

# Log canonical thresholds of high multiplicity reduced plane curves

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We compute log canonical thresholds of reduced plane curves of degree  $d$  at points of multiplicity  $d - 1$ . As a consequence, we describe all possible values of log canonical threshold that are less than  $2/(d - 1)$  for reduced plane curves of degree  $d$ . In addition, we compute log canonical thresholds for all reduced plane curves of degree less than 6.

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## 1 Introduction

Given a hypersurface  $Z := \{F(x_1, \dots, x_n) = 0\} \in \mathbb{A}^n$ , a classical question in singularity theory is to understand the complexity of singularities of  $Z$ . An invariant that can be used to measure the singularities of  $Z$  is the *log canonical threshold* which analyses the resolution of each singularity, revealing its complexity.

Log canonical thresholds appear in different problems pertaining to differential and algebraic geometry. For instance the greatest root of the Bernstein polynomial of a given polynomial  $F$ , which is another invariant of the singularities of  $F$ , is the negative of the log canonical threshold (see [Kol97, Theorem 10.6]). Another important application is in establishing the existence of Kähler-Einstein metrics on Fano varieties using the Tian's

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criterion which is an asymptotic version of the log canonical threshold ([Tia87]). In this regard, log canonical thresholds also play an important role in stability problems (see for example [Zan22]).

In this paper we study the log canonical thresholds of plane curves  $C \subset \mathbb{A}^2$  at a point  $P \in C$ . A pair  $(\mathbb{A}^2, C)$  is said to be *log canonical* at  $P$  when it has a *log resolution* over  $P$  such that locally the coefficients of all the prime divisors of the log pullback of  $C$  are at most 1 (Definition 2.7). We define

$$\text{lct}_P(\mathbb{A}^2, C) = \sup\left\{\lambda \mid (\mathbb{A}^2, \lambda C) \text{ is log canonical at the point } P\right\}.$$

There are many results in the literature on computing other invariants such as *Milnor number* and *multiplicity*. Note that the log canonical threshold can be considered to be a refinement of multiplicity (see [KSC04, Exercise 6.18 and Lemma 6.35]):

$$\frac{1}{\text{mult}_P(C)} \leq \text{lct}_P(\mathbb{A}^2, C) \leq \frac{2}{\text{mult}_P(C)}. \quad (1.0.1)$$

By [Var82], log canonical threshold is constant in  $\mu$ -constant strata, so log canonical threshold is a finer invariant than Milnor number. For instance, while the plane curves  $x^3y + xy^4 + x^2y^3$  and  $x^4 + y^5 + x^2y^3$  both have multiplicity 4 and Milnor number 12, the log canonical thresholds are respectively  $\frac{5}{11}$  and  $\frac{9}{20}$ , distinguishing the singularities.

Equation (1.0.1) shows that a high multiplicity corresponds to a low log canonical threshold. To put it more sharply, by [Che01, Theorem 4.1], the least log canonical threshold of a reduced plane curve of degree  $d$  is  $2/d$ , which happens precisely when the multiplicity at the point is  $d$ , meaning that the curve is the union of  $d$  lines. By [Che17] or [Vis20, Proposition 4.5], if  $\text{mult}_P(C) \leq d - 2$ , then  $\text{lct}_P(\mathbb{A}^2, C) \geq 2/(d - 1)$ . Therefore, the lowest log canonical thresholds, meaning the values between  $2/d$  and  $2/(d - 1)$ , happen when the multiplicity at the point is high, meaning at least  $d - 1$ .

Our main result is giving a simple formula for the log canonical thresholds for all reduced plane curves of multiplicity  $d - 1$ :

**Theorem 1.1** (= Theorem 3.2). *Let  $C$  be a reduced plane curve of degree  $d$  and  $P$  a point of  $C$  of multiplicity  $d - 1$ . Let  $C^1$  be the strict transform of  $C$  under the blowup of the plane  $\mathbb{A}^2$  at  $P$  and let  $E$  be the exceptional divisor. Then*

$$\text{lct}_P(\mathbb{A}^2, C) < \frac{2}{d - 1} \iff \exists Q \in C^1: \text{mult}_Q(C^1 \cdot E) > \frac{d - 1}{2}.$$

*If the point  $Q$  exists, then it is unique. In this case,*

$$\text{lct}_P(\mathbb{A}^2, C) = \begin{cases} \frac{2 \cdot \text{mult}_Q(C^1 \cdot E) - 1}{d \cdot (\text{mult}_Q(C^1 \cdot E) - 1) + 1} & \text{if } L_Q \text{ is an irreducible} \\ & \text{component of } C, \\ \frac{2 \cdot \text{mult}_Q(C^1 \cdot E) + 1}{d \cdot \text{mult}_Q(C^1 \cdot E)} & \text{otherwise,} \end{cases}$$

*where  $L_Q$  is the unique line on the affine plane containing  $P$  such that its strict transform contains  $Q$ .*

It was proved in [Che17, Theorem 1.10] that for  $d \geq 4$ , the five smallest log canonical thresholds are

$$\left\{ \frac{2}{d}, \frac{2d - 3}{(d - 1)^2}, \frac{2d - 1}{d(d - 1)}, \frac{2d - 5}{d^2 - 3d + 1}, \frac{2d - 3}{d(d - 2)} \right\},$$

and in [Vis20, Theorem 1.8], that for  $d \geq 5$  the sixth smallest log canonical threshold is  $\frac{2d-7}{d^2-4d+1}$ . A keen eye will notice a pattern in the six rational numbers above, namely that they contain two simple subsequences. We show that these subsequences can be extended. Moreover we describe all log canonical thresholds for points at multiplicity  $d - 1$ :

**Corollary 1.2** (= Corollary 3.4). *Let  $\Lambda_{d,d-1}$  denote the set of log canonical thresholds of pairs  $(\mathbb{A}^2, C)$  at a point of multiplicity  $d - 1$  of a reduced plane curve  $C$  of degree  $d$ . Then for every  $d \geq 3$ ,*

$$\Lambda_{d,d-1} = \left\{ \frac{2}{d-1} \right\} \cup \left\{ \frac{2k+1}{kd+1} \mid k \in \left\{ \left\lfloor \frac{d-1}{2} \right\rfloor, \dots, d-2 \right\} \right\} \\ \cup \left\{ \frac{2k+1}{kd} \mid k \in \left\{ \left\lfloor \frac{d+1}{2} \right\rfloor, \dots, d-1 \right\} \right\},$$

where  $\lfloor x \rfloor$  denotes the greatest integer not greater than  $x$ .

