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RESEARCH ARTICLE

Lowest log canonical thresholds of a reduced plane curve of degree *d*

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Abstract
We describe the sixth worst singularity that a plane curve of degree $d \geqslant 5$ could have, using its log canonical threshold at the point of singularity. This is an extension of a result due to Cheltsov (J Geom Anal 27(3):2302–2338, 2017) wherein the five lowest values of log canonical thresholds of a plane curve of degree $d \geqslant 3$ were computed. These six small log canonical thresholds, in order, are $2/d$, $(2d-3)/(d-1)^2$, $(2d-1)/(d^2-d)$, $(2d-5)/(d^2-3d+1)$, $(2d-3)/(d^2-2d)$ and $(2d-7)/(d^2-4d+1)$. We give examples of curves with these values as their log canonical thresholds using illustrations.

Keywords Log canonical threshold · Plane curves · α-invariant of Tian singularity

Mathematics Subject Classification 14H20 · 14H50 · 14J70 · 14E05

1 Introduction

Let $C_d \subset \mathbb{P}^2$ be a reduced plane curve of degree d over $\mathbb{C}$ and P be a point on C_d. We aim to address the following question:

Question 1.1 Given a curve C_d of fixed degree d, what is the worst singularity that the curve can have at the point P?

We can use various parameters to measure the singularity at the point P, such as multiplicity of the curve at P, $\mathrm{mult}_P(C_d)$, Milnor number, $\mu(P)$, or log canonical threshold of the curve at P, $\mathrm{lct}_P(\mathbb{P}^2, C_d)$. In this paper, we will use $\mathrm{lct}_P(\mathbb{P}^2, C_d)$ to answer the above question. Recall that

$$\mathrm{lct}_P(\mathbb{P}^2, C_d) = \sup\bigl\{\lambda \in \mathbb{Q} \mid \text{the log pair } (\mathbb{P}^2, \lambda C_d) \text{ is log canonical at } P\bigr\}.$$

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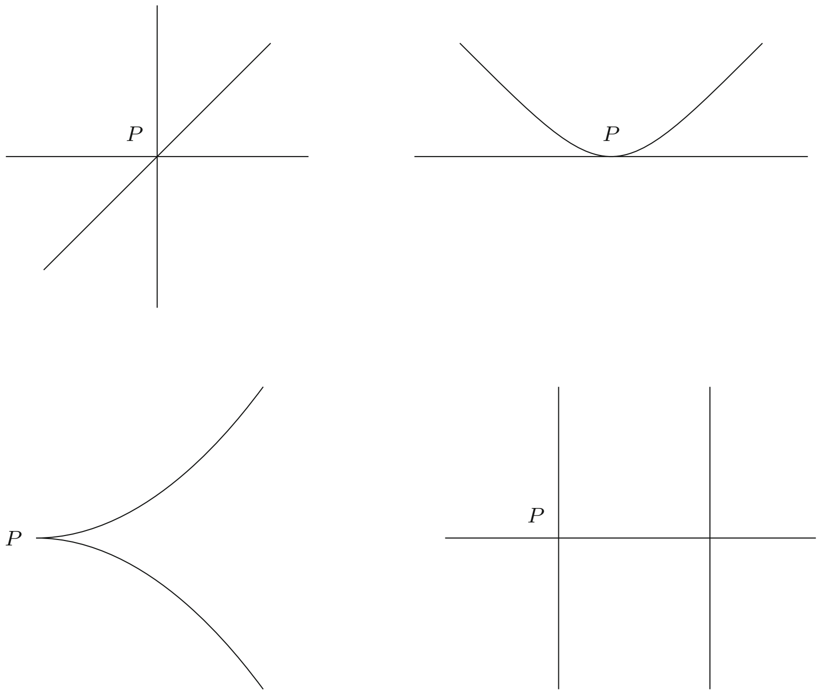

Fig. 1 (i) $C_3 : xy(x-y)=0$; (ii) $C_3 : y(x^2-y)=0$; (iii) $C_3 : x^2-y^3=0$; (iv) $C_3 : xy(x-1)=0$

By [4, Exercise 6.18] and [4, Lemma 6.35], we have

$$\frac{1}{\operatorname{mult}_P(C_d)} \leqslant \operatorname{lct}_P(\mathbb{P}^2, C_d) \leqslant \frac{2}{\operatorname{mult}_P(C_d)}.$$

This implies that the smaller the value of $\operatorname{lct}_P(\mathbb{P}^2, C_d)$, the worse the singularity of the curve C_d at P.

In order to answer Question 1.1 for $d \leqslant 4$, the values of log canonical threshold of a given reduced curve C_d at P were computed.

Example 1.2 If $d=1$ or $d=2$, then $\operatorname{lct}_P(\mathbb{P}^2, C_d)=1$.

Example 1.3 If $d=3$, then $\operatorname{lct}_P(\mathbb{P}^2, C_3)$ is one of $\{1, 5/6, 3/4, 2/3\}$. The worst singularity corresponds to $\operatorname{lct}_P(\mathbb{P}^2, C_3)=2/3$ and in this case, C_d is a union of three lines intersecting at P (example of such a curve is $xy(x-y)=0$). Examples of curves with the given values of log canonical threshold (2/3, 3/4, 5/6, 1, resp.) are illustrated in Fig. 1.

Example 1.4 (*Erik Paemurru*) Let C_4 be a quartic curve. Then $\operatorname{lct}_P(\mathbb{P}^2, C_4)$ is one of $\{1, 5/6, 3/4, 7/10, 9/14, 5/8, 2/3, 3/5, 7/12, 5/9, 1/2\}$. The worst singularity occurs when $\operatorname{lct}_P(\mathbb{P}^2, C_4)=1/2$ and in this case C_4 is a union of four lines passing through P (example of such a curve is $xy(x-y)(x+y)=0$) as illustrated in Fig. 2.

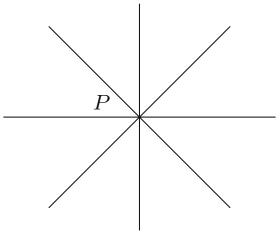

Fig. 2 $C_4 : xy(x^2 - y^2) = 0$

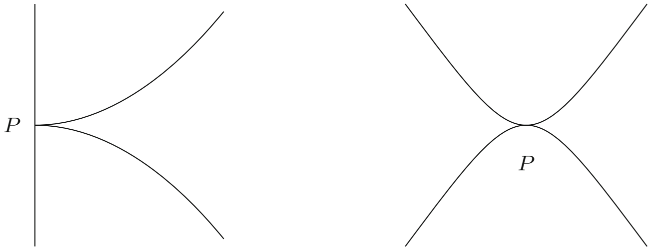

Fig. 3 (i) $C_4 : y(x^2 - y^3) = 0$; (ii) $C_4 : (x + x^2 + y^2)(x + y^2) = 0$

For curves C_d with $d \leqslant 3$, $\mathrm{lct}_P(\mathbb{P}^2, C_d)$ corresponds uniquely to a type of singularity of C_d at the point P. When $d = 4$, this is not the case. For example, both the curves illustrated in Fig. 3 have $\mathrm{lct}_P(\mathbb{P}^2, C_4) = 5/8$ but have very different singularities at P.

All the three parameters mentioned earlier give the same answer to Question 1.1, since $\mathrm{mult}_P(C_d) \leqslant d$, $\mu(P) \leqslant (d-1)^2$, with $\mathrm{mult}_P(C_d) = d$, $\mu(P) = (d-1)^2$ if and only if C_d is a union of d lines. The following theorem proves that computing the log canonical threshold of the curve at P also gives the same answer to the above question.

