frac{1}{2}, 1), \end{cases}$$

$$\sup_{h \in \mathcal{H}_{\beta} \cap \mathcal{F}_{\beta}} \left| \mathbb{E}h(S_n) - \mu(h) \right| \leqslant C \left( n^{-1} + \left| \int_{\mathbb{S}^{d-1}} \theta \nu(d\theta) \right| n^{\frac{\alpha - 1}{\alpha}} \right), \quad \alpha \in (0, \frac{1}{2}].$$

*Proof* By definition 2, we obtain

$$p_{\tau_i}(r)dr\nu(d\theta) = \begin{cases} \frac{\alpha}{r^{\alpha+1}}dr\nu(d\theta), & r \geqslant 1, \\ 0, & r < 1. \end{cases}$$

Let  $\zeta_{n,i}=l_n^{-1/\alpha}\tau_i$  and  $\eta_{n,i}=l_n^{1/\alpha}\zeta_{n,i}=\tau_i,$  it follows that

$$p_{\eta_{n,i}}(r)dr\nu(d\theta) = \begin{cases} \frac{\alpha}{r^{\alpha+1}}dr\nu(d\theta), & r \geqslant 1, \\ 0, & r < 1. \end{cases}$$

According to Theorem 5,

i) When  $\alpha \in (1, 2)$ , we have

$$n^{-\frac{2}{\alpha}}\mathbb{E}|\eta_{n,i}|^{2-\alpha} + n^{-\frac{2}{\alpha}} \left(\mathbb{E}|\eta_{n,i}|\right)^2 = n^{-\frac{2}{\alpha}}\mathbb{E}|\tau_i|^{2-\alpha} + n^{-\frac{2}{\alpha}} \left(\mathbb{E}|\tau_i|\right)^2 \leqslant Cn^{-\frac{2}{\alpha}},$$

$$n^{-\frac{2}{\alpha}} \int_{0}^{\frac{1}{n}} r^{2} \left| \frac{\alpha}{r^{\alpha+1}} - p_{\eta_{n,i}}(r) \right| dr = n^{-\frac{2}{\alpha}} \int_{0}^{1} \frac{\alpha}{r^{\alpha-1}} dr = \frac{\alpha}{2 - \alpha} n^{-\frac{2}{\alpha}}$$

and

$$n^{-\frac{1}{\alpha}} \int_{l_n^{\frac{1}{\alpha}}}^{\infty} \left| \frac{\alpha}{r^{\alpha}} - r p_{\eta_{n,i}}(r) dr \right| = 0.$$

These inequalities imply  $d_W(\mathcal{L}(S_n), \pi) \leq n^{\frac{\alpha-2}{\alpha}}$ .

ii) When  $\alpha = 1$  and  $\int_{\mathbb{S}^{d-1}}^{\infty} \theta \nu(d\theta) = 0$ , we have

$$n^{-2} \int_0^{l_n} r (1 - \log(l_n^{-1}r)) p_{\eta_{n,i}}(r) dr \leqslant C n^{-2} (1 + \log n + (\log n)^2),$$

$$n^{-\beta} \int_l^{\infty} t^{\beta} |d[\frac{1}{t} - t p_{\eta_{n,i}}(t)]| = 0, \quad n^{-1} \int_l^{\infty} p_{\eta_{n,i}}(r) dr = n^{-2},$$

and

$$n^{-2} \int_0^{l_n} r^2 (2 - \log(l_n^{-1}r)) \left| \frac{1}{r^2} - p_{\eta_{n,i}}(r) \right| dr \leqslant C n^{-2} (1 + \log n).$$

Hence, we have

$$d_{W_{\beta}}(\mathcal{L}(S_n), \mu) \leqslant Cn^{-1}(\log n)^2$$
.