Lastly, we concentrate on low degree curves. Singularities of low degree plane curves have been intensively studied from various points of view. We fill a gap in this direction by computing the log canonical thresholds for all reduced plane curves of degree at most 5. Since log canonical threshold is constant in  $\mu$ -constant strata ([Var82]), we can make use of the existing classification lists of singularities to compute log canonical thresholds.

Table 1.1: Log canonical thresholds of reduced plane curves

| degree | $\text{lct}_P(\mathbb{A}^n, C)$  |
|--------|--|
| 1      | 1  |
| 2      | 1  |
| 3      | 1, $\frac{5}{6}$ , $\frac{3}{4}$ , $\frac{2}{3}$   |
| 4      | 1, $\frac{5}{6}$ , $\frac{3}{4}$ , $\frac{7}{10}$ , $\frac{2}{3}$ , $\frac{9}{14}$ , $\frac{5}{8}$ , $\frac{3}{5}$ , $\frac{7}{12}$ , $\frac{5}{9}$ , $\frac{1}{2}$  |
| 5      | 1, $\frac{5}{6}$ , $\frac{3}{4}$ , $\frac{7}{10}$ , $\frac{2}{3}$ , $\frac{9}{14}$ , $\frac{5}{8}$ , $\frac{11}{18}$ , $\frac{3}{5}$ , $\frac{13}{22}$ ,<br>$\frac{7}{12}$ , $\frac{4}{7}$ , $\frac{9}{16}$ , $\frac{5}{9}$ , $\frac{11}{20}$ , $\frac{6}{11}$ , $\frac{1}{2}$ , $\frac{7}{15}$ , $\frac{5}{11}$ , $\frac{9}{20}$ , $\frac{7}{16}$ , $\frac{2}{5}$ |

Table 1.1 gives an exhaustive list of all possible log canonical thresholds of reduced plane curves  $C$  of degree at most 5 at a given point  $P$ . In addition, Table 4.2 lists all the singularities that reduced plane curves of degree at most 5 can have and Table 4.1 lists the normal forms and the log canonical thresholds for each singularity.

## 2 Preliminaries

A **variety** is an integral separated scheme of finite type over the complex numbers  $\mathbb{C}$ . A **curve** is a reduced separated scheme of finite type over  $\mathbb{C}$  of pure dimension 1. All morphisms of varieties and curves are over  $\mathbb{C}$ . A polynomial  $f \in \mathbb{C}[x_1, \dots, x_n]$ , respectively power series  $f \in \mathbb{C}\{x_1, \dots, x_n\}$ , is **square-free** if no square of a non-unit in  $\mathbb{C}[x_1, \dots, x_n]$ , respectively  $\mathbb{C}\{x_1, \dots, x_n\}$ , divides  $f$ . A **plane curve of degree  $d$**  is a scheme which is isomorphic to an open dense subscheme of  $\text{Proj } \mathbb{C}[x, y, z]/(f)$  for a square-free polynomial  $f \in \mathbb{C}[x, y, z]$  homogeneous of degree  $d$ , where  $d$  is a positive integer.

## 2.1 Power series

We use the definitions below in Section 4.

**Definition 2.1** ([GLS07, Definitions I.1.1, I.1.47 and I.2.1]). Let  $n$  be a positive integer. We denote:

- $\mathbb{C}\{x_1, \dots, x_n\}$  — the  $\mathbb{C}$ -algebra of power series in variables  $x_1, \dots, x_n$  that are absolutely convergent in a neighbourhood of the origin,
- $(\mathbb{V}(f), \mathbf{0})$  — the (possibly non-reduced) complex space subgerm of  $(\mathbb{C}^n, \mathbf{0})$  defined by  $f \in \mathbb{C}\{x_1, \dots, x_n\}$ .
- $\mu(f)$  — the **Milnor number** of  $f \in \mathbb{C}\{x_1, \dots, x_n\}$ , defined by

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_1, \dots, x_n\}}{\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)}.$$

**Definition 2.2** ([GLS07, Section I.0.B and Definitions I.2.11 and I.2.17]). Let  $n$  be a positive integer. Let  $(w_1, \dots, w_n)$  be positive rational numbers corresponding to the variables  $x_1, \dots, x_n$  and let  $f \in \mathbb{C}\{x_1, \dots, x_n\}$ . We denote:

- $\text{wt}(f)$  — the **weight** of  $f$ , defined by

$$\text{wt}(f) := \min \left\{ i_1 w_1 + \dots + i_n w_n \mid \begin{array}{l} i_1, \dots, i_n \in \mathbb{Z}_{\geq 0}, \text{ the coefficient} \\ \text{of } x_1^{i_1} \cdots x_n^{i_n} \text{ in } f \text{ is non-zero} \end{array} \right\},$$

- $\text{mult}(f)$  — the **multiplicity of  $f$** , defined to be the weight of  $f$  with respect to the weights  $(1, \dots, 1)$ .

We say that  $f$  is **quasihomogeneous** if all the monomials with a non-zero coefficient in  $f$  have the same weight. We say that  $f$  is **semiquasihomogeneous** if the least weight part of  $f$  has an isolated singularity at the origin.

