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Hierarchy relaxations for robust equilibrium constrained polynomial problems and applications to electric vehicle charging scheduling

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Abstract

In this paper, we consider a polynomial problem with equilibrium constraints in which the constraint functions and the equilibrium constraints involve data uncertainties. Employing a robust optimization approach, we examine the uncertain equilibrium constrained polynomial optimization problem by establishing lower bound approximations and asymptotic convergences of bounded degree diagonally dominant sum-of-squares (DSOS), scaled diagonally dominant sum-of-squares (SDSOS) and sum-of-squares (SOS) polynomial relaxations for the robust equilibrium constrained polynomial optimization problem. We also provide numerical examples to illustrate how the optimal value of a robust equilibrium constrained problem can be calculated by solving associated relaxation problems. Furthermore, an application to electric vehicle charging scheduling problems under uncertain discharging supplies shows that for the lower relaxation degrees, the DSOS, SDSOS and SOS relaxations obtain reasonable charging costs and for the higher relaxation degrees, the SDSOS relaxation scheme has the best performance, making it desirable for practical applications.

Keywords Global optimization · Robust optimization · Equilibrium constraint · Semidefinite programming · Electric vehicle charging

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1 Introduction

Equilibrium constrained optimization problems (or mathematical programs with equilibrium constraints, MPECs for short) are constrained optimization models that are defined via a number of constraints together with *equilibrium* conditions such as variational inequalities or complementarity conditions; see e.g., [6, 9, 18, 29, 30, 37, 38, 41, 44, 51] and the references therein. MPECs are difficult constrained optimization problems since they are often nonconvex and admit combinatorial features that violate almost all commonly used constraint qualification conditions. However, MPECs have received much attention because they belong to a broad class of structured decision making optimization problems that encompass all important and popular mathematical models including a subclass of *bilevel* optimization problems where their constraints are determined by the solution set of another parametric mathematical program (see e.g., [11, 12, 15, 17, 27, 43, 48]). Originally, MPECs are related to the Stackelberg game in economic sciences [38, 44] and nowadays, the applications of equilibrium constrained optimization programs appear naturally in almost real-life implementations in engineering, industries and governments such as traffic networks [47], energy networks [20], contact mechanics, or taxation and subsidies; see [3, 19, 21] and other references therein. The interested reader is referred to [15, 18, 38, 44] for various aspects of the research in equilibrium constrained optimization.

The classical equilibrium constrained optimization problems often require *accurate values* or *detailed information* of the inputs or problem parameters (see e.g., [18, 38, 44]), while the data of real-world problems would mostly be *uncertain* due to unknown environments, imprecise estimations, disturbances, measurement errors, or noisy information. Therefore, it is more important than ever for decision making to propose new classes of MPECs and associated methods that are capable of handling optimization problems involving *data uncertainties*. Such equilibrium constrained optimization programs are expected to provide global optimal values/solutions that can be *immunized* against uncertainty data. *Robust optimization* techniques (see e.g., [4, 5]) have emerged as promising and efficient paradigms and settings for studying mathematical programming problems under data uncertainties. Consider, for example, an *electric vehicle* (EV) *charging scheduling* model (see e.g., [50] for a model with *certain* data), which is described as follows: When an EV user arrives at a charging station, the EV charging demand should be satisfied with the minimal charging cost during a parking time. On one hand, with the help of *vehicle to grid* (V2G) technology, when the EV parking time is more than the EV charging time, the EV user has the ability to reduce its utility cost by deciding on how much for charging amount and how much for discharging the electricity to the power grid at each time slot. In each time slot, the EV user should decide on taking charging an amount or discharging an amount in every time slot to achieve a minimized utility cost. On the other hand, the charging station operator should arrange the charging and discharging amounts for the EV user to maximize its trading profits by supplying the charging service consistently and by selling the discharging amount (and in this scenario, the charging station operator would reduce the charging price for the EV user) from the EV user to other EV users timely. Consequently, in the electric vehicle charging scheduling model, each EV user and the charging station operator share an *equilibrium condition* on the charging amount and the discharging amount as the EV user can only in charging or discharging state in each time slot with the *uncertain/fluctuated* electricity price.

Motivated by the above considerations, we study *uncertain* and *robust* equilibrium constrained polynomial problems as follows.

Uncertain equilibrium constrained polynomial programs. Let $f, f_j, f_{i,j} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $j = 1, \dots, s_1, i = 1, \dots, l$, $g_j, g_{i,j} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $i = 1, \dots, r$, $j = 1, \dots, s_2$ be polynomials and $\Gamma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be a polynomial mapping. We consider an *uncertain* equilibrium constrained polynomial program of the form:

$$\min_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m} \Big\{ f(x, y) \mid f_j(x, y) + \sum_{i=1}^{l} u_i f_{i,j}(x, y) \le 0, \ \ j = 1, \dots, s_1,$$

$$y \in Z(x, v) := \{z \in \mathbb{R}^m \mid g_j(x, z) + \sum_{i=1}^{r} v_i g_{i,j}(x, z) \le 0, \ \ j = 1, \dots, s_2\},$$

$$\langle \Gamma(x, y), z - y\rangle \ge 0, \ \forall z \in Z(x, v)\Big\}, \tag{UEP}$$

where $u := (u_1, \dots, u_l) \in U, v := (v_1, \dots, v_r) \in V$ are *uncertain vectors*, $U \subset \mathbb{R}^l$, $V \subset \mathbb{R}^r$ are *uncertainty sets*, and the notation $\langle\cdot, \cdot\rangle$ denotes the inner product in $\mathbb{R}^m$. It is assumed here that the feasibility set of (UEP) is nonempty and note that (UEP) can be viewed as a *parametric* program because its optimal value/global solutions are well-defined on a certain pair of (u, v) in $U \times V$.

Robust equilibrium constrained polynomial programs. To handle the uncertain equilibrium constrained polynomial program (UEP), we examine a *robust* problem as follows:

$$\min_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m} \Big\{ f(x, y) \mid f_j(x, y) + \sum_{i=1}^{l} u_i f_{i,j}(x, y) \le 0, \ \ j = 1, \dots, s_1, \ \forall u \in U,$$

$$y \in Z(x) := \{z \in \mathbb{R}^m \mid g_j(x, z) + \sum_{i=1}^{r} v_i g_{i,j}(x, z) \le 0, \ \ j = 1, \dots, s_2, \ \forall v \in V\},$$

$$\langle \Gamma(x, y), z - y\rangle \ge 0, \ \ \forall z \in Z(x)\Big\}, \tag{REP}$$

where uncertain parameters u and v are enforced for every possible values in the uncertainty sets U and V, respectively.

In general, robust equilibrium constrained polynomial problems of the form (REP) are considered as a class of hard problems in global optimization because of the complexities posed in the structures of variables and equilibrium/variational inequalities as well as the involvement of uncertainty data. Therefore, we stipulate a blanket assumption that the feasible set of problem (REP) is nonempty and the associated set of the variational inequality $Z(x)$ is a *convex* set for each $x \in \mathbb{R}^n$. We also assume that the uncertainty sets are given by $U := \mathrm{conv}\,\{\bar{u}^1, \dots, \bar{u}^{l_1}\}$ with $\bar{u}^j := (\bar{u}^j_1, \dots, \bar{u}^j_l) \in \mathbb{R}^l$, $j = 1, \dots, l_1$ and $V := \mathrm{conv}\,\{\bar{v}^1, \dots, \bar{v}^{r_1}\}$ with $\bar{v}^j := (\bar{v}^j_1, \dots, \bar{v}^j_r) \in \mathbb{R}^r$, $j = 1, \dots, r_1$, which are known as the polytope uncertainty sets in [4]. In the setting of *no uncertain parameters* in the problem description, the paper [28] presented a hierarchy of semidefinite programming relaxations in the sense of Lasserre [33] to calculate the global optimal value as well as global optimal solutions for a more general mathematical program with equilibrium constraints in which the equilibrium constraint is defined by way of a polynomial function. Meanwhile, the authors in [52] provided a method based on a Lasserre's type of semidefinite relaxation [33] to solve a polynomial optimization problem with second-order cone complementarity constraints.

In this paper, we utilize the robust optimization approach (cf. [4, 5]) to investigate the *uncertain* equilibrium constrained polynomial problem (UEP) by establishing lower bound approximations and asymptotic convergences of bounded degree *diagonally dominant sum-of-squares* (DSOS), *scaled diagonally dominant sum-of-squares* (SDSOS) and *sum-of-squares* (SOS) polynomial relaxations for the robust equilibrium constrained polynomial optimization problem (REP). In each relaxation problem, there are two parameters of degrees: the first is the degree of DSOS, SDSOS or SOS constraints that is fixed and the other is the degree of approximated polynomials in the hierarchy that is varied. More precisely, we prove that optimal values of these relaxation problems are lower bounds and they tend to the optimal value of the robust equilibrium constrained problem (REP) when the first parameter degrees are fixed and the second parameter degrees go to infinity. To this end, we first translate the robust equilibrium constrained problem (REP) into a nonconvex polynomial problem by employing a dual characterization for the equilibrium constraints under a qualification condition and then employ the convergences of linear programming hierarchy [35] and/or bounded degree hierarchies [13, 36] in polynomial optimization to present DSOS, SDSOS and SOS relaxation convergences.

Numerical examples are provided to show how the global optimal value of a robust equilibrium constrained polynomial problem can be found by solving corresponding relaxations using commonly available programming packages such as the polynomial optimization toolbox SPOT [40]. Furthermore, we perform an application to *uncertain electric vehicle charging scheduling models*, where each electric vehicle user wants to lower her/him utility cost in a competitive market condition that the underlying charging station operator seeks to raise its trading profits under the setting of vehicle to grid technology and *uncertain discharging* supplies from other electric vehicles. It is worth noting that the proposed uncertain model is more dynamic than a *certain/standard* electric vehicle charging scheduling problem (cf. [50]), where the problem parameters are fixed. In this application, we also compare the proposed bounded degree hierarchies with a *linear programming* hierarchy and the moment-SOS relaxation hierarchy [33] implemented by Gloptipoly 3 [23]. The experiment results illustrate that for the lower relaxation degrees, the DSOS, SDSOS and SOS relaxations obtain reasonable charging costs and for the higher relaxation degrees, the SDSOS relaxation scheme has the best performance, which is recommended for similar practical scenarios, while the compared/existing methods run out of memory for our current simulation computer system.

The organization of the paper is as follows. Section 2 is devoted to providing preliminaries and a dual characterization of feasibility for the robust equilibrium constrained polynomial program (REP). In Sect. 3, we establish bounded degree DSOS, SDSOS and SOS hierarchy relaxation schemes for the robust equilibrium constrained polynomial problem (REP).

Section 4 presents an application in electric vehicle charging scheduling. The last section summarizes results and provides research perspectives.

2 Preliminaries and dual characterizations

We begin with some preliminary definitions and present a dual characterization of feasibility for the robust equilibrium constrained polynomial program (REP) that plays an important role in establishing our hierarchy relaxation schemes later. The notation $\mathbb{R}^n$ indicates the Euclidean space whose norm is denoted by $\|\cdot\|$ for each $n \in I\!N := \{1, 2, \ldots\}$. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^T y$ for $x, y \in \mathbb{R}^n$. The notation $\alpha \in (I\!N_0)^m$ means $\alpha := (\alpha_1, \ldots, \alpha_m)$ with $\alpha_i \in I\!N_0$, $i = 1, \ldots, m$ and $|\alpha| := \sum_{i=1}^m \alpha_i$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Let $\mathbb{R}[x]$ (or $\mathbb{R}[x_1, \ldots, x_n]$) be the ring of polynomials in $x := (x_1, \ldots, x_n)$ with real coefficients. Then, $f \in \mathbb{R}[x]$ is called a *sum-of-squares* (SOS) polynomial (see e.g., [33]) if there exist polynomials $f_i \in \mathbb{R}[x]$, $i = 1, \ldots, k$ such that $f = \sum_{i=1}^k f_i^2$. The set of all SOS polynomials in x with degree at most $d \in I\!N_0$ is denoted by $\mathbf{SOS}_d[x]$. As usual, the symbols $\nabla f(\bar{y})$ and $\nabla_z h(\bar{y}, \bar{z})$ stand for the derivates/gradients of $f : \mathbb{R}^m \to \mathbb{R}$ at $\bar{y} \in \mathbb{R}^m$ and $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ with respect to the second variable z at $(\bar{y}, \bar{z}) \in \mathbb{R}^n \times \mathbb{R}^m$, respectively, while the symbol $(\nabla f(\bar{y}))_j$ signifies the jth component of $\nabla f(\bar{y})$ for $j = 1, \ldots, m$. A polynomial $f : \mathbb{R}^n \to \mathbb{R}$ is *coercive* on $\mathbb{R}^n$ if $\liminf_{||x|| \to \infty} f(x) = +\infty$.

Definition 2.1 (DSOS and SDSOS polynomials [1, 2]) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial.

(i) f is called a *diagonally dominant sum-of-squares* (DSOS) polynomial if one can find $r \in \mathbb{N}$ and nonnegative scalars α_i, β_{ij} and γ_{ij} such that

$$f(x) = \sum_{i=1}^r \alpha_i m_i^2(x) + \sum_{i,j=1}^r \beta_{ij}(m_i(x) + m_j(x))^2 + \sum_{i,j=1}^r \gamma_{ij}(m_i(x) - m_j(x))^2,$$

where m_i and m_j are monomials in the variable x. We denote by $\mathbf{DSOS}_d[x]$ the set of all DSOS polynomials on $\mathbb{R}^n$ with degree at most $d \in I\!N_0$.

(ii) f is called a *scaled diagonally dominant sum-of-squares* (SDSOS) polynomial if one can find $r \in \mathbb{N}$ and scalars α_i, β_{ij}^+, β_{ij}^- and γ_{ij}^+, γ_{ij}^- with $\alpha_i \geq 0$ such that

$$f(x) = \sum_{i=1}^r \alpha_i m_i^2(x) + \sum_{i,j=1}^r (\beta_{ij}^+ m_i(x) + \beta_{ij}^- m_j(x))^2 + \sum_{i,j=1}^r (\gamma_{ij}^+ m_i(x) - \gamma_{ij}^- m_j(x))^2,$$

where m_i and m_j are monomials in the variable x. We denote by $\mathbf{SDSOS}_d[x]$ the set of all SDSOS polynomials on $\mathbb{R}^n$ with degree at most $d \in I\!N_0$.

Note by definition that $\mathbf{DSOS}_d[x] \subset \mathbf{SDSOS}_d[x] \subset \mathbf{SOS}_d[x]$ and their inverse inclusions are not true in general as shown in [2].

The forthcoming proposition presents a dual characterization of feasibility for the robust equilibrium constrained polynomial problem (REP). To this end, we define a following regularity condition for the associated sets of the variational inequality constraint of the problem (REP).

Definition 2.2 We say that the associated sets of the variational inequality of problem (REP) satisfy the *Mangasarian-Fromovitz constraint qualification* (MFCQ) if for $\tilde{x} \in \mathbb{R}^n$ and $\tilde{y} \in Z(\tilde{x})$, one can find $\tilde{\omega} \in \mathbb{R}^m$ such that

$$\left\langle \nabla_y g_j(\tilde{x}, \tilde{y}) + \sum_{i=1}^r \bar{v}_i^k \nabla_y g_{i,j}(\tilde{x}, \tilde{y}), \tilde{\omega} \right\rangle < 0 \text{ for all } (j,k) \in A(\tilde{x}, \tilde{y}), \tag{2.1}$$

where $A(\tilde{x}, \tilde{y}) := \left\{(j,k) \in \{1, \ldots, s_2\} \times \{1, \ldots, r_1\} \mid g_j(\tilde{x}, \tilde{y}) + \sum_{i=1}^r \bar{v}_i^k g_{i,j}(\tilde{x}, \tilde{y}) = 0\right\}$.