Theorem 1.5 ([2, Theorem 4.1]) *One has* $\mathrm{lct}_P(\mathbb{P}^2, C_d) \geqslant 2/d$ *and the equality holds if and only if* C_d *is a union of* d *lines passing through* P.

We can then ask the following question:

Question 1.6 What is the second worst singularity at the point P?

While examples given above answer this question for curves of degree $d \leqslant 4$, [1] answers this question for degree $d \geqslant 5$ curves. To present this answer, we introduce certain types of singularities in Sect. 2 and we call these types of singularities $\mathbb{K}_n, \mathbb{T}_n, \widetilde{\mathbb{K}}_n, \widetilde{\mathbb{T}}_n, \mathbb{M}_n, \widetilde{\mathbb{M}}_n, \widehat{\mathbb{M}}_n$, where $n = \mathrm{mult}_P(C_d)$. In [1], the following result was obtained.

Theorem 1.7 *Suppose* $d \geqslant 5$ *and* $2/d < \mathrm{lct}_P(\mathbb{P}^2, C_d) \leqslant (2d-3)/(d^2-2d)$. *Then the curve* C_d *has singularities of type* $\mathbb{T}_{d-1}, \mathbb{K}_{d-1}, \widetilde{\mathbb{T}}_{d-1}, \widetilde{\mathbb{K}}_{d-1}$ *at* P *and the values of their log canonical thresholds at* P *are* $(2d-3)/(d-1)^2 < (2d-1)/(d^2-d) < (2d-5)/(d^2-3d+1) < (2d-3)/(d^2-2d)$, *respectively.*

This result and Theorem 1.5 give the five worst singularities of the curve C_d. In this paper, we describe the sixth worst one. To be precise we prove

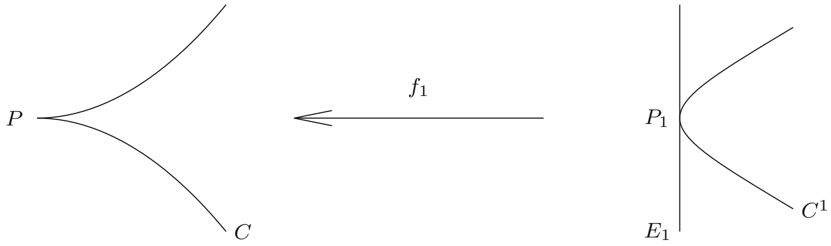

Fig. 4 C with a $\mathbb{K}_n$ singularity at P and its blow-up at P

Theorem 1.8 *Suppose $d \geqslant 5$ and*

$$\frac{2d-3}{d(d-2)} < \operatorname{lct}_P(\mathbb{P}^2, C_d) \leqslant \frac{2d-7}{d^2-4d+1}.$$

Then the curve C_d has singularity of type $\mathbb{M}_{d-1}$, $\widetilde{\mathbb{M}}_{d-1}$ or $\widehat{\mathbb{M}}_{d-1}$ at the point P with $\operatorname{lct}_P(\mathbb{P}^2, C_d) = (2d-7)/(d^2-4d+1)$.

In the case of $d = 5$, one can hope to determine all possible values of log canonical threshold of quintic curves, like in the case of $d = 4$ this was done in Example 1.4.

In Sect. 3 we present preliminary results used in the Proof of Theorem 1.8, while the proof itself is given in Sect. 4.

2 Cusps and other singularities

Let C be a reduced curve on a smooth surface S and P be a point on C. We are interested in singularities of the curve C at the point P. In this section, we introduce various types of singularities which we denote by $\mathbb{T}_n$, $\mathbb{K}_n$, $\widetilde{\mathbb{T}}_n$, $\widetilde{\mathbb{K}}_n$, $\mathbb{M}_n$, $\widetilde{\mathbb{M}}_n$ and $\widehat{\mathbb{M}}_n$, where $n = \operatorname{mult}_P(C)$. We aim to describe geometric properties of the curve C having one of these types of singularities at P.

Let $f_1\colon S_1 \to S$ be the blow-up of S at the point P. Let C^1 be the proper transform in S_1 of the curve C and E_1 be the exceptional divisor of the blow-up.

2.1 Singularities of type $\mathbb{K}_n$ (cusps)

A curve C having singularity of type $\mathbb{K}_n$ can be defined with the help of its geometric properties as given below. These singularities are also called cusps.

- $\operatorname{mult}_P(C) = n \geqslant 2$,
- $C^1 \cap E_1 = P_1$,
- C^1 intersects E_1 tangentially at P_1 and is smooth at this point (Fig. 4).

Recall from [5, Theorem 1.1] that the log canonical threshold of a cuspidal curve is

$$\operatorname{lct}_P(S, C) = \frac{1}{n} + \frac{1}{n+1}.$$

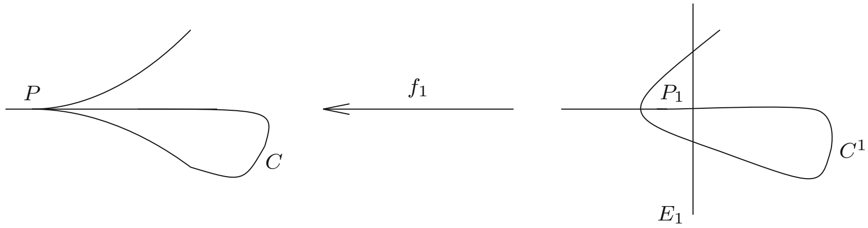

Fig. 5 C with a $\mathbb{T}_n$ singularity at P and its blow-up at P

Remark 2.1 Suppose $S = \mathbb{P}^2$. Let C be a curve of degree $d \geqslant 3$ having a $\mathbb{K}_n$ singularity at P. Then $n \leqslant d-1$. If $n = d-1$, then the curve C is irreducible. Such curves do exist. For example, the curve given by $zx^{d-1} + y^d = 0$ has singularity of type $\mathbb{K}_{d-1}$ at the point $P = [0:0:1]$.

2.2 Singularities of type $\mathbb{T}_n$

A curve C having singularity of type $\mathbb{T}_n$ at P can be defined using the following geometric properties:

- $\text{mult}_P(C) = n \geqslant 3$,
- $C^1 \cap E_1 = P_1$,
- the point P_1 is an ordinary double point of C^1 (Fig. 5).

Remark 2.2 Suppose $S = \mathbb{P}^2$ and C is a curve of degree d. Let L be a line in $\mathbb{P}^2$ passing through P, whose proper transform L^1 in S_1 passes through P_1. If the curve C has singularity of type $\mathbb{T}_n$, then $C = L + Z$, where Z is an irreducible curve of degree $d-1$ that does not contain L as an irreducible component. Since if not, then

$$d \geqslant L.C = d - 1 + 2 = d + 1$$

which is absurd. Thus, $C = L + Z$ and $L \cap Z = P$ where Z has singularity of type $\mathbb{K}_{d-2}$ at the point P.

2.3 Singularities of type $\widetilde{\mathbb{T}}_n$

A curve C having singularity of type $\widetilde{\mathbb{T}}_n$ at P can be defined using the following geometric properties:

- $\text{mult}_P(C) = n \geqslant 4$,
- $C^1 \cap E_1 = \{P_1, Q_1\}$,
- the point P_1 is an ordinary double point of C^1,
- C^1 intersects E_1 transversally at Q_1 and is smooth at this point (Fig. 6).

Remark 2.3 Suppose $S = \mathbb{P}^2$ and C is a curve of degree d. Let L be a line in $\mathbb{P}^2$ passing through the point P, whose proper transform L^1 in S_1 passes through the point P_1.