**Definition 2.3** ([GLS07, Definition I.2.9] and [AGZV85, Introduction to Part II]). Let  $n \leq m$  be positive integers. Let  $f, g \in \mathbb{C}\{x_1, \dots, x_n\}$  and  $h \in \mathbb{C}\{y_1, \dots, y_m\}$ .

- We say that  $f$  and  $g$  are **right equivalent** if there exists an automorphism  $\Phi$  of  $\mathbb{C}\{x_1, \dots, x_n\}$  such that  $\Phi(f) = g$ . In simpler terms,  $f$  and  $g$  are right equivalent if they coincide up to local analytic coordinate changes.
- We say that  $f$  and  $h$  are **stably right equivalent** if there exists a non-negative integer  $k$  and an isomorphism  $\Psi: \mathbb{C}\{x_1, \dots, x_{n+k}\} \rightarrow \mathbb{C}\{y_1, \dots, y_{n+k}\}$  such that  $\Psi(f + x_{n+1}^2 + \dots + x_{n+k}^2) = h + y_{m+1}^2 + \dots + y_{n+k}^2$ .

*Remark 2.4.* Power series  $f, g \in \mathbb{C}\{x_1, \dots, x_n\}$  are stably right equivalent if and only if they are right equivalent, see [AGZV85, Remark in Section 11.1].

## 2.2 Log resolutions

We use the definitions below in Section 3.

**Definition 2.5.** (a) Let  $C$  be a curve,  $P \in C$  a closed point such that the complex space germ  $(C^{\text{an}}, P)$  is isomorphic to the complex space subgerm  $(\mathbb{V}(f), \mathbf{0})$  of  $(\mathbb{C}^2, \mathbf{0})$ , where  $C^{\text{an}}$  denotes the analytification of  $C$  and  $f$  is a convergent power series in  $\mathbb{C}\{x, y\}$ . Then the **multiplicity of  $C$  at  $P$** , denoted  $\text{mult}_P(C)$ , is defined to be  $\text{mult}(f)$ .

- (b) The **intersection multiplicity** of  $X$  and  $V$  along  $Z$ , which we denote  $\text{mult}_Z(X \cdot V)$ , is defined in [Ful98, Definition 7.1]. For computations, we use [Ful98, Example 7.1.10(b)].

**Definition 2.6** ([KM98, Notation 0.4]). Let  $S$  be a smooth variety. A  $\mathbb{Q}$ -**divisor** on  $S$  is a formal  $\mathbb{Q}$ -linear combination  $\sum \lambda_i D_i$  of prime divisors  $D_i$  where  $\lambda_i \in \mathbb{Q}$ . A  $\mathbb{Q}$ -divisor  $\sum \lambda_i D_i$  is **snc** if all of the following are satisfied:

- for all  $i$ ,  $D_i$  is smooth,
- for all  $i \neq j$ , the prime divisors  $D_i$  and  $D_j$  intersect transversally, and
- for all pairwise different  $i, j, k$ , the intersection  $D_i \cap D_j \cap D_k$  is empty.

Let  $D$  be a  $\mathbb{Q}$ -divisor on a smooth variety  $S$ . A **log resolution of  $(S, D)$  over  $P$**  is a proper birational morphism  $\pi: S' \rightarrow S$  from a scheme  $S'$  such that there exists an open neighbourhood  $U \subseteq S$  of  $P$  such that  $\pi^{-1}U$  is a smooth variety, the exceptional locus  $E$  of  $\pi|_{\pi^{-1}U}$  is of pure codimension 1 and  $E \cup \pi|_{\pi^{-1}U}^{-1}(\text{Supp}(D \cap U))$  is an snc divisor of  $\pi^{-1}U$ .

Let  $D$  be a  $\mathbb{Q}$ -divisor on a smooth variety  $S$  and  $\varphi: S' \rightarrow S$  a proper birational morphism from a smooth variety  $S'$ . The **relative canonical divisor** of  $\pi$ , denoted  $K_\pi$ , is the unique  $\mathbb{Q}$ -divisor that is linearly equivalent to  $\pi^*(K_S) - K_{S'}$  and supported on the exceptional locus of  $\pi$ , where  $K_S$  and  $K_{S'}$  are the canonical classes of respectively  $S$  and  $S'$ . The **log pullback** of  $D$  with respect to  $\pi$  is the  $\mathbb{Q}$ -divisor  $D' = K_\pi + \pi^*D$  on  $S'$ . We have the linear equivalence  $K_{S'} + D' \sim \varphi^*(K_S + D)$ .

**Definition 2.7** ([Kol97, Definition 3.5] or [KM98, Definition 2.34]). Let  $D$  be a  $\mathbb{Q}$ -divisor on a smooth variety  $S$  and let  $P \in S$  be a point. The pair  $(S, D)$  is **log canonical at  $P$**  if we can restrict  $(S, D)$  to an open neighbourhood of  $P$  such that there exists a log resolution such that all the coefficients of the prime divisors in the log pullback of  $D$  are at most 1. The **log canonical threshold of  $(S, D)$  at  $P$**  is

$$\text{lct}_P(S, D) := \sup\left\{\lambda \in (0, 1] \cap \mathbb{Q} \mid (S, \lambda D) \text{ is log canonical at } P\right\}.$$

**Proposition 2.8** ([Mat02, Proposition 4-4-4]). *The log canonical threshold at a closed point is a local analytic invariant.*

### 3 High multiplicity curves

In this section we classify log canonical thresholds at points of multiplicity  $d - 1$  for reduced plane curves of degree  $d$ . The notation we use is given in Setting 3.1.