(iii) When  $\alpha \in (0,1)$ , we have

$$n^{-\frac{3\alpha+1}{2\alpha}} \int_0^{\frac{1}{\alpha}} r^{\frac{\alpha+1}{2}} p_{\eta_{n,i}}(r) dr + n^{-1} \int_{l_n^{\frac{1}{\alpha}}}^{\infty} p_{\eta_{n,i}}(r) dr \leqslant C n^{-2},$$

$$n^{-\frac{\beta}{\alpha}} \int_{l_n^{\frac{1}{\alpha}}}^{\infty} t^{\beta} \left| d \left[ \frac{\alpha}{t^{\alpha}} - t p_{\eta_{n,i}}(t) \right] \right| = 0,$$

$$n^{-\frac{1+\alpha}{\alpha}} \int_0^{l_n^{\frac{1}{\alpha}}} r^{\alpha+1} \left| \frac{\alpha}{r^{\alpha+1}} - p_{\eta_{n,i}}(r) \right| dr = \alpha n^{-\frac{1+\alpha}{\alpha}},$$

and

$$\mathcal{R}_{n,\alpha,i} \leqslant \frac{\alpha}{1-\alpha} \Big| \int_{\mathbb{S}^{d-1}} \theta \nu(d\theta) \Big| n^{-\frac{1}{\alpha}}.$$

Therefore, one can derive that

$$\sup_{h \in \mathcal{H}_{\beta} \cap \mathcal{F}_{\beta}} \left| \mathbb{E}h(S_n) - \mu(h) \right| \leqslant C \left( n^{-1} + \left| \int_{\mathbb{S}^{d-1}} \theta \nu(d\theta) \left| n^{\frac{\alpha - 1}{\alpha}} \right| \right), \quad \alpha \in (0, \frac{1}{2}]$$

and

$$d_W(\mathcal{L}(S_n), \mu) \leqslant C\left(n^{-1} + \left| \int_{\mathbb{S}^{d-1}} \theta \nu(d\theta) \left| n^{\frac{\alpha-1}{\alpha}} \right| \right), \quad \alpha \in (\frac{1}{2}, 1).$$

The proof is complete.

Remark 4 Let us compare our result with the known results in literatures. When the spectral measure  $\nu$  is symmetric, the authors of [13] obtained a rate  $n^{-\frac{\alpha}{d+\alpha}}$  for d dimensional stable law in total variation distance and conjectured that the rate can be improved to  $n^{-\min\{1,\frac{2-\alpha}{\alpha}\}}$  in  $L^1$  or total variation distance. Our results gives a positive answer to their conjecture for the Wasserstein(-type) distance.

# Appendix A Some auxiliary estimates

#### A.1 Moment estimate

**Lemma 17** Let  $(Z_t)_{t\geqslant 0}$  be the strictly  $\alpha$ -stable Lévy process with characteristic function  $\widehat{\pi}$ , which is defined in (1). Suppose that the assumption of Lemma 2 holds. Then for any  $\beta \in (0, \alpha)$ , there exists a constant C > 0 such that

$$\mathbb{E}|Z_1|^{\beta} < C.$$

Proof When  $d = \gamma$ , for any  $\beta \in (0, \alpha)$ , (12) implies

$$\mathbb{E}|Z_1|^{\beta} = \int_{\mathbb{R}^d} |x|^{\beta} p(x) dx \leqslant \int_{\mathbb{R}^d} \frac{|x|^{\beta}}{(1+|x|)^{\alpha+d}} dx$$

$$= \int_{\mathbb{S}^{d-1}} \int_0^1 \frac{r^{\beta} r^{d-1}}{(1+r)^{\alpha+d}} dr d\theta + \int_{\mathbb{S}^{d-1}} \int_1^{\infty} \frac{r^{\beta} r^{d-1}}{(1+r)^{\alpha+d}} dr d\theta$$

$$\leqslant C \left( \int_0^1 r^{\beta+d-1} dr + \int_1^{\infty} r^{\beta-\alpha-1} dr \right) \leqslant C.$$

For the general case, since

$$\int_{\mathbb{S}^{d-1}} \int_1^\infty \frac{(r\vee 1)^\beta}{r^{\alpha+1}} dt d\theta = \int_1^\infty \frac{r^\beta}{r^{\alpha+1}} dr = \frac{1}{\alpha-\beta},$$

according to [18, Theorem 25.3, Proposition 25.4 (ii) and (iii)], there exists a constants C>0 such that

$$\mathbb{E}\left(|Z_1|\vee 1\right)^{\beta}\leqslant C.$$

Hence, the desired result follow from the fact  $|x|^{\beta} \leq (|x| \vee 1)^{\beta}$  for any  $x \in \mathbb{R}^d$ .