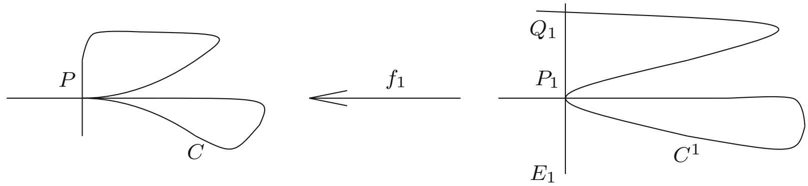

Fig. 6 C with a $\widetilde{\mathbb{T}}_n$ singularity at P and its blow-up at P

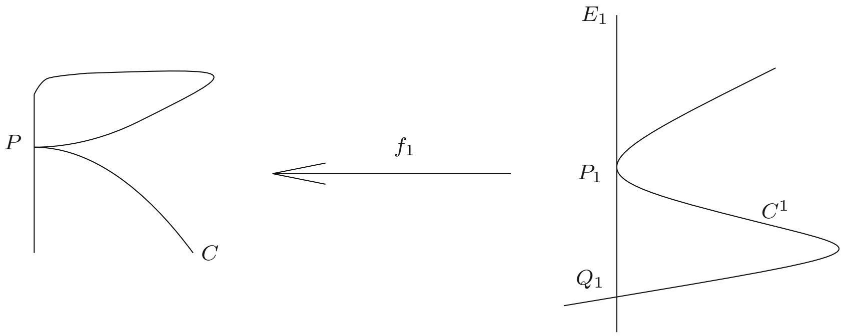

Fig. 7 C with a $\widetilde{\mathbb{K}}_n$ singularity at P and its blow-up at P

Similar computations as in Remark 2.2 imply $C = Z + L$ so that $L \cap Z = P$, where Z is an irreducible curve of degree $d-1$ that does not contain L as an irreducible component and Z has singularity of type $\widetilde{\mathbb{K}}_{d-2}$ at the point P, which is introduced in the next subsection.

2.4 Singularities of type $\widetilde{\mathbb{K}}_n$

A curve C with singularity of type $\widetilde{\mathbb{K}}_n$ can be defined using the following geometric properties:

- $\mathrm{mult}_P(C) = n \geqslant 3$,
- $C^1 \cap E_1 = \{P_1, Q_1\}$,
- C^1 intersects E_1 tangentially at the point P_1 and is smooth at this point,
- C^1 is smooth at Q_1 and intersects E_1 transversally at this point (Fig. 7).

Remark 2.4 Suppose $S = \mathbb{P}^2$ and C is a curve of degree d. Then C with a $\widetilde{\mathbb{K}}_{d-1}$ singularity at P exists. Such a curve can be reducible, for example, $y(x^{d-2} - y^{d-1}) = 0$ or can be irreducible, for example, $x^{d-2}y + y^d + x^d = 0$. If C is reducible, then $C = L + Z$ where Z is a curve of degree $d-1$ which does not contain L as an irreducible component and has singularity of type $\mathbb{K}_{d-2}$ at the point P.

2.5 Singularities of type $\mathbb{M}_n$

A curve C with singularity of type $\mathbb{M}_n$ at P can be defined using the following geometric properties:

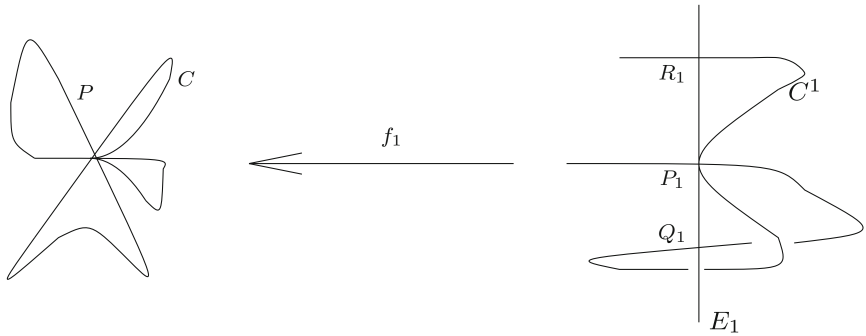

Fig. 8 C with an $\mathbb{M}_n$ singularity at P and its blow-up at P

- $\mathrm{mult}_P(C) = n \geqslant 5$,
- $C^1 \cap E_1 = \{P_1, Q_1, R_1\}$,
- C^1 is smooth at the points Q_1 and R_1 where it intersects transversally with E_1,
- the point P_1 is an ordinary double point of C^1 (Fig. 8).

Remark 2.5 Suppose $S = \mathbb{P}^2$ and C is a curve of degree d. A curve having singularity of type $\mathbb{M}_{d-1}$ at P exists, for example, $x(x^2 - y^2)(x^{d-4} - y^{d-3}) = 0$. It is reducible and thus $C = Z + L$ where L is a line in S that contains the point P so that its proper transform L^1 in S_1 contains the point P_1 and Z is an irreducible curve of degree $d-1$ which does not contain L as an irreducible component.

2.6 Singularities of type $\widetilde{\mathbb{M}}_n$

A curve C with singularity of type $\widetilde{\mathbb{M}}_n$ at P can be defined using the following geometric properties:

- $\mathrm{mult}_P(C) = n \geqslant 5$,
- $C^1 \cap E_1 = \{P_1, Q_1\}$,
- P_1 is an ordinary double point of C^1 with $(C^1.E_1)_{P_1} = n - 2$,
- C^1 intersects E_1 tangentially at the point Q_1 with $(C^1.E_1)_{Q_1} = 2$ and is smooth at this point (Fig. 9).

Remark 2.6 Suppose $S = \mathbb{P}^2$ and C is a curve of degree d. Then $n = d - 1$ is possible and a curve with singularity of type $\widetilde{\mathbb{M}}_{d-1}$ exists. For example, $y(zx^2 + y^3)(zy^{d-4} + x^{d-3}) = 0$ has an $\widetilde{\mathbb{M}}_{d-1}$ singularity at the point $P = [0:0:1]$. In this case, C is reducible and thus, $C = L + Z$ where L is the line in S containing P such that its proper transform L^1 passes through the point P_1 in S_1 and Z is a $d-1$ degree irreducible curve that does not contain L as an irreducible component.

2.7 Singularities of type $\widehat{\mathbb{M}}_n$

A curve C with singularity of type $\widehat{\mathbb{M}}_n$ at P can be defined using the following geometric properties:

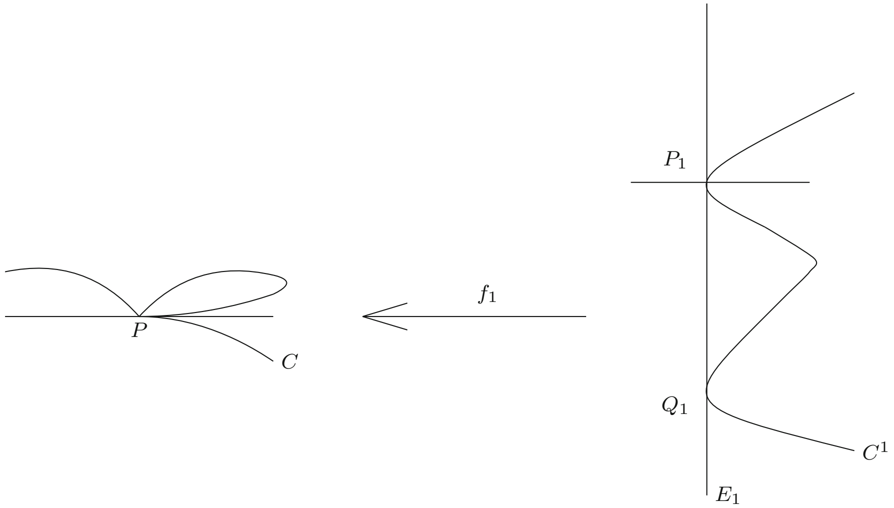

Fig. 9 C with an $\widetilde{\mathbb{M}}_n$ singularity at P and its blow-up at P