**Setting 3.1.** Let  $d \geq 3$  be an integer. Let  $P$  be a point of a reduced affine plane curve  $C$  of degree  $d$  such that  $\text{mult}_P C = d - 1$ . Let  $C^1$  be the strict transform of  $C$  under the blowup of the affine plane along  $P$  with exceptional divisor  $E_1$ . For every point  $Q \in C^1$  such that  $\text{mult}_Q(C^1 \cdot E_1) > 1$ , let  $L_Q$  be the line on the affine plane through  $P$  such that its strict transform passes through  $Q$ . For every such  $Q \in C^1$ , let  $C_Q^1$  be the strict transform of the Zariski closure  $C_Q$  of  $C \setminus L_Q$ . Define the positive integer  $k_Q$  by

$$k_Q := \text{mult}_Q(C_Q^1 \cdot E_1)$$

and define the positive rational number  $l_Q$  by

$$l_Q := \begin{cases} \frac{2k_Q + 1}{k_Q d + 1} & \text{if } L_Q \text{ is an irreducible component of } C, \\ \frac{2k_Q + 1}{k_Q d} & \text{otherwise.} \end{cases}$$

By Equation (1.0.1),  $\text{lct}_P(\mathbb{A}^2, C) \leq 2/(d-1)$ . The main theorem of this section is as follows.

**Theorem 3.2.** *In Setting 3.1,*

$$\text{lct}_P(\mathbb{A}^2, C) < \frac{2}{d-1} \iff \exists Q \in C^1: \text{mult}_Q(C^1 \cdot E_1) > \frac{d-1}{2}.$$

*If the point  $Q$  exists, then it is unique and  $\text{lct}_P(\mathbb{A}^2, C) = l_Q$ .*

To prove Theorem 3.2, we first describe  $C$  using equations:

**Lemma 3.3.** *We say that two triples  $(C, P, Q)$  and  $(C', P', Q')$  are isomorphic if there exists an isomorphism  $C \rightarrow C'$  of curves that takes the point  $P$  to  $P'$  and  $Q$  to  $Q'$ . Let  $\text{Spec } \mathbb{C}[x_1, y] \cong \text{Spec } \mathbb{C}[x/y, y] \leftarrow \mathbb{C}[x, y]$  be one of the affine opens of the blowup of  $\mathbb{A}^2 := \text{Spec } \mathbb{C}[x, y]$  at  $\mathbf{0}$ . Then, up to isomorphism, the triples  $(C, P, Q)$  in Setting 3.1 are precisely given by*

$$\begin{aligned} C &= \mathbb{V}(f) \subseteq \mathbb{A}^2, \\ P &= \mathbf{0} \in \mathbb{A}^2, \\ Q &= \mathbf{0} \in \text{Spec } \mathbb{C}[x_1, y], \end{aligned}$$

where  $d \in \mathbb{Z}_{\geq 3}$ ,  $a_i, b_j \in \mathbb{C}$ ,  $a_{k_Q} \neq 0$ ,  $b_0 \neq 0$ ,  $f$  is square-free and where one of the following holds:

- $L_Q$  is an irreducible component of  $C$ ,  $k_Q \in \{1, \dots, d-2\}$  and

$$f := x \left( \sum_{i \in \{k_Q, k_Q+1, \dots, d-2\}} a_i x^i y^{d-2-i} + \sum_{j \in \{0, 1, \dots, d-1\}} b_j x^j y^{d-1-j} \right),$$

or

- $L_Q$  is not an irreducible component of  $C$ ,  $k_Q \in \{2, \dots, d-1\}$  and

$$f := \sum_{i \in \{k_Q, k_Q+1, \dots, d-1\}} a_i x^i y^{d-1-i} + \sum_{j \in \{0, 1, \dots, d\}} b_j x^j y^{d-j}.$$

In both cases,  $L_Q$  and  $E_1 \cap \text{Spec } \mathbb{C}[x_1, y]$  from Setting 3.1 correspond respectively to  $\mathbb{V}(x) \subseteq \mathbb{A}^2$  and  $\mathbb{V}(y) \subseteq \text{Spec } \mathbb{C}[x_1, y]$ .

*Proof.* We show how every  $(C, P, Q)$  from Setting 3.1 is given by some  $(\mathbb{V}(f), \mathbf{0}, \mathbf{0})$ . First, translate  $P$  to  $\mathbf{0}$  on  $\mathbb{A}^2$ . Then use a linear invertible map on  $\mathbb{A}^2$  fixing  $\mathbf{0}$  to move  $Q$  to  $\mathbf{0} \in \text{Spec } \mathbb{C}[x_1, y]$ . It follows that  $L_Q = \mathbb{V}(x)$  and  $E_1 \cap \text{Spec } \mathbb{C}[x_1, y] = \mathbb{V}(y)$ . The curve  $C$  is given by  $\mathbb{V}(g) \subseteq \mathbb{A}^2$  where  $g = g_{d-1} + g_d \in \mathbb{C}[x, y]$ , where  $g_{d-1}$  and  $g_d$  are respectively homogeneous of degrees  $d-1$  and  $d$ . The curve  $C^1 \cap \text{Spec } \mathbb{C}[x_1, y]$  is given by  $g_{d-1}(x_1, 1) + yg_d(x_1, 1)$ . We see that  $C^1 \cdot E = \text{mult } g_{d-1}(x_1, 1)$ . This shows that  $g$  is equal to  $f$  for some choice of  $a_i$  and  $b_j$ .