Observe here that if the associated set $Z(\tilde{x})$ does *not* involve uncertainties and does *not depend* on $\tilde{x}$, i.e., $Z(\tilde{x}) \equiv Z$, where

$$Z := \left\{z \in \mathbb{R}^m \mid g_j(z) \leq 0, \;\; j = 1, \ldots, s_2\right\} \tag{2.2}$$

for some polynomials $g_j : \mathbb{R}^m \to \mathbb{R}$, $j = 1, \ldots, s_2$, then the condition in (2.1) collapses to

$$\langle \nabla g_j(z), \tilde{\omega} \rangle < 0 \text{ for all } j \in A(z), \tag{2.3}$$

where $A(z) := \{j \in \{1, \ldots, s_2\} \mid g_j(z) = 0\}$ and $z \in Z$, which means that the Mangasarian-Fromovitz constraint qualification holds at $z \in Z$ in the classical sense (cf. [39]).

It is worth emphasizing that under our framework the set Z in (2.2) is convex, and so the condition (2.3) amounts to an assertion that (cf. [10, Corollary 2.1]) $\nabla g_j(z) \neq 0$ for all $z \in Z$ and $j \in A(z)$ (*the nondegeneracy condition* [34]) and there is $\tilde{z} \in \mathbb{R}^m$ satisfying $g_j(\tilde{z}) < 0$, $j = 1, \ldots, s_2$ (*the Slater qualification*). In particular, if the polynomials g_j, $j = 1, \ldots, s_2$ are convex, the nondegeneracy condition is automatically satisfied at any $z \in Z$ under the validation of the Slater qualification. In what follows, we denote by K the feasible set of problem (REP); i.e.,

$$K := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid f_j(x, y) + \sum_{i=1}^{l} u_i f_{i,j}(x, y) \leq 0,$$
$$j = 1, \ldots, s_1, \forall u := (u_1, \ldots, u_l) \in U, y \in Y(x)\}, \tag{2.4}$$

where $Y(x) := \big\{y \in Z(x) \mid \langle \Gamma(x, y), z - y\rangle \geq 0,\ \forall z \in Z(x)\big\}$ with $Z(x) := \{z \in \mathbb{R}^m \mid g_j(x, z) + \sum_{i=1}^{r} v_i g_{i,j}(x, z) \leq 0,\ j = 1, \ldots, s_2,\ \forall v := (v_1, \ldots, v_r) \in V\}$.

Note also that, in forthcoming main results, we require (MFCQ) to hold for the associated sets (i.e., for $Z(x)$ with $x \in \mathbb{R}^n$) of the variational inequality constraint of problem (REP). This type of assumption is widely used in convex and smooth optimization as the related functions of an associated set $Z(x)$ are polynomials and $Z(x)$ is a convex set. (We do not impose (MFCQ) on the feasible set K of problem (REP) in (2.4) as the involved functions and sets in K often violate almost all known constraint qualification conditions.)

Proposition 2.3 *Let $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, and assume that the associated sets of the variational inequality of problem (REP) satisfy (MFCQ). Then, $(x, y) \in K$ if and only if there exists $\lambda := (\lambda_0, \lambda_1^1, \ldots, \lambda_{s_2}^1, \ldots, \lambda_1^{r_1}, \ldots, \lambda_{s_2}^{r_1}) \in \mathbb{R}^{r_1 s_2 + 1}$ such that*

$$G_\gamma(x, y, \lambda) \leq 0,\ \gamma \in J_G, \quad H_\gamma(x, y, \lambda) = 0,\ \gamma \in J_H, \tag{2.5}$$

where

$$G_\gamma(x, y, \lambda) := \begin{cases} f_j(x, y) + \sum_{i=1}^{l} \bar{u}_i^k f_{i,j}(x, y), & \gamma = (j, k), j = 1, \ldots, s_1, k = 1, \ldots, l_1, \\ g_j(x, y) + \sum_{i=1}^{r} \bar{v}_i^k g_{i,j}(x, y), & \gamma = (s_1 + j, l_1 + k), j = 1, \ldots, s_2, k = 1, \ldots, r_1, \\ -\lambda_j^k, & \gamma = (s_1 + s_2 + j, l_1 + r_1 + k), j = 1, \ldots, s_2, k = 1, \ldots, r_1, \\ -\lambda_0, & \gamma = (s_1 + 2s_2 + 1, l_1 + 2r_1 + 1), \end{cases} \tag{2.6}$$

$$H_\gamma(x, y, \lambda) := \begin{cases} \lambda_j^k\Big[g_j(x, y) + \sum_{i=1}^{r} \bar{v}_i^k g_{i,j}(x, y)\Big], & \gamma = (j, k), j = 1, \ldots, s_2, k = 1, \ldots, r_1, \\ \Big(\lambda_0 \Gamma(x, y) + \sum_{j=1}^{s_2} \sum_{k=1}^{r_1} \lambda_j^k\Big[\nabla_y g_j(x, y) + \sum_{i=1}^{r} \bar{v}_i^k \nabla_y g_{i,j}(x, y)\Big]\Big)_\ell, & \gamma = (s_2 + \ell, r_1 + \ell), \ell = 1, \ldots, m, \\ \|\lambda\|^2 - 1, & \gamma = (s_2 + m + 1, r_1 + m + 1), \end{cases} \tag{2.7}$$

$$\begin{aligned} J_G := &\{(j, k) \mid j = 1, \ldots, s_1, k = 1, \ldots, l_1\} \\ &\cup \{(s_1 + j, l_1 + k), (s_1 + s_2 + j, l_1 + r_1 + k) \mid j = 1, \ldots, s_2, k = 1, \ldots, r_1\} \\ &\cup \{(s_1 + 2s_2 + 1, l_1 + 2r_1 + 1)\} \end{aligned}$$

and

$$J_H := \{(j, k) \mid j = 1, \ldots, s_2, k = 1, \ldots, r_1\} \cup \{(s_2 + \ell, r_1 + \ell) \mid \ell = 1, \ldots, m + 1\}.$$

Proof [$\Longrightarrow$] Let $(x, y) \in K$. We have

$$f_j(x, y) + \sum_{i=1}^{l} u_i f_{i,j}(x, y) \leq 0, \ \ j = 1, \ldots, s_1, \ \forall u := (u_1, \ldots, u_l) \in U, \tag{2.8}$$

$$y \in Z(x), \ \langle \Gamma(x, y), z - y \rangle \geq 0, \ \forall z \in Z(x), \tag{2.9}$$

where $Z(x) := \{z \in \mathbb{R}^m \mid g_j(x, z) + \sum_{i=1}^{r} v_i g_{i,j}(x, z) \leq 0, \ \ j = 1, \ldots, s_2, \ \forall v \in V\}$.

Since $U = \text{conv}\,\{\bar{u}^1, \ldots, \bar{u}^{l_1}\}$ with $\bar{u}^j := (\bar{u}_1^j, \ldots, \bar{u}_l^j) \in \mathbb{R}^l$, $j = 1, \ldots, l_1$, (2.8) is equivalent to the following inequalities

$$f_j(x, y) + \sum_{i=1}^{l} \bar{u}_i^k f_{i,j}(x, y) \leq 0, k = 1, \ldots, l_1, j = 1, \ldots, s_1. \tag{2.10}$$

Similarly, by $V := \text{conv}\,\{\bar{v}^1, \ldots, \bar{v}^{r_1}\}$ and $\bar{v}^j := (\bar{v}_1^j, \ldots, \bar{v}_r^j) \in \mathbb{R}^r$, $j = 1, \ldots, r_1$, we see that

$$Z(x) = \left\{z \in \mathbb{R}^m \mid g_j(x, z) + \sum_{i=1}^{r} \bar{v}_i^k g_{i,j}(x, z) \leq 0, \ k = 1, \ldots, r_1, j = 1, \ldots, s_2\right\}.$$

Then, by (2.9), it yields $y \in Z(x)$ and

$$L(y) \leq L(z), \ \forall z \in Z(x), \tag{2.11}$$

where $L(z) := \langle \Gamma(x, y), z - y \rangle$ for $z \in \mathbb{R}^m$, and so y is a solution of the following program:

$$\min_{z \in \mathbb{R}^m} \left\{L(z) \mid g_j(x, z) + \sum_{i=1}^{r} \bar{v}_i^k g_{i,j}(x, z) \leq 0, \ k = 1, \ldots, r_1, j = 1, \ldots, s_2\right\}. \tag{2.12}$$

Observe that L is a convex function and the feasible set of problem (2.12) is a convex set. Under the validation of (MFCQ) at y, we invoke the above remark to assert that (cf. [34, Theorem 2.3]) y is a Karush-Kuhn-Tucker/KKT point of (2.12). This means that there exist $\tilde{\lambda}_j^k \geq 0$, $j = 1, \ldots, s_2$, $k = 1, \ldots, r_1$, such that

$$\Gamma(x, y) + \sum_{j=1}^{s_2} \sum_{k=1}^{r_1} \tilde{\lambda}_j^k \left[\nabla_y g_j(x, y) + \sum_{i=1}^{r} \bar{v}_i^k \nabla_y g_{i,j}(x, y)\right] = 0,$$

$$\tilde{\lambda}_j^k \left[g_j(x, y) + \sum_{i=1}^{r} \bar{v}_i^k g_{i,j}(x, y)\right] = 0, \ k = 1, \ldots, r_1, j = 1, \ldots, s_2.$$

Hence, by letting $\lambda_0 := \frac{1}{\sqrt{1+\sum_{j=1}^{s_2}\sum_{k=1}^{r_1}(\tilde{\lambda}_j^k)^2}}$ and $\lambda_j^k := \frac{\tilde{\lambda}_j^k}{\sqrt{1+\sum_{j=1}^{s_2}\sum_{k=1}^{r_1}(\tilde{\lambda}_j^k)^2}}$, $j = 1, \ldots, s_2, k = 1, \ldots, r_1$, we arrive at

$$\lambda_0 \Gamma(x, y) + \sum_{j=1}^{s_2} \sum_{k=1}^{r_1} \lambda_j^k \left[\nabla_y g_j(x, y) + \sum_{i=1}^{r} \bar{v}_i^k \nabla_y g_{i,j}(x, y)\right] = 0, \tag{2.13}$$

$$\lambda_j^k \left[g_j(x, y) + \sum_{i=1}^{r} \bar{v}_i^k g_{i,j}(x, y)\right] = 0, \ k = 1, \ldots, r_1, j = 1, \ldots, s_2, \tag{2.14}$$

$$\|\lambda\| = 1, \tag{2.15}$$

where $\lambda := \big(\lambda_0, \lambda_1^1, \ldots, \lambda_{s_2}^1, \ldots, \lambda_1^{r_1}, \ldots, \lambda_{s_2}^{r_1}\big)$. Now, observing by (2.10), (2.11), (2.13), (2.14) and (2.15) that there exists $\lambda \in \mathbb{R}^{r_1 s_2+1}$ such that (2.5) holds.

[$\Longleftarrow$] Conversely, let $\lambda := (\lambda_0, \lambda_1^1, \ldots, \lambda_{s_2}^1, \ldots, \lambda_1^{r_1}, \ldots, \lambda_{s_2}^{r_1}) \in \mathbb{R}^{r_1 s_2+1}$ satisfy (2.5). Proceeding similarly, we get by $G_\gamma(x, y, \lambda) \leq 0, \gamma = (j, k), j = 1, \ldots, s_1, k = 1, \ldots, l_1$ that

$$f_j(x, y) + \sum_{i=1}^{l} u_i f_{i,j}(x, y) \leq 0,\ \ j = 1, \ldots, s_1,\ \forall u := (u_1, \ldots, u_l) \in U, \tag{2.16}$$

and by $G_\gamma(x, y, \lambda) \leq 0, \gamma = (s_1 + j, l_1 + k), j = 1, \ldots, s_2, k = 1, \ldots, r_1$ that $y \in Z(x)$. The inequalities $G_\gamma(x, y, \lambda) \leq 0, \gamma = (s_1+s_2+j, l_1+r_1+k), j = 1, \ldots, s_2, k = 1, \ldots, r_1$ and $\gamma = (s_1 + 2s_2 + 1, l_1 + 2r_1 + 1)$, together with the equations $H_\gamma(x, y, \lambda) = 0, \gamma \in J_H$, show that $\lambda_0 \geq 0, \lambda_j^k \geq 0, j = 1, \ldots, s_2, k = 1, \ldots, r_1$, and that

$$\lambda_0 \Gamma(x, y) + \sum_{j=1}^{s_2}\sum_{k=1}^{r_1} \lambda_j^k \Big[\nabla_y g_j(x, y) + \sum_{i=1}^{r} \bar{v}_i^k \nabla_y g_{i,j}(x, y)\Big] = 0, \tag{2.17}$$

$$\lambda_j^k \Big[g_j(x, y) + \sum_{i=1}^{r} \bar{v}_i^k g_{i,j}(x, y)\Big] = 0,\ \ j = 1, \ldots, s_2, k = 1, \ldots, r_1, \tag{2.18}$$

$$\|\lambda\| = 1. \tag{2.19}$$

We claim that $\lambda_0 \neq 0$. To see this, assume on the contrary that $\lambda_0 = 0$. Then, from (2.19) and (2.18) there exists $(j, k) \in \{1, \ldots, s_2\} \times \{1, \ldots, r_1\}$ such that $\lambda_j^k > 0$ and that $g_j(x, y) + \sum_{i=1}^{r} \bar{v}_i^k g_{i,j}(x, y) = 0$, which also shows that $(j, k) \in A(x, y)$, where $A(x, y) := \{(j, k) \in \{1, \ldots, s_2\} \times \{1, \ldots, r_1\} \mid g_j(x, y) + \sum_{i=1}^{r} \bar{v}_i^k g_{i,j}(x, y) = 0\}$. Under the validation of (MFCQ), we find $\tilde{w} \in \mathbb{R}^m$ such that

$$\Big\langle \nabla_y g_j(x, y) + \sum_{i=1}^{r} \bar{v}_i^k \nabla_y g_{i,j}(x, y), \tilde{w} \Big\rangle < 0$$

for all $(j, k) \in A(x, y)$ and thus, we arrive at

$$\Big\langle \sum_{j=1}^{s_2}\sum_{k=1}^{r_1} \lambda_j^k \Big[\nabla_y g_j(x, y) + \sum_{i=1}^{r} \bar{v}_i^k \nabla_y g_{i,j}(x, y)\Big], \tilde{w} \Big\rangle < 0,$$

which contradicts (2.17). So, our claim is valid, which means that $\lambda_0 > 0$. Dividing both sides of the relations in (2.17) and (2.18) by λ_0 and denoting $\tilde{\lambda}_j^k := \frac{\lambda_j^k}{\lambda_0} \geq 0, j = 1, \ldots, s_2, k = 1, \ldots, r_1$, we arrive at

$$\Gamma(x, y) + \sum_{j=1}^{s_2}\sum_{k=1}^{r_1} \tilde{\lambda}_j^k \Big[\nabla_y g_j(x, y) + \sum_{i=1}^{r} \bar{v}_i^k \nabla_y g_{i,j}(x, y)\Big] = 0,$$

$$\tilde{\lambda}_j^k \Big[g_j(x, y) + \sum_{i=1}^{r} \bar{v}_i^k g_{i,j}(x, y)\Big] = 0,\ \ k = 1, \ldots, r_1, j = 1, \ldots, s_2,$$

which shows that y is a KKT point for the following problem:

$$\min_{z \in \mathbb{R}^m} \Big\{L(z) \mid g_j(x, z) + \sum_{i=1}^{r} \bar{v}_i^k g_{i,j}(x, z) \leq 0,\ \ j = 1, \ldots, s_2, k = 1, \ldots, r_1\Big\}, \tag{2.20}$$

where $L(z) := \langle \Gamma(x, y), z - y \rangle$ for $z \in \mathbb{R}^m$. As L is an affine function and the feasible set of problem (2.20) is a convex set, it follows that y is a global optimal solution of problem (2.20) under the fulfilment of (MFCQ) (cf. [34, Theorem 2.3]). Then $L(y) \le L(z)$ for all $z \in Z(x)$, which means that $\langle \Gamma(x, y), z - y \rangle \ge 0$ for all $z \in Z(x)$. Combining this and (2.16) concludes that $(x, y) \in K$, and so the proof is complete. □

3 Bounded degree hierarchy relaxations

This section is devoted to presenting bounded degree *diagonally dominant sum-of-squares* (DSOS), *scaled diagonally dominant sum-of-squares* (SDSOS) and *sum-of-squares* (SOS) relaxations for the robust equilibrium constrained polynomial problem (REP). This shows how one can compute the global optimal value of problem (REP) by solving a sequence of associated bounded degree DSOS, SDSOS or SOS relaxation problems.