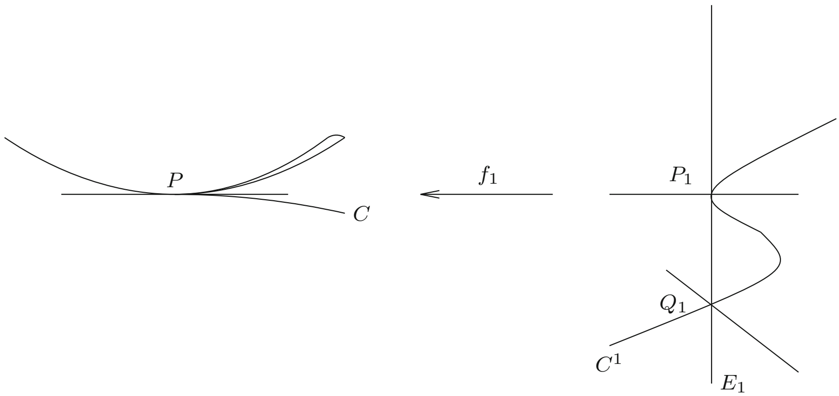

Fig. 10 C with an $\widehat{\mathbb{M}}_n$ singularity at P and its blow-up at P

- $\mathrm{mult}_P(C) = n \geqslant 5$,
- $C^1 \cap E_1 = \{P_1, Q_1\}$,
- P_1 is an ordinary double point of C^1 with $(C_1.E_1)_{P_1} = n - 2$,
- Q_1 is an ordinary double point of C^1 with $(C_1.E_1)_{Q_1} = 2$ (Fig. 10).

Remark 2.7 Suppose $S = \mathbb{P}^2$ and C is a curve of degree d with singularity of type $\widehat{\mathbb{M}}_n$ at the point P. Then $n = d - 1$ is possible, for example, C given by $x(zx^{d-4} + y^{d-3})(z^2y^2 + x^4) = 0$ has an $\widehat{\mathbb{M}}_{d-1}$ singularity at the point $P = [0:0:1]$. That is, $C = L + Z$ where L is a line in S that passes through the point P whose proper transform contains the point P_1 and Z is an irreducible curve in S of degree $d - 1$ which does not contain L as an irreducible component.

2.8 Defining equations

In this section, we describe a curve C having any of the above types of singularities using local equations. These descriptions actually are not essential to prove Theorem 1.8. Up to analytic change of coordinates, the equations of the curve C with the respective singularities are given below:

- $\mathbb{K}_n$: $x^n + y^{n+1} + \sum_{i=1}^{n+1} a_i x^i y^{n+1-i} + \text{H.O.T.} = 0$,
- $\mathbb{T}_n$: $x\left(x^{n-1} - y^n + \sum_{i=2}^{n+1} a_i x^{i-1} y^{n+1-i} + \text{H.O.T}\right) = 0$,
- $\widetilde{T}_n$: $x\left(y(y^{n-1} - x^{n-2}) + \sum_{i=2}^{n+1} a_i x^{i-1} y^{n+1-i} + \text{H.O.T}\right) = 0$,
- $\widetilde{K}_n$: $y(x^{n-1} - y^n) + \sum_{i=1}^{n+1} a_i x^{i-1} i y^{n+1-i} + \text{H.O.T} = 0$,
- $\mathbb{M}_n$:$x\left((x^2 - y^2)(x^{n-3} - y^{n-2}) + \sum_{i=2, i\neq 3}^{n} a_i x^i y^{n+1-i} + \text{H.O.T}\right) = 0$,
- $\widetilde{\mathbb{M}}_n$: $y\left((x^2+y^3)(y^{n-3}+x^{n-2})+\sum_{i=1}^{n-1} a_i x^i y^{n-i}+\sum_{i=0, i\neq n-2}^{n+1} b_i x^i y^{n+1-i}+\text{H.O.T}\right) = 0$,
- $\widehat{\mathbb{M}}_n$: $x\Bigg((x^{n-3} + y^{n-2})(y^2 + x^4) + \sum_{i=0, i\neq 1}^{n} a_i x^{i-1} y^{n+1-i} + \sum_{i=0}^{n} b_i x^{i-1} y^{n+2-i} +$
  $\sum_{i=0, i\neq 5}^{n+2} c_i x^{i-1} y^{n+3-i} + \text{H.O.T}\Bigg) = 0.$

The above set of equations comprise an exhaustive list of curves C of a given degree with the various types of singularities, up to analytic change of coordinates and include the curves missing from the list in [1, Definition 1.9], as pointed out by the referee.

3 Preliminaries

Let S be a smooth surface and P be a point in S. Let D be an effective non-zero $\mathbb{Q}$-divisor on the surface S. Then,

$$D = \sum_{i=1}^{r} a_i C_i$$

where each C_i is an irreducible curve on S and $a_i \in \mathbb{Q}_{\geqslant 0}$.

Let $\pi : \widetilde{S} \to S$ be a birational morphism such that $\widetilde{S}$ is smooth. One can then conclude that π is a composition of n blow-ups of points. For each C_i, we denote its proper transform by $\widetilde{C}_i$ and the exceptional curves of the blow-up by $F_1, F_2, \dots, F_n$. Then,

$$K_{\tilde{S}} + \sum_{i=1}^{r} a_i \widetilde{C}_i + \sum_{j=1}^{n} b_j F_j \sim_{\mathbb{Q}} \pi^*(K_S + D)$$

where $b_i \in \mathbb{Q}$. Suppose $\sum_{i=1}^{r} \widetilde{C}_i + \sum_{j=1}^{n} F_j$ is a divisor with simple normal crossings (SNC).

Definition 3.1 ([4, Definition 6.16]) The log pair (S, D) is said to be *log canonical at* P if the following conditions are satisfied:

- $a_i \leqslant 1$ for every C_i such that $P \in C_i$,
- $b_j \leqslant 1$ for every F_j such that $\pi(F_j) = P$.

Similarly, the log pair (S, D) is said to be *Kawamata log terminal at* P if

- $a_i < 1$ for every C_i such that $P \in C_i$,
- $b_j < 1$ for every F_j such that $\pi(F_j) = P$.

Let $\pi_1 \colon S_1 \to S$ be the blow-up of S at the point P and E_1 be the exceptional curve of the blow-up. Let D^1 be the proper transform of the divisor D on the surface S_1 after blow-up. Let

$$D^{S_1} = D^1 + (\mathrm{mult}_P(D) - 1)E_1.$$

This is called the *log pull-back* of the log pair (S, D). Observe that

$$K_{S_1} + D^1 + (\mathrm{mult}_P(D) - 1)E_1 \sim_{\mathbb{Q}} \pi_1^*(K_S + D).$$

This implies that the log pair (S, D) is not log canonical at P if $\mathrm{mult}_P(D) > 2$, and is not Kawamata log terminal if $\mathrm{mult}_P(D) \geqslant 2$.

Remark 3.2 The log pair (S, D) is log canonical at the point P if and only if (S_1, D^{S_1}) is log canonical at every point in E_1. Similarly, the log pair (S, D) is Kawamata log terminal at the point P if and only if (S_1, D^{S_1}) is Kawamata log terminal at every point in E_1.

Lemma 3.3 ([4, Exercise 6.18]) *Suppose* (S, D) *is not log canonical at P, then* $\mathrm{mult}_P(D) > 1$. *Similarly, if* (S, D) *is not Kawamata log terminal at* P*, then* $\mathrm{mult}_P(D) \geqslant 1$.