# A.2 Heat Kernel Estimates of Rotationally Invariant α-stable Lévy process

Let p(t,x) be the transition probability density of rotationally invariant  $\alpha$ stable process  $Z_t$ , which has characteristic function  $e^{-t|\lambda|^{\alpha}}$ . It is well known
that

$$p(t,x) = t^{-d/\alpha} p(1, t^{-1/\alpha} x), \quad t > 0, \quad x \in \mathbb{R}^d.$$

Then, we have the following estimates:

**Lemma 18** Let p(x) be the probability density of  $Z_1$ , we have

$$p(x) \leqslant 2^{-d+1} \pi^{-\frac{d}{2}} \frac{\Gamma(d/\alpha)}{\alpha \Gamma(d/2)}, \qquad p(x) \leqslant \frac{\alpha 2^{\alpha - 1} \sin \frac{\alpha \pi}{2} \Gamma((d+\alpha)/2) \Gamma(\alpha/2)}{\pi^{d/2 + 1} |x|^{d + \alpha}},$$
$$|\nabla p(x)| \leqslant 2\pi |x| p_{(d+2)}(\tilde{x}) \tag{A1}$$

and

$$\|\nabla^2 p(x)\|_{\text{op}} \le 2\pi p_{(d+2)}(1,\tilde{x}) + 4\pi^2 |x|^2 p_{(d+4)}(\hat{x}),$$
 (A2)

where  $\tilde{x} \in \mathbb{R}^{d+2}$  satisfies  $|\tilde{x}| = |x|$ ,  $p_{d+2}(\tilde{x})$  is the density of the rotationally invariant  $\alpha$ -stable process  $Z_1$  in dimension d+2,  $\hat{x} \in \mathbb{R}^{d+4}$  satisfies  $|\hat{x}| = |x|$  and  $p_{d+4}(\hat{x})$  is the density of the rotationally invariant  $\alpha$ -stable process  $Z_1$  in dimension d+4.

Proof By the [18, Proposition 2.3 (XII)], we have

$$p(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\langle x, \lambda \rangle} e^{-|\lambda|^{\alpha}} dt = (2\pi)^{-d} \int_{\mathbb{R}^d} \cos(\langle x, \lambda \rangle) e^{-|\lambda|^{\alpha}} d\lambda,$$

since  $|\cos(\langle x, \lambda \rangle)| \leq 1$ , we have

$$\begin{split} p(x) &\leqslant (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{-|\lambda|^\alpha} \mathrm{d}\lambda \\ &= (2\pi)^{-d} \int_{\mathbb{S}^{d-1}} \int_0^\infty r^{d-1} \mathrm{e}^{-r^\alpha} \mathrm{d}r \mathrm{d}\theta \\ &= (2\pi)^{-d} V(\mathbb{S}^{d-1}) \int_0^\infty r^{d-1} \mathrm{e}^{-r^\alpha} \mathrm{d}r \\ &= (2\pi)^{-d} V(\mathbb{S}^{d-1}) \frac{1}{\alpha} \int_0^\infty y^{\frac{d}{\alpha} - 1} \mathrm{e}^{-y} \mathrm{d}y \\ &= (2\pi)^{-d} V(\mathbb{S}^{d-1}) \frac{\Gamma(d/\alpha)}{\alpha}, \end{split}$$

where the last second equality is by taking  $y = r^{\alpha}$ . Recall  $V(\mathbb{S}^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ , we have

$$p(x) \leqslant 2^{-d+1} \pi^{-\frac{d}{2}} \frac{\Gamma(d/\alpha)}{\alpha \Gamma(d/2)}.$$

Using the Fourier inversion theorem for radial functions [6, (2.1)], we have

$$p(x) = (2\pi)^{-\frac{d}{2}}|x|^{-\frac{d}{2}+1} \int_0^\infty e^{-t^{\alpha}} t^{\frac{d}{2}} J_{(d-2)/2}(|x|t) dt,$$

where  $J_m$  is the Bessel function of first kind of order m. Then, let r = |x|t in the above integral term, we have

$$p(x) = (2\pi)^{-\frac{d}{2}} |x|^{-d} \int_0^\infty e^{-(\frac{r}{|x|})^{\alpha}} r^{\frac{d}{2}} J_{(d-2)/2}(r) dr$$