3.1 Relaxations with the boundedness of feasibility

We first consider the robust equilibrium constrained problem (REP) with the boundedness of its feasible set. More precisely, we impose the following boundedness assumption:

Assumption A. There exists $M_K > 0$ satisfying $||(\tilde{x}, \tilde{y})|| \le \sqrt{M_K}$ for all $(\tilde{x}, \tilde{y}) \in K$, where K is the feasible set of problem (REP) given as in (2.4).

We also use a commonly used hypothesis in polynomial optimization (see e.g., [33, 36]) as follows:

Assumption B. The polynomials $\left\{1,\ f_j + \sum_{i=1}^{l} \bar{u}_i^k f_{i,j},\ j = 1, \ldots, s_1, k = 1, \ldots, l_1, g_j + \sum_{i=1}^{r} \bar{v}_i^k g_{i,j},\ \ j = 1, \ldots, s_2, k = 1, \ldots, r_1 \right\}$ generates the ring $\mathbb{R}[x, y]$.

It is worth mentioning here that verifying Assumption A would cost additionally computational effort in general. However, in a particular framework, where the constraints of the equilibrium constrained problem (REP) involve the quadratic function $(x, y) \mapsto \|(x, y)\|^2 - \tau$ for a given $\tau > 0$, we can simply take M_K as τ. Moreover, in some practical applications, we often know the bounds of budget constraints and so it is convenient to choose M_K based on the budget bounds and other known factors. For instance, let see an example in Sect. 4 below on how to verify Assumption A for a practical application. Note also that if Assumption A holds, then Assumption B can be easily made valid by adding *redundant constraints* of $\tilde{f}_j(x, y) \le 0, \tilde{g}_j(x, y) \le 0$ into the problem (REP) with $\tilde{f}_j(x, y) := x_j - \sqrt{M_K}, j = 1, \ldots, n$ and $\tilde{g}_j(x, y) := y_j - \sqrt{M_K}, j = 1, \ldots, m.$

Consider a positive number R defined by

$$R \ge \max_{(x,y,\lambda)\in\Omega} \{-G_\gamma(x, y, \lambda), \gamma \in J_G\}, \tag{3.1}$$

where $\Omega := \{(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{r_1 s_2+1} \mid \|(x, y, \lambda)\|^2 \le 1 + M_K\}$ and $G_\gamma, \gamma \in J_G$ are given as in (2.6). In what follows, we use the following functions

$$\widehat{G}_p := \begin{cases} -\frac{1}{R} G_\gamma, & p = \gamma, \gamma \in J_G, \\ -H_\gamma, & p = (s_1 + 2s_2 + 1 + \gamma_1, l_1 + 2r_1 + 1 + \gamma_2), \gamma := (\gamma_1, \gamma_2) \in J_H, \\ H_\gamma, & p = (s_1 + 3s_2 + m + 2 + \gamma_1, l_1 + 3r_1 + m + 2 + \gamma_2), \gamma := (\gamma_1, \gamma_2) \in J_H, \end{cases} \tag{3.2}$$

where $H_\gamma, \gamma \in J_H$ are given in (2.7). Denoting by

$$J := J_G \cup \{(s_1 + 2s_2 + 1 + \gamma_1, l_1 + 2r_1 + 1 + \gamma_2),\\ (s_1 + 3s_2 + m + 2 + \gamma_1, l_1 + 3r_1 + m + 2 + \gamma_2) \mid (\gamma_1, \gamma_2) \in J_H\},$$

we see that the cardinality of J is $4r_1s_2+l_1s_1+2m+3$ (i.e., $|J| = 4r_1s_2+l_1s_1+2m+3$). Note that the definition of $\widehat{G}_p$ in (3.2) is merely a convenient way to construct the forthcoming bounded degree relaxation problems. In fact, one can define other analogous functions to formulate corresponding relaxations as long as such similar functions would be defined based on the characterization functions G_γ in (2.6) and H_γ in (2.7) for the feasibility set of problem (REP).

We are now ready to propose three types of relaxation problems for the equilibrium constrained problem (REP) as follows.

Bounded degree SOS relaxation problems. Fix a positive even number $d \in I\!N_0$ and define a hierarchy of *SOS relaxations* for the equilibrium constrained problem (REP) as

$$\sup_{(t,c_{\alpha,\beta})} \Big\{ t \mid f - \sum_{\alpha,\beta\in(\mathbb{N}_0)^{|J|}, |\alpha|+|\beta|\le k} c_{\alpha,\beta} \prod_{p\in J} (\widehat{G}_p)^{\alpha_p}(1-\widehat{G}_p)^{\beta_p} \\ - t \in \mathbf{SOS}_d[x, y, \lambda], t \in \mathbb{R}, c_{\alpha,\beta} \ge 0 \Big\}, \tag{SDP1$_k^d$}$$

where $k \in \mathbb{N}$.

Bounded degree SDSOS relaxation problems. Fix a positive even number $d \in I\!N_0$ and define a hierarchy of *SDSOS relaxations* for the equilibrium constrained problem (REP) as

$$\sup_{(t,c_{\alpha,\beta})} \Big\{ t \mid f - \sum_{\alpha,\beta\in(\mathbb{N}_0)^{|J|}, |\alpha|+|\beta|\le k} c_{\alpha,\beta} \prod_{p\in J} (\widehat{G}_p)^{\alpha_p}(1-\widehat{G}_p)^{\beta_p} \\ - t \in \mathbf{SDSOS}_d[x, y, \lambda], t \in \mathbb{R}, c_{\alpha,\beta} \ge 0 \Big\}, \tag{SOCP1$_k^d$}$$

where $k \in \mathbb{N}$.

Bounded degree DSOS relaxation problems. Fix a positive even number $d \in I\!N_0$ and define a hierarchy of *DSOS relaxations* for the equilibrium constrained problem (REP) as

$$\sup_{(t,c_{\alpha,\beta})} \Big\{ t \mid f - \sum_{\alpha,\beta\in(\mathbb{N}_0)^{|J|}, |\alpha|+|\beta|\le k} c_{\alpha,\beta} \prod_{p\in J} (\widehat{G}_p)^{\alpha_p}(1-\widehat{G}_p)^{\beta_p} \\ - t \in \mathbf{DSOS}_d[x, y, \lambda], t \in \mathbb{R}, c_{\alpha,\beta} \ge 0 \Big\}, \tag{LP1$_k^d$}$$

where $k \in \mathbb{N}$.

For the sake of completeness and comparison, we also consider a hierarchy of *linear programming* (LP) relaxation problems for the robust equilibrium constrained polynomial problem (REP).

Linear programming relaxation problems. Define a hierarchy of *LP relaxations* for the equilibrium constrained problem (REP) as

$$\sup_{(t,c_{\alpha,\beta})}\Big\{t \mid f-\sum_{\alpha,\beta\in(\mathbb{N}_0)^{|J|},|\alpha|+|\beta|\le k} c_{\alpha,\beta}\prod_{p\in J}(\widehat{G}_p)^{\alpha_p}(1-\widehat{G}_p)^{\beta_p}-t=0, t\in\mathbb{R}, c_{\alpha,\beta}\ge 0\Big\}, \tag{LP1$_k$}$$

where $k\in\mathbb{N}$.

Remark 3.1 (i) It is worth noting that the problem (SDP1$_k^d$) for $d\in I\!N_0$ and $k\in I\!N$ is regarded as a sum-of-squares (SOS) relaxation problem for the robust equilibrium constrained polynomial problem (REP). Since checking whether a given polynomial is SOS can be equivalently reformulated as a feasibility problem of *semidefinite programming* (SDP) problem (see e.g., [33]), the problem (SDP1$_k^d$) can be rewritten and solved as an SDP problem. Similarly, checking whether a polynomial is SDSOS (respectively, DSOS) can be equivalently reformulated as a *second-order cone programming* (SOCP) feasibility problem (respectively, *linear programming* (LP) feasibility problem) [2]. So, each problem (SOCP1$_k^d$) (respectively, (LP1$_k^d$)) is equivalently solved as an SOCP problem (respectively, an LP problem). Note also that for $k\in I\!N$, the relaxation problem (LP1$_k$) can be solved as a *linear programming* problem (cf. [35]).

(ii) Since the polynomial 0 is DSOS, any DSOS polynomial is SDSOS and any SDSOS polynomial is SOS, it holds by definition that

$$\mathrm{val}(\mathrm{LP1}_k)\le \mathrm{val}(\mathrm{LP1}_k^d)\le \mathrm{val}(\mathrm{SOCP1}_k^d)\le \mathrm{val}(\mathrm{SDP1}_k^d)\ \text{ for all } d\in I\!N_0 \text{ and } k\in I\!N, \tag{3.3}$$

where val(LP1$_k$), val(LP1$_k^d$), val(SOCP1$_k^d$) and val(SDP1$_k^d$) denote the global optimal value of problems (LP1$_k$), (LP1$_k^d$), (SOCP1$_k^d$) and (SDP1$_k^d$), respectively.

(iii) By the definition of the proposed DSOS, SDSOS or SOS hierarchy, the degree d of the DSOS, SDSOS or SOS constraint in the relaxations is *independent* of the level k of the approximation problems in the corresponding hierarchy. However, the interplay between d and k should be taken into consideration when solving relaxations as if d is essentially smaller than k, the corresponding DSOS, SDSOS or SOS relaxation problem might be infeasible. The interested reader is referred to [13] for bounded degree hierarchies with SDSOS and DSOS polynomial relaxations and to [36] for a bounded degree hierarchy with SOS polynomial relaxations for the class of *standard polynomial* programs.

(iv) To construct the relaxation hierarchies of (LP1$_k$), (LP1$_k^d$), (SOCP1$_k^d$) and (SDP1$_k^d$), we need to know a *scaled* parameter $R>0$ that satisfies (3.1). A general way of choosing a scaled parameter R is to solve the problem in the right-hand side of (3.1) as finding the maximum value of the polynomials $-G_\gamma, \gamma\in J_G$ over the ball Ω, and solving this problem would be NP-hard in general even in a particular setting, where $G_\gamma, \gamma\in J_G$ are nonconvex quadratic polynomials (see e.g., [45]). One practical possibility of selecting a scaled parameter R is to take an upper bound of upper estimations from the polynomials $-G_\gamma, \gamma\in J_G$ on the ball Ω. This can be easily seen because for each $(x,y,\lambda)\in\Omega$, $|x_i|\le\sqrt{1+M_K}, i=1,\dots,n$, $|y_j|\le\sqrt{1+M_K}, j=1,\dots,m$, $|\lambda_\ell|\le\sqrt{1+M_K}, \ell=1,\dots,r_1s_2+1$, we can evaluate the polynomials $-G_\gamma, \gamma\in J_G$ up to the upper bounds of the variables x_i, y_j and λ_ℓ with absolute values of the coefficients.

Under suitable conditions, we now present the solution existence of the equilibrium constrained problem (REP). We also show that the optimal values of the relaxations (LP1$_k$), (SDP1^d_k), (SOCP1^d_k) and (LP1^d_k) are convergent to the optimal value of problem (REP) whenever k tends to infinity. Similarly, val(REP) denotes the optimal value of problem (REP).

Theorem 3.2 (Convergence of Relaxations) *Let Assumptions A and* **B** *hold for the equilibrium constrained problem* (REP). *Assume that the associated sets of the variational inequality of problem* (REP) *satisfy (MFCQ). Then, the equilibrium constrained problem* (REP) *possesses a solution, say* $(\bar{x}, \bar{y})$, *satisfying*

$$\text{val}(\text{SDP1}^d_k) \le \text{val}(\text{SDP1}^d_{k+1}) \le \text{val}(REP) = f(\bar{x}, \bar{y}) \ \textit{for} \ k \in I\!N, \tag{3.4}$$

$$\text{val}(\text{SOCP1}^d_k) \le \text{val}(\text{SOCP1}^d_{k+1}) \le \text{val}(REP) \ \textit{for} \ k \in I\!N, \tag{3.5}$$

$$\text{val}(\text{LP1}^d_k) \le \text{val}(\text{LP1}^d_{k+1}) \le \text{val}(REP) \ \textit{for} \ k \in I\!N, \tag{3.6}$$

$$\text{val}(\text{LP1}_k) \le \text{val}(\text{LP1}_{k+1}) \le \text{val}(REP) \ \textit{for} \ k \in I\!N, \tag{3.7}$$

and

$$\lim_{k\to\infty} \text{val}(\text{LP1}_k) = \lim_{k\to\infty} \text{val}(\text{LP1}^d_k) = \lim_{k\to\infty} \text{val}(\text{SOCP1}^d_k) = \lim_{k\to\infty} \text{val}(\text{SDP1}^d_k) = \text{val}(REP), \tag{3.8}$$

where $d \in I\!N_0$ *is given.*

Proof We first claim that the feasibility set K in (2.4) is a closed set. Indeed, let $\{(x_q, y_q)\}_{q\in I\!N} \subset K$ be a sequence such that $(x_q, y_q) \to (x_0, y_0)$ as $q \to \infty$. Then, for each $q \in I\!N$, it holds that

$$f_j(x_q, y_q) + \sum_{i=1}^{l} u_i f_{i,j}(x_q, y_q) \le 0, \ j = 1, \ldots, s_1, \ \forall u := (u_1, \ldots, u_l) \in U, \tag{3.9}$$

$$y_q \in Y(x_q), \tag{3.10}$$

where $Y(x) := \{y \in Z(x) \mid \langle \Gamma(x, y), z - y\rangle \ge 0, \ \forall z \in Z(x)\}$ with $Z(x) := \{z \in \mathbb{R}^m \mid g_j(x, z) + \sum_{i=1}^{r} v_i g_{i,j}(x, z) \le 0, \ j = 1, \ldots, s_2, \ \forall v := (v_1, \ldots, v_r) \in V\}$. As $f_j, f_{i,j} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $j = 1, \ldots, s_1, i = 1, \ldots, l$, are continuous functions, we pass to the limit in (3.9) as $q \to \infty$ and obtain that

$$f_j(x_0, y_0) + \sum_{i=1}^{l} u_i f_{i,j}(x_0, y_0) \le 0, \ j = 1, \ldots, s_1, \ \forall u := (u_1, \ldots, u_l) \in U. \tag{3.11}$$

Observe by (3.10) that $y_q \in Z(x_q)$ for all $q \in I\!N$ and similarly, we get

$$g_j(x_0, y_0) + \sum_{i=1}^{r} v_i g_{i,j}(x_0, y_0) \le 0, \ j = 1, \ldots, s_2, \ \forall v := (v_1, \ldots, v_r) \in V,$$

which shows that $y_0 \in Z(x_0)$.