Let Z be an irreducible curve on S that contains the point P and is smooth at P. Suppose that Z is not contained in $\mathrm{Supp}(D)$. Let μ be a non-negative rational number.

Theorem 3.4 ([3, Theorem 7], [4, Exercise 6.31], [6, Corollary 3.12]) *Suppose the log pair* $(S, \mu Z + D)$ *is not log canonical* (*not Kawamata log terminal, resp.*) *at* P *and* $\mu \leqslant 1$ ($\mu < 1$, *resp.*). *Then* $\mathrm{mult}_P(D.Z) > 1$.

Lemma 3.5 *If (S, D) is not log canonical at P and* $\text{mult}_P(D) \leqslant 2$*, then there exists a unique point in E_1 such that (S_1, D^{S_1}) is not log canonical at it. Similarly, if (S, D) is not Kawamata log terminal at P, and* $\text{mult}_P(D) < 2$*, then there exists a unique point in E_1 such that (S_1, D^{S_1}) is not Kawamata log terminal at it.*

Proof Suppose (S, D) is not log canonical at P and $\text{mult}_P(D) \leqslant 2$ and suppose there exist two distinct points P_1 and P_2 in E_1 at which (S_1, D^{S_1}) is not log canonical. Then,

$$2 \geqslant \text{mult}_P(D) = D^1.E_1 \geqslant \text{mult}_{P_1}(D^1.E_1) + \text{mult}_{P_2}(D^1.E_1) > 2$$

by Theorem 3.4. Thus, Remark 3.2 proves the first assertion. Similarly we can prove the second assertion. □

Lemma 3.6 ([1, Lemma 2.14]) *Suppose (S, D) is not Kawamata log terminal at P, and (S, D) is Kawamata log terminal in a punctured neighbourhood of the point P, then* $\text{mult}_P(D) > 1$.

Proof Suppose $\text{mult}_P(D) \leqslant 1$. Let us seek for a contradiction. Since (S, D) is not Kawamata log terminal at P, we have that $(S^1, D^1 + (1 - \text{mult}_P(D))E_1)$ is not Kawamata log terminal at some point $P_1 \in E_1$. From Lemma 3.5 we have that this point P_1 is unique. This implies that $\text{mult}_P(D) > 1$, by Lemma 3.3, which in turn contradicts our assumption. □

Let Z_1 and Z_2 be irreducible curves on the surface S such that Z_1 and Z_2 are not contained in $\text{Supp}(D)$ and $P \in Z_1 \cap Z_2$. Also, suppose that Z_1 and Z_2 are smooth at P and intersect transversally at P. Let μ_1 and μ_2 be non-negative rational numbers.

Theorem 3.7 ([3, Theorem 13]) *If the log pair $(S, \mu_1 Z_1 + \mu_2 Z_2 + D)$ is not log canonical at the point P, and* $\text{mult}_P(D) \leqslant 1$*, then* $\text{mult}_P(D.Z_1) > 2(1 - \mu_2)$ *or* $\text{mult}_P(D.Z_2) > 2(1-\mu_1)$ *(or both). Similarly, if the log pair $(S, \mu_1 Z_1+\mu_2 Z_2+D)$ is not Kawamata log terminal at the point P, and* $\text{mult}_P(D) < 1$*, then* $\text{mult}_P(D.Z_1) \geqslant 2(1 - \mu_2)$ *or* $\text{mult}_P(D.Z_2) \geqslant 2(1 - \mu_1)$ *(or both).*

4 Proof of the main result

Let us now prove the main result of the paper. Let C_d be a reduced curve of degree $d \geqslant 5$ on a smooth surface S such that $P \in C_d$ and let $m_0 = \text{mult}_P(C_d)$. Suppose $(2d - 3)/(d^2 - 2d) < \text{lct}_P(S, C_d) < (2d - 7)/(d^2 - 4d + 1)$. This means that there exists $\lambda < (2d - 7)/(d^2 - 4d + 1)$ such that $(S, \lambda C_d)$ is not Kawamata log terminal at P. Let us also assume that $m_0 \neq d$ and thus C_d is not a union of d lines. We want to show that the curve C_d has singularity of type $\mathbb{M}_{d-1}$, $\widetilde{\mathbb{M}}_{d-1}$ or $\widehat{\mathbb{M}}_{d-1}$ at the point P. It is important to notice that the arguments presented below are very similar to the arguments in [1]. Also, these are local arguments, i.e., it is not necessary for the curve C_d to be smooth everywhere outside of P. We assume that the respective divisors on the surface S at various levels are Kawamata log terminal (or log canonical) at a punctured neighbourhood of P.

Lemma 4.1 *The following inequalities are used in the proof of the main result:*

(i) $\lambda < 2/(d-1)$,
(ii) $\lambda < (2k+1)/kd$, *for* $k \in \mathbb{Z}_{>0}$ *such that* $k \leqslant d-3$,
(iii) $\lambda < (2k+1)/(kd+1)$ *for* $k \in \mathbb{Z}_{>0}$ *such that* $k \leqslant d-5$,
(iv) $\lambda < 3/d$,
(v) $\lambda < 2/(d-2)$,
(vi) $\lambda < 6/(3d-4)$,
(vii) $\lambda < 5/2d$.

The proof is straightforward.

We will now introduce some notations. Let $S = S_0 = \mathbb{P}^2$ and $D = (\mathbb{P}^2, \lambda C_d)$. Consider a sequence of blow-ups $f_i \colon S_i \to S_{i-1}$ such that f_1 is the blow-up of $P_0 = P$, f_2 is the blow-up of the point P_1, and so on, i.e., f_i is the blow-up of the point $P_{i-1} \in S_{i-1}$. We have

$$\cdots \to S_{k+1} \xrightarrow{f_{k+1}} S_k \to \cdots \xrightarrow{f_4} S_3 \xrightarrow{f_3} S_2 \xrightarrow{f_2} S_1 \xrightarrow{f_1} S_0.$$

Also, let $f\colon S_{k+1} \to S$ be the composition of the blow-ups, i.e., $f = f_1 \circ f_2 \circ \cdots \circ f_{k+1}$. The f_i-exceptional divisor during each blow-up is denoted by E_i. The proper transform of the exceptional divisors E_j in S_i is denoted by E_j^i for all $j < i$. Also, after the f_i blow-up, the curve C_d is denoted by C_d^i in S_i. The divisors comprising of the curve and the exceptional curves on every floor S_i are together denoted by D^{S_i}. We will explicitly describe how each of these points of blow-up are chosen.

Since $(S, \lambda C_d)$ is not Kawamata log terminal at the point $P \in C_d$, by Remark 3.2 one has that $(S_1, \lambda C_d^1 + (\lambda m_0 - 1)E_1)$ is not Kawamata log terminal at some point in E_1. Let this point be P_1.

Lemma 4.2 $\lambda m_0 < 2$.

Proof Since $m_0 \leqslant d-1$, we have $\lambda m_0 \leqslant \lambda(d-1)$. Using Lemma 4.1 (i), we get $\lambda m_0 < 2$. □

From Lemma 3.5 this implies that the point P_1 is a unique point on E_1 at which (S_1, D^{S_1}), that is, $(S_1, \lambda C_d^1 + (\lambda m_0 - 1)E_1)$ is not Kawamata log terminal.

Let L be the line in $\mathbb{P}^2$ whose proper transform, L^1 in S_1, contains the point P_1.

Lemma 4.3 *Suppose* $m_0 = d-1$. *Then* L *is an irreducible component of* C_d.

Proof Observe that

$$C_d^1 \sim_{\mathbb{Q}} f_1^*(C_d) - m_0 E_1.$$

If L is not an irreducible component of the curve C_d, then we have

$$m_1 \leqslant C_d^1.L^1 = C_d.L - m_0 E_1.L^1 = d - m_0, \tag{1}$$

where $m_1 = \mathrm{mult}_{P_1}(C_d^1)$. Thus we have $m_0 + m_1 \leqslant d$. $P_1 \in C_d^1$, since if not, then $(S_1, (\lambda m_0 - 1)E_1)$ is not log canonical at the point P_1, which is not possible since $\lambda m_0 - 1 < 1$ from Lemma 4.2. Since $m_0 = d - 1$, $m_1 = 1$. Therefore, C_d^1 is smooth at P_1.

Let $k = \mathrm{mult}_{P_1}(C_d^1.E_1)$. We claim that $k > d - 3$. Indeed, suppose $k \leqslant d - 3$. Since (S_1, D^{S_1}) is not Kawamata log terminal at the point $P_1 \in E_1$, we have that (S_2, D^{S_2}) is not Kawamata log terminal at some point in E_2, where

$$D^{S_2} = \lambda C_d^2 + (\lambda(d-1) - 1)E_1^2 + (\lambda d - 2)E_2.$$

Let this point be P_2. Note that all the coefficients of the curves in D^{S_2} are less than 1. In particular, we have

$$\lambda d - 2 = \frac{d-2}{d^2 - 4d + 1} < 1.$$

Thus, by Definition 3.1, (S_3, D^{S_3}) is not Kawamata log terminal at some point in E_3. Let this point be P_3. Thus, f_2 is the blow-up of $P_1 \in C^1 \cap E_1$, f_3 is the blow-up of $P_2 \in C^2 \cap E_2$, and so on. One then has the sequence as mentioned in (i).

We require $k + 1$ blow-ups to ensure simple normal crossing of the elements of the divisor over the point P. Here the points of blow-up are such that $P_i = C^i \cap E_i$ and using the notations described earlier, we have

$$\begin{aligned} &K_{S_{k+1}} + \lambda C_d^{k+1} + (\lambda(d-1) - 1)E_1^{k+1} \\ &\quad + (\lambda d - 2)E_2^{k+1} + \cdots + (\lambda k d - 2k)E_{k+1} \sim_{\mathbb{Q}} f^*(K_S + \lambda C_d). \end{aligned} \tag{2}$$

Let the coefficients of E_i^{k+1} in (2) be denoted by b_i. Then since

$$\lambda C^{k+1} + \sum_{i=1}^{k} b_i E_i^{k+1} + b_{k+1} E_{k+1}$$

is a divisor with simple normal crossings over P, at least one of $b_i > 1$ or $b_{k+1} > 1$. But the coefficients b_i are such that $b_j < b_i$ for all $j < i$ and we have

$$\lambda k d - 2k \leqslant \frac{d^2 - 5d + 6}{d^2 - 4d + 1} < 1$$

from Lemma 4.1 (ii). That is, in particular $b_j < b_{k+1} < 1$ for all $j < k + 1$.

This contradiction implies that $k > d - 3$. We also know that

$$k = \mathrm{mult}_{P_1}(C_d^1.E_1) \leqslant (C_d^1.E_1) = m_0 = d - 1.$$

Therefore, these inequalities imply $k = d - 1$ or $d - 2$. Thus, when

- $k = d - 1$, then C_d has singularity of type $\mathbb{K}_{d-1}$ at P (see Sect. 2.1).

- $k = d-2$, then C_d has singularity of type $\widetilde{\mathbb{K}}_{d-1}$ at P (see Sect. 2.4).

If the curve C_d has either of the above singularities at the point P, then $\mathrm{lct}_P(\mathbb{P}^2, C_d) = (2d-1)/(d^2-d)$ or $\mathrm{lct}_P(\mathbb{P}^2, C_d) = (2d-3)/(d^2-2d)$, respectively. Since we assume $\mathrm{lct}_P(\mathbb{P}^2, C_d) > (2d-3)/(d^2-2d)$, neither of these values for k are possible. Therefore, this contradiction implies that L is an irreducible component of the curve C_d. ◻

Lemma 4.4 *Suppose $m_0 = d-1$. Then C_d has singularity of type $\mathbb{M}_{d-1}$, $\widetilde{\mathbb{M}}_{d-1}$ or $\widehat{\mathbb{M}}_{d-1}$ at the point P.*

Proof From Lemma 4.3 we know that L is an irreducible component of the curve C_d, i.e., we have $C_d = C_{d-1} + L$ where C_{d-1} is an irreducible curve of degree $d-1$ which does not contain L as an irreducible component. Let $n_0 = \mathrm{mult}_P(C_{d-1})$. Since $m_0 = \mathrm{mult}_P(C_d) = d-1$, we have $n_0 = m_0 - 1 = d-2$.

Let $f_1 : S_1 \to S$ be the blow-up at the point P and $n_1 = \mathrm{mult}_{P_1}(C^1_{d-1})$. We have $n_1 = m_1 - 1$. We also have $P_1 \in C^1_{d-1}$ since if not, it would mean that $(S_1, \lambda L^1 + (\lambda(d-1)-1)E_1)$ is not log canonical at the point P_1 which is a contradiction since $\lambda < 1$ and $\lambda(d-1)-1 < 1$ and L^1, E_1 are SNC divisors over P_1. Thus, $n_1 \geqslant 1$.

Consider

$$n_1 \leqslant L^1.C^1_{d-1} = d-1-n_0,$$

that is, $n_0 + n_1 \leqslant d-1$ and since $n_0 = d-2$, we have $n_1 = 1$. Thus the curve C^1_{d-1} is smooth at P_1.

Let $k = \mathrm{mult}_{P_1}(C^1_{d-1}.E_1)$. We claim $k > d-5$. Instead, suppose $k \leqslant d-5$, then using similar computations as in Lemma 4.3, after $k+1$ blow-ups, we get

$$\begin{aligned} K_{S_{k+1}} + \lambda C^{k+1}_{d-1} + \lambda L^{k+1} + (\lambda(d-1)-1)E^{k+1}_1 \\ + \cdots + (\lambda(kd+1)-2k)E_{k+1} \sim_{\mathbb{Q}} f^*(K_S + \lambda C_d) \end{aligned}$$

where $(S_{k+1}, D^{S_{k+1}})$ is not Kawamata log terminal at some point in E_{k+1}, which we take to be P_{k+1}. Here again, f is a composition of $k+1$ blow-ups and b_i are the coefficients of E^{k+1}_i in the above equation.

Since the curves in the divisor

$$\lambda C^{k+1}_{d-1} + \lambda L^{k+1} + \sum_{i=1}^{k} b_i E^{k+1}_i + b_{k+1}E_{k+1}$$

intersect at simple normal crossing at the point P after $k+1$ blow-ups, one of these coefficients should be such that $b_i > 1$ or $b_{k+1} > 1$ but we have

$$\lambda(kd+1) - 2k = \frac{(k+2)d - 2k - 7}{d^2 - 4d + 1} < 1$$

from Lemma 4.1 (iii) and the coefficients are such that $b_j < b_i$ for all $j < i$. In particular, $b_j < b_{k+1} < 1$ for all $j < k+1$. This contradiction implies $k > d-5$.

We also know that

$$k = \mathrm{mult}_{P_1}(C^1_{d-1}.E_1) \leqslant (C^1_{d-1}.E_1) = \mathrm{mult}_P(C_{d-1}) = n_0 = d-2.$$

Thus, these inequalities imply that $k = d-2$ or $d-3$ or $d-4$. Thus, when

- $k = d-2$, C_d has singularity of type $\mathbb{T}_{d-1}$ at P (see Sect. 2.2),
- $k = d-3$, C_d has singularity of type $\widetilde{\mathbb{T}}_{d-1}$ at P (see Sect. 2.3).