From [5, section 7.2.8 (50)], we have  $\frac{\partial}{\partial t}(t^m J_m(t)) = t^m J_{m-1}(t)$ . Hence, we use integration by parts

$$\begin{split} p(x) &= \alpha (2\pi)^{-\frac{d}{2}} |x|^{-d-\alpha} \int_0^\infty \mathrm{e}^{-\left(\frac{r}{|x|}\right)^\alpha} r^{\frac{d}{2}+\alpha-1} J_{d/2}(r) \mathrm{d}r \\ &\leqslant \alpha (2\pi)^{-\frac{d}{2}} |x|^{-d-\alpha} \int_0^\infty r^{\frac{d}{2}+\alpha-1} J_{d/2}(r) \mathrm{d}r \\ &= \frac{\alpha 2^{\alpha-1} \sin \frac{\alpha \pi}{2} \Gamma((d+\alpha)/2) \Gamma(\alpha/2)}{\pi^{d/2+1} |x|^{d+\alpha}}, \end{split}$$

where the last equality comes from the proof of [6, Theorem 2.1].

Furthermore, from [7, (11)], we have  $\nabla p(x) = -2\pi x \hat{p}_{(d+2)}(\tilde{x})$ , so

$$|\nabla p(x)| = 2\pi |x| p_{(d+2)}(\tilde{x}),$$

and by the same argument as the proof of [7, (11)], we can also obtain

$$\|\nabla^2 p(x)\|_{\text{op}} \le 2\pi p_{(d+2)}(\tilde{x}) + 4\pi^2 |x|^2 p_{(d+4)}(\hat{x}),$$

the desired conclusions follow.

# Appendix B Proof of Proposition 1

**Lemma 19** Let  $(Q_t)_{t\geqslant 0}$  be a Markovian semigroup with transition density  $q(t,x,y)=p_{1-e^{-t}}(y-e^{-\frac{t}{\alpha}}x)$ . Then for any  $h\in Lip(1)$  in the case  $\alpha>1$  and  $h\in\mathcal{H}_{\beta}$  for some  $\beta<\alpha$  in the case  $\alpha\leqslant 1$ , we have

$$\partial_t Q_t h(x) = \mathcal{A}^{\alpha,\nu} Q_t h(x).$$

Proof Recall that  $q(t, x, y) = p_{1-e^{-t}}(y - e^{-\frac{t}{\alpha}}x)$  and  $s(t) = 1 - e^{-t}$ . Then

$$\begin{split} \left| \frac{\partial}{\partial t} q(t,x,y) \right| &= \left| e^{-t} \frac{\partial}{\partial s(t)} p_{1-e^{-t}} (y - e^{-\frac{t}{\alpha}} x) + \alpha^{-1} e^{-\frac{t}{\alpha}} x \frac{\partial}{\partial y} p_{1-e^{-t}} (y - e^{-\frac{t}{\alpha}} x) \right| \\ &\leq \frac{C(1-e^{-t})^{(\gamma-d)/\alpha}}{((1-e^{-t})^{1/\alpha} + |y - e^{-\frac{t}{\alpha}} x|)^{\alpha+\gamma}} \\ &\quad + \frac{|x|}{\alpha} e^{-\frac{t}{\alpha}} \frac{C(1-e^{-t})^{(\gamma-d)/\alpha}}{((1-e^{-t})^{1/\alpha} + |y - e^{-\frac{t}{\alpha}} x|)^{\alpha+\gamma}} \\ &\leq \frac{C(1+\frac{|x|}{\alpha} e^{-\frac{t}{\alpha}})(1-e^{-t})^{(\gamma-d)/\alpha}}{((1-e^{-t})^{1/\alpha} + |y - e^{-\frac{t}{\alpha}} x|)^{\alpha+\gamma}}, \end{split}$$

where the first inequality above follows from [8, Theorem 1.2]. Thus, for t > 0, s > 0 small enough such that  $(1 - e^{-s/\alpha})|x| \le \frac{1}{2}(e^t - 1)^{1/\alpha}$ ,

$$|q(t+s,x,y) - q(t,x,y)| \le s \frac{C(1 + \frac{|x|}{\alpha}e^{-\frac{t}{\alpha}})(1 - e^{-t})^{(\gamma - d)/\alpha}}{((1 - e^{-t})^{1/\alpha} + |y - e^{-\frac{t}{\alpha}}x|)^{\alpha + \gamma}}.$$