Arguing as in the proof of Proposition 2.3, we see that (3.10) amounts to saying that y_q is a global solution of the program:

$$\min_{z\in\mathbb{R}^m} \left\{ L_q(z) \mid g_j(x_q, z) + \sum_{i=1}^{r} \bar{v}_i^k g_{i,j}(x_q, z) \le 0, \ k = 1, \ldots, r_1, j = 1, \ldots, s_2 \right\}, \tag{3.12}$$

where $L_q(z) := \langle \Gamma(x_q, y_q), z - y_q\rangle$ for $z \in \mathbb{R}^m$. Observe that L_q is a convex function and the feasible set of problem (3.12) is a convex set. This, under the validation of (MFCQ) at y_q, guarantees that y_q is a KKT point of problem (3.12), and then there exist $\lambda_{0,q} \geq 0, \lambda^k_{j,q} \geq 0$, $j = 1, \ldots, s_2, k = 1, \ldots, r_1$, such that

$$\lambda_{0,q}\Gamma(x_q, y_q) + \sum_{j=1}^{s_2}\sum_{k=1}^{r_1} \lambda^k_{j,q}\left[\nabla_y g_j(x_q, y_q) + \sum_{i=1}^{r} \bar{v}^k_i \nabla_y g_{i,j}(x_q, y_q)\right] = 0, \tag{3.13}$$

$$\lambda^k_{j,q}\left[g_j(x_q, y_q) + \sum_{i=1}^{r} \bar{v}^k_i g_{i,j}(x_q, y_q)\right] = 0, \ \ j = 1, \ldots, s_2, k = 1, \ldots, r_1, \tag{3.14}$$

$$\|\lambda_q\| = 1, \tag{3.15}$$

where $\lambda_q := \left(\lambda_{0,q}, \lambda^1_{1,q}, \ldots, \lambda^1_{s_2,q}, \ldots, \lambda^{r_1}_{1,q}, \ldots, \lambda^{r_1}_{s_2,q}\right)$. By (3.15), we may assume without loss of generality that $\lambda_{0,q} \to \lambda_0$ and $\lambda^k_{j,q} \to \lambda^k_j$, $j = 1, \ldots, s_2, k = 1, \ldots, r_1$, as $q \to \infty$. Therefore, by letting $q \to \infty$ in (3.13), (3.14) and (3.15), we arrive at

$$\lambda_0\Gamma(x_0, y_0) + \sum_{j=1}^{s_2}\sum_{k=1}^{r_1} \lambda^k_j\left[\nabla_y g_j(x_0, y_0) + \sum_{i=1}^{r} \bar{v}^k_i \nabla_y g_{i,j}(x_0, y_0)\right] = 0, \tag{3.16}$$

$$\lambda^k_j\left[g_j(x_0, y_0) + \sum_{i=1}^{r} \bar{v}^k_i g_{i,j}(x_0, y_0)\right] = 0, \ \ j = 1, \ldots, s_2, k = 1, \ldots, r_1, \tag{3.17}$$

$$\|\lambda\| = 1, \tag{3.18}$$

where $\lambda := (\lambda_0, \lambda^1_1, \ldots, \lambda^1_{s_2}, \ldots, \lambda^{r_1}_1, \ldots, \lambda^{r_1}_{s_2})$. Let $L_0(z) := \langle\Gamma(x_0, y_0), z - y_0\rangle$ for $z \in \mathbb{R}^m$. Using similar arguments as in the proof of Proposition 2.3, we deduce from (3.16), (3.17) and (3.18) that $L_0(y_0) \leq L_0(z)$ for all $z \in Z(x_0)$, which means that $\langle\Gamma(x_0, y_0), z - y_0\rangle \geq 0$ for all $z \in Z(x_0)$. It shows that $y_0 \in Y(x_0)$. Combining this with (3.11), we see that $(x_0, y_0) \in K$, which concludes that K is closed.

Furthermore, K is bounded due to Assumption A, and so K is compact. This together with the continuity of f implies that there exists $(\bar{x}, \bar{y}) \in K$ with the property:

$$f(\bar{x}, \bar{y}) \leq f(x, y) \ \text{ for all } \ (x, y) \in K.$$

So $(\bar{x}, \bar{y})$ is a global solution of the equilibrium constrained problem (REP) and then,

$$\text{val}(REP) = f(\bar{x}, \bar{y}).$$

Now, we set

$$\begin{aligned} X := \Big\{&(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{r_1 s_2+1} \mid -G_\gamma(x, y, \lambda) \geq 0, \gamma \in J_G, \\ &H_\gamma(x, y, \lambda) \geq 0, -H_\gamma(x, y, \lambda) \geq 0, \gamma \in J_H\Big\}, \end{aligned} \tag{3.19}$$

where $G_\gamma, \gamma \in J_G$ and $H_\gamma, \gamma \in J_H$, are defined respectively as in (2.6) and (2.7). By Assumption A, $||(x, y)|| \leq \sqrt{M_K}$ for all $(x, y) \in K$, we claim that

$$X \subseteq \Omega := \left\{(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{r_1 s_2+1} \mid \|(x, y, \lambda)\|^2 \leq 1 + M_K\right\}. \tag{3.20}$$

To see this, let $(x, y, \lambda) \in X$, where $\lambda := \left(\lambda_0, \lambda^1_1, \ldots, \lambda^1_{s_2}, \ldots, \lambda^{r_1}_1, \ldots, \lambda^{r_1}_{s_2}\right) \in \mathbb{R}^{r_1 s_2+1}$. In view of Proposition 2.3, it holds that

$$(x, y) \in K, \quad \|\lambda\| = 1. \tag{3.21}$$

So $\|(x, y, \lambda)\|^2 \leq 1 + M_K$; i.e., $(x, y, \lambda) \in \Omega$ and hence our claim in (3.20) holds. So we have

$$R \geq \max_{(x,y,\lambda)\in X} \{-G_\gamma(x, y, \lambda), \gamma \in J_G\},$$

where R is given as in (3.1). This ensures that $0 \leq \widehat{G}_p \leq 1$ on X for all $p \in J_G$ and $\widehat{G}_p = 0$ on X for all $p \in J \backslash J_G$, where $\widehat{G}_p$, $p \in J$ are given by (3.2).

Let $d \in I\!N_0$. By the construction of relaxation problems, it is clear that val(SDP1_k^d)$\leq$ val(SDP1_{k+1}^d), val(SOCP1_k^d)$\leq$ val(SOCP1_{k+1}^d), val(LP1_k^d)$\leq$ val(LP1_{k+1}^d) and val(LP1_k) $\leq$ val(LP1_{k+1}) for all $k \in I\!N$. Moreover, by Remark 3.1, we have

$$\text{val}(\text{LP1}_k) \leq \text{val}(\text{LP1}_k^d) \leq \text{val}(\text{SOCP1}_k^d) \leq \text{val}(\text{SDP1}_k^d) \text{ for all } k \in I\!N. \tag{3.22}$$

To justify (3.4)–(3.7), it is sufficient to show that

$$\text{val}(\text{SDP1}_k^d) \leq f(\bar{x}, \bar{y}) \text{ for all } k \in I\!N. \tag{3.23}$$

Let $k \in I\!N$. If the feasible set of problem (SDP1_k^d) is empty, then (3.23) holds trivially as val(SDP1_k^d)$= -\infty$. Now, let $(t, c_{\alpha,\beta})$ be a feasible point of problem (SDP1_k^d). This means that $c_{\alpha,\beta} \geq 0$ for all $\alpha, \beta \in (\mathbb{N}_0)^{|J|}$, $|\alpha| + |\beta| \leq k$ and

$$f - \sum_{\alpha,\beta\in(\mathbb{N}_0)^{|J|}, |\alpha|+|\beta|\leq k} c_{\alpha,\beta} \prod_{p\in J} (\widehat{G}_p)^{\alpha_p}(1 - \widehat{G}_p)^{\beta_p} - t \in \mathbf{SOS}_d[x, y, \lambda]. \tag{3.24}$$

Since $(\bar{x}, \bar{y}) \in K$ and (MFCQ) holds, we invoke Proposition 2.3 to assert that there exists $\bar{\lambda} \in \mathbb{R}^{r_1 s_2+1}$ such that

$$G_\gamma(\bar{x}, \bar{y}, \bar{\lambda}) \leq 0, \ \gamma \in J_G, \ H_\gamma(\bar{x}, \bar{y}, \bar{\lambda}) = 0, \ \gamma \in J_H,$$

which shows that $(\bar{x}, \bar{y}, \bar{\lambda}) \in X$ and thus $0 \leq \widehat{G}_p(\bar{x}, \bar{y}, \bar{\lambda}) \leq 1$ for all $p \in J_G$ and $\widehat{G}_p(\bar{x}, \bar{y}, \bar{\lambda}) = 0$ for all $p \in J \setminus J_G$. Therefore, by recalling the nonnegativity of SOS polynomials, we evaluate (3.24) at $(\bar{x}, \bar{y}, \bar{\lambda})$ to obtain that $f(\bar{x}, \bar{y}) \geq t$. This yields val($\text{SDP1}_k^d$)$\leq f(\bar{x}, \bar{y})$; i.e., (3.23) has been proved.

By virtue of Assumption B, the polynomials $\{1, -G_\gamma, \gamma \in J_G, H_\gamma, -H_\gamma, \gamma \in J_H\}$ generates the ring $\mathbb{R}[x, y, \lambda]$. We can employ the convergence of linear programming hierarchy in polynomial optimization (see [35] or also, [13, Theorem 3.2 and Remark 3.3(ii)]) applied to our problem with the objective f over the feasible set X to assert that

$$\lim_{k\to\infty} \text{val}(\text{LP1}_k) = \text{val}(REP).$$

This, together with (3.22) and (3.23), justifies (3.8), and so the proof is complete. □

The next example illustrates that without the validation of the MFCQ condition given in (2.1), the bounded degree polynomial convergences obtained in Theorem 3.2 may fail.

Example 3.3 (The importance of MFCQ) Consider the following robust equilibrium constrained polynomial problem

$$\min_{(x,y)\in\mathbb{R}^2\times\mathbb{R}^2} \Big\{x_1^4 + 2x_2^2 + y_1^3 + 2y_2 + 4 \mid -u_1(x_1^2 + x_2^4) - u_2 y_1 - 1 \leq 0, \tag{EP$_1$}$$

$$u_1 x_1 + u_2 x_2 \leq 0, \ \forall u := (u_1, u_2) \in U,$$

$$y \in Z(x) := \Big\{z \in \mathbb{R}^2 \mid v_1(x_1^4 + x_2^2 + 1)z_1^4 + v_2 z_2^2 \leq 0, z_2 \leq 0, \ \forall v := (v_1, v_2) \in V\Big\},$$

$$\langle (x_1^3 + y_1^5, y_2^4), z - y\rangle \geq 0,\ \forall z \in Z(x)\Big\},$$

where $U := \text{conv}\{(-1,0),(0,1)\}$ and $V := \text{conv}\{(-1,0),(1,0)\}$ are uncertainty sets.

The robust problem (EP$_1$) can be viewed in terms of (REP), where $f(x,y) := x_1^4 + 2x_2^2 + y_1^3 + 2y_2 + 4$, $f_1(x,y) := -1$, $f_{1,1}(x,y) := -x_1^2 - x_2^4$, $f_{2,1}(x,y) := -y_1$, $f_2(x,y) := 0$, $f_{1,2}(x,y) := x_1$, $f_{2,2}(x,y) := x_2$, $g_1(x,z) := 0$, $g_{1,1}(x,z) := (x_1^4 + x_2^2 + 1)z_1^4$, $g_{2,1}(x,z) := z_2^2$, $g_2(x,z) := z_2$, $g_{1,2}(x,z) := 0$, $g_{2,2}(x,z) := 0$, $\Gamma(x,y) := (x_1^3 + y_1^5, y_2^4)$ for $x := (x_1,x_2) \in \mathbb{R}^2$, $y := (y_1,y_2) \in \mathbb{R}^2$ and $z := (z_1,z_2) \in \mathbb{R}^2$, and $\bar{u}^1 := (-1,0)$, $\bar{u}^2 := (0,1)$, $\bar{v}^1 := (-1,0)$, $\bar{v}^2 := (1,0)$.

Observe by $-u_1(x_1^2 + x_2^4) - u_2 y_1 - 1 \leq 0$ for all $u := (u_1,u_2) \in U$ that $x_1^2 + x_2^4 \leq 1$, and then $||x|| \leq \sqrt{2}$. Furthermore, we can verify directly that for $x \in \mathbb{R}^2$, one has $Z(x) = \{0\} \times (-\infty, 0]$ and $Y(x) = \{(0,0)\}$, where $Y(x) := \{y \in Z(x) \mid \langle (x_1^3 + y_1^5, y_2^4), z - y\rangle \geq 0,\ \forall z \in Z(x)\}$. Hence, by setting $M_K := 2$, it yields $||(x,y)|| \leq \sqrt{M_K}$ for all $(x,y) \in K$, where K is the feasibility set of (EP$_1$). This shows that Assumption A is valid. Similarly, we see that Assumption B also holds. Moreover, we can check that $(\bar{x},\bar{y})$, where $\bar{x} := (0,0)$ and $\bar{y} := (0,0)$, is a solution of problem (EP$_1$).

In this setting, the functions $G_\gamma, \gamma \in J_G$ and $H_\gamma, \gamma \in J_H$ defined respectively in (2.6) and (2.7) reduce to the following ones:

$$\begin{cases} G_{1,1}(x,y,\lambda) & = x_1^2 + x_2^4 - 1,\ G_{1,2}(x,y,\lambda) = -1 - y_1,\ G_{2,1}(x,y,\lambda) = -x_1,\ G_{2,2}(x,y,\lambda) = x_2, \\ G_{3,3}(x,y,\lambda) & = -(x_1^4 + x_2^2 + 1)y_1^4,\ G_{3,4}(x,y,\lambda) = (x_1^4 + x_2^2 + 1)y_1^4,\ G_{4,3}(x,y,\lambda) = G_{4,4}(x,y,\lambda) = y_2, \\ G_{5,5}(x,y,\lambda) & = -\lambda_1^1,\ G_{5,6}(x,y,\lambda) = -\lambda_1^2, G_{6,5}(x,y,\lambda) = -\lambda_2^1,\ G_{6,6}(x,y,\lambda) = -\lambda_2^2, G_{7,7}(x,y,\lambda) = -\lambda_0, \end{cases}$$

$$\begin{cases} H_{1,1}(x,y,\lambda) & = -\lambda_1^1(x_1^4 + x_2^2 + 1)y_1^4,\ H_{1,2}(x,y,\lambda) = \lambda_1^2(x_1^4 + x_2^2 + 1)y_1^4, H_{2,1}(x,y,\lambda) = \lambda_2^1 y_2, \\ H_{2,2}(x,y,\lambda) & = \lambda_2^2 y_2, H_{3,3}(x,y,\lambda) = \lambda_0(x_1^3 + y_1^5) + 4(\lambda_1^2 - \lambda_1^1)y_1^3(x_1^4 + x_2^2 + 1), \\ H_{4,4}(x,y,\lambda) & = \lambda_0 y_2^4 + \lambda_2^1 + \lambda_2^2,\ H_{5,5}(x,y,\lambda) = \|\lambda\|^2 - 1, \end{cases}$$

where $(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$, $\lambda := (\lambda_0, \lambda_1^1, \lambda_1^2, \lambda_2^1, \lambda_2^2) \in \mathbb{R}^5$, $J_G := \{(j,k) \mid j = 1,2, k = 1,2\} \cup \{(2+j, 2+k), (4+j, 4+k) \mid j = 1,2, k = 1,2\} \cup \{(7,7)\}$ and $J_H := \{(j,k) \mid j = 1,2, k = 1,2\} \cup \{(2+\ell, 2+\ell) \mid \ell = 1,2,3\}$.

Let R be a positive number such that $R \geq \max_{(x,y,\lambda)\in\Omega}\{-G_\gamma(x,y,\lambda), \gamma \in J_G\}$, where $\Omega := \{(x,y,\lambda) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^5 \mid \|(x,y,\lambda)\|^2 \leq 1 + M_K\}$, and define the following functions

$$\widehat{G}_p := \begin{cases} -\frac{1}{R}G_\gamma, & p = \gamma, \gamma \in J_G, \\ -H_\gamma, & p = (7+\gamma_1, 7+\gamma_2), \gamma := (\gamma_1,\gamma_2) \in J_H \\ H_\gamma, & p = (12+\gamma_1, 12+\gamma_2), \gamma := (\gamma_1,\gamma_2) \in J_H. \end{cases}$$

Now, the relaxation problems (SDP1$_k^d$) for $k \in I\!N$ become the following problems

$$\sup_{(t, c_{\alpha,\beta})} \Big\{ t \mid f - \sum_{\alpha,\beta \in (\mathbb{N}_0)^{27}, |\alpha|+|\beta| \leq k} c_{\alpha,\beta} \prod_{p\in J} (\widehat{G}_p)^{\alpha_p}(1 - \widehat{G}_p)^{\beta_p} - t \in \mathbf{SOS}_d[x,y,\lambda], t \in \mathbb{R}, c_{\alpha,\beta} \geq 0 \Big\}, \tag{ED$_k^d$}$$

where $d \in I\!N_0$ is given and $J := J_G \cup \{(7+\gamma_1, 7+\gamma_2), (12+\gamma_1, 12+\gamma_2) \mid (\gamma_1,\gamma_2) \in J_H\}$.

We claim that the representation of the SOS polynomial in (ED$_k^d$) is not valid for any $k \in I\!N$ and $d \in I\!N_0$, and any $t \in (2, +\infty)$. If this is not the case, we would find $k \in I\!N$, $d \in I\!N_0$, $t \in (2, +\infty)$, $c_{\alpha,\beta} \geq 0$, where $\alpha, \beta \in (\mathbb{N}_0)^{27}$, $|\alpha| + |\beta| \leq k$ and $\sigma \in \mathbf{SOS}_d[x,y,\lambda]$

such that

$$f-\sum_{\alpha,\beta\in(\mathbb{N}_0)^{27},|\alpha|+|\beta|\le k} c_{\alpha,\beta}\prod_{p\in J}(\widehat{G}_p)^{\alpha_p}(1-\widehat{G}_p)^{\beta_p}-t=\sigma, \tag{3.25}$$

where we should note that $|J|=27$. Picking $\tilde{y}:=(0,-1), \bar{x}:=(0,0)$ and $\tilde{\lambda}:=(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0)$, it holds that $-1\le G_\gamma(\bar{x},\tilde{y},\tilde{\lambda})\le 0$ for all $\gamma\in J_G$ and $H_\gamma(\bar{x},\tilde{y},\tilde{\lambda})=0$ for all $\gamma\in J_H$. Note further that $(\bar{x},\tilde{y},\tilde{\lambda})\in\Omega$, which ensures that $R\ge -G_{1,1}(\bar{x},\tilde{y},\tilde{\lambda})=1$. Consequently, we arrive at $0\le \widehat{G}_p(\bar{x},\tilde{y},\tilde{\lambda})\le 1$ for $p\in J_G$ and $\widehat{G}_p(\bar{x},\tilde{y},\tilde{\lambda})=0$ for $p\in J\setminus J_G$. Now, we can deduce from (3.25) that

$$\sigma(\bar{x},\tilde{y},\tilde{\lambda})\le f(\bar{x},\tilde{y})-t.$$

This means that $\sigma(\bar{x},\tilde{y},\tilde{\lambda})\le 2-t<0$, which is impossible as $\sigma\in\mathbf{SOS}_d[x,y,\lambda]$ and then $\sigma(\bar{x},\tilde{y},\tilde{\lambda})\ge 0$.

The above observation entails that val(ED_k^d)≤ 2 for any $k\in\mathbb{N}$ and $d\in\mathbb{N}_0$. So the conclusion (3.8) of Theorem 3.2 is not valid for this setting as val(EP_1)$=f(\bar{x},\bar{y})=4$. The reason is that the MFCQ condition is violated for the associated sets $Z(x)$ with $x\in\mathbb{R}^2$ of the underlying problem. To see this, just take any $\hat{x}\in\mathbb{R}^2$ and $\bar{y}:=(0,0)\in Z(\hat{x})$. Then $g_1(\hat{x},\bar{y})+\bar{v}_1^1 g_{1,1}(\hat{x},\bar{y})+\bar{v}_2^1 g_{2,1}(\hat{x},\bar{y})=0$ and

$$\nabla_y g_1(\hat{x},\bar{y})+\bar{v}_1^1\nabla_y g_{1,1}(\hat{x},\bar{y})+\bar{v}_2^1\nabla g_{2,1}(\hat{x},\bar{y})=(0,0),$$

which shows the violation of the strict inequalities in (2.1).

3.2 Relaxations with the objective coercivity

We now consider the robust equilibrium constrained polynomial problem (REP) in which the feasibility set is arbitrary and the objective function is *coercive*. Namely, we impose the following assumption of coercivity:

Assumption C. Let f be coercive on $\mathbb{R}^n\times\mathbb{R}^m$.

Let $\tau\in\mathbb{R}$ be such that $\tau\ge f(x_0,y_0)$ for some $(x_0,y_0)\in K$, where K is the feasible set of problem (REP) given as in (2.4). Define a function $G_{0,0}:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{r_1s_2+1}\to\mathbb{R}$ given by $G_{0,0}(x,y,\lambda):=f(x,y)-\tau$ for $(x,y,\lambda)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{r_1s_2+1}$, and let R_0 be a positive number such that

$$R_0\ge\max_{(x,y,\lambda)\in\Omega_0}\big\{-G_\gamma(x,y,\lambda),\gamma\in J_G\cup\{(0,0)\}\big\}, \tag{3.26}$$

where $\Omega_0:=\{(x,y,\lambda)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{r_1s_2+1}\mid f(x,y)+\|\lambda\|^2\le 1+\tau\}$ and $G_\gamma,\gamma\in J_G$ are given as in (2.6). Under the coercivity of f, the set Ω_0 is compact and thus R_0 in (3.26) is well-defined, and some possible ways of choosing R_0 can be similarly done as in Remark 3.1(iv).

Then we can define the following functions

$$\widehat{G}_p:=\begin{cases}-\frac{1}{R_0}G_\gamma, & p=\gamma,\gamma\in J_G\cup\{(0,0)\},\\ -H_\gamma, & p=(s_1+2s_2+1+\gamma_1,l_1+2r_1+1+\gamma_2),\gamma:=(\gamma_1,\gamma_2)\in J_H,\\ H_\gamma, & p=(s_1+3s_2+m+2+\gamma_1,l_1+3r_1+m+2+\gamma_2),\gamma:=(\gamma_1,\gamma_2)\in J_H,\end{cases} \tag{3.27}$$

where $H_\gamma,\gamma\in J_H$ are given in (2.7). Denoting by

$$J_0:=J_G\cup\{(0,0)\}\cup\{(s_1+2s_2+1+\gamma_1,l_1+2r_1+1+\gamma_2),$$

$$(s_1 + 3s_2 + m + 2 + \gamma_1, l_1 + 3r_1 + m + 2 + \gamma_2) \mid (\gamma_1, \gamma_2) \in J_H\},$$

we see that the cardinality of J_0 is $4r_1s_2 + l_1s_1 + 2m + 4$ (i.e., $|J_0| = 4r_1s_2 + l_1s_1 + 2m + 4$).

Similarly as above, we address three types of relaxations for the equilibrium constrained problem (REP), and these relaxation problems are constructed via the functions $\widehat{G}_\gamma$ given in (3.27).

Bounded degree SOS relaxation problems. Fix a positive even number $d \in I\!N_0$ and define a hierarchy of *SOS relaxations* for the equilibrium constrained problem (REP) as

$$\sup_{(t, c_{\alpha,\beta})} \Big\{ t \mid f - \sum_{\alpha,\beta \in (\mathbb{N}_0)^{|J_0|}, |\alpha|+|\beta| \le k} c_{\alpha,\beta} \prod_{p \in J_0} (\widehat{G}_p)^{\alpha_p} (1 - \widehat{G}_p)^{\beta_p} - t \in \mathbf{SOS}_d[x, y, \lambda], t \in \mathbb{R}, c_{\alpha,\beta} \ge 0 \Big\}, \tag{SDP2$_k^d$}$$

where $k \in \mathbb{N}$.

Bounded degree SDSOS relaxation problems. Fix a positive even number $d \in I\!N_0$ and define a hierarchy of *SDSOS relaxations* for the equilibrium constrained problem (REP) as

$$\sup_{(t, c_{\alpha,\beta})} \Big\{ t \mid f - \sum_{\alpha,\beta \in (\mathbb{N}_0)^{|J_0|}, |\alpha|+|\beta| \le k} c_{\alpha,\beta} \prod_{p \in J_0} (\widehat{G}_p)^{\alpha_p} (1 - \widehat{G}_p)^{\beta_p} - t \in \mathbf{SDSOS}_d[x, y, \lambda], t \in \mathbb{R}, c_{\alpha,\beta} \ge 0 \Big\}, \tag{SOCP2$_k^d$}$$

where $k \in \mathbb{N}$.

Bounded degree DSOS relaxation problems. Fix a positive even number $d \in I\!N_0$ and define a hierarchy of *DSOS relaxations* problems for the equilibrium constrained problem (REP) as

$$\sup_{(t, c_{\alpha,\beta})} \Big\{ t \mid f - \sum_{\alpha,\beta \in (\mathbb{N}_0)^{|J_0|}, |\alpha|+|\beta| \le k} c_{\alpha,\beta} \prod_{p \in J_0} (\widehat{G}_p)^{\alpha_p} (1 - \widehat{G}_p)^{\beta_p} - t \in \mathbf{DSOS}_d[x, y, \lambda], t \in \mathbb{R}, c_{\alpha,\beta} \ge 0 \Big\}, \tag{LP2$_k^d$}$$

where $k \in \mathbb{N}$.

It is worth noting that there is a slight difference between the definitions of the bounded degree relaxations (SDP2$_k^d$), (SOCP2$_k^d$) and (LP2$_k^d$) in this subsection and those (SDP1$_k^d$), (SOCP1$_k^d$) and (LP1$_k^d$) in the previous subsection. More precisely, the relaxations of this subsection involve the polynomial $G_{0,0}$, which relates to an upper bound τ of the coercive objective, while the previous ones do not have $G_{0,0}$. This in turn results in the variant manners of choosing R over the set Ω in (3.1) and choosing R_0 on the set Ω_0 in (3.26).

We now show that, under the *coercivity of the objective* and regularity, the robust equilibrium constrained polynomial problem (REP) possesses a global solution and its optimal value is the limit of bounded degree hierarchy relaxation problems (SDP2$_k^d$), (SOCP2$_k^d$) and (LP2$_k^d$), where $d \in I\!N_0$ is fixed and $k \in I\!N$ goes to infinity.

Theorem 3.4 *(Convergence of Relaxations) Let Assumptions B and C hold for the equilibrium constrained problem (REP). Assume that the associated sets of the variational inequality of problem (REP) satisfy (MFCQ). Then, the equilibrium constrained problem (REP) possesses a solution, say* $(\bar{x}, \bar{y})$*, satisfying*

$$\mathrm{val}(\mathrm{SDP2}_k^d) \le \mathrm{val}(\mathrm{SDP2}_{k+1}^d) \le \mathrm{val}(REP) = f(\bar{x}, \bar{y}) \ \ for \ \ k \in I\!N, \tag{3.28}$$

$$\mathrm{val}(\mathrm{SOCP2}_k^d) \le \mathrm{val}(\mathrm{SOCP2}_{k+1}^d) \le \mathrm{val}(REP) \ \ for \ \ k \in I\!N, \tag{3.29}$$

$$\mathrm{val}(\mathrm{LP2}_k^d) \le \mathrm{val}(\mathrm{LP2}_{k+1}^d) \le \mathrm{val}(REP) \ \ for \ \ k \in I\!N, \tag{3.30}$$

and

$$\lim_{k\to\infty} \mathrm{val}(\mathrm{LP2}_k^d) = \lim_{k\to\infty} \mathrm{val}(\mathrm{SOCP2}_k^d) = \lim_{k\to\infty} \mathrm{val}(\mathrm{SDP2}_k^d) = \mathrm{val}(REP), \tag{3.31}$$

where $d \in I\!N_0$ *is given.*

Proof Let $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$ be a feasible point of (REP) mentioned in Assumption C. Then, we see that $\tau \ge f(x_0, y_0)$, and invoke Proposition 2.3 to find $\bar{\lambda} \in \mathbb{R}^{r_1 s_2+1}$ such that

$$G_\gamma(x_0, y_0, \bar{\lambda}) \le 0, \ \gamma \in J_G, \ H_\gamma(x_0, y_0, \bar{\lambda}) = 0, \ \gamma \in J_H,$$

where $G_\gamma, \gamma \in J_G$ and $H_\gamma, \gamma \in J_H$, are defined in (2.6) and (2.7), respectively. Let

$$\begin{aligned} X_0 := \{(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{r_1 s_2+1} \mid \quad & -G_\gamma(x, y, \lambda) \ge 0, \gamma \in J_G \cup \{(0,0)\}, \\ H_\gamma(x, y, \lambda) \ge 0, -H_\gamma(x, y, \lambda) \ge 0, \gamma \in J_H\}, & \end{aligned} \tag{3.32}$$

where $G_{0,0}(x, y, \lambda) := f(x, y) - \tau$ for $(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{r_1 s_2+1}$ as above. It is easy to see that X_0 is a closed set and $X_0 \ne \emptyset$ because of $(x_0, y_0, \bar{\lambda}) \in X_0$.

Furthermore, for each $(x, y, \lambda) \in X_0$, where $\lambda := \left(\lambda_0, \lambda_1^1, \ldots, \lambda_{s_2}^1, \ldots, \lambda_1^{r_1}, \ldots, \lambda_{s_2}^{r_1}\right) \in \mathbb{R}^{r_1 s_2+1}$, it holds that $G_{0,0}(x, y, \lambda) \le 0$ and $H_{s_2+m+1, r_1+m+1}(x, y, \lambda) = 0$, which means that

$$f(x, y) \le \tau, \quad ||\lambda|| = 1. \tag{3.33}$$

This ensures that $(x, y, \lambda) \in \Omega_0$; i.e., $X_0 \subset \Omega_0$, where $\Omega_0 := \{(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{r_1 s_2+1} \mid f(x, y) + \|\lambda\|^2 \le 1 + \tau\}$. Since f is coercive on $\mathbb{R}^n \times \mathbb{R}^m$, Ω_0 is a compact set, and so is X_0. We now consider a function $f_0 : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{r_1 s_2+1} \to \mathbb{R}$ given by $f_0(x, y, \lambda) := f(x, y)$ for $(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{r_1 s_2+1}$. The continuity of f_0 and the compactness of X_0 guarantee that there exists $(\bar{x}, \bar{y}, \hat{\lambda}) \in X_0$ such that

$$f_0(\bar{x}, \bar{y}, \hat{\lambda}) \le f_0(x, y, \lambda) \ \text{ for all } (x, y, \lambda) \in X_0. \tag{3.34}$$

We now show that $(\bar{x}, \bar{y})$ is a global solution of problem (REP). Observe by $(\bar{x}, \bar{y}, \hat{\lambda}) \in X_0$ that

$$\tau \ge f(\bar{x}, \bar{y}), \tag{3.35}$$

and that

$$G_\gamma(\bar{x}, \bar{y}, \hat{\lambda}) \le 0, \ \gamma \in J_G, \ H_\gamma(\bar{x}, \bar{y}, \hat{\lambda}) = 0, \ \gamma \in J_H. \tag{3.36}$$

Thus, by (MFCQ), we invoke Proposition 2.3 to conclude that $(\bar{x}, \bar{y})$ is feasible for (REP). Let $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ be an arbitrary feasible point of problem (REP). In view of Proposition 2.3 again, there exists $\lambda \in \mathbb{R}^{r_1 s_2+1}$ such that

$$G_\gamma(x, y, \lambda) \le 0, \ \gamma \in J_G, \ H_\gamma(x, y, \lambda) = 0, \ \gamma \in J_H.$$

In the case of $\tau \geq f(x, y)$, it holds $(x, y, \lambda) \in X_0$, and so we get by (3.34) that $f(\bar{x}, \bar{y}) \leq f(x, y)$. In the case of $\tau < f(x, y)$, we conclude by (3.35) that $f(x, y) > f(\bar{x}, \bar{y})$. In conclusion, $(\bar{x}, \bar{y})$ is a solution of problem (REP). Then, it holds that

$$\mathrm{val}(REP) = f(\bar{x}, \bar{y}). \tag{3.37}$$

By $X_0 \subset \Omega_0$, we see that

$$R_0 \geq \max_{(x,y,\lambda)\in X_0} \big\{ -G_\gamma(x, y, \lambda), \gamma \in J_G \cup \{(0,0)\}\big\},$$

where R_0 is given in (3.26). This ensures that $0 \leq \widehat{G}_p \leq 1$ on X_0 for all $p \in J_G \cup \{(0,0)\}$ and $\widehat{G}_p = 0$ on X_0 for all $p \in J_0 \backslash (J_G \cup \{(0,0)\})$, where $\widehat{G}_p$, $p \in J_0$ are given by (3.27).

Let $d \in I\!N_0$. By the construction of relaxation problems, it is clear that val(SDP2^d_k)$\leq$ val(SDP2^d_{k+1}), val(SOCP2^d_k)$\leq$ val(SOCP2^d_{k+1}) and val(LP2^d_k)$\leq$ val(LP2^d_{k+1}) for all $k \in I\!N$. Furthermore, it holds that

$$\mathrm{val}(\mathrm{LP}2^d_k) \leq \mathrm{val}(\mathrm{SOCP}2^d_k) \leq \mathrm{val}(\mathrm{SDP}2^d_k) \ \text{ for all } k \in I\!N. \tag{3.38}$$

To justify (3.28) to (3.30), it is sufficient to show that

$$\mathrm{val}(\mathrm{SDP}2^d_k) \leq f(\bar{x}, \bar{y}) \ \text{ for all } k \in I\!N. \tag{3.39}$$

Let $k \in I\!N$. If the feasible set of problem (SDP2^d_k) is empty, then (3.39) holds trivially as val (SDP2^d_k)$= -\infty$. Now, let $(t, c_{\alpha,\beta})$ be feasible for (SDP2^d_k). This means that $c_{\alpha,\beta} \geq 0$ for all $\alpha, \beta \in (\mathbb{N}_0)^{|J_0|}$, $|\alpha| + |\beta| \leq k$ and

$$f - \sum_{\alpha,\beta\in(\mathbb{N}_0)^{|J_0|},|\alpha|+|\beta|\leq k} c_{\alpha,\beta} \prod_{p\in J_0} (\widehat{G}_p)^{\alpha_p}(1 - \widehat{G}_p)^{\beta_p} - t \in \mathbf{SOS}_d[x, y, \lambda]. \tag{3.40}$$

Since $(\bar{x}, \bar{y}) \in K$ and (MFCQ) holds, we invoke Proposition 2.3 to assert that there exists $\bar{\lambda} \in \mathbb{R}^{r_1 s_2+1}$ such that

$$G_\gamma(\bar{x}, \bar{y}, \bar{\lambda}) \leq 0,\ \gamma \in J_G,\ H_\gamma(\bar{x}, \bar{y}, \bar{\lambda}) = 0,\ \gamma \in J_H.$$

Moreover, by (3.35), $\tau \geq f(\bar{x}, \bar{y})$ and so $(\bar{x}, \bar{y}, \bar{\lambda}) \in X_0$. This guarantees that $0 \leq \widehat{G}_p(\bar{x}, \bar{y}, \bar{\lambda}) \leq 1$ for all $p \in J_G \cup \{(0,0)\}$ and $\widehat{G}_p(\bar{x}, \bar{y}, \bar{\lambda}) = 0$ for all $p \in J_0 \backslash (J_G \cup \{(0,0)\})$. Therefore, by recalling the nonnegativity of SOS polynomials, we evaluate (3.40) at $(\bar{x}, \bar{y}, \bar{\lambda})$ to obtain that $f(\bar{x}, \bar{y}) \geq t$. This yields val(SDP$2^d_k$)$\leq f(\bar{x}, \bar{y})$; i.e., (3.39) has been proved.

Note that Assumption B ensures that the polynomials $\{1, -G_\gamma, \gamma \in J_G \cup \{(0,0)\}, H_\gamma, -H_\gamma, \gamma \in J_H\}$ generates the ring $\mathbb{R}[x, y, \lambda]$. We can employ the convergence of linear programming hierarchy in polynomial optimization (see [13, Theorem 3.2 and a remark on page 901]) applied to our problem with the objective f over the feasible set X_0 to assert that

$$\lim_{k\to\infty} \mathrm{val}(\mathrm{LP}2^d_k) = \mathrm{val}(REP).$$

This, together with (3.38) and (3.39), justifies (3.31), and so the proof is complete. □

The following example illustrates how one can employ the bounded degree polynomial relaxations to identify the optimal value of a robust equilibrium constrained problem.

Example 3.5 (Finding optimal values using relaxations) Consider a robust equilibrium constrained problem given by

$$\min_{(x,y)\in\mathbb{R}\times\mathbb{R}^2} \Big\{x^3 + xy_1^2 - 3y_1y_2 + y_2^4 - 3 \mid -u_1x^2 - u_2y_2^2 - 1 \leq 0, u_1x \leq 0,\ \forall u := (u_1, u_2) \in U, \tag{EP$_2$}$$

$$y \in Z(x) := \Big\{z \in \mathbb{R}^2 \mid v_1(x^2+1)z_1 + v_2 z_2^2 - 1 \le 0, -v_1 z_1 - v_2 z_2 \le 0,\ \forall v := (v_1, v_2) \in V\Big\},$$
$$\langle (x^2 + y_1^2, 2y_2^2), z - y\rangle \ge 0,\ \forall z \in Z(x)\Big\},$$

where $U := \text{conv}\,\{(-1,0),(0,1)\}$ and $V := \text{conv}\,\{(0,1),(1,0)\}$ are uncertain sets. Note that the problem (EP$_2$) can be viewed by way of (REP), where $f(x,y) := x^3 + xy_1^2 - 3y_1y_2 + y_2^4 - 3$, $f_1(x,y) := -1$, $f_{1,1}(x,y) := -x^2$, $f_{2,1}(x,y) := -y_2^2$, $f_2(x,y) := 0$, $f_{1,2}(x,y) := x$, $f_{2,2}(x,y) := 0$, $g_1(x,z) := -1$, $g_{1,1}(x,z) := (x^2+1)z_1$, $g_{2,1}(x,z) := z_2^2$, $g_2(x,z) := 0$, $g_{1,2}(x,z) := -z_1$, $g_{2,2}(x,z) := -z_2$, $\Gamma(x,y) := (x^2+y_1^2, 2y_2^2)$ for $x \in \mathbb{R}$, $y := (y_1,y_2) \in \mathbb{R}^2$ and $z := (z_1,z_2) \in \mathbb{R}^2$, and $\bar{u}^1 := (-1,0)$, $\bar{u}^2 := (0,1)$, $\bar{v}^1 := (0,1)$, $\bar{v}^2 := (1,0)$. We can verify directly that $(\bar{x},\bar{y})$, where $\bar{x} := 0$ and $\bar{y} := (0,0)$, is a solution of (EP$_2$) and therefore, the optimal value of (EP$_2$) is val(EP$_2$)$= -3$.

Let us now employ bounded degree polynomial relaxation schemes to check the optimal value of (EP$_2$). Observe first that the associated sets of the variational inequality of problem (EP$_2$) satisfy (MFCQ). From $-u_1x^2 - u_2y_2^2 - 1 \le 0$ for all $u := (u_1,u_2) \in U$, it holds that $x^2 \le 1$ and hence, $|x| \le 1$. In addition, we can calculate directly that for $x \in \mathbb{R}$, $Z(x) = \big[0, \frac{1}{1+x^2}\big] \times [0,1]$ and $Y(x) = \{(0,0)\}$, where $Y(x) := \big\{y \in Z(x) \mid \langle (x^2 + y_1^2, 2y_2^2), z - y\rangle \ge 0,\ \forall z \in Z(x)\big\}$. Thus, by letting $M_K := 1$, one has $||(x,y)|| \le \sqrt{M_K}$ for every $(x,y) \in K$, where $K := \Big\{(x,y) \in \mathbb{R} \times \mathbb{R}^2 \mid f_j(x,y) + \sum_{i=1}^{2} u_i f_{i,j}(x,y) \le 0, \forall u \in U, j = 1,2, y \in Y(x)\Big\}$ is the feasibility set of (EP$_2$). This shows that Assumption A is true. Similarly, we can verify that Assumption B also holds.

In this setting, the functions $G_\gamma, \gamma \in J_G$ and $H_\gamma, \gamma \in J_H$ defined respectively in (2.6) and (2.7) reduce to the following ones:

$$\begin{cases} G_{1,1}(x,y,\lambda) &= x^2 - 1,\ G_{1,2}(x,y,\lambda) = -1 - y_2^2,\ G_{2,1}(x,y,\lambda) = -x,\ G_{2,2}(x,y,\lambda) = 0,\\ G_{3,3}(x,y,\lambda) &= y_2^2 - 1,\ G_{3,4}(x,y,\lambda) = (x^2+1)y_1 - 1,\ G_{4,3}(x,y,\lambda) = -y_2, G_{4,4}(x,y,\lambda) = -y_1,\\ G_{5,5}(x,y,\lambda) &= -\lambda_1^1,\ G_{5,6}(x,y,\lambda) = -\lambda_1^2, G_{6,5}(x,y,\lambda) = -\lambda_2^1,\ G_{6,6}(x,y,\lambda) = -\lambda_2^2, G_{7,7}(x,y,\lambda) = -\lambda_0, \end{cases}$$

$$\begin{cases} H_{1,1}(x,y,\lambda) &= \lambda_1^1(y_2^2 - 1),\ H_{1,2}(x,y,\lambda) = \lambda_1^2[(x^2+1)y_1 - 1], H_{2,1}(x,y,\lambda) = -\lambda_2^1 y_2,\\ H_{2,2}(x,y,\lambda) &= -\lambda_2^2 y_1, H_{3,3}(x,y,\lambda) = \lambda_0(x^2+y_1^2) + \lambda_1^2(x^2+1) - \lambda_2^2,\\ H_{4,4}(x,y,\lambda) &= 2\lambda_0 y_2^2 + 2\lambda_1^1 y_2 - \lambda_2^1,\ H_{5,5}(x,y,\lambda) = \|\lambda\|^2 - 1, \end{cases}$$

where $(x,y) \in \mathbb{R} \times \mathbb{R}^2$, $\lambda := \big(\lambda_0, \lambda_1^1, \lambda_1^2, \lambda_2^1, \lambda_2^2\big) \in \mathbb{R}^5$, $J_G := \{(j,k) \mid j = 1,2, k = 1,2\} \cup \{(2+j, 2+k), (4+j, 4+k) \mid j = 1,2, k = 1,2\} \cup \{(7,7)\}$ and $J_H := \{(j,k) \mid j = 1,2, k = 1,2\} \cup \{(2+\ell, 2+\ell) \mid \ell = 1,2,3\}$.

Let R be a positive number such that

$$R \ge \max_{(x,y,\lambda)\in\Omega} \{-G_\gamma(x,y,\lambda), \gamma \in J_G\}, \tag{3.41}$$

where $\Omega := \{(x,y,\lambda) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^5 \mid \|(x,y,\lambda)\|^2 \le 1 + M_K\}$, and define the following functions

$$\widehat{G}_p := \begin{cases} -\frac{1}{R} G_\gamma, & p = \gamma, \gamma \in J_G,\\ -H_\gamma, & p = (7+\gamma_1, 7+\gamma_2), \gamma := (\gamma_1,\gamma_2) \in J_H\\ H_\gamma, & p = (12+\gamma_1, 12+\gamma_2), \gamma := (\gamma_1,\gamma_2) \in J_H. \end{cases}$$

Let $J := J_G \cup \{(7+\gamma_1, 7+\gamma_2), (12+\gamma_1, 12+\gamma_2) \mid (\gamma_1,\gamma_2) \in J_H\}$, it holds that $|J| = 27$.

In this setting, the relaxation problems (SDP1$_k^d$), (SOCP1$_k^d$) and (LP1$_k^d$) for $k \in I\!N$ are respectively given as

$$\sup_{(t,c_{\alpha,\beta})} \Big\{ t \mid f - \sum_{\alpha,\beta\in(\mathbb{N}_0)^{27},|\alpha|+|\beta|\le k} c_{\alpha,\beta} \prod_{p\in J} (\widehat{G}_p)^{\alpha_p}(1-\widehat{G}_p)^{\beta_p} - t \in \mathbf{SOS}_d[x,y,\lambda],\ t\in\mathbb{R},\ c_{\alpha,\beta}\ge 0 \Big\}, \tag{SDE$_k^d$}$$

$$\sup_{(t,c_{\alpha,\beta})} \Big\{ t \mid f - \sum_{\alpha,\beta\in(\mathbb{N}_0)^{27},|\alpha|+|\beta|\le k} c_{\alpha,\beta} \prod_{p\in J} (\widehat{G}_p)^{\alpha_p}(1-\widehat{G}_p)^{\beta_p} - t \in \mathbf{SDSOS}_d[x,y,\lambda],\ t\in\mathbb{R},\ c_{\alpha,\beta}\ge 0 \Big\}, \tag{SOCE$_k^d$}$$

$$\sup_{(t,c_{\alpha,\beta})} \Big\{ t \mid f - \sum_{\alpha,\beta\in(\mathbb{N}_0)^{27},|\alpha|+|\beta|\le k} c_{\alpha,\beta} \prod_{p\in J} (\widehat{G}_p)^{\alpha_p}(1-\widehat{G}_p)^{\beta_p} - t \in \mathbf{DSOS}_d[x,y,\lambda],\ t\in\mathbb{R},\ c_{\alpha,\beta}\ge 0 \Big\}, \tag{LE$_k^d$}$$

where $d \in I\!N_0$. Taking a given number $d \in I\!N_0$, Theorem 3.2 tells us that the equilibrium constrained problem (EP$_2$) admits a global solution, and its optimal value satisfies the following relations

$$\mathrm{val}(\mathrm{SDE}_k^d) \le \mathrm{val}(\mathrm{SDE}_{k+1}^d) \le \mathrm{val}(\mathrm{EP}_2),\ \ \mathrm{val}(\mathrm{SOCE}_k^d) \le \mathrm{val}(\mathrm{SOCE}_{k+1}^d) \le \mathrm{val}(\mathrm{EP}_2),$$
$$\mathrm{val}(\mathrm{LE}_k^d) \le \mathrm{val}(\mathrm{LE}_{k+1}^d) \le \mathrm{val}(\mathrm{EP}_2)$$

for all $k \in I\!N$ and

$$\lim_{k\to\infty} \mathrm{val}(\mathrm{LE}_k^d) = \lim_{k\to\infty} \mathrm{val}(\mathrm{SOCE}_k^d) = \lim_{k\to\infty} \mathrm{val}(\mathrm{SDE}_k^d) = \mathrm{val}(\mathrm{EP}_2).$$

Now, take an arbitrary $R > 0$ satisfying (3.41) (for instance, $R := 5$), we solve the relaxation problems (SDE$_k^d$), (SOCE$_k^d$) and (LE$_k^d$) with $d = 4$ for $k = 1, 2, 3$ within the DSOS, SDSOS and SOS hierarchies using the polynomial optimization toolbox SPOT [40] and the conic program solver MOSEK [42]. More explicitly, the polynomial optimization toolbox SPOT allows us to first convert relaxation problems within the SOS, SDSOS or DSOS hierarchy into corresponding semidefinite programming (SDP), second-order cone programming (SOCP) or linear programming (LP) problems, and then solve the converted programs to find lower bounds/optimal values. We also compare the proposed bounded degree hierarchies with the LP relaxation hierarchy defined by (LP1$_k$) for $k = 1, 2, 3$ and the moment-SOS relaxation hierarchy [33] implemented by Gloptipoly 3 [23]. The numerical tests are conducted on a computer with a Quad-Core Intel Core $i5$-$8279U$ CPU2.40GHz, 16GB 2133MHz LPDDR3 memory, equipped with MATLAB R2021b.

Table 1 summarizes the computed optimal values of relaxation problems as well as the CPU time used in seconds. In this table, "Output=infeasible" means that the corresponding method provides trivial lower bound $-\infty$, and particularly, for the case of Gloptipoly 3, this means that the corresponding SOS relaxation is infeasible. As we can see from the table that, for the relaxation order $k = 2$, the SOS relaxation provides a better lower bound than the SDSOS relaxation, and in its turn the SDSOS relaxation provides a better lower bound than

Table 1 Computing optimal values of relaxation problems

k	LP hierarchy	DSOS hierarchy	SDSOS hierarchy	SOS hierarchy	Gloptipoly 3
1	Output = infeasible	Output = infeasible	Output = infeasible	Output = infeasible	Output = infeasible
2	Output = infeasible	Output = − 7.7500	Output = − 6.2834	Output = − 3.9401	Output = infeasible
		Time = 2.258319	Time = 2.719339	Time = 2.675706	
		Status: lower bound	Status: lower bound	Status: lower bound	
3	Out of memory	Output = − 4.1250	Output = − 3.0000	Output = − 3.0000	Output = − 3.0000
		Time = 31.498645	Time = 39.059717	Time = 105.968693	Time = 225.281314
		Status: lower bound	Status: optimal value	Status: optimal value	Status: optimal value

the DSOS relaxation while the SOS and SDSOS relaxations occupy more CPU time than the DSOS relaxation. Moreover, for the relaxation order $k = 3$, while the LP relaxation runs out of memory, the SDSOS relaxation, the SOS relaxation and Gloptipoly 3 return the optimal value of the testing problem, where Gloptipoly 3 requires the most CPU time.

Observe that the robust equilibrium constrained polynomial problems considered in Examples 3.3 and 3.5 could be expressed in terms of robust *bilevel* polynomial programs [14] by using related reformulation techniques (see e.g., [15, 16]). It would be interesting to examine whether a randomly given equilibrium constrained polynomial problem can be conveniently transferred into a corresponding robust bilevel format. If this is the case, we can exploit the *semidefinite programming* (SDP) hierarchies for a robust bilevel polynomial program in [14], which were established by means of Lasserre hierarchy of semidefinite programming relaxations [33], to find the global optimal value of a robust equilibrium constrained polynomial problem.

We close this section with some remarks on the proposed bounded degree DSOS, SDSOS and SOS polynomial relaxations.

Remark 3.6 (Asymptotic and Finite convergences & Global optimal solutions)

(i) The proposed bounded degree SOS, SDSOS and DSOS hierarchies allow one to calculate *lower bounds/optimal value* of a robust equilibrium constrained program by way of the so-called *asymptotic convergences*. We recall, for instance, a result of asymptotic convergence in Theorem 3.2 as

$$\lim_{k\to\infty} \mathrm{val}(\mathrm{LP1}_k^d) = \lim_{k\to\infty} \mathrm{val}(\mathrm{SOCP1}_k^d) = \lim_{k\to\infty} \mathrm{val}(\mathrm{SDP1}_k^d) = \mathrm{val}(REP),$$

which shows that the optimal values of these relaxations tend to the optimal value of the considered robust equilibrium constrained problem whenever the degrees of the approximating polynomials in the hierarchies go to infinity ($k \to \infty$). So, in general, we do not know when the optimal values of relaxations in the hierarchies *reach* the optimal value of the underlying robust equilibrium constrained program. Moreover, since these relaxation problems are built from the view of nonnegative polynomials, the proposed schemes are generally not able to provide a *solution* for the considered robust equilibrium constrained program.

(ii) If one can find $k_0 \in \mathbb{N}$ such that

$$\mathrm{val}(\mathrm{LP1}_{k_0}^d) = \mathrm{val}(\mathrm{SOCP1}_{k_0}^d) = \mathrm{val}(\mathrm{SDP1}_{k_0}^d) = \mathrm{val}(REP),$$

then the corresponding results are called *finite convergences*. For a class of standard *polynomial programs*, a sufficient criterion in terms of *rank conditions* [22] and certain commonly used constraint qualification conditions for the bounded degree SOS polynomial relaxation to have a finite convergence can be found in [36]. For some special classes of *convex* polynomial programs, other sufficient criteria for finite convergences to occur at the first relaxation (i.e., $k_0 = 1$) in the hierarchies can be found in [35, 36] for the setting of SOS polynomial relaxations and in [13] for the settings of SDSOS and DSOS polynomial relaxations. As the above-mentioned sufficient criteria are checked via the Lagrangian dual problems of relaxations, a *global optimal solution* can also be extracted as long as a finite convergence takes place.
However, for the class of our *robust equilibrium* constrained polynomial programs, a question on how to find verifiable sufficient conditions that guarantee *finite convergences* for the proposed bounded degree DSOS, SDSOS and SOS polynomial relaxations deserves a further investigation due to the fact that the related functions and sets in the constraints (such as K in (2.4)) of equilibrium constrained programs often violate almost all commonly used constraint qualification conditions [18, 44].

4 Applications to electric vehicle charging under uncertain discharging supplies

In this section, we employ the bounded degree SOS, SDSOS and DSOS relaxation hierarchies and the LP hierarchy to solve a practical problem in *Electric Vehicle Charging Scheduling* (EVCS). In this EVCS model, each electric vehicle (EV) user seeks to minimize her/him utility cost in a competitive market condition that the underlying station operator wants to maximize its trading profits under the setting of vehicle to grid (V2G) technology and *uncertain* discharging supplies. By making use of *uncertainty* data, the considered EV charging scheduling model below is more dynamic than a *certain*/standard electric vehicle charging scheduling problem (cf. [50]), which has been examined in [8] by using a noncooperative game model and recently in [25] by using a multiobjective approach.

A mathematical model for electric vehicle charging scheduling. Consider an *electric vehicle charging scheduling* model, where the objective is to achieve a minimized utility cost for each EV user and a maximized trading profit for the station operator within the framework of competitive markets. During the parking time $\mathcal{T}$, with the help of V2G technology, if the EV parking time is more than the EV charging time, the EV user has the ability to reduce its utility cost by deciding on how much for charging amount x_t and how much for discharging amount y_t of the electricity to the power grid at time slot t. Meanwhile, the charging station operator should arrange the charging and discharging amounts for the EV user to maximize its trading profits by supplying the charging service consistently and by selling the discharging amount from the EV user to other EV users timely. So, in this scheduling model, each EV user and the charging station operator share an *equilibrium condition* on the charging x_t amount and the discharging y_t amount. Moreover, under the consideration of discharging supplies, which are *uncertain* amounts from other EV users, the electricity price is hardly determined at every charging time slot t.

With the EV charging and discharging constraints described as above, a new electric vehicle charging scheduling problem under *uncertain* discharging supplies can be mathematically modeled as follows:

$$\min_{x\in\mathbb{R}^{\mathcal{T}},y\in\mathbb{R}^{\mathcal{T}}} \sum_{t=1}^{\mathcal{T}}(L_t + x_t - y_t)(x_t - y_t), \tag{4.1}$$

$$s.t.\ x_t \le X,\ t = 1,\ldots,\mathcal{T}, \tag{4.2}$$

$$y_t \le Y,\ t = 1,\ldots,\mathcal{T}, \tag{4.3}$$

$$L_t + u_t x_t \le M,\ t = 1,\ldots,\mathcal{T}, \tag{4.4}$$

$$\sum_{t=1}^{T}(x_t - y_t) = D, \tag{4.5}$$

$$y_t \le \sum_{t'=1}^{t-1}(x_{t'} - y_{t'}),\ t = 2,\ldots,\mathcal{T}, \tag{4.6}$$

$$\sum_{t'=1}^{t-1}(x_{t'} - y_{t'}) + x_t \le C,\ t = 2,\ldots,\mathcal{T}, \tag{4.7}$$

$$x_t \ge 0,\ y_t \ge 0,\ x_t y_t = 0,\ t = 1,\ldots,\mathcal{T}, \tag{4.8}$$

where $x := (x_1,\ldots,x_T)$ and $y := (y_1,\ldots,y_T)$. The L_t indicates the baseload at the charging station at t. $(L_t + x_t - y_t)(x_t - y_t)$ denotes the electricity price at t. Equations (4.2) and (4.3) constrain the charging and discharging efficiencies, where X and Y are the maximal charging and discharging amounts at t, respectively. Equation (4.4) describes the power transmission from the power grid to the charging station at t. u_t is the discharged electricity from other EV users. Parameter u_t is *uncertain* and $u_t \in [\alpha_1, \gamma_1]$ for given $\alpha_1 \in \mathbb{R}$ and $\gamma_1 \in \mathbb{R}$. M is the maximal power transmission at t. Equation (4.5) explains the charging demand that must be satisfied. Equation (4.6) describes that the discharging amount is equal or less than the sum of earlier charged amounts. Equation (4.7) constrains the total charging amounts, where C is the EV battery capacity. The EV user and the charging station operator share an (implicitly) equilibrium constraint in (4.8), where in each time slot t, the EV user can either charge with x_t or discharge with y_t, while the charging station operator can either provide with x_t or sell with y_t from the EV user to others.

As we can see from the above model, which contains *uncertain* discharging supplies from other EV users, the electricity price is often fluctuated and is unable to be precisely determined at every charging time slot. This leads to a fact that both the EV user and the charging station operator face challenging on how to effectively schedule EV charging and discharging amounts to achieve their goals. Below, we employ our robust equilibrium constrained polynomial program to handle the above-mentioned electric vehicle charging scheduling model with data uncertainties.

Transforming into equilibrium constrained polynomial programs. Denote by $U := [\alpha_1, \gamma_1]^{\mathcal{T}}$, which is a box in $\mathbb{R}^{\mathcal{T}}$. Then this set can be presented as $U = \operatorname{conv}\{\bar{u}^k \mid k = 1,\ldots,2^{\mathcal{T}}\}$, where $\bar{u}^k := (\bar{u}^k_1,\ldots,\bar{u}^k_{\mathcal{T}}), k = 1,\ldots,2^{\mathcal{T}}$ are extreme points of the box $[\alpha_1, \gamma_1]^{\mathcal{T}}$. For $x := (x_1,\ldots,x_{\mathcal{T}}) \in \mathbb{R}^{\mathcal{T}}$ and $y := (y_1,\ldots,y_{\mathcal{T}}) \in \mathbb{R}^{\mathcal{T}}$, we let $\Gamma : \mathbb{R}^{\mathcal{T}} \times \mathbb{R}^{\mathcal{T}} \to \mathbb{R}^{\mathcal{T}}$ be given by $\Gamma(x, y) := x$ and let $f(x, y) := \sum_{t=1}^{\mathcal{T}}(L_t + x_t - y_t)(x_t - y_t)$,

$$f_j(x,y) := \begin{cases} x_j - X, & j = 1,\dots,\mathcal{T}, \\ y_{j-\mathcal{T}} - Y & j = \mathcal{T}+1,\dots,2\mathcal{T}, \\ L_{j-2\mathcal{T}} - M, & j = 2\mathcal{T}+1,\dots,3\mathcal{T}, \\ \sum_{t=1}^{\mathcal{T}}(x_t - y_t) - D, & j = 3\mathcal{T}+1, \\ -\sum_{t=1}^{\mathcal{T}}(x_t - y_t) + D, & j = 3\mathcal{T}+2, \\ y_{j-3\mathcal{T}-1} - \sum_{t=1}^{j-3\mathcal{T}-2}(x_t - y_t), & j = 3\mathcal{T}+3,\dots,4\mathcal{T}+1, \\ \sum_{t=1}^{j-4\mathcal{T}-1}(x_t - y_t) + x_{j-4\mathcal{T}} - C, & j = 4\mathcal{T}+2\dots,5\mathcal{T}, \end{cases} \tag{4.9}$$

$$f_{i,j}(x,y) := \begin{cases} x_i, & i = j - 2\mathcal{T},\ j = 2\mathcal{T}+1,\dots,3\mathcal{T}, \\ 0 & \text{others}, \end{cases} \tag{4.10}$$

$$g_j(x,y) := -y_j,\ j = 1,\dots,\mathcal{T},\ g_{i,j}(x,y) := 0,\ i = 1,\dots,\mathcal{T},\ j = 1,\dots,\mathcal{T}. \tag{4.11}$$

Now, the problem of (4.1)–(4.8) is rewritten in an *uncertain* equilibrium constrained problem:

$$\min_{(x,y)\in\mathbb{R}^{\mathcal{T}}\times\mathbb{R}^{\mathcal{T}}} \Bigg\{ f(x,y) \mid f_j(x,y) + \sum_{i=1}^{\mathcal{T}} u_i f_{i,j}(x,y) \le 0,\ j = 1,\dots,5\mathcal{T}, \tag{UVP}$$

$$y \in Z(x,v) := \Bigg\{ z \in \mathbb{R}^{\mathcal{T}} \mid g_j(x,z) + \sum_{i=1}^{\mathcal{T}} v_i g_{i,j}(x,z) \le 0,\ j = 1,\dots,\mathcal{T} \Bigg\},$$

$$\langle \Gamma(x,y), z - y \rangle \ge 0,\ \ \forall z \in Z(x,v) \Bigg\},$$

where $u := (u_1,\dots,u_{\mathcal{T}}) \in U$ and $v := (v_1,\dots,v_{\mathcal{T}}) \in V := \mathrm{conv}\,\{0_{\mathcal{T}}\}$, are uncertain sets. To deal with the uncertain equilibrium constrained program (UVP), we consider its associated *robust* problem:

$$\min_{(x,y)\in\mathbb{R}^{\mathcal{T}}\times\mathbb{R}^{\mathcal{T}}} \Bigg\{ f(x,y) \mid f_j(x,y) + \sum_{i=1}^{\mathcal{T}} u_i f_{i,j}(x,y) \le 0, \forall u \in U,\ j = 1,\dots,5\mathcal{T}, \tag{RVP}$$

$$y \in Z(x) := \Bigg\{ z \in \mathbb{R}^{\mathcal{T}} \mid g_j(x,z) + \sum_{i=1}^{\mathcal{T}} v_i g_{i,j}(x,z) \le 0, \forall v \in V,\ j = 1,\dots,\mathcal{T} \Bigg\},$$

$$\langle \Gamma(x,y), z - y \rangle \ge 0,\ \ \forall z \in Z(x) \Bigg\}.$$

Verifying assumptions. Letting $M_K := 2\mathcal{T}(\max\{X,Y\})^2 > 0$, it holds that $||(x,y)|| \le \sqrt{M_K}$ for all $(x,y) \in K$, where $K := \Big\{ (x,y) \in \mathbb{R}^{\mathcal{T}} \times \mathbb{R}^{\mathcal{T}} \mid f_j(x,y) + \sum_{i=1}^{\mathcal{T}} u_i f_{i,j}(x,y) \le 0, \forall u \in U,\ \ j = 1,\dots,5\mathcal{T}, y \in Z(x), \langle \Gamma(x,y), z - y \rangle \ge 0, \forall z \in Z(x) \Big\}$ is the feasibility set of (RVP). This shows that Assumption A is valid. Since the polynomials $\Big\{ 1,\ f_j +$

$\left.\sum_{i=1}^{\mathcal{T}} \bar{u}_i^k f_{i,j}, j = 1, \dots, 5\mathcal{T}, k = 1, \dots, 2^{\mathcal{T}}\right\}$ generates the ring $\mathbb{R}[x, y]$, Assumption B is also valid. Moreover, we can see by (4.11) that (MFCQ) holds for the associated sets of the variational inequality of problem (RVP).