Conversely, if  $(C, P, Q)$  are given by some  $(\mathbb{V}(f), \mathbf{0}, \mathbf{0})$  as above, then  $C$  is a reduced affine plane curve of degree  $d$ ,  $P \in C$  a point of multiplicity  $d-1$ ,  $\text{mult}_Q(C_Q^1 \cdot E_1) = k_Q$  and  $\text{mult}_Q(C^1 \cdot E_1) > 1$ .  $\square$

*Proof of Theorem 3.2.* Let  $\pi_1: S_1 \rightarrow \mathbb{A}^2$  be the blowup along  $P$ . If for every point  $Q \in C^1 \cap E_1$  we have that  $\text{mult}_Q(C^1 \cdot E_1) = 1$ , then  $\text{lct}_P(\mathbb{A}^2, C) = 2/(d-1)$ . Otherwise,

let  $Q$  be any point of  $C^1$  such that  $\text{mult}_Q(C^1 \cdot E_1) > 1$ . By Lemma 3.3,  $C_Q^1 \cap \text{Spec } \mathbb{C}[x_1, y]$  is given by

$$f_1 := \sum_{i \in \{k_Q, k_Q+1, \dots, d-2\}} a_i x_1^i + y \sum_{j \in \{0, 1, \dots, d-1\}} b_j x_1^j, \quad (3.3.1)$$

if  $L_Q$  is an irreducible component of  $C$  and by

$$f_1 := \sum_{i \in \{k_Q, k_Q+1, \dots, d-1\}} a_i x_1^i + y \sum_{j \in \{0, 1, \dots, d\}} b_j x_1^j. \quad (3.3.2)$$

otherwise. Every irreducible component of  $C$  that is not a line has degree strictly greater than its multiplicity at  $P$ . Therefore, the curve  $C$  has exactly one irreducible component which is not a line. Therefore,  $C_Q^1$  has exactly one irreducible component passing through  $Q$ . The strict transform  $L_Q^1$  of  $L_Q$  under  $\pi_1$  is given on  $\text{Spec } \mathbb{C}[x_1, y]$  by  $\mathbb{V}(x_1)$ . Using Equations (3.3.1) and (3.3.2), we see that  $L_Q^1$  and  $C_Q^1$  intersect in exactly one point, namely  $Q$ , and the intersection is transversal. In particular,  $Q$  is a smooth point of  $C_Q^1$ .

For every rational number  $\lambda > 1/(d-1)$ , let  $D_\lambda$  denote the effective  $\mathbb{Q}$ -divisor

$$D_\lambda := \lambda C^1 + (\lambda(d-1) - 1)E_1$$

on  $S_1$ . Then  $D_\lambda$  is the log pullback of  $\lambda C$  under  $\pi_1$ .

We describe a log resolution  $\pi_2 \circ \dots \circ \pi_{k_Q+1}$  over  $Q$  of  $(S_1, D_\lambda)$ . Let  $\pi_2: S_2 \rightarrow S_1$  be the blowup along  $Q$ . For every  $r \in \{2, \dots, k_Q\}$ , let  $\pi_{r+1}: S_{r+1} \rightarrow S_r$  be the blowup along the point  $Q_r := \mathbb{V}(x_1, y_r)$  of the affine open  $\text{Spec } \mathbb{C}[x_1, y_r] \cong \text{Spec } \mathbb{C}[x_1, y_{r-1}/x_1]$  of  $S_r$ , where  $y_1 := y$ . Let  $C_Q^r$  denote the strict transform of  $C_Q$  under  $\pi_1 \circ \dots \circ \pi_r$ . We see using Equations (3.3.1) and (3.3.2) that  $C_Q^r \cap \text{Spec } \mathbb{C}[x_1, y_r]$  is given by

$$f_r := \sum_{i \in \{k_Q, k_Q+1, \dots, d-2\}} a_i x_1^{i-r+1} + y_r \sum_{j \in \{0, 1, \dots, d-1\}} b_j x_1^j, \quad (3.3.3)$$

if  $L_Q$  is an irreducible component of  $C$  and by

$$f_r := \sum_{i \in \{k_Q, k_Q+1, \dots, d-1\}} a_i x_1^{i-r+1} + y_r \sum_{j \in \{0, 1, \dots, d\}} b_j x_1^j. \quad (3.3.4)$$

otherwise. Let  $E_r$  be the exceptional divisor of  $\pi_r$  and let  $E_i^r$  be the strict transform of the exceptional divisor of  $\pi_i$  under  $\pi_i \circ \dots \circ \pi_r$ . We have

$$\begin{aligned} E_r \cap \text{Spec } \mathbb{C}[x_1, y_r] &= \mathbb{V}(x_1), \\ E_1^r \cap \text{Spec } \mathbb{C}[x_1, y_r] &= \mathbb{V}(y_r). \end{aligned}$$

We find that  $E_r$  and  $E_1^r$  intersect in exactly one point, namely  $Q_r$ , and the intersection is transversal. Moreover,  $Q_r \notin E_2^r \cup E_3^r \cup \dots \cup E_{r-1}^r$ . Therefore,  $\sum_{i \in \{1, \dots, r\}} E_i^r$  is snc. Using Equations (3.3.3) and (3.3.4), we see that  $C_Q^r$  and  $E_r$  intersect in exactly one point, namely  $Q_r$ , and the intersection is transversal. Therefore,  $(E_2^r \cup E_3^r \cup \dots \cup E_{r-1}^r) \cap C_Q^r$  is empty. From Equations (3.3.3) and (3.3.4), we find

$$\text{mult}_{Q_r}(C_Q^r \cdot E_1^r) = k_Q - r + 1.$$

The varieties  $C_Q^{k_Q}$ ,  $E_{k_Q}$  and  $E_1^{k_Q}$  have pairwise transversal intersections at  $Q_{k_Q}$ . Therefore,  $\pi_2 \circ \dots \circ \pi_{k_Q+1}$  is a log resolution of  $(S_1, D_\lambda)$  over  $Q$ .