In addition, according to (7) and (4), we have

$$\partial_t q(t, x, y) = \mathcal{A}^{\alpha, \nu} q(t, x, y).$$
 (B3)

Hence, since  $\alpha + \gamma > d$ , using dominated convergence theorem, (B3) and Fubini's theorem, we have

$$\begin{split} \partial_t Q_t h(x) &= \partial_t \int_{\mathbb{R}^d} q(t,x,y) h(y) dy = \int_{\mathbb{R}^d} \partial_t q(t,x,y) h(y) dy \\ &= \int_{\mathbb{R}^d} \mathcal{A}^{\alpha,\nu} q(t,x,y) h(y) dy = \mathcal{A}^{\alpha,\nu} \int_{\mathbb{R}^d} q(t,x,y) h(y) dy = \mathcal{A}^{\alpha,\nu} Q_t h(x), \end{split}$$

the desired conclusion follows.

*Proof of Proposition* 1. From Remark 1 (i), we know that f is well defined. Observing

$$Q_{t}h(x) = \int_{\mathbb{R}^{d}} p_{1-e^{-t}}(y - e^{-\frac{t}{\alpha}}x)h(y)dy = \int_{\mathbb{R}^{d}} p_{1}(y)h((1 - e^{-t})^{\frac{1}{\alpha}}y + e^{-\frac{t}{\alpha}}x)dy.$$
(B4)

When  $\alpha \in (1,2)$ , since  $h \in \text{Lip}_1$ 

$$\left| h\left( (1 - e^{-t})^{\frac{1}{\alpha}} y + e^{-\frac{t}{\alpha}} (x+z) \right) - h\left( (1 - e^{-t})^{\frac{1}{\alpha}} y + e^{-\frac{t}{\alpha}} x \right) \right| \leqslant e^{-\frac{t}{\alpha}} |z|,$$

which immediately implies

$$\left| Q_t h(x+z) - Q_t h(x) \right| \leqslant \int_{\mathbb{R}^d} p_1(y) e^{-\frac{t}{\alpha}} |z| dy = e^{-\frac{t}{\alpha}} |z|.$$
 (B5)

Recall  $\mathcal{A}^{\alpha,\nu}f(x) = \mathcal{L}^{\alpha,\nu}f(x) - \frac{1}{\alpha}\langle x, \nabla f(x)\rangle$ . By (B5), using the dominated convergence theorem, we get that

$$\nabla f(x) = -\int_0^\infty \nabla Q_t h(x) \, dt.$$

Furthermore, we have by (13)

$$\begin{split} \left| \nabla_x p_{1-e^{-t}}(y - e^{-\frac{t}{\alpha}} x) \right| = & \left| e^{-\frac{t}{\alpha}} \nabla_y p_{1-e^{-t},\beta}(y - e^{-\frac{t}{\alpha}} x) \right| \\ \leqslant & \frac{C(e^t - 1)^{-1/\alpha} (1 - e^{-t})}{((1 - e^{-t})^{1/\alpha} + |y - e^{-\frac{t}{\alpha}} x|)^{\alpha + \gamma}}, \end{split}$$

then for  $x \in \mathbb{R}^d$ ,  $z \in \mathbb{R}^d$  such that  $|z| \leqslant \frac{1}{2}(e^t - 1)^{\frac{1}{\alpha}}$ ,

$$\begin{aligned} & \left| p_{1-e^{-t}} \left( y - e^{-\frac{t}{\alpha}} (x+z) \right) - p_{1-e^{-t}} (y - e^{-\frac{t}{\alpha}} x) \right| \\ \leqslant & \left| z \right| \frac{C 2^{d+\alpha} (e^t - 1)^{-1/\alpha} (1 - e^{-t})}{((1 - e^{-t})^{1/\alpha} + |y - e^{-\frac{t}{\alpha}} x|)^{\alpha + \gamma}}. \end{aligned}$$