Formulating relaxation problems. For $(x, y) \in \mathbb{R}^{\mathcal{T}} \times \mathbb{R}^{\mathcal{T}}$ and $\lambda := (\lambda_0, \lambda_1, \dots, \lambda_{\mathcal{T}}) \in \mathbb{R}^{\mathcal{T}+1}$, we define the following functions:

$$G_\gamma(x, y, \lambda) := \begin{cases} f_j(x, y) + \sum_{i=1}^{\mathcal{T}} \bar{u}_i^k f_{i,j}(x, y), & \gamma = (j, k), j = 1, \dots, 5\mathcal{T}, k = 1, \dots, 2^{\mathcal{T}}, \\ g_j(x, y), & \gamma = (5\mathcal{T} + j, 2^{\mathcal{T}} + j), j = 1, \dots, \mathcal{T}, \\ -\lambda_j, & \gamma = (6\mathcal{T} + j, 2^{\mathcal{T}} + \mathcal{T} + j), j = 1, \dots, \mathcal{T}, \\ -\lambda_0, & \gamma = (7\mathcal{T} + 1, 2^{\mathcal{T}} + 2\mathcal{T} + 1), \end{cases} \tag{4.12}$$

and

$$H_\gamma(x, y, \lambda) := \begin{cases} \lambda_j g_j(x, y), & \gamma = (j, j), j = 1, \dots, \mathcal{T}, \\ \left(\lambda_0 \Gamma(x, y) + \sum_{i=1}^{\mathcal{T}} \lambda_i \nabla_y g_i(x, y)\right)_j, & \gamma = (\mathcal{T} + j, \mathcal{T} + j), j = 1, \dots, \mathcal{T}, \\ \|\lambda\|^2 - 1, & \gamma = (2\mathcal{T} + 1, 2\mathcal{T} + 1). \end{cases} \tag{4.13}$$

Let R be a positive number such that

$$R \geq \max_{(x,y,\lambda)\in\Omega} \{-G_\gamma(x, y, \lambda), \gamma \in J_G\}, \tag{4.14}$$

where $\Omega := \{(x, y, \lambda) \in \mathbb{R}^{\mathcal{T}} \times \mathbb{R}^{\mathcal{T}} \times \mathbb{R}^{\mathcal{T}+1} \mid \|(x, y, \lambda)\|^2 \leq 1 + M_K\}$ and $J_G := \{(j, k) \mid j = 1, \dots, 5\mathcal{T}, k = 1, \dots, 2^{\mathcal{T}}\} \cup \{(5\mathcal{T} + j, 2^{\mathcal{T}} + j) \mid j = 1, \dots, 2\mathcal{T} + 1\}$. Similarly, we denote $J_H := \{(j, j) \mid j = 1, \dots, 2\mathcal{T} + 1\}$ and put

$$\widehat{G}_p := \begin{cases} -\frac{1}{R} G_\gamma, & p = \gamma, \gamma \in J_G, \\ -H_\gamma, & p = (7\mathcal{T} + 1 + j, 2^{\mathcal{T}} + 2\mathcal{T} + 1 + j), (j, j) \in J_H, \\ H_\gamma, & p = (9\mathcal{T} + 2 + j, 2^{\mathcal{T}} + 4\mathcal{T} + 2 + j), (j, j) \in J_H. \end{cases} \tag{4.15}$$

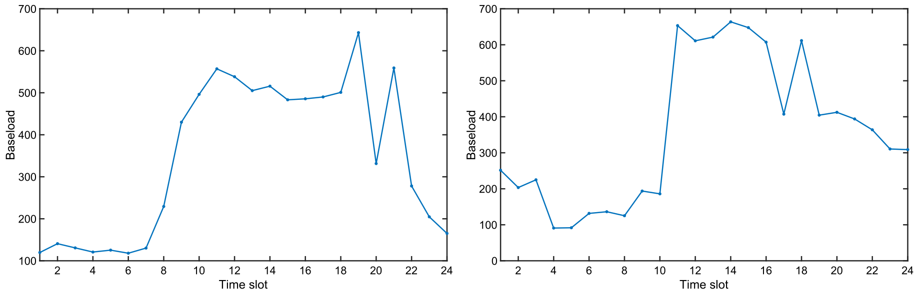
Let $J := J_G \cup \{(7\mathcal{T}+1+j, 2^{\mathcal{T}}+2\mathcal{T}+1+j), (9\mathcal{T}+2+j, 2^{\mathcal{T}}+4\mathcal{T}+2+j) \mid (j, j) \in J_H\}$, it holds that $|J| = 2^{\mathcal{T}} 5\mathcal{T} + 6\mathcal{T} + 3$.

We now state bounded degree polynomial relaxations for the equilibrium constrained problem (RVP).

Bounded degree relaxation problems. Fix a positive even number $d \in I\!N_0$ and define a hierarchy of *SOS relaxations* for the equilibrium constrained problem (RVP) as

$$\sup_{(t, c_{\alpha,\beta})} \left\{ t \mid f - \sum_{\alpha,\beta \in (\mathbb{N}_0)^{|J|}, |\alpha|+|\beta| \leq k} c_{\alpha,\beta} \prod_{p \in J} (\widehat{G}_p)^{\alpha_p} (1 - \widehat{G}_p)^{\beta_p} - t \in \mathbf{SOS}_d[x, y, \lambda], t \in \mathbb{R}, c_{\alpha,\beta} \geq 0 \right\}, \tag{SDPV$_k^d$}$$

where $k \in \mathbb{N}$. Similarly, in the problem (SDPV$_k^d$), if the cone $\mathbf{SOS}_d[x, y, \lambda]$ is replaced by the cone $\mathbf{SDSOS}_d[x, y, \lambda]$ (respectively, $\mathbf{DSOS}_d[x, y, \lambda]$), we obtain the *SDSOS relax-*



(a) The 24 hour-baseload for Day 1. (b) The 24 hour-baseload for Day 2.

Fig. 1 The 24 h-baseload for an office building for two days

ation problem (SOCPE$_k^d$) (respectively, the *DSOS relaxation* (LPE$_k^d$) problem) for the robust equilibrium constrained polynomial problem (RVP).

EV charging and discharging environment settings. We apply the Tesla Model 3[1] to be the EV example, which is equipped with a 75 kWh capacity battery, the maximum charging rate is 30 kWh per hour, and the maximum discharging rate is 45 kWh per hour. The EV is parking at the charging station for three hours and the initial state of charge is 6.6% (5 kWh power in the battery). Namely, the charging demand is 70 kWh. We randomly select an office building electricity load [7] for two days from Chongqing City of China that is shown in Fig. 1. For the uncertain parameter, we set 0% to 50% charging load that can be covered by other EV discharging loads. The maximal power transmission is 500 kWh per hour.

We test the proposed EV charging scheduling problem of (4.1)–(4.8) by using our relaxation problems in (SDPV$_k^d$), (SOCPE$_k^d$) and (LPE$_k^d$) for 8 different baseload scenarios in 24 h. In other words, there are 8 parking sessions in one day and 3 h for each parking session. The relaxation problems are solved by the DSOS hierarchy, the SDSOS hierarchy, and the SOS hierarchy, respectively. We generate feasible values, which play the roles as *upper bounds* of the optimal costs, to evaluate the tightness of the DSOS hierarchy, the SDSOS hierarchy, and the SOS hierarchy. The interested reader is referred to [31] for a heuristic method that employs SDSOS and DSOS polynomial relaxations to generate favorable feasible solutions. We also compare the proposed bounded degree hierarchies with the LP relaxation hierarchy defined by (LP1$_k$) for $k = 1, 2, 3$ and the moment-SOS relaxation hierarchy implemented by Gloptipoly 3. The outputs of LP and Gloptipoly 3 are trivial lower bound $-\infty$ for the relaxation orders of $k = 1$ and $k = 2$. Unfortunately, for the relaxation order $k = 3$, both LP and Gloptipoly 3 do not give us any helpful information as they run out of memory for our current simulation computer system.

The optimized EV charging cost is shown in Tables 2 and 3. As we can see from the table, the charging cost is more when the EV is charged at the peak load parking sessions. The DSOS relaxation scheme always obtains a (minimal) lower charging cost than the SDSOS and SOS ones do in each session. For the relaxation order $k = 2$, SOS cannot obtain any results with the restrained computing memory. Moreover, in the relaxation order of $k = 2$, the DSOS and SDSOS relaxation schemes take more time on calculation and obtain a higher charging cost than $k = 1$'s cases. Notably, the SOS relaxation problems in Session 5 achieve the best result in terms of higher charging cost and less computing time compared to other relaxation ones for the relaxation order of $k = 1$. Consequently, the SOS relaxations with

1 https://en.wikipedia.org/wiki/Tesla_Model_3.

Table 2 The optimized EV charging cost in 8 sessions on Day 1

k	Sessions		1	2	3	4	5	6	7	8
	Upper bounds		17.620	36.648	42.300	18.120	34.950	81.032	61.530	64.320
		Cost	1.0000	1.0000	1.0000	15.7709	10.1220	10.7037	16.9859	0.6722
	SOS	Time	0.5141	0.5235	0.5278	0.7629	0.7336	0.6862	0.8632	0.5410
		Cost	0.0014	0.0005	0.9996	13.8710	8.4170	8.9190	15.0850	0.0006
	SDSOS	Time	0.5291	0.5276	0.5772	0.7930	0.7384	0.7818	0.7730	0.5356
		Cost	0.0014	0.0005	0.9996	13.8710	8.4170	8.9190	15.0850	0.0006
1	DSOS	Time	0.5941	0.5876	0.6810	0.8320	0.8523	0.9240	0.8340	0.7324
		Cost	Out of memory							
	SOS	Time	59.0894	57.6158	84.8312	90.9339	71.2964	67.6654	71.0360	81.6045
		Cost	1.1066	1.1386	1.1178	17.6763	12.5736	11.5637	18.4314	0.9138
	SDSOS	Time	52.5728	51.9841	64.7232	60.0645	63.8659	50.9398	60.9498	59.0917
		Cost	0.6865	0.2702	1.0000	14.4295	8.9700	9.1224	15.696	0.2152
2	DSOS	Time	50.6919	52.3267	52.4467	53.9142	54.069	54.0523	51.2345	51.1222

Table 3 The optimized EV charging cost in 8 sessions on Day 2

k	Sessions		1	2	3	4	5	6	7	8
	Upper bounds		12.82	64.46	41.44	36.27	36.51	48.48	49.86	46.53
		Cost	2.8263	2.0450	2.0186	1.5629	1.4571	1.3855	1.3329	1.2928
	SOS	Time	1.9755	1.2102	1.0422	1.1728	1.1317	1.0956	1.0082	1.0256
		Cost	1.1028	1.2573	1.0319	1.0000	1.0000	1.0000	1.0000	1.0000
	SDSOS	Time	1.5559	1.2444	1.2109	1.1836	1.2080	1.1343	1.1164	1.0747
		Cost	1.0000	1.0000	1.0000	0.6589	0.9491	0.5943	0.5892	0.7278
1	DSOS	Time	1.4227	1.3145	1.2289	1.1992	1.3066	1.1710	1.1580	1.1267
		Cost	Out of memory							
	SOS	Time	52.1544	72.8037	74.0027	69.6755	62.0868	55.1067	58.4149	63.3937
		Cost	3.2324	3.7782	3.5848	2.2717	1.7147	1.6788	2.2398	1.9308
	SDSOS	Time	79.3250	72.5282	57.9678	74.2287	54.5370	99.5889	96.6429	54.9770
		Cost	1.4752	1.8263	1.6733	0.8584	1.3707	1.1862	1.2206	0.9767
2	DSOS	Time	52.1566	51.6146	50.5009	51.1079	54.5234	55.4089	54.5169	54.0463

$k = 1$ and the SDSOS relaxations with $k = 2$ have the best performance on the proposed EV charging scheduling problem in an acceptable computing time.

5 Conclusions and outlook

In this paper, we have exploited the robust approach to examine an equilibrium constrained polynomial problem, where the constraint functions and the equilibrium constraints involve uncertainty data. We have established bounded degree DSOS, SDSOS and SOS polynomial relaxation problems for solving the considered robust equilibrium constrained program. It has been shown that the optimal value of the considered equilibrium constrained problem is the limit of sequences of optimal values of bounded degree DSOS, SDSOS or SOS relaxations whenever the degrees of the approximated polynomials go to infinity. The bounded degree hierarchical relaxation convergences have been established by reformulating the robust equilibrium constrained polynomial program into a resulting nonconvex polynomial program with the help of a dual characterization of the equilibrium constraints of the underlying program and the convergences of linear programming hierarchy or bounded degree hierarchies in polynomial optimization [13, 35, 36].

We have also presented numerical examples that show how lower bounds and the optimal value of a robust equilibrium constrained polynomial problem can be calculated by way of the obtained relaxation schemes with the polynomial optimization toolbox SPOT [40]. An application to electric vehicle charging scheduling problems under uncertain discharging supplies shows that for the lower relaxation degrees, the DSOS, SDSOS and SOS relaxations obtain reasonable charging costs and for the higher relaxation degrees, the SDSOS relaxation scheme has the best performance, which is recommended for similar practical scenarios.

In the process of reformulating the robust equilibrium constrained polynomial problem into our relaxation hierarchies, many new variables have shown up in the relaxation models, and as a result, this leads to a computational burden for the proposed relaxation shemes when applying to higher dimensional practical problems. The *sparse* convergent SDP-relaxations [32] or the *sparse* moment-SOS hierarchy (TSSOS) [49], which have demonstrated better performances in terms of efficiency and scalability for polynomial optimization problems, would be good candidates to reduce the computational cost. Moreover, it is worth exploring how we can develop and apply the proposed DSOS, SDSOS and SOS relaxations schemes to solve other practical problems, such as the process engineering models as in [46] or the electric power market problems as in [24, 26], where the problem data often contain uncertainty factors. These aspects will be our future research topics.

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