Let  $D_\lambda^{k_Q+1}$  denote the strict transform of  $D_\lambda$  under  $\pi_2 \circ \dots \circ \pi_{k_Q+1}$ . The log pullback of  $D_\lambda$  under the composition  $\pi_2 \circ \dots \circ \pi_{k_Q+1}$  is given by

$$D_\lambda^{k_Q+1} + \sum_{j \in \{1, \dots, k_Q\}} (\lambda(jd + 1) - 2j) E_{j+1}^{k_Q+1}$$

if  $L_Q$  is an irreducible component of  $C$  and

$$D_\lambda^{k_Q+1} + \sum_{j \in \{1, \dots, k_Q\}} (\lambda jd - 2j) E_{j+1}^{k_Q+1}$$

otherwise.

We have the following equivalences:

$$\begin{aligned} \lambda(d-1) - 1 \leq 1 &\iff \lambda \leq \frac{2}{d-1}, \\ \lambda(jd+1) - 2j \leq 1 &\iff \lambda \leq \frac{2j+1}{jd+1}, \\ \lambda jd - 2j \leq 1 &\iff \lambda \leq \frac{2j+1}{jd}. \end{aligned}$$

Note that

$$l_Q = \begin{cases} \min \left\{ \frac{2j+1}{jd+1} \mid j \in \{1, \dots, k_Q\} \right\} & \text{if } L_Q \text{ is an irreducible} \\ & \text{component of } C, \\ \min \left\{ \frac{2j+1}{jd} \mid j \in \{1, \dots, k_Q\} \right\} & \text{otherwise.} \end{cases}$$

Let  $\pi$  be the composition of the blowup  $\pi_1$  with the  $\sum_Q k_Q$  blowups above, where the sum is over points  $Q \in C^1$  such that  $\text{mult}_Q(C^1 \cdot E_1) > 1$ . Then  $\pi$  is a log resolution of  $(\mathbb{A}^2, C)$  over  $P$ . The greatest positive rational number  $\lambda$  such that all the coefficients of the prime divisors in the log pullback of  $\lambda C$  with respect to  $\pi$  are at most 1 is

$$\lambda = \min \left( \left\{ \frac{2}{d-1} \right\} \cup \left\{ l_Q \mid Q \in C^1, \text{mult}_Q(C^1 \cdot E_1) > 1 \right\} \right),$$

which is by definition the log canonical threshold of  $(\mathbb{A}^2, C)$  at  $P$ .

Finally, it is easy to compute that for any point  $Q$ ,  $l_Q < 2/(d-1)$  if and only if  $\text{mult}_Q(C^1 \cdot E_1) > (d-1)/2$ . If such a point  $Q$  exists, then it is necessarily unique since

$$\sum_{Q \in C^1 \cap E_1} \text{mult}_Q(C^1 \cdot E_1) = C^1 \cdot E_1 = \text{mult}_P(C) = d-1. \quad \square$$

**Corollary 3.4.** *Let  $\Lambda_{d,d-1}$  denote the set of log canonical thresholds of pairs  $(\mathbb{A}^2, C)$  at a point of multiplicity  $d-1$  of a reduced plane curve  $C$  of degree  $d$ . Then for every  $d \geq 3$ ,*

$$\begin{aligned} \Lambda_{d,d-1} = & \left\{ \frac{2}{d-1} \right\} \cup \left\{ \frac{2k+1}{kd+1} \mid k \in \left\{ \left\lfloor \frac{d-1}{2} \right\rfloor, \dots, d-2 \right\} \right\} \\ & \cup \left\{ \frac{2k+1}{kd} \mid k \in \left\{ \left\lfloor \frac{d+1}{2} \right\rfloor, \dots, d-1 \right\} \right\}, \end{aligned} \quad (3.4.1)$$

where  $\lfloor x \rfloor$  denotes the greatest integer not greater than  $x$ .

*Proof.* If  $f \in \mathbb{C}[x, y]$  is a multiplicity  $d - 1$  polynomial such that its homogeneous degree  $d - 1$  part is square-free, then  $\text{lct}_P(V(f)) = 2/(d - 1)$ . This shows that  $2/(d - 1) \in \Lambda_{d,d-1}$ .

In Setting 3.1 by Theorem 3.2, we have

$$\text{lct}_P(Q) \leq 2/(d - 1) \quad (3.4.2)$$

and the strict inequality holds in Equation (3.4.2) if and only if there exists a point  $Q \in C^1$  such that  $\text{mult}_Q(C^1 \cdot E_1) > 1$  one of the following holds:

(3.4.3)  $k_Q \geq \lfloor (d - 1)/2 \rfloor$  and  $L_Q$  is an irreducible component of  $C$ , or

(3.4.4)  $k_Q \geq \lfloor (d + 1)/2 \rfloor$  and  $L_Q$  is not an irreducible component of  $C$ .

Moreover, if (3.4.3) or (3.4.4) holds for some  $Q \in C^1$ , then  $\text{lct}_P(Q) = l_Q$ . Denoting the right-hand side of Equation (3.4.1) by RHS, we find  $\Lambda_{d,d-1} \subseteq \text{RHS}$ . On the other hand, it is easy to construct examples of reduced plane curves satisfying (3.4.3) or (3.4.4) using Lemma 3.3. This proves  $\text{RHS} \subseteq \Lambda_{d,d-1}$ .  $\square$

*Remark 3.5.* The sets

$$\left\{ \frac{2k + 1}{kd + 1} \mid k \in \left\{ \left\lfloor \frac{d - 1}{2} \right\rfloor, \dots, d - 2 \right\} \right\}$$

and

$$\left\{ \frac{2k + 1}{kd} \mid k \in \left\{ \left\lfloor \frac{d + 1}{2} \right\rfloor, \dots, d - 1 \right\} \right\}$$

that appear in Corollary 3.4 are disjoint for every integer  $d \geq 3$ .

## 4 Low degree curves

### 4.1 Table of normal forms of singularities

We define normal forms for the  $\mu$ -constant stratum. Note that in [AGZV85, Section 15.0], normal forms are defined more generally for any class of singularities, not only the  $\mu$ -constant stratum, and the image of  $\Phi$  is the whole polynomial ring  $\mathbb{C}[x_1, \dots, x_n]$ , not a jet space. The reason it suffices to consider a jet space here is that a convergent power series  $f$  of finite Milnor number  $\mu(f)$  is  $(\mu(f) + 1)$ -determined, see [GLS07, Corollary I.2.24].

**Definition 4.1** ([AGZV85, Section 15.0]). Let  $n$  be a positive integer, let  $m$  be a non-negative integer and let  $f \in \mathbb{C}\{x_1, \dots, x_n\}$  have finite Milnor number  $\mu(f)$ . The  $m$ -**jet** of  $f$  is the sum over  $k \in \{0, \dots, m\}$  of the homogeneous degree  $k$  parts of  $f$ . The  $m$ -**jet space**, denoted  $\mathbb{C}[x_1, \dots, x_n]_{\leq m}$ , is the  $\mathbb{C}$ -vector space of polynomials in  $\mathbb{C}[x_1, \dots, x_n]$  of degree at most  $m$ . As a  $\binom{n+m-1}{m}$ -dimensional vector space over  $\mathbb{C}$ , the  $m$ -jet space has a natural structure of a smooth complex space. The  $\mu$ -**constant stratum of  $f$**  is the connected component of the  $(\mu(f) + 1)$ -jet space of polynomials with Milnor number  $\mu(f)$  which contains the  $(\mu(f) + 1)$ -jet of  $f$ . A **normal form of  $f$**  is a holomorphic map  $\mathbb{C}^m \rightarrow \mathbb{C}[x_1, \dots, x_n]_{\leq \mu(f)+1}$  such that all of the following hold:

- (1)  $\Phi(\mathbb{C}^m)$  intersects the right equivalence class of every polynomial in the  $\mu$ -constant stratum of  $f$ ,
- (2) the inverse image under  $\Phi$  of every right equivalence class in  $\Phi(\mathbb{C}^m)$  is finite, and
- (3) the inverse image under  $\Phi$  of the complement of the  $\mu$ -constant stratum of  $f$  is contained in a proper closed analytic subset of  $\mathbb{C}^m$ .

A **normal form** is a holomorphic map  $\mathbb{C}^m \rightarrow \mathbb{C}[x_1, \dots, x_n]_{\leq k}$ , where  $k$  is a positive integer, which is a normal form of some polynomial in its image. A **polynomial normal form** is a normal form  $\Phi$  such that  $\Phi$  is a polynomial map. The  **$\mu$ -constant stratum of a normal form**  $\Phi$  is the  $\mu$ -constant stratum of a polynomial  $f$  such that  $\Phi$  is a normal form of  $f$ .

All of the normal forms below are polynomial normal forms. Table 4.1 contains the notation from [AGZV85, Sections 15] (or [Arn75, Section 13]) for the normal forms that we use. In Table 4.1,  $a, b$  and  $c$  are complex numbers,  $k, q$  and  $r$  are positive integers, *mult* stands for multiplicity,  $\mu$  for Milnor number, *lct* for log canonical threshold, *restrictions* describes the domain of the indices and  *$\mu$ -constant stratum* describes the intersection of the image and the  $\mu$ -constant stratum of the normal form.

Table 4.1: Notation for normal forms

| Symbol      | Indices                                   | Normal form                               | $\mu$ -constant stratum | mult | $\mu$       | lct                  |
|-------------|---|---|-------------------------|------|-------------|----------------------|
| $A_k$       | $k \geq 1$                                | $x^2 + y^{k+1}$                           |                         | 2    | $k$         | $\frac{k+3}{2(k+1)}$ |
| $D_k$       | $k \geq 4$                                | $x^2y + y^{k-1}$                          |                         | 3    | $k$         | $\frac{k}{2(k-1)}$   |
| $E_6$       |   | $x^3 + y^4$                               |                         | 3    | 6           | $\frac{7}{12}$       |
| $E_7$       |   | $x^3 + xy^3$                              |                         | 3    | 7           | $\frac{5}{9}$        |
| $E_8$       |   | $x^3 + y^5$                               |                         | 3    | 8           | $\frac{8}{15}$       |
| $T_{2,3,6}$ |   | $ax^2y^2 + x^3 + y^6$                     | $4a^3 + 27 \neq 0$      | 3    | 10          | $\frac{1}{2}$        |
| $T_{2,4,4}$ |   | $ax^2y^2 + x^4 + y^4$                     | $a^2 \neq 4$            | 4    | 9           | $\frac{1}{2}$        |
| $T_{2,q,r}$ | $\frac{1}{q} + \frac{1}{r} < \frac{1}{2}$ | $ax^2y^2 + x^q + y^r$                     | $a \neq 0$              | 4    | $q + r + 1$ | $\frac{1}{2}$        |
| $Z_{11}$    |   | $x^3y + y^5 + axy^4$                      |                         | 4    | 11          | $\frac{7}{15}$       |
| $Z_{12}$    |   | $x^3y + xy^4 + ax^2y^3$                   |                         | 4    | 12          | $\frac{5}{11}$       |
| $W_{12}$    |   | $x^4 + y^5 + ax^2y^3$                     |                         | 4    | 12          | $\frac{9}{20}$       |
| $W_{13}$    |   | $x^4 + xy^4 + ay^6$                       |                         | 4    | 13          | $\frac{7}{16}$       |
| $N_{16}$    |   | $x^5 + ax^3y^2 + bx^2y^3 + y^5 + cx^3y^3$ | $f_5$ square-free       | 5    | 16          | $\frac{2}{5}$        |

*Remark 4.2.* (a) We have added the polynomial for  $N_{16}$  in Table 4.1 which does not appear in [AGZV85, Sections 15]. The polynomial for  $N_{16}$  defines a normal form by [BMP20, Theorem 3.20]. The  $\mu$ -constant stratum of  $N_{16}$  is the open dense subset where the homogeneous degree 5 part is a product of five pairwise coprime linear forms.

- (b) By  $A_k$  *singularity*,  $D_k$  *singularity*,  $\dots$ ,  $N_{16}$  *singularity*, we mean a complex space germ  $(X, P)$  isomorphic to a complex space subgerm  $(\mathbb{V}(f), \mathbf{0})$  of  $(\mathbb{C}^n, \mathbf{0})$  where the stable right equivalence class of  $f \in \mathbb{C}\{x_1, \dots, x_n\}$  contains a polynomial which is in the  $\mu$ -constant stratum of respectively  $A_k, D_k, \dots, N_{16}$ .
- (c) Normal forms are usually considered up to stable right equivalence, meaning that if  $f$  and  $g$  are stably right equivalent, then the normal forms of  $f$  and  $g$  are considered to be the same. Due to this, there are several different notations for some of the normal forms in Table 4.1:

- (1)  $T_{2,3,6} = J_{10}$ ,
- (2)  $T_{2,3,6+k} = J_{10+k}$  for all positive integers  $k$ ,
- (3)  $T_{2,4,4} = X_9$ ,
- (4)  $T_{2,4,4+k} = X_{9+k} = X_{1,k}$  for all positive integers  $k$ ,
- (5)  $T_{2,4+r,4+s} = Y_{4+r,4+s} = Y_{r,s}^1$  for all positive integers  $r$  and  $s$ .

We compute the log canonical thresholds of the polynomials in Table 4.1 using Lemma 4.3.

**Lemma 4.3** ([Kol97, Remark 8.14.1] or [Kuw99, Proposition 2.1]). *Let  $f \in \mathbb{C}\{x_1, \dots, x_n\}$ . Assign positive rational weights  $w_i$  to the variables. Let  $f_w$  denote the weighted homogeneous leading term of  $f$ . Define  $b := \sum_i w_i / \text{wt}(f)$ . If the pair  $(\mathbb{C}^n, b\mathbb{V}(f_w))$  is log canonical outside the origin, then  $\text{lct}_{\mathbf{0}}(\mathbb{C}^n, \mathbb{V}(f)) = b$ , where we consider the spaces as complex space germs around  $\mathbf{0}$ .*

**Lemma 4.4.** *Let  $f$  be one of the polynomials in the column normal form in Table 4.1, satisfying the corresponding restrictions in column  $\mu$ -constant stratum. Then  $f$  has multiplicity  $\text{mult}$  and Milnor number  $\mu$  and  $(\mathbb{A}^2, f)$  has log canonical threshold  $\text{lct}$  at the origin as given in Table 4.1.*

*Proof.* The power series for  $T_{2,q,r}$  are Newton non-degenerate and the power series for the other singularities in Table 4.1 are semiquasihomogeneous. There are combinatorial formulas for the Milnor number in these cases, see [GLS07, Proposition 2.16 and Corollary 2.18].

Choose the weights  $(2, q-2)$  for  $(x, y)$  and let  $f$  be a power series for  $T_{2,q,r}$  in Table 4.1. Since the pair  $(\mathbb{C}^2, \frac{1}{2}\mathbb{V}(ax^2y^2 + x^q))$  is log canonical outside the origin, the log canonical threshold of  $f$  at the origin is  $\frac{1}{2}$ . The power series for all the other singularities in Table 4.1 are semiquasihomogeneous and Lemma 4.3 gives a combinatorial formula for the log canonical threshold.  $\square$

## 4.2 Tables of low degree curves

Table 4.2: Singularities of reduced plane curves of given degree

| Degree $d$ | Possible singularities  |
|------------|---|
| 2          | $A_1$   |
| 3          | $A_1, A_2, A_3, D_4$  |
| 4          | $A_1, \dots, A_7, D_4, D_5, D_6, E_6, E_7, T_{2,4,4}$   |
| 5          | $A_1, \dots, A_{12}, D_4, \dots, D_{12}, E_6, E_7, E_8, T_{2,3,6}, \dots, T_{2,3,10}, T_{2,4,4}, T_{2,4,5}, T_{2,4,6}, T_{2,5,5}, T_{2,5,6}, T_{2,6,6}, Z_{11}, Z_{12}, W_{12}, W_{13}, N_{16}$ |

**Proposition 4.5.** *Every singularity of every reduced affine plane curve of degree  $d \leq 5$  is of one of the types given in row  $d$  of Table 4.2. Conversely, for every singularity type given in row  $d$  of Table 4.2, there exists a square-free degree  $d$  polynomial  $f \in \mathbb{C}[x, y]$  such that its right equivalence class has non-empty intersection with the  $\mu$ -constant stratum of the normal form given in Table 4.1 of the singularity.*

*Proof.* The normal forms for  $d \leq 5$  are well-known. The lists for  $d = 4$  are given in [WW09, Section 2] and the lists for  $d = 5$  are given in [WW09, Section 3] or [Wal96]. For every normal form  $\Phi$  in the degree  $d = 4$  row of Table 4.2, [WW09, Appendix A] contains an example of a quartic polynomial  $f$  such that  $f$  belongs to the  $\mu$ -constant stratum of  $\Phi$ . For every normal form  $\Phi$  in the degree  $d = 5$  row of Table 4.2, [WW09, Section 3] and its erratum describe all quintic polynomials  $f$  such that  $f$  belongs to the  $\mu$ -constant stratum of  $\Phi$ .  $\square$

*Remark 4.6.* [WW09] actually proves more, namely the classification of real normal forms. Considered as complex normal forms, the normal forms with a star symbol are the same as without the star, for example  $A_k = A_k^*$ ,  $D_k = D_k^*$ ,  $X_9 = X_9^* = X_9^{**}$ , etc.

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