If C_d has either one of the above singularities at the point P, then $\mathrm{lct}_P(\mathbb{P}^2, C_d) = (2d-3)/(d-1)^2$ or $\mathrm{lct}_P(\mathbb{P}^2, C_d) = (2d-5)/(d^2-3d+1)$, respectively. Since we assume that $\mathrm{lct}_P(\mathbb{P}^2, C_d) > (2d-3)/(d^2-2d)$, these values of k are not possible. Thus $k = d-4$, i.e., C_d has singularity of type $\mathbb{M}_{d-1}$, $\widetilde{\mathbb{M}}_{d-1}$ or $\widehat{\mathbb{M}}_{d-1}$ at P. □

Observe that Lemmas 4.3 and 4.4 complete the proof of the main result if $m_0 = d-1$. In the remaining part of the section, we will prove that $m_0 \leqslant d-2$ is not possible. In particular, we prove the following proposition.

Proposition 4.5 *If $m_0 \leqslant d-2$, then $\mathrm{lct}_P(S, C_d) \geqslant 2/(d-1)$.*

This in turn proves that for our choice of λ and the assumption that $(S, \lambda C_d)$ is not Kawamata log terminal at P, $m_0 \leqslant d-2$ is not possible, since $\lambda < 2/(d-1)$. Let us prove this proposition by the method of contradiction.

Proof Suppose $m_0 \leqslant d-2$ and $\mathrm{lct}(S, C_d) < 2/(d-1)$. Let $\mu = 2/(d-1)$. Then $(S, \mu C_d)$ is not log canonical, in particular, is not Kawamata log terminal at a point, say P. Let us now obtain the necessary contradiction.

Claim 1 *The line L is not an irreducible component of the curve C_d.*

Proof We shall prove this by contradiction. Suppose L is an irreducible component of the curve C_d. Then $C_d = L + C_{d-1}$, where C_{d-1} is an irreducible curve of degree $d-1$ in $\mathbb{P}^2$ and does not contain L as an irreducible component. Let $f_1\colon S_1 \to S$ be the blow-up at the point P in C_d. Let $n_0 = \mathrm{mult}_P(C_{d-1})$.

Since $(S, \mu C_{d-1} + \mu L)$ is not log canonical at P, we have that $(S_1, \mu C^1_{d-1} + \mu L^1 + (\mu(n_0+1)-1)E_1)$ is not log canonical at some point in E_1. We choose this point to be P_1. Let $n_1 = \mathrm{mult}_{P_1}(C^1_{d-1})$. Consider

$$d-1-n_0 = C^1_{d-1}.L^1 \geqslant n_1$$

which implies that $n_0+n_1 \leqslant d-1$. But $n_0 = m_0 - 1 \leqslant d-3$, using our assumption.

Also, $2n_1 \leqslant n_0+n_1$ which implies $2n_1 \leqslant d-1$. We can then conclude that $\mu n_1 \leqslant 1$. We also have L^1 and E_1 smooth at P_1 and intersecting transversally at P_1. Thus applying Theorem 3.7, we get

$$\mu(d-1-n_0) = \mu C^1_{d-1}.L^1 > 2(2-\mu(n_0+1)) \tag{3}$$

which implies that $\mu(d-1) > 2$ or

$$\mu n_0 = \mu C^1_{d-1}.E_1 > 2(1-\mu) \tag{4}$$

which implies that $\mu(n_0+2) > 2$.

The two inequalities in (3) and (4) imply that $\mu(d-1) > 2$ which is absurd. Thus, L is not an irreducible component of C_d. ■

Since L is not an irreducible component of the curve C_d, from the computations in (1) we can also assume that $m_0 + m_1 \leqslant d$.

Since $(S, \mu C_d)$ is not log canonical at the point P and since $\mu < 1$, we have that $(S_1, \mu C^1_d + (\mu m_0 - 1)E_1)$ is not log canonical at some point in E_1, say P_1. We also have

$$\mu m_0 - 1 \leqslant \mu(d-2) - 1 = \frac{2}{d-1}(d-2) - 1 = \frac{d-3}{d-1} < 1,$$

for $d \geqslant 5$. Thus, from Lemma 3.5 there exists a unique point in E_2, say P_2, such that $(S_2, \mu C^2_d + (\mu m_0 - 1)E^2_1 + (\mu(m_0+m_1) - 2)E_2)$ is not log canonical at P_2.

We know that $P_2 \in C^2_d$, since if not, it would imply $(S_2, (\mu m_0 - 1)E^2_1 + (\mu(m_0 + m_1) - 2)E_2)$ is not log canonical at the point P_2. This is not possible since $\mu m_0 - 1 < 1$, $\mu(m_0+m_1) - 2 \leqslant \mu d - 2 < 1$, and E^2_1, E_2 are SNC divisors at P_2.

Claim 2 $P_2 \notin E^2_1$.

Proof Suppose $P_2 \in E^2_1$. Observe that

$$C^2_d \sim f^*_2(C^1_d) - m_1 E_2$$

so that $C^2_d.E^2_1 = C^1_d.E_1 - m_1 E_2.E^2_1 = m_0 - m_1$.

Also, since $P_2 \in C^2_d \cap E^2_1 \cap E_2$, we have $m_2 = \mathrm{mult}_{P_2}(C^2_d) \leqslant (C^2_d.E^2_1)$. Therefore, $m_2 \leqslant m_0 - m_1$. Since $m_2 \leqslant m_1$, we have $2m_2 \leqslant m_1 + m_2 \leqslant m_0$ which implies

$$m_2 \leqslant \frac{m_0}{2}.$$

From Lemma 4.1 (v), we have

$$\mu m_2 \leqslant \mu \frac{m_0}{2} \leqslant \mu \frac{d-2}{2} < 1.$$

We also know that E^2_1 and E_2 are smooth at P_2 and intersect transversally at P_2.

Since $(S_2, \mu C^2_d + (\mu m_0 - 1)E^2_1 + (\mu(m_0+m_1) - 2)E_2)$ is not log canonical at the point P_2, Theorem 3.7 implies $\mu(m_0 - m_1) = \mu C^2_d.E^2_1 > 2(3 - \mu(m_0+m_1))$ or $\lambda m_1 = \lambda C^2_d.E_2 > 2(2 - \lambda m_0)$. That is, $\mu(3m_0+m_1) > 6$ or $\mu(2m_0+m_1) > 4$. We have

$$3m_0 + m_1 = 2m_0 + m_0 + m_1 \leqslant 2(d-2) + d = 3d - 4, \tag{5}$$

$$2m_0 + m_1 \leqslant d - 2 + d = 2d - 2. \tag{6}$$

Using the above equations (5) and (6), both of the above mentioned inequalities obtained from using Theorem 3.7 result in contradiction, hence proving our claim that $P_2 \notin E_1^2$. ■

Claim 3 $P_2 \notin L^2$.

Proof Suppose $P_2 \in L^2$. Since L is not an irreducible component of C_d, we have $d - m_0 - m_1 = L^2.E_2 \geqslant m_2$ and this implies that $m_0 + m_1 + m_2 \leqslant d$. Also, applying Lemma 3.3, we get $\mu d \geqslant \mu(m_0 + m_1 + m_2) > 3$ which results in a contradiction since $\mu d < 3$. Thus, $P_2 \neq L^2 \cap E_2$. ■

Thus, we have that $(S_2, \mu C_d^2 + (\mu(m_0+m_1)-2)E_2)$ is not log canonical at the point P_2. Then from Remark 3.2, $(S_3, \mu C_d^3 + (\mu(m_0+m_1)-2)E_2^3 + (\mu(m_0+m_1+m_2)-3)E_3)$ is not log canonical at some point in E_3, say P_3.

We have $2m_1 \leqslant m_0 + m_1 \leqslant d$, and

$$\mu(m_0 + m_1 + m_2) \leqslant \mu(m_0 + 2m_1) \leqslant \mu(d - 2 + d) = 4. \tag{7}$$

Therefore $\mu(m_0 + m_1 + m_2) - 3 \leqslant 1$.

$P_3 \in C_d^3$, since if not, then this would imply that $(S_3, (\mu(m_0+m_1)-2)E_2^3 + (\mu(m_0+m_1+m_2)-3)E_3)$ is not log canonical at the point P_3. But since the coefficients of $E_i \leqslant 1$ and E_2^3, E_3 are SNC divisors over the point P_3, this is not possible.

Claim 4 $P_3 \notin E_2^3$.

Proof Suppose $P_3 \in E_2^3$. Observe that

$$C_d^3 \sim_{\mathbb{Q}} f_3^*(C_d^2) - m_2 E_3.$$

We thus have

$$C_d^3.E_2^3 = f_3^*(C_d^2.E_2) - m_2(E_3.E_2^3) = m_1 - m_2.$$

Therefore, Theorem 3.4 implies

$$\mu(m_1 - m_2) + (\mu(m_0 + m_1 + m_2) - 3) = (\mu C_d^3 + (\mu(m_0 + m_1 + m_2) - 3)E_3).E_2^3 > 1,$$

which implies $\mu(m_0 + 2m_1) > 4$. From (7) we know that $\mu(m_0 + 2m_1) \leqslant 4$. This contradiction proves our claim. ■

Therefore, the log pair $(S_3, \mu C_d^3 + (\mu(m_0 + m_1 + m_2) - 3)E_3)$ is not log canonical, at the point P_3. Thus from Remark 3.2, $(S_4, \mu C_d^4 + (\mu(m_0 + m_1 + m_2) - 3)E_3^4 + (\mu(m_0 + m_1 + m_2 + m_3) - 4)E_4)$ is not log canonical at a point $P_4 \in E_4$. We have

$$m_2 + m_3 \leqslant 2m_2 \leqslant m_0 + m_1 \leqslant d.$$

Thus,

$$\mu(m_0+m_1+m_2+m_3) \leqslant \mu(2(m_0+m_1)) \leqslant \frac{2}{d-1}2d < 5. \tag{8}$$

Claim 5 $P_4 \notin E_3^4$.

Proof Suppose $P_4 \in E_3^4$. From inequality (8), $m_0+m_1+2m_2 < 5/\mu$. Also, observe that

$$C_d^4 \sim_{\mathbb{Q}} f_4^*(C_d^3) - m_3 E_4$$

so that $C_d^4.E_3^4 = m_2 - m_3$. Then from Theorem 3.4 we have

$$(\mu C_d^4 + (\mu(m_0+m_1+m_2+m_3)-4)E_4).E_3^4 > 1,$$

which implies

$$m_0+m_1+2m_2 > \frac{5}{\mu}.$$

This contradicts inequality in (8). Thus $P_4 \notin E_3^4$. ∎

Thus, $(S_4, \mu C_d^4 + (\mu(m_0+m_1+m_2+m_3)-4)E_4)$ is not log canonical at the point P_4. From Lemma 3.6, we have

$$\mu \mathrm{mult}_{P_4}(C_d^4) + (\mu(m_0+m_1+m_2+m_3)-4) > 1,$$

which implies

$$\mathrm{mult}_{Q_4}(C_d^4) + m_0+m_1+m_2+m_3 > \frac{5}{\mu}. \tag{9}$$

Now using (9) and a geometric construction of a special curve in S_4 we will try to arrive at a contradiction. We may assume that the line L is given by $x=0$ and $P=[0:0:1]$. Let $\mathcal{C}$ be the conic in $\mathbb{P}^2$ that is given by

$$xz + Axy + By^2 = 0,$$

where $A, B \in \mathbb{C}$ and $B \neq 0$. Then $\mathcal{C}$ is smooth and is tangent to the line L. Denote the proper transform of $\mathcal{C}$ in S^i by $\mathcal{C}^i$. It follows from Claims 2, 3, 4 and 5, that there exist A and $B \neq 0$ such that $\mathcal{C}^i$ on S^i contain P_i for $i=1,2,3$. So we can assume that A, B are chosen this way. Then we have

$$\mathcal{C}^4 \sim 2L^4 + E_1^4 + 2E_2^4 + E_3^4.$$

Thus the pencil $|\mathcal{C}^4|$ does not have base points.

Also, let $\mathcal{L}$ be a pencil of conics in $\mathbb{P}^2$ given by

$$sx^2 + t(x + Axy + By^2) = 0,$$

where $s, t \in \mathbb{C}$. It is generated by $2L$ and $\mathcal{C}$.

Let $\phi_{|\mathcal{C}^4|} : S^4 \to \mathbb{P}^1$ be the morphism defined by the pencil $|\mathcal{C}^4|$. Similarly, let $\phi_{\mathcal{L}} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be the rational map defined by the pencil $\mathcal{L}$. These make the following diagram commutative:

$$\begin{tikzcd}
\mathbb{P}^2 \arrow[dr, "\phi_{\mathcal{L}}"'] & S_1 \arrow[l, "f_1"'] & S_2 \arrow[l, "f_2"'] & S_3 \arrow[l, "f_3"'] & S_4 \arrow[l, "f_4"'] \arrow[dlll, "\phi_{|\mathcal{C}^4|}"] \\
& \mathbb{P}^1. & & &
\end{tikzcd}$$

Choose a curve Z^4 in $|\mathcal{C}^4|$ that passes through the point P_4. Then Z^4 is a smooth irreducible curve. Let the proper transform of Z^4 in $\mathbb{P}^2$ be denoted by Z. Thus Z is a smooth conic in the pencil $\mathcal{L}$. Suppose Z is not an irreducible component of the curve C_d, then we have

$$2d - (m_0 + m_1 + m_2 + m_3) = C_d^4.Z^4 \geqslant \mathrm{mult}_{P_4}(C_d^4). \tag{10}$$

Equations (9) and (10) result in a contradiction since $\mu < 5/(2d)$.

Thus, $C_d = Z + C_{d-2}$ where C_{d-2} is an irreducible curve of degree $d-2$ which does not contain the conic Z as an irreducible component.

Let $C^1_{d-2}, C^2_{d-2}, C^3_{d-2}, C^4_{d-2}$ be the proper transforms of the curve C_{d-2} on the surfaces S_1, S_2, S_3 and S_4, respectively. Denote by $n_0 = \mathrm{mult}_P(C_{d-2})$, $n_1 = \mathrm{mult}_{P_1}(C^1_{d-2})$, $n_2 = \mathrm{mult}_{P_2}(C^2_{d-2})$, $n_3 = \mathrm{mult}_{P_3}(C^3_{d-2})$, and $n_4 = \mathrm{mult}_{P_4}(C^4_{d-2})$. Thus $(S_4, \mu C^4_{d-2} + \mu Z^4 + ((\mu(n_0 + n_1 + n_2 + n_3 + 4) - 4)E_4)$ is not log canonical at P_4.

Applying Theorem 3.4 to the above gives $\mu(2(d-2) - n_0 - n_1 - n_2 - n_3) + \mu(n_0 + n_1 + n_2 + n_3 + 4) - 4 > 1$, which implies $\mu > 5/(2d)$. But $\mu < 5/(2d)$ and thus this contradiction proves the proposition. $\square$

This in turn proves that $m_0 \leqslant d-2$ is not possible for the chosen value of λ, hence completing the Proof of Theorem 1.8.

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