Hence, one can derive from the dominated convergence theorem and integration by parts that

$$\begin{aligned} \left| \nabla_x Q_t \big( h(x) - \pi(h) \big) \right| &= \left| \int_{\mathbb{R}^d} \nabla_x p_{1-e^{-t}} (y - e^{-\frac{t}{\alpha}} x) \big( h(y) - \pi(h) \big) dy \right| \\ &= \left| e^{-\frac{t}{\alpha}} \int_{\mathbb{R}^d} \nabla_y p_{1-e^{-t}} (y - e^{-\frac{t}{\alpha}} x) \big( h(y) - \pi(h) \big) dy \right| \\ &= \left| e^{-\frac{t}{\alpha}} \int_{\mathbb{R}^d} p_{1-e^{-t}} (y - e^{-\frac{t}{\alpha}} x) \nabla h(y) dy \right| \leqslant \|\nabla h\|_{\infty} e^{-\frac{t}{\alpha}}, \end{aligned}$$

and similarly by (13)

$$\left| \nabla_x^2 Q_t \big( h(x) - \pi(h) \big) \right| = \left| e^{-\frac{2t}{\alpha}} \int_{\mathbb{R}^d} \nabla_y p_{(1 - e^{-t})^{\frac{1}{\alpha}}} (y - e^{-\frac{t}{\alpha}} x) \cdot \nabla h(y)^T ) dy \right|$$
  
$$\leq (1 - e^{-t})^{-\frac{1}{\alpha}} e^{-\frac{2t}{\alpha}} \leq t^{-\frac{1}{\alpha}} e^{-\frac{t}{\alpha}}.$$

These imply

$$\left|\mathcal{L}^{\alpha,\nu}f(x)\right|$$

$$\begin{split} &\leqslant d_{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \int_{0}^{\infty} \frac{\left| Q_{t}h(x+r\theta) - Q_{t}h(x) - \langle r\theta, \nabla Q_{t}h(x) \rangle \right|}{r^{\alpha+1}} dt dr \nu(d\theta) \\ &\leqslant d_{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{1} \int_{0}^{\infty} \int_{0}^{1} \int_{0}^{1} \frac{sr^{2} \left| \nabla^{2}Q_{t}h(x+sur\theta) \right|}{r^{\alpha+1}} du ds dt dr \nu(d\theta) \\ &+ d_{\alpha} \int_{\mathbb{S}^{d-1}} \int_{1}^{\infty} \int_{0}^{\infty} \int_{0}^{1} \frac{r \left| \nabla Q_{t}h(x+sr\theta) - \nabla Q_{t}h(x) \right|}{r^{\alpha+1}} ds dt dr \nu(d\theta) \leqslant C. \end{split}$$

When  $\alpha \in (0,1]$ , since  $h \in \mathcal{H}_{\beta}$ 

$$\left|h\big((1-e^{-t})^{\frac{1}{\alpha}}y+e^{-\frac{t}{\alpha}}(x+z)\big)-h\big((1-e^{-t})^{\frac{1}{\alpha}}y+e^{-\frac{t}{\alpha}}x\big)\right|\leqslant e^{-\frac{\beta t}{\alpha}}(|z|\wedge|z|^{\beta}).$$

By (B4), we immediately have

$$\left| Q_t h(x+z) - Q_t h(x) \right| \leqslant \int_{\mathbb{R}^d} p(1,y) e^{-\frac{\beta t}{\alpha}} (|z| \wedge |z|^{\beta}) dy = e^{-\frac{\beta t}{\alpha}} (|z| \wedge |z|^{\beta}).$$
(B6)

Recall  $\mathcal{A}^{\alpha,\nu}f(x) = \mathcal{L}^{\alpha,\nu}f(x) - \frac{1}{\alpha}\langle x, \nabla f(x)\rangle$ . By (B6), using the dominated convergence theorem, we get that

$$\nabla f(x) = -\int_0^\infty \nabla Q_t h(x) \, dt.$$

Furthermore, if  $\alpha = 1$ , we have

$$\mathcal{L}^{1,\nu}f(x)$$

$$= -d_1 \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_0^\infty \frac{Q_t h(x+r\theta) - Q_t h(x) - \langle r\theta, \nabla Q_t h(x) \mathbf{1}_{(0,1)}(r) \rangle}{r^2} dt dr \nu(d\theta)$$

$$= -d_1 \int_{\mathbb{S}^{d-1}} \int_0^1 \int_0^\infty \int_0^1 \frac{\langle r\theta, \nabla Q_t h(x+sr\theta) \rangle - \langle r\theta, \nabla Q_t h(x) \rangle}{r^2} ds dt dr \nu(d\theta)$$

$$+ -d_1 \int_{\mathbb{S}^{d-1}} \int_1^\infty \int_0^\infty \frac{Q_t h(x+r\theta) - Q_t h(x)}{r^2} dt dr \nu(d\theta),$$

and by integration by parts,

$$\left| \nabla Q_t h(x+zs) - \nabla Q_t h(x) \right| \\
\leq e^{-t} \int_{\mathbb{R}^d} \left| p \left( y - (1-e^{-t})^{-1} e^{-t} (x+zs) \right) - p \left( y - (1-e^{-t})^{-1} e^{-t} x \right) \right| \\
\cdot \left| \nabla h \left( (1-e^{-t})y \right) \right| dy \\
\leq C e^{-t} \left( (1-e^{-t})^{-1} e^{-t} |zs| \wedge 1 \right) \leq C e^{-t} \left( t^{-1} |zs| \wedge 1 \right),$$

these imply

$$\int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \int_{0}^{\infty} \frac{\left| Q_{t}h(x+r\theta) - Q_{t}h(x) - \langle r\theta, \nabla Q_{t}h(x)\mathbf{1}_{(0,1)}(r) \rangle \right|}{r^{2}} dt dr \nu(d\theta)$$

$$\leqslant C \int_{\mathbb{S}^{d-1}} \int_{0}^{1} \int_{0}^{\infty} \frac{e^{-t}(t^{-1}r \wedge 1)}{r} dt dr \nu(d\theta) + \int_{\mathbb{S}^{d-1}} \int_{1}^{\infty} \int_{0}^{\infty} \frac{e^{-\frac{\beta t}{\alpha}r\beta}}{r^{2}} dt dr \nu(d\theta)$$

$$= C \int_{0}^{1} \int_{0}^{\infty} \frac{e^{-t}(t^{-1}r \wedge 1)}{r} dt dr + \int_{1}^{\infty} \int_{0}^{\infty} \frac{e^{-\frac{\beta t}{\alpha}r\beta}}{r^{2}} dt dr \leqslant C.$$

If  $\alpha \in (0,1)$ , we have

$$\mathcal{L}^{\alpha,\nu}f(x) = -d_{\alpha} \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \int_{0}^{\infty} \frac{Q_{t}h(x+r\theta) - Q_{t}h(x)}{r^{\alpha+1}} dt dr \nu(d\theta),$$

and we have by (B6)

$$\int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \int_{0}^{\infty} \frac{\left| Q_{t}h(x+r\theta) - Q_{t}h(x) \right|}{r^{\alpha+1}} dt dr \nu(d\theta)$$

$$\leqslant \int_{\mathbb{S}^{d-1}} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\frac{\beta t}{\alpha}} \frac{r \wedge r^{\beta}}{r^{\alpha+1}} dt dr \nu(d\theta)$$

$$= \int_{0}^{\infty} \frac{r \wedge r^{\beta}}{r^{\alpha+1}} dr \int_{0}^{\infty} e^{-\frac{\beta t}{\alpha}} dt \leqslant C.$$

Therefore, by Fubini's theorem, we have

$$\mathcal{L}^{\alpha,\nu}f(x) = -\int_0^\infty \mathcal{L}^{\alpha,\nu}Q_t h(x)dt.$$

Hence, according to Lemma 19, we can obtain

$$\mathcal{A}^{\alpha,\nu}f = -\int_0^\infty \mathcal{A}^{\alpha,\nu}Q_thdt = -\int_0^\infty \partial_tQ_thdt = Q_0h - Q_\infty h,$$

here  $Q_{\infty} = \pi$ , the unique invariant distribution of the semigroup  $(Q_t)_{t\geq 0}$ associated with  $\mathcal{A}^{\alpha}$  by [18, Cor. 17.9]. The proof is complete.

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#### Declarations

#### Conflict of interest

There are no competing interests to declare which arose during the preparation or publication process of this article.

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