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Solving Two-stage Quadratic Multiobjective Problems via Optimality and Relaxations

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Abstract

This paper focuses on the study of *robust two-stage* quadratic multiobjective optimization problems. We formulate new necessary and sufficient optimality conditions for a robust two-stage multiobjective optimization problem. The obtained optimality conditions are presented by means of linear matrix inequalities and thus they can be numerically validated by using a semidefinite programming problem. The proposed optimality conditions can be elaborated further as second-order conic expressions for robust two-stage quadratic multiobjective optimization problems with separable functions and ellipsoidal uncertainty sets. We also propose relaxation schemes for finding a (weak) efficient solution of the robust two-stage multiobjective problem by employing associated semidefinite programming or second-order cone programming relaxations. Moreover, numerical examples are given to demonstrate the solution variety of our flexible models and the numerical verifiability of the proposed schemes.

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1 Introduction

The data or system parameters of practical decision making problems are often noisy or fluctuating because of errors in estimation, prediction or misinformation. *Robust optimization* has become a powerful deterministic approach to handle effectively such uncertain real-life decision making models, see e.g., [3, 4, 8, 10, 13, 20, 22] and the references therein. *Adjustable* robust optimization [2], where some decision variables can be adjusted after the realization of uncertain parameters, is known as an extension of robust optimization. The adjustable robust optimization handles flexibly dynamic decision-making models under data uncertainties by allowing some decision variables to evolve over time in phases/stages based on the updated information of uncertainty data. This is often seen in practice, for instance, the cost of producing a product is just an estimated value until the product is actually made [23], and so the production decision maker needs to wait and possibly adjust the investment strategy according to the actual production cost. More generally, the decision variables of such a model would *depend* on the uncertainty factors and a robust counterpart of the underlying model is called a robust *two-stage* (or more general *multi-stage*) decision making problem [2, 5, 11, 14–16, 27, 33].

Furthermore, many real-world decision making problems are in the face of two-stage/multi-stage nature with *multiple objectives* (called *two-stage/multi-stage multiobjective* optimization programs) such as the integrated community management model [26], the reliability growth planning problem [25], the problem of energy retrofit of buildings [18, 19] or the electric-vehicle charging station placement problem [31]. As an illustration, in the operation of coordinated gas and electricity networks, the system operators may face the conflicting benefits between the demands of the natural gas network and the electricity network under an uncertain environment evolving over time [34]. Therefore, a more flexible two-stage multiobjective procedure was employed to schedule the coordinated community energy systems (see [26]). One important feature of the above-mentioned models is that, beside (standard) *here-and-now* decision variables, the problem data also contain *wait-and-see* decision variables that can be adjusted after some of uncertain parameters have revealed their values. These observations motivate us to investigate the forthcoming two-stage multiobjective problems involving data uncertainties.

An *uncertain* quadratic multiobjective problem is given (see e.g., [13]) by

$$\min_{x\in\mathbb{R}^q} \left\{\left(f_1(x,v),\ldots,f_m(x,v)\right) \mid g_j(x,v)\le 0,\ j=1,\ldots,n\right\}, \tag{P}$$

where $v\in V$ is an *uncertain parameter*, $V\subset\mathbb{R}^r$ is a nonempty compact *uncertainty* set, and $f_i:\mathbb{R}^q\times\mathbb{R}^r\to\mathbb{R}$, $i=1,\ldots,m$, $g_j:\mathbb{R}^q\times\mathbb{R}^r\to\mathbb{R}$, $j=1,\ldots,n$ are quadratic functions defined by, for $x\in\mathbb{R}^q$ and $v:=(v_1,\ldots,v_r)\in V$,

$$f_i(x,v) := x^\top Q_1^i x + (\xi_1^i)^\top x + \beta_1^i + \sum_{l=1}^{r} v_l\big((\xi_{1,l}^i)^\top x + \beta_{1,l}^i\big),$$

$$g_j(x,v) := x^\top Q_2^j x + (\xi_2^j)^\top x + \beta_2^j + \sum_{l=1}^{r} v_l\big((\xi_{2,l}^j)^\top x + \beta_{2,l}^j\big) \tag{1}$$

with $Q_1^i \succeq 0$, $Q_2^j \succeq 0$, $\xi_1^i \in \mathbb{R}^q$, $\xi_{1,l}^i \in \mathbb{R}^q$, $\xi_2^j \in \mathbb{R}^q$, $\xi_{2,l}^j \in \mathbb{R}^q$, $\beta_1^i \in \mathbb{R}$, $\beta_{1,l}^i \in \mathbb{R}$, $\beta_2^j \in \mathbb{R}$, $\beta_{2,l}^j \in \mathbb{R}$, $i = 1,\dots,m$, $j = 1,\dots,n$, $l = 1,\dots,r$ fixed. Here, the notation $A \succeq 0$ means that the matrix A is positive semidefinite.

We study an *uncertain* two-stage quadratic multiobjective program of (P) as follows

$$\min_{x,y} \big(f_1(x,v) + (\theta_1^1)^\top y(v), \dots, f_m(x,v) + (\theta_1^m)^\top y(v)\big) \tag{UT}$$
$$\text{s.t. } g_j(x,v) + (\theta_2^j)^\top y(v) \le 0, \;\; j = 1,\dots,n,$$

where $x \in \mathbb{R}^q$ is the *first-stage* or *here-and-now* decision variable, $y : V \to \mathbb{R}^p$ is the *second-stage* or *wait-and-see* decision variable and $\theta_1^i \in \mathbb{R}^p$, $i = 1,\dots,m$ and $\theta_2^j \in \mathbb{R}^p$, $j = 1,\dots,n$ are given parameters. Note that the wait-and-see variable y is *adjustable* and it is depending on uncertain values.

According to the two-stage terminology of (scalar) programming problems (cf. [2, 33]), the first-stage variable x is determined before the uncertain parameter v is realized, while the second-stage variable y is determined after some of uncertainties have shown up their values. As the wait-and-see variable y is an arbitrary map involving uncertain parameters from a general uncertainty set V, examining numerically the two-stage multiobjective program (UT) is generally challenging and moreover, expected optimality criteria would not be tractable/verifiable. To this end, we assume that the uncertainty set V is a nonempty compact set, which is of a *spectrahedral* form (see e.g., [28]) given by

$$V := \left\{ v := (v_1,\dots,v_r) \in \mathbb{R}^r \mid A + \sum_{l=1}^{r} v_l A_l \succeq 0 \right\}, \tag{2}$$

where A, A_l, $l = 1,\dots,r$ are symmetric $(m_0 \times m_0)$ matrices and the second-stage variable y is an affine rule (cf. [2, Page 356] or [33, Equation (8)]) given by

$$y(v) := y^0 + Yv, \;\; v \in V,$$

where $y^0 \in \mathbb{R}^p$ and $Y \in \mathbb{R}^{p\times r}$ are *nonadjustable* variables.

A *robust two-stage* multiobjective problem is defined via the *robust* counterpart of (UT) as

$$\min_{x\in\mathbb{R}^q, y^0\in\mathbb{R}^p, Y\in\mathbb{R}^{p\times r}} \big(\max_{v\in V}\{f_1(x,v) + (\theta_1^1)^\top y(v)\}, \dots, \max_{v\in V}\{f_m(x,v) + (\theta_1^m)^\top y(v)\}\big) \tag{RT}$$
$$\text{s.t. } g_j(x,v) + (\theta_2^j)^\top y(v) \le 0,\, j = 1,\dots,n,\, y(v) = y^0 + Yv, \forall v \in V.$$

Note that, in the model (RT), the constraints are stipulated for all possible values of the *uncertain* parameter v within the corresponding uncertainty set V.

To state the solution notions of worst-case efficiency in multiobjective optimization (see e.g., [17, 24]) for our robust two-stage multiobjective setting, we put $\mathcal{F}_i(x, y^0, Y) := \max_{v \in V}\{f_i(x, v) + (\theta_1^i)^\top(y^0 + Yv)\}$, $i = 1, \ldots, m$ for $x \in \mathbb{R}^q$, $y^0 \in \mathbb{R}^p$ and $Y \in \mathbb{R}^{p \times r}$ and denote by $\mathcal{C}$ the set of all feasible points of problem (RT), i.e.,

$$\mathcal{C} := \Big\{(x, y^0, Y) \in \mathbb{R}^{q+p+p\times r} \mid g_j(x, v) + (\theta_2^j)^\top y(v) \le 0, j = 1, \ldots, n, \\ y(v) = y^0 + Yv, \forall v \in V\Big\}.$$

Definition 1.1 (*Weak/Efficient Solutions*) For the problem (RT), let $(\widetilde{x}, \widetilde{y}^0, \widetilde{Y}) \in \mathcal{C}$.
(i) $(\widetilde{x}, \widetilde{y}^0, \widetilde{Y})$ is said to be a weak efficient solution of (RT) if there does not exist $(x, y^0, Y) \in \mathcal{C}$ such that

$$\mathcal{F}_i(x, y^0, Y) < \mathcal{F}_i(\widetilde{x}, \widetilde{y}^0, \widetilde{Y}),\ i = 1, \ldots, p.$$

(ii) $(\widetilde{x}, \widetilde{y}^0, \widetilde{Y})$ is called an *efficient solution* of (RT) if there does not exist $(x, y^0, Y) \in \mathcal{C}$ such that

$$\mathcal{F}_i(x, y^0, Y) \le \mathcal{F}_i(\widetilde{x}, \widetilde{y}^0, \widetilde{Y}),\ i = 1, \ldots, p \ \text{ and} \\ \mathcal{F}_i(x, y^0, Y) < \mathcal{F}_i(\widetilde{x}, \widetilde{y}^0, \widetilde{Y}) \ \text{ for some } i \in \{1, \ldots, p\}.$$

Despite there is a great deal of recent research on robust two-stage (scalar) optimization (see e.g., [2–4, 6, 9, 15] and the references therein), an answer to a *question* on how to establish verifiable optimality conditions as well as associated methods for solving numerically a nonlinear robust two-stage multiobjective optimization problem such as (RT) is currently unavailable, which is because of the numerical nontractability inherent in multiobjective and multi-stage structures. In this paper, we provide an answer to the above question by examining tractable optimality conditions and associated relaxation schemes for solving numerically the robust two-stage multiobjective program (RT). A deeper understanding on these optimality conditions and relaxation schemes could help us improve modeling formulations and corresponding computational methods for solving a broader class of nonlinear robust two-stage multiobjective optimization problems.

More precisely, the first aim of this paper is to establish new necessary and sufficient optimality conditions for the robust two-stage multiobjective optimization problem (RT). An advantage feature of the obtained optimality conditions is that these optimality criteria are linear matrix inequalities and hence they can be numerically validated by using a semidefinite programming problem. It is also shown that such optimality conditions can be elaborated further as second-order conic expressions for robust two-stage multiobjective optimization problems with separable functions and ellipsoidal uncertainty sets. The second aim of this paper is to propose novel relaxation schemes that allow one to calculate a (weak) efficient solution of the robust two-stage multiobjective problem (RT) by means of semidefinite programming (SDP) or second-order cone programming (SOCP) relaxation problems. The third aim of this paper is to provide numerical examples, which demonstrate that the proposed SDP or SOCP relaxations can be employed to locate (weak) efficient solutions of concrete robust two-stage multiobjective problems including those arisen from practical applications. In particular, these numerical examples also illustrate the solution variety of the considered models and the numerical tractability of the associated relaxation schemes. The interested reader is refered to [12] for some related results for an adjustable robust *linear* multiobjective optimization problem.

The structure of this paper is as follows. In Sect. 2, after providing basic definitions and notations, we first present necessary and sufficient linear matrix inequality optimality conditions for the robust two-stage multiobjective optimization problem (RT). We then expound second-order cone optimality conditions for a special robust two-stage multiobjective optimization problem. Section 3 shows how (weak) efficient solutions of the two-stage multiobjective optimization problem (RT) can be calculated by using associated SDP or SOCP relaxation problems. In Sect. 4, we present numerical examples including those emerged from practical applications. Section 5 is devoted to providing conclusions and research perspectives.

2 Optimality for Two-stage Multiobjective Problems

In this section, we provide necessary conditions and sufficient conditions for (weak) efficient solutions of the robust two-stage multiobjective optimization problem (RT).

Let us start by providing notations and definitions. We denote by $\mathbb{R}^q$ the Euclidean space whose norm is denoted by $\|\cdot\|$ for each $q \in I\!N := \{1, 2, \ldots\}$. We use 0 to denote the origin of a space, but 0_q is sometimes used for the origin of $\mathbb{R}^q$ for more clarification. For each $k \in \{1, \ldots, q\}$, e_k^q is the unit vector in $\mathbb{R}^q$ whose kth element is one and the others are all zero. The inner product in $\mathbb{R}^q$ is defined by $\langle x, y\rangle := x^\top y$ for all $x, y \in \mathbb{R}^q$. For a nonempty set $\Omega \subset \mathbb{R}^q$, $\operatorname{conv}\Omega$ denotes the convex hull of Ω and $\operatorname{cl}\Omega$ stands for the closure of Ω, while $\operatorname{int}\Omega$ is the interior of Ω.

An $(m \times n)$ real matrix A is denoted by $A \in \mathbb{R}^{m\times n}$. $A \in \mathbb{R}^{n\times n}$ is symmetric if $A^\top = A$, where $A^\top$ is the transpose of A. The set of all symmetric $(n\times n)$ real matrices is denoted by S^n. For $A \in S^n$, the notation A^{-1} is the inverse of A. As usual, the symbol $I_n \in \mathbb{R}^{n\times n}$ stands for the identity $(n\times n)$ matrix. A matrix $A \in S^n$ is said to be positive semidefinite, denoted by $A \succeq 0$, whenever $x^\top Ax \geq 0$ for all $x \in \mathbb{R}^n$. If $x^\top Ax > 0$ for all $x \in \mathbb{R}^n \setminus \{0_n\}$, then A is called positive definite, denoted by $A \succ 0$. The trace of a square matrix A is denoted by $\operatorname{Tr}(A)$. Given $v := (v_1, \ldots, v_n)$, the notation $\operatorname{diag}(v)$ or $\operatorname{diag}(v_1, \ldots, v_n)$ denotes a diagonal matrix with entries $v_1, \ldots, v_n$ along the diagonal and zeros elsewhere. Similarly, $\operatorname{diag}(A_1, \ldots, A_n)$ denotes the block diagonal matrix with submatrices $A_1, \ldots, A_n$ along the diagonal and zero submatrices elsewhere.

The following theorem states necessary and sufficient optimality conditions for the robust two-stage multiobjective optimization problem (RT). These optimality conditions are exhibited in terms of *linear matrix inequalities* (LMIs) and so they can be numerically validated by using a semidefinite programming problem.

Theorem 2.1 (Linear matrix inequality optimality) *For the problem* (RT)*, let* $(\overline{x}, \overline{y}^0, \overline{Y}) \in \mathbb{R}^{q+p+p\times r}$ *be a feasible point.*
*(i) (**Necessary optimality**) Assume that the Slater qualification condition holds for the problem* (RT)*, i.e., there exists* $(\widehat{x}, \widehat{y}^0, \widehat{Y}) \in \mathbb{R}^{q+p+p\times r}$ *such that*

$$g_j(\widehat{x}, v) + (\theta_2^j)^\top(\widehat{y}^0 + \widehat{Y}v) < 0, \forall v \in V, j = 1, \ldots, n. \tag{3}$$

Let $(\overline{x}, \overline{y}^0, \overline{Y})$ *be a weak efficient solution of* (RT)*. Then there exist* $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+ \setminus \{0\}$, $\alpha_i^s \in \mathbb{R}, i = 1, \ldots, m, s = 1, \ldots, r$ *and* $\lambda_j \geq 0, \lambda_j^s \in \mathbb{R}, j = 1, \ldots, n,$ $s = 1, \ldots, r$ *such that*

$$\sum_{i=1}^m \alpha_i\theta_1^i + \sum_{j=1}^n \lambda_j\theta_2^j = 0, \ \sum_{i=1}^m \alpha_i^s\theta_1^i + \sum_{j=1}^n \lambda_j^s\theta_2^j = 0, s = 1, \ldots, r, \tag{4}$$

$$\alpha_i A + \sum_{s=1}^r \alpha_i^s A_s \succeq 0, i = 1, \ldots, m, \ \lambda_j A + \sum_{s=1}^r \lambda_j^s A_s \succeq 0, j = 1, \ldots, n, \tag{5}$$

$$\begin{pmatrix} \mathcal{M}_1 & \frac{1}{2}\mathcal{M}_2 \\ \frac{1}{2}(\mathcal{M}_2)^\top & \mathcal{M}_3 \end{pmatrix} \succeq 0, \tag{6}$$

where $\mathcal{M}_1 := \sum_{i=1}^m \alpha_i Q_1^i + \sum\limits_{j=1}^n \lambda_j Q_2^j$, $\mathcal{M}_2 := \sum_{i=1}^m(\alpha_i\xi_1^i + \sum_{s=1}^r \alpha_i^s\xi_{1,s}^i) + \sum_{j=1}^n(\lambda_j\xi_2^j + \sum_{s=1}^r \lambda_j^s\xi_{2,s}^j)$ *and* $\mathcal{M}_3 := \sum_{i=1}^m(\alpha_i\beta_1^i + \sum_{s=1}^r \alpha_i^s\beta_{1,s}^i) + \sum\limits_{j=1}^n(\lambda_j\beta_2^j + \sum_{s=1}^r \lambda_j^s\beta_{2,s}^j) - \sum_{i=1}^m \alpha_i\mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y})$ *with* $\mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}) := \max_{v\in V}\{f_i(\overline{x}, v) + (\theta_1^i)^\top(\overline{y}^0 + \overline{Y}v)\}$ *for* $i = 1, \ldots, m$.
*(ii) (**Sufficient conditions for weak efficient solutions**) Let* $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+ \setminus \{0\}$, $\alpha_i^s \in \mathbb{R}, i = 1, \ldots, m, s = 1, \ldots, r$ *and* $\lambda_j \geq 0, \lambda_j^s \in \mathbb{R}, j = 1, \ldots, n, s = 1, \ldots, r$ *satisfy* (4)–(6)*. Then* $(\overline{x}, \overline{y}^0, \overline{Y})$ *is a weak efficient solution of* (RT).
*(iii) (**Sufficient conditons for efficient solutions**) Let* $(\alpha_1, \ldots, \alpha_m) \in \text{int}\mathbb{R}^m_+$, $\alpha_i^s \in \mathbb{R}, i = 1, \ldots, m, s = 1, \ldots, r$ *and* $\lambda_j \geq 0, \lambda_j^s \in \mathbb{R}, j = 1, \ldots, n, s = 1, \ldots, r$ *satisfy* (4)–(6)*. Then* $(\overline{x}, \overline{y}^0, \overline{Y})$ *is an efficient solution of* (RT).

Proof (i) Let the Slater qualification condition in (3) hold, and assume that $(\overline{x}, \overline{y}^0, \overline{Y})$ is a weak efficient solution of problem (RT). Denoting $G_j(x, y^0, Y) := \max\limits_{v\in V}\{g_j(x, v) + (\theta_2^j)^\top(y^0 + Yv)\}$ for $j = 1, \ldots, n$ and $(x, y^0, Y) \in \mathbb{R}^{q+p+p\times r}$, one can verify that

$$\Big\{(x, y^0, Y) \in \mathbb{R}^{q+p+p\times r} \ |\mathcal{F}_i(x, y^0, Y) - \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}) < 0, i = 1, \ldots, m, \\ G_j(x, y^0, Y) < 0, j = 1, \ldots, n\Big\} = \emptyset,$$

where $\mathcal{F}_i(x, y^0, Y) := \max\limits_{v\in V}\{f_i(x, v) + (\theta_1^i)^\top(y^0 + Yv)\}, i = 1, \ldots, m$ are defined as above. Since $\mathcal{F}_i, i = 1, \ldots, m$ and $G_j, j = 1, \ldots, n$ are convex functions with finite values on $\mathbb{R}^{q+p+p\times r}$, we invoke an alternative theorem in convex analysis (cf.

[29, Theorem 21.1]) to find $\alpha_i \geq 0, i = 1, \ldots, m, \lambda_j \geq 0, j = 1, \ldots, n$, not all zero, such that

$$\sum_{i=1}^{m} \alpha_i \big(\mathcal{F}_i(x, y^0, Y) - \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y})\big) + \sum_{j=1}^{n} \lambda_j G_j(x, y^0, Y) \geq 0 \tag{7}$$

for all $(x, y^0, Y) \in \mathbb{R}^{q+p+p\times r}$. If $\alpha_i = 0$ for all $i = 1, \ldots, m$, then there exists $j_0 \in \{1, \ldots, n\}$ such that $\lambda_{j_0} > 0$. Then, we get by (3) that $\sum_{j=1}^{n} \lambda_j G_j(\widehat{x}, \widehat{y}^0, \widehat{Y}) < 0$. This together with (7) establishes a contradiction. So, there exists $i_0 \in \{1, \ldots, m\}$ such that $\alpha_{i_0} > 0$.

Note that (7) can be rewritten as

$$\inf_{(x,y^0,Y)\in\mathbb{R}^{q+p+p\times r}} \Bigg\{ \sum_{i=1}^{m} \alpha_i \max_{v_1^i \in V} \{f_i(x, v_1^i) + (\theta_1^i)^\top (y^0 + Y v_1^i)\} + \sum_{j=1}^{n} \lambda_j \max_{v_2^j \in V} \{g_j(x, v_2^j) + (\theta_2^j)^\top (y^0 + Y v_2^j)\} \Bigg\} \geq \sum_{i=1}^{m} \alpha_i \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}),$$

which turns out to be the following inequality

$$\inf_{(x,y^0,Y)\in\mathbb{R}^{q+p+p\times r}} \max_{v_1^i\in V, i=1,\ldots,m, v_2^j \in V, j=1,\ldots,n} \Bigg\{ \sum_{i=1}^{m} \alpha_i \Big(f_i(x, v_1^i) + (\theta_1^i)^\top (y^0 + Y v_1^i)\Big) + \sum_{j=1}^{n} \lambda_j \big(g_j(x, v_2^j) + (\theta_2^j)^\top (y^0 + Y v_2^j)\big) \Bigg\} \geq \sum_{i=1}^{m} \alpha_i \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}). \tag{8}$$

Letting $\Omega := V^m \times V^n$ and considering a function $H : \mathbb{R}^{q+p+p\times r} \times \mathbb{R}^{(m+n)\times r} \to \mathbb{R}$ defined by

$$H(\widetilde{x}, \widetilde{v}) := \sum_{i=1}^{m} \alpha_i \big(f_i(x, v_1^i) + (\theta_1^i)^\top (y^0 + Y v_1^i)\big) + \sum_{j=1}^{n} \lambda_j \big(g_j(x, v_2^j) + (\theta_2^j)^\top (y^0 + Y v_2^j)\big)$$

for $\widetilde{x} := (x, y^0, Y) \in \mathbb{R}^{q+p+p\times r}$ and $\widetilde{v} := (v_1^1, \ldots, v_1^m, v_2^1, \ldots, v_2^n) \in \mathbb{R}^{(m+n)\times r}$, we see that Ω is a convex compact set in $\mathbb{R}^{(m+n)\times r}$, and H is a convex function with respect to $\widetilde{x}$ and is an affine function with respect to $\widetilde{v}$. Thus, we can apply a minimax theorem (see e.g., [30, Theorem 4.2]) to (8) and obtain that

$$\max_{\widetilde{v}\in\Omega} \inf_{\widetilde{x}\in\mathbb{R}^{q+p+p\times r}} H(\widetilde{x}, \widetilde{v}) = \inf_{\widetilde{x}\in\mathbb{R}^{q+p+p\times r}} \max_{\widetilde{v}\in\Omega} H(\widetilde{x}, \widetilde{v}) \geq \sum_{i=1}^{p} \alpha_i \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}).$$

This allows us to find $\widetilde{v}^* := (v_1^{1*}, \ldots, v_1^{m*}, v_2^{1*}, \ldots, v_2^{n*})$, where $v_1^{i*} := (v_{1,1}^{i*}, \ldots, v_{1,r}^{i*}) \in V$, $i = 1, \ldots, m$ and $v_2^{j*} := (v_{2,1}^{j*}, \ldots, v_{2,r}^{j*}) \in V$, $j = 1, \ldots, n$, such that

$$\inf_{\widetilde{x} \in \mathbb{R}^{q+p+p\times r}} H(\widetilde{x}, \widetilde{v}^*) \geq \sum_{i=1}^{p} \alpha_i \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}). \tag{9}$$

For $i \in \{1, \ldots, m\}$, let $\alpha_i^s := \alpha_i v_{1,s}^{i*}$, $s = 1, \ldots, r$. We get by $v_1^{i*} \in V$ that

$$\alpha_i A + \sum_{l=1}^{r} \alpha_i^l A_l = \alpha_i \big(A + \sum_{l=1}^{r} v_{1,l}^{i*} A_l\big) \succeq 0,$$

where it is noted that $\alpha_i \geq 0$. Similarly, by letting $\lambda_j^s := \lambda_j v_{2,s}^{j*}$, $s = 1, \ldots, r$ for $j = 1, \ldots, n$, we can verify that

$$\lambda_j A + \sum_{l=1}^{r} \lambda_j^l A_l \succeq 0, \ j = 1, \ldots, n.$$

Let $Y_1, \ldots, Y_r$ denote the columns of the matrix $Y \in \mathbb{R}^{p\times r}$. Then, it holds that $Y_s \in \mathbb{R}^p$ for all $s = 1, \ldots, r$ and

$$Y v_1^{i*} = \sum_{l=1}^{r} v_{1,l}^{i*} Y_l, \ \ i = 1, \ldots, m, \quad Y v_2^{j*} = \sum_{l=1}^{r} v_{2,l}^{j*} Y_l, \ \ j = 1, \ldots, n,$$

and so we can rewrite (9) as

$$h_1(x) + h_2(y^0, Y_1, \ldots, Y_r) \geq 0 \ \text{ for all } \ x \in \mathbb{R}^q, y^0 \in \mathbb{R}^p, Y_s \in \mathbb{R}^p, s = 1, \ldots, r, \tag{10}$$

where h_1 and h_2 are given respectively by

$$\begin{aligned} h_1(x) &:= x^\top \Big(\sum_{i=1}^{m} \alpha_i Q_1^i + \sum_{j=1}^{n} \lambda_j Q_2^j \Big) x \\ &+ \Bigg(\sum_{i=1}^{m} \Big(\alpha_i \xi_1^i + \sum_{l=1}^{r} \alpha_i^l \xi_{1,l}^i \Big) + \sum_{j=1}^{n} \Big(\lambda_j \xi_2^j + \sum_{s=1}^{r} \lambda_j^s \xi_{2,s}^j \Big) \Bigg)^\top x \\ &+ \sum_{i=1}^{m} \Big(\alpha_i \beta_1^i + \sum_{l=1}^{r} \alpha_i^l \beta_{1,l}^i \Big) + \sum_{j=1}^{n} \Big(\lambda_j \beta_2^j + \sum_{s=1}^{r} \lambda_j^s \beta_{2,s}^j \Big) - \sum_{i=1}^{m} \alpha_i \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}), \\ &= x^\top \mathcal{M}_1 x + (\mathcal{M}_2)^\top x + \mathcal{M}_3, \ \ x \in \mathbb{R}^n, \end{aligned} \tag{11}$$

$$h_2(y^0, Y_1, \ldots, Y_r) := \Big(\sum_{i=1}^{m} \alpha_i \theta_1^i + \sum_{j=1}^{n} \lambda_j \theta_2^j \Big)^\top y^0$$

$$+\sum_{l=1}^{r}\left(\sum_{i=1}^{m}\alpha_i^l\theta_1^i+\sum_{j=1}^{n}\lambda_j^l\theta_2^j\right)^\top Y_l,\, y^0\in\mathbb{R}^p,\quad Y_s\in\mathbb{R}^p, s=1,\ldots,r, \tag{12}$$

where $\mathcal{M}_1 := \sum_{i=1}^m \alpha_i Q_1^i + \sum_{j=1}^n \lambda_j Q_2^j$, $\mathcal{M}_2 := \sum_{i=1}^m(\alpha_i\xi_1^i + \sum_{s=1}^r \alpha_i^s \xi_{1,s}^i) + \sum_{j=1}^n(\lambda_j\xi_2^j + \sum_{s=1}^r \lambda_j^s\xi_{2,s}^j)$ and $\mathcal{M}_3 := \sum_{i=1}^m(\alpha_i\beta_1^i + \sum_{s=1}^r \alpha_i^s\beta_{1,s}^i) + \sum_{j=1}^n(\lambda_j\beta_2^j + \sum_{s=1}^r \lambda_j^s\beta_{2,s}^j) - \sum_{i=1}^m \alpha_i\mathcal{F}_i(\overline{x},\overline{y}^0,\overline{Y})$.

Since h_2 is a linear function, (10) entails that

$$\sum_{i=1}^{m}\alpha_i\theta_1^i+\sum_{j=1}^{n}\lambda_j\theta_2^j=0,\ \sum_{i=1}^{m}\alpha_i^l\theta_1^i+\sum_{j=1}^{n}\lambda_j^l\theta_2^j=0, l=1,\ldots,r$$

and

$$h_1(x)\geq 0 \text{ for all } x\in\mathbb{R}^n. \tag{13}$$

Note further that (13) can be written as the following matrix inequality (cf. [1, Simple lemma, p. 163]):

$$\begin{pmatrix}\mathcal{M}_1 & \frac{1}{2}\mathcal{M}_2\\ \frac{1}{2}(\mathcal{M}_2)^\top & \mathcal{M}_3\end{pmatrix}\succeq 0.$$

So the assertions (4)–(6) are valid, which substantiates (i).

(ii) Assume that there exist $(\alpha_1,\ldots,\alpha_m)\in\mathbb{R}_+^m\setminus\{0\}$, $\alpha_i^s\in\mathbb{R}, i=1,\ldots,m,$ $s=1,\ldots,r$ and $\lambda_j\geq 0, \lambda_j^s\in\mathbb{R}, j=1,\ldots,n,$ $s=1,\ldots,r$ such that (4)–(6) hold.

As shown in the proof of (i), (6) is equivalent to the inequalities $h_1(x)\geq 0$ for all $x\in\mathbb{R}^n$, where h_1 is given as in (11). Therefore, by taking (4) into account, we arrive at

$$h_1(x)+h_2(y^0,Y_1,\ldots,Y_r)\geq 0 \text{ for all } x\in\mathbb{R}^q, y^0\in\mathbb{R}^p, Y_s\in\mathbb{R}^p, s=1,\ldots,r, \tag{14}$$

where h_2 is given as in (12) and $Y_1,\ldots,Y_r$ stand for the columns of the matrix $Y\in\mathbb{R}^{p\times r}$.

Consider any $i\in\{1,\ldots,m\}$. As V is a compact set, we assert by (5) that if $\alpha_i=0$, then $\alpha_i^s=0$ for all $s=1,\ldots,r$. Indeed, suppose by contradiction that $\alpha_i=0$ and there exists $s_0\in\{1,\ldots,r\}$ such that $\alpha_i^{s_0}\neq 0$. Then, we get by (5) that $\sum_{l=1}^r\alpha_i^l A_l\succeq 0$. Taking $\overline{v}:=(\overline{v}_1,\ldots,\overline{v}_r)\in V$, we see that

$$A+\sum_{l=1}^{r}(\overline{v}_l+\gamma\alpha_i^l)A_l=\left(A+\sum_{l=1}^{r}\overline{v}_lA_l\right)+\gamma\sum_{l=1}^{r}\alpha_i^lA_l\succeq 0 \text{ for all } \gamma>0,$$

which shows that $\overline{v}+\gamma(\alpha_i^1,\ldots,\alpha_i^r)\in V$ for all $\gamma>0$. This contradicts the fact that $(\alpha_i^1,\ldots,\alpha_i^r)\neq 0$ and V is a bounded set. So our assertion above must be valid. Let us take $\widehat{v}^i := (\widehat{v}^i_1, \ldots, \widehat{v}^i_r) \in V$ and define $v^{i*}_1 := (v^{i*}_{1,1}, \ldots, v^{i*}_{1,r})$ by

$$v^{i*}_{1,s} := \begin{cases} \widehat{v}^i_s & \text{if } \alpha_i = 0, \\ \frac{\alpha^s_i}{\alpha_i} & \text{if } \alpha_i \neq 0, \end{cases} \quad s = 1, \ldots, r,$$

which by virtue of (5) shows that $v^{i*}_1 \in V$ and $\alpha^s_i = \alpha_i v^{i*}_{1,s}, s = 1, \ldots, r$. Similarly, we can derive from (5) and the bounded property of V that for each $j \in \{1, \ldots, n\}$, there exists $v^{j*}_2 := (v^{j*}_{2,1}, \ldots, v^{j*}_{2,r}) \in V$ such that $\lambda^s_j = \lambda_j v^{j*}_{2,s}, s = 1, \ldots, r$. Now, (14) is rewritten as

$$\sum_{i=1}^m \alpha_i \big(f_i(x, v^{i*}_1) + (\theta^i_1)^\top (y^0 + Y v^{i*}_1) \big) + \sum_{j=1}^n \lambda_j \big(g_j(x, v^{j*}_2) + (\theta^j_2)^\top (y^0 + Y v^{j*}_2) \big)$$
$$- \sum_{i=1}^m \alpha_i \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}) \geq 0 \text{ for all } x \in \mathbb{R}^q, y^0 \in \mathbb{R}^p, Y \in \mathbb{R}^{p\times r}. \tag{15}$$

To proceed, let $(\widehat{x}, \widehat{y}^0, \widehat{Y}) \in \mathbb{R}^{q+p+p\times r}$ be an arbitrary feasible point of problem (RT), i.e., $(\widehat{x}, \widehat{y}^0, \widehat{Y}) \in \mathcal{C}$. Then, $g_j(\widehat{x}, v^{j*}_2) + (\theta^j_2)^\top (\widehat{y}^0 + \widehat{Y} v^{j*}_2) \leq 0$ for $j = 1, \ldots, n$ and so we estimate (15) at $(\widehat{x}, \widehat{y}^0, \widehat{Y})$ to arrive at $\sum_{i=1}^m \alpha_i \big(f_i(\widehat{x}, v^{i*}_1) + (\theta^i_1)^\top (\widehat{y}^0 + \widehat{Y} v^{i*}_1) \big) \geq \sum_{i=1}^m \alpha_i \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y})$. This in turn entails that

$$\sum_{i=1}^m \alpha_i \mathcal{F}_i(\widehat{x}, \widehat{y}^0, \widehat{Y}) \geq \sum_{i=1}^m \alpha_i \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}) \tag{16}$$

inasmuch as $\alpha_i \geq 0$ and $\mathcal{F}_i(\widehat{x}, \widehat{y}^0, \widehat{Y}) := \max\limits_{v\in V} \{ f_i(\widehat{x}, v) + (\theta^i_1)^\top (\widehat{y}^0 + \widehat{Y} v) \}$ for all $i = 1, \ldots, m$. By virtue of $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+ \setminus \{0\}$, (16) implies that there does not exist $(\widehat{x}, \widehat{y}^0, \widehat{Y}) \in \mathcal{C}$ such that

$$\mathcal{F}_i(\widehat{x}, \widehat{y}^0, \widehat{Y}) < \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}), \ i = 1, \ldots, m,$$

which means that the triple $(\overline{x}, \overline{y}^0, \overline{Y})$ is a weak efficient solution of (RT).

(iii) Let $(\alpha_1, \ldots, \alpha_m) \in \text{int}\mathbb{R}^m_+$, $\alpha^s_i \in \mathbb{R}, i = 1, \ldots, m, s = 1, \ldots, r$ and $\lambda_j \geq 0, \lambda^s_j \in \mathbb{R}, j = 1, \ldots, n, s = 1, \ldots, r$ be such that (4)–(6) hold. Similarly to the proof of (ii), we are able to arrive at the assertion in (16). Granting this, we conclude by $(\alpha_1, \ldots, \alpha_m) \in \text{int}\mathbb{R}^m_+$ that $(\overline{x}, \overline{y}^0, \overline{Y})$ is an efficient solution of problem (RT), which completes the proof of the theorem. □

In the following example, we show that the Slater qualification condition (3) is *essential* for obtaining the neccessary LMI optimality in (i) of Theorem 2.1.

Example 2.1 (The role of Slater qualification) Let us consider an *uncertain two-stage* multiobjective problem:

$$\min_{x,y} \big\{ (x_1 + x_2^2 - v_1, 3x_1 + v_2) \mid v_1 x_1 + (v_2 + 2) x_2 \leq 0, \ x_1 + \theta^\top y(v) \leq 1 \big\}, \tag{E1}$$

where $\theta := (2, 2)$ is fixed, $x := (x_1, x_2) \in \mathbb{R}^2$ is the first-stage variable, y is the second-stage variable, $v := (v_1, v_2) \in \mathbb{R}^2$ is an uncertain parameter, which resides in an uncertainty set V. In this setting, we assume that the uncertainty set V is defined by

$$V := \{v := (v_1, v_2) \in \mathbb{R}^2 \mid \frac{v_1^2}{5} + \frac{v_2^2}{4} \leq 1\}$$

and the second-stage variable y is given by

$$y(v) := y^0 + Yv, \;\; v \in V,$$

where $y^0 \in \mathbb{R}^2$ and $Y \in \mathbb{R}^{2\times 2}$ are nonadjustable variables.

Consider now a *robust* counterpart of problem (E1) that is given by

$$\min_{x\in\mathbb{R}^2, y^0\in\mathbb{R}^2, Y\in\mathbb{R}^{2\times 2}} \big\{\big(\max_{v\in V}\{x_1 + x_2^2 - v_1\}, \max_{v\in V}\{3x_1 + v_2\}\big) \mid v_1x_1 + (v_2+2)x_2 \leq 0, \quad x_1 + \theta^\top y(v) \leq 1, \, y(v) = y^0 + Yv, \;\; \forall v \in V\big\}. \tag{R1}$$

Clearly, the problem (R1) is of the form of problem (RT), where $\theta_1^1 := \theta_1^2 := \theta_2^1 := 0_2$, $\theta_2^2 := \theta$, $f_i : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, i = 1, 2, g_j : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, j = 1, 2$ are defined by $Q_1^1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $Q_1^2 := 0_{2\times 2}$, $\xi_1^1 := (1, 0)$, $\xi_1^2 := (3, 0)$, $\xi_{1,l}^1 := \xi_{1,l}^2 := 0_2, l = 1, 2$, $\beta_1^1 := \beta_1^2 := \beta_{1,2}^1 := \beta_{1,1}^2 := 0$, $\beta_{1,1}^1 := -1$, $\beta_{1,2}^2 := 1$ and $Q_2^1 := Q_2^2 := 0_{2\times 2}$, $\xi_2^1 := (0, 2)$, $\xi_{2,1}^2 := \xi_{2,2}^2 := 0_2$, $\xi_{2,1}^1 := \xi_2^2 := (1, 0)$, $\xi_{2,2}^1 := (0, 1)$, $\beta_2^1 := \beta_{2,1}^1 := \beta_{2,2}^1 := \beta_{2,1}^2 := \beta_{2,2}^2 := 0$, $\beta_2^2 := -1$, and V is a spectrahedron described by

$$A := \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_1 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, A_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Denoting $\overline{x} := \overline{y} := 0_2$ and $\overline{Y} := 0_{2\times 2}$, we claim that $(\overline{x}, \overline{y}^0, \overline{Y})$ an *efficient solution* of (R1). Otherwise, we would find a feasible point of (R1), denoted by $(\widetilde{x}, \widetilde{y}^0, \widetilde{Y})$, such that

$$\max_{v\in V}\{\widetilde{x}_1 + \widetilde{x}_2^2 - v_1\} \leq \max_{v\in V}\{\overline{x}_1 + \overline{x}_2^2 - v_1\}, \;\max_{v\in V}\{3\widetilde{x}_1 + v_2\} \leq \max_{v\in V}\{3\overline{x}_1 + v_2\}, \tag{17}$$

$$\big(\max_{v\in V}\{\widetilde{x}_1 + \widetilde{x}_2^2 - v_1\}, \max_{v\in V}\{3\widetilde{x}_1 + v_2\}\big) \neq \big(\max_{v\in V}\{\overline{x}_1 + \overline{x}_2^2 - v_1\}, \max_{v\in V}\{3\overline{x}_1 + v_2\}\big). \tag{18}$$

As $(\widetilde{x}, \widetilde{y}^0, \widetilde{Y})$ is a feasible point of problem (R1), it follows that $v_1\widetilde{x}_1 + (v_2+2)\widetilde{x}_2 \leq 0$ for all $v := (v_1, v_2) \in V$. This guarantees that $\widetilde{x}_1 = 0$ and $\widetilde{x}_2 \leq 0$. Hence, we get by (17) that $\widetilde{x}_2 = 0$, which contradicts (18).

In this setting, we show that the assertions in (4)–(6) are *not valid* at $(\overline{x}, \overline{y}^0, \overline{Y})$. Otherwise, one can find $(\alpha_1, \alpha_2) \in \mathbb{R}^2_+\backslash\{0\}$, $\alpha_i^r \in \mathbb{R}$, $i = 1, 2$, $r = 1, 2$ and $\lambda_j \geq 0$, $\lambda_j^r \in \mathbb{R}$, $j = 1, 2$, $r = 1, 2$ such that

$$\sum_{i=1}^{2} \alpha_i \theta_1^i + \sum_{j=1}^{2} \lambda_j \theta_2^j = 0,\ \sum_{i=1}^{2} \alpha_i^l \theta_1^i + \sum_{j=1}^{2} \lambda_j^l \theta_2^j = 0, l = 1, 2, \tag{19}$$

$$\alpha_i A + \sum_{l=1}^{2} \alpha_i^l A_l \succeq 0, i = 1, 2,\ \lambda_j A + \sum_{l=1}^{2} \lambda_j^l A_l \succeq 0,\ j = 1, 2, \tag{20}$$

$$\begin{pmatrix} \mathcal{M}_1 & \frac{1}{2}\mathcal{M}_2 \\ \frac{1}{2}(\mathcal{M}_2)^\top & \mathcal{M}_3 \end{pmatrix} \succeq 0, \tag{21}$$

where $\mathcal{M}_1 := \sum_{i=1}^2 \alpha_i Q_1^i + \sum_{j=1}^2 \lambda_j Q_2^j$, $\mathcal{M}_2 := \sum_{i=1}^2(\alpha_i \xi_1^i + \sum_{l=1}^2 \alpha_i^l \xi_{1,l}^i) + \sum_{j=1}^2(\lambda_j \xi_2^j + \sum_{r=1}^2 \lambda_j^r \xi_{2,r}^j)$ and $\mathcal{M}_3 := \sum_{i=1}^2(\alpha_i \beta_1^i + \sum_{l=1}^2 \alpha_i^l \beta_{1,l}^i) + \sum_{j=1}^2(\lambda_j \beta_2^j + \sum_{r=1}^2 \lambda_j^r \beta_{2,r}^j) - \sum_{i=1}^2 \alpha_i \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y})$ with $\mathcal{F}_1(\overline{x}, \overline{y}^0, \overline{Y}) := \max_{v \in V}\{\overline{x}_1 + \overline{x}_2^2 - v_1\} = \sqrt{5}$ and $\mathcal{F}_2(\overline{x}, \overline{y}^0, \overline{Y}) := \max_{v \in V}\{3\overline{x}_1 + v_2\} = 2$. Observe that (20) is equivalent to the following inequalities

$$\frac{(\alpha_i^1)^2}{5} + \frac{(\alpha_i^2)^2}{4} \leq (\alpha_i)^2, i = 1, 2,\ \frac{(\lambda_j^1)^2}{5} + \frac{(\lambda_j^2)^2}{4} \leq (\lambda_j)^2,\ j = 1, 2. \tag{22}$$

We get by (21) that

$$\alpha_1 x_2^2 + (\alpha_1 + 3\alpha_2 + \lambda_1^1 + \lambda_2)x_1 + (2\lambda_1 + \lambda_1^2)x_2 - \alpha_1^1 + \alpha_2^2 - \lambda_2 - \sqrt{5}\alpha_1 - 2\alpha_2 \geq 0$$

for all $x_1 \in \mathbb{R}$ and all $x_2 \in \mathbb{R}$. This implies, in particular, that

$$(\alpha_1 + 3\alpha_2 + \lambda_1^1 + \lambda_2)x_1 - \alpha_1^1 + \alpha_2^2 - \lambda_2 - \sqrt{5}\alpha_1 - 2\alpha_2 \geq 0 \text{ for all } x_1 \in \mathbb{R}, \tag{23}$$

$$\alpha_1 x_2^2 + (2\lambda_1 + \lambda_1^2)x_2 - \alpha_1^1 + \alpha_2^2 - \lambda_2 - \sqrt{5}\alpha_1 - 2\alpha_2 \geq 0 \text{ for all } x_2 \in \mathbb{R}. \tag{24}$$

Observe now by (23) that $\alpha_1 + 3\alpha_2 + \lambda_1^1 + \lambda_2 = 0$ and

$$-\alpha_1^1 + \alpha_2^2 - \lambda_2 - \sqrt{5}\alpha_1 - 2\alpha_2 \geq 0.$$

Furthermore, it is easy to see that $-\alpha_1^1 - \sqrt{5}\alpha_1 \leq |\alpha_1^1| - \sqrt{5}\alpha_1 \leq 0$, where the validation of the last inequality is due to (22). Similarly, it holds that $\alpha_2^2 - 2\alpha_2 \leq 0$, and so we arrive at

$$-\alpha_1^1 + \alpha_2^2 - \lambda_2 - \sqrt{5}\alpha_1 - 2\alpha_2 \leq 0$$

by virtue of $\lambda_2 \geq 0$. Consequently, $-\alpha_1^1 + \alpha_2^2 - \lambda_2 - \sqrt{5}\alpha_1 - 2\alpha_2 = 0$. This, together with (24), ensures that $2\lambda_1 + \lambda_1^2 = 0$.

Next, we substitute $\lambda_1^1 = -\lambda_2 - \alpha_1 - 3\alpha_2$ and $\lambda_1^2 = -2\lambda_1$ into (22), we arrive at $(\lambda_2 + \alpha_1 + 3\alpha_2)^2 \leq 0$, which is impossible as $(\alpha_1, \alpha_2) \in \mathbb{R}^2_+ \backslash \{0\}$ and $\lambda_2 \geq 0$.

In conlusion, the linear matrix inequality conditions in (19)–(21) *go awry* for the above Pareto solution $(\overline{x}, \overline{y}^0, \overline{Y})$. This is because the Slater qualification condition (3) is *violated* for this setting.

Second-Order Cone Optimality Conditions. We now consider a special setting of the robust two-stage multiobjective optimization problem (RT), where the objectives $f_i, i = 1, \ldots, m$ and the constraints $g_j, j = 1, \ldots, n$ are *separable* quadratic functions in the first-state variable defined by, for $x := (x_1, \ldots, x_q) \in \mathbb{R}^q$ and $v := (v_1, \ldots, v_r) \in V$,

$$\begin{aligned} f_i(x, v) &:= \sum_{k=1}^{q} \omega^i_{1,k} x_k^2 + (\xi^i_1)^\top x + \beta^i_1 + \sum_{l=1}^{r} v_l \big((\xi^i_{1,l})^\top x + \beta^i_{1,l} \big), \\ g_j(x, v) &:= \sum_{k=1}^{q} \omega^j_{2,k} x_k^2 + (\xi^j_2)^\top x + \beta^j_2 + \sum_{l=1}^{r} v_l \big((\xi^j_{2,l})^\top x + \beta^j_{2,l} \big) \end{aligned} \tag{25}$$

with $\omega^i_{1,k} \geq 0$, $\omega^j_{2,k} \geq 0$, $\xi^i_1 := (\xi^{i,1}_1, \ldots, \xi^{i,q}_1) \in \mathbb{R}^q$, $\xi^i_{1,l} := (\xi^{i,1}_{1,l}, \ldots, \xi^{i,q}_{1,l}) \in \mathbb{R}^q$, $\xi^j_2 := (\xi^{j,1}_2, \ldots, \xi^{j,q}_2) \in \mathbb{R}^q$, $\xi^j_{2,l} := (\xi^{j,1}_{2,l}, \ldots, \xi^{j,q}_{2,l}) \in \mathbb{R}^q$, $\beta^i_1 \in \mathbb{R}$, $\beta^i_{1,l} \in \mathbb{R}$, $\beta^j_2 \in \mathbb{R}$, $\beta^j_{2,l} \in \mathbb{R}$, $k = 1, \ldots, q$, $i = 1, \ldots, m$, $l = 1, \ldots, r$, $j = 1, \ldots, n$ fixed, and the uncertainty set V is given by the following *ellipsoid*

$$V := \left\{ v \in \mathbb{R}^r \mid v^\top E v \leq 1 \right\} \tag{26}$$

with a symmetric $(r \times r)$ matrix $E \succ 0$. Let E^d be an $(r \times r)$ matrix such that $E = (E^d)^\top E^d$.

In this framework, we obtain necessary/sufficient optimality conditions by way of *second-order cone* (SOC) expressions for the robust two-stage multiobjective problem (RT).

Corollary 2.1 (SOC optimality) *Consider the problem* (RT)*, where* $f_i, i = 1, \ldots, m$ *and* $g_j, j = 1, \ldots, n$ *in* (1) *are replaced by those in* (25)*, and* V *in* (2) *is replaced by the one in* (26)*. Let* $(\overline{x}, \overline{y}^0, \overline{Y}) \in \mathbb{R}^{q+p+p\times r}$ *be a feasible point of this problem.*
*(i) (**Necessary optimality**) Assume that the Slater qualification condition* (3) *holds for this setting. If* $(\overline{x}, \overline{y}^0, \overline{Y})$ *is a weak efficient solution of this problem, then there exist* $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+ \setminus \{0\}$, $\alpha^s_i \in \mathbb{R}, i = 1, \ldots, m, s = 1, \ldots, r$, $\lambda_j \geq 0, \lambda^s_j \in \mathbb{R}$, $j = 1, \ldots, n, s = 1, \ldots, r$ *and* $t_k \geq 0, k = 1, \ldots, q$ *such that*

$$\sum_{i=1}^{m} \alpha_i \theta^i_1 + \sum_{j=1}^{n} \lambda_j \theta^j_2 = 0, \ \sum_{i=1}^{m} \alpha^l_i \theta^i_1 + \sum_{j=1}^{n} \lambda^l_j \theta^j_2 = 0, l = 1, \ldots, r, \tag{27}$$

$$\left\| E^d \begin{pmatrix} \alpha^1_i \\ \vdots \\ \alpha^r_i \end{pmatrix} \right\| \leq \alpha_i, i = 1, \ldots, m, \ \left\| E^d \begin{pmatrix} \lambda^1_j \\ \vdots \\ \lambda^r_j \end{pmatrix} \right\| \leq \lambda_j, j = 1, \ldots, n, \tag{28}$$

$$\sum_{i=1}^{m}\left(\alpha_i\beta_1^i+\sum_{l=1}^{r}\alpha_i^l\beta_{1,l}^i\right)+\sum_{j=1}^{n}\left(\lambda_j\beta_2^j+\sum_{s=1}^{r}\lambda_j^s\beta_{2,s}^j\right)$$
$$-\sum_{i=1}^{m}\alpha_i\mathcal{F}_i(\overline{x},\overline{y}^0,\overline{Y})-\sum_{k=1}^{q}t_k\geq 0, \tag{29}$$
$$\left\|\left(t_k-\left(\sum_{i=1}^{m}\alpha_i\omega_{1,k}^i+\sum_{j=1}^{n}\lambda_j\omega_{2,k}^j\right),\sum_{i=1}^{m}\left(\alpha_i\xi_1^{i,k}+\sum_{l=1}^{r}\alpha_i^l\xi_{1,l}^{i,k}\right)\right.\right.$$
$$\left.\left.+\sum_{j=1}^{n}\left(\lambda_j\xi_2^{j,k}+\sum_{s=1}^{r}\lambda_j^s\xi_{2,s}^{j,k}\right)\right)\right\|\leq$$
$$t_k+\sum_{i=1}^{m}\alpha_i\omega_{1,k}^i+\sum_{j=1}^{n}\lambda_j\omega_{2,k}^j,\ k=1,\ldots,q, \tag{30}$$

where $\mathcal{F}_i(\overline{x},\overline{y}^0,\overline{Y}):=\max_{v\in V}\{f_i(\overline{x},v)+(\theta_1^i)^\top(\overline{y}^0+\overline{Y}v)\}$ *for* $i=1,\ldots,m$.
*(ii) (**Sufficient conditions for weak efficient solutions**) Let* $(\alpha_1,\ldots,\alpha_m)\in\mathbb{R}_+^m\setminus\{0\}$, $\alpha_i^s\in\mathbb{R}, i=1,\ldots,m, s=1,\ldots,r,\ \lambda_j\geq 0, \lambda_j^s\in\mathbb{R}, j=1,\ldots,n, s=1,\ldots,r$ *and* $t_k\geq 0, k=1,\ldots,q$ *satisfy* (27)–(30). *Then, we assert that* $(\overline{x},\overline{y}^0,\overline{Y})$ *is a weak efficient solution of this problem.*
*(iii) (**Sufficient conditions for efficient solutions**) Let* $(\alpha_1,\ldots,\alpha_m)\in\text{int}\mathbb{R}_+^m$, $\alpha_i^s\in\mathbb{R}, i=1,\ldots,m, s=1,\ldots,r,\ \lambda_j\geq 0, \lambda_j^s\in\mathbb{R}, j=1,\ldots,n, s=1,\ldots,r$ *and* $t_k\geq 0, k=1,\ldots,q$ *satisfy* (27)–(30). *Then* $(\overline{x},\overline{y}^0,\overline{Y})$ *is an efficient solution of this problem.*

Proof Considering $(r+1)\times(r+1)$ matrices A and A_l for $l=1,...,r$ as

$$A:=\begin{pmatrix}E^{-1}&0\\0&1\end{pmatrix},\quad A_l:=\begin{pmatrix}0&e_l^r\\(e_l^r)^\top&0\end{pmatrix},\ l=1,\ldots,r, \tag{31}$$

it holds that the ellipsoid in (26) is in the form of (2) with $A, A_l, l=1,...,r$ in (31). In this case, for $(\alpha_1,\ldots,\alpha_m)\in\mathbb{R}_+^m\setminus\{0\}$, $\alpha_i^s\in\mathbb{R}, i=1,\ldots,m, s=1,\ldots,r$, one can check that

$$\alpha_iA+\sum_{l=1}^{r}\alpha_i^lA_l\succeq 0\Leftrightarrow\left\|E^d\begin{pmatrix}\alpha_i^1\\\vdots\\\alpha_i^r\end{pmatrix}\right\|\leq\alpha_i.$$

Similarly, for $\lambda_j\geq 0, \lambda_j^s\in\mathbb{R}, j=1,\ldots,n, s=1,\ldots,r$, we have

$$\lambda_jA+\sum_{l=1}^{r}\lambda_j^lA_l\succeq 0\Leftrightarrow\left\|E^d\begin{pmatrix}\lambda_j^1\\\vdots\\\lambda_j^r\end{pmatrix}\right\|\leq\lambda_j.$$

Note that our problem here is a particular case of problem (RT), where $Q_1^i := \operatorname{diag}(\omega_{1,1}^i, \ldots, \omega_{1,q}^i), i = 1, \ldots, m$, $Q_2^j := \operatorname{diag}(\omega_{2,1}^j, \ldots, \omega_{2,q}^j)$, $j = 1, \ldots, n$, and V is defined by the matrices A, A_l, $l = 1, \ldots, r$ in (31).

Now, in view of Theorem 2.1, the proof will be completed if we can show that (6) in this case is equivalent to (29) and (30) for some $t_k \geq 0$, $k = 1, \ldots, q$.

To this end, we first assume that there exist $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}_+^m \setminus \{0\}$, $\alpha_i^s \in \mathbb{R}$, $i = 1, \ldots, m$, $s = 1, \ldots, r$ and $\lambda_j \geq 0$, $\lambda_j^s \in \mathbb{R}$, $j = 1, \ldots, n$, $s = 1, \ldots, r$ such that (6) holds with $Q_1^i := \operatorname{diag}(\omega_{1,1}^i, \ldots, \omega_{1,q}^i)$, $i = 1, \ldots, m$ and $Q_2^j := \operatorname{diag}(\omega_{2,1}^j, \ldots, \omega_{2,q}^j)$, $j = 1, \ldots, n$. We show that there exist $t_k \geq 0$, $k = 1, \ldots, q$ such that (29) and (30) hold. To see this, we consider two possibilities as below.

Case 1: $\sum\limits_{i=1}^m \alpha_i \omega_{1,k}^i + \sum\limits_{j=1}^n \lambda_j \omega_{2,k}^j = 0$ for all $k = 1, \ldots, q$. In this case, (6) entails that

$$\sum_{i=1}^m \left(\alpha_i \xi_1^i + \sum_{l=1}^r \alpha_i^l \xi_{1,l}^i\right) + \sum_{j=1}^n \left(\lambda_j \xi_2^j + \sum_{s=1}^r \lambda_j^s \xi_{2,s}^j\right) = 0,$$

$$\sum_{i=1}^m \left(\alpha_i \beta_1^i + \sum_{l=1}^r \alpha_i^l \beta_{1,l}^i\right) + \sum_{j=1}^n \left(\lambda_j \beta_2^j + \sum_{s=1}^r \lambda_j^s \beta_{2,s}^j\right) - \sum_{i=1}^m \alpha_i \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}) \geq 0,$$

which shows that (29) and (30) hold by choosing $t_k := 0$, $k = 1, \ldots, q$.

Case 2: There exists $k \in \{1, \ldots, q\}$ such that $\sum_{i=1}^m \alpha_i \omega_{1,k}^i + \sum_{j=1}^n \lambda_j \omega_{2,k}^j > 0$. In this case, we denote $\mathcal{I} := \{k \in \{1, \ldots, q\} \mid \sum_{i=1}^m \alpha_i \omega_{1,k}^i + \sum_{j=1}^n \lambda_j \omega_{2,k}^j > 0\}$, which is a nonempty set. Observe similarly as in *Case 1* that if $\sum_{i=1}^m \alpha_i \omega_{1,k}^i + \sum_{j=1}^n \lambda_j \omega_{2,k}^j = 0$ (i.e., $k \in \{1, \ldots, q\} \setminus \mathcal{I}$), then (6) guarantees that $\sum_{i=1}^m (\alpha_i \xi_1^{i,k} + \sum_{l=1}^r \alpha_i^l \xi_{1,l}^{i,k}) + \sum_{j=1}^n (\lambda_j \xi_2^{j,k} + \sum_{s=1}^r \lambda_j^s \xi_{2,s}^{j,k}) = 0$. For each $k \in \{1, \ldots, q\}$, put

$$t_k := \begin{cases} \dfrac{1}{4\left(\sum\limits_{i=1}^m \alpha_i \omega_{1,k}^i + \sum\limits_{j=1}^n \lambda_j \omega_{2,k}^j\right)} \left(\sum\limits_{i=1}^m (\alpha_i \xi_1^{i,k} + \sum\limits_{l=1}^r \alpha_i^l \xi_{1,l}^{i,k}) + \sum\limits_{j=1}^n \left(\lambda_j \xi_2^{j,k} + \sum\limits_{s=1}^r \lambda_j^s \xi_{2,s}^{j,k}\right)\right)^2 & \text{if } k \in \mathcal{I}, \\ 0 & \text{otherwise.} \end{cases}$$

We see that $t_k \geq 0$ and

$$\frac{1}{4}\left(\sum_{i=1}^m (\alpha_i \xi_1^{i,k} + \sum_{l=1}^r \alpha_i^l \xi_{1,l}^{i,k}) + \sum_{j=1}^n (\lambda_j \xi_2^{j,k} + \sum_{s=1}^r \lambda_j^s \xi_{2,s}^{j,k})\right)^2 \leq t_k \left(\sum_{i=1}^m \alpha_i \omega_{1,k}^i + \sum_{j=1}^n \lambda_j \omega_{2,k}^j\right) \quad (32)$$

for $k = 1, \ldots, q$. Letting $a := \sum_{i=1}^m \alpha_i \omega_{1,k}^i + \sum_{j=1}^n \lambda_j \omega_{2,k}^j$ and $b := \frac{1}{2}\Big(\sum_{i=1}^m (\alpha_i \xi_1^{i,k} + \sum_{l=1}^r \alpha_i^l \xi_{1,l}^{i,k}) + \sum_{j=1}^n (\lambda_j \xi_2^{j,k} + \sum_{s=1}^r \lambda_j^s \xi_{2,s}^{j,k})\Big)$, we rewrite the inequalities in (30) as

$$\|(t_k - a, 2b)\| \leq t_k + a, \ \ k = 1, \ldots, q. \quad (33)$$

As $a \geq 0$ and $t_k \geq 0, k = 1, \ldots, q$, the inequalities in (33) are equivalent to the following ones:

$$b^2 \leq t_k a, \ k = 1, \ldots, q.$$

This shows that (32) and (30) are equivalent to each other. Denote $v_{\mathcal{I}} := \big(\sum_{i=1}^m (\alpha_i \xi_1^{i,k} + \sum_{l=1}^r \alpha_i^l \xi_{1,l}^{i,k}) + \sum_{j=1}^n (\lambda_j \xi_2^{j,k} + \sum_{s=1}^r \lambda_j^s \xi_{2,s}^{j,k}), k \in \mathcal{I}\big)$ and consider a diagonal $(|\mathcal{I}| \times |\mathcal{I}|)$ matrix $M_{\mathcal{I}} := \operatorname{diag}(\sum_{i=1}^m \alpha_i \omega_{1,k}^i + \sum_{j=1}^n \lambda_j \omega_{2,k}^j, k \in \mathcal{I})$. We get by (6) that

$$\begin{pmatrix} M_{\mathcal{I}} & \frac{1}{2} v_{\mathcal{I}} \\ \frac{1}{2} v_{\mathcal{I}}^\top & \sum_{i=1}^m (\alpha_i \beta_1^i + \sum_{l=1}^r \alpha_i^l \beta_{1,l}^i) + \sum_{j=1}^n (\lambda_j \beta_2^j + \sum_{s=1}^r \lambda_j^s \beta_{2,s}^j) - \sum_{i=1}^m \alpha_i \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}) \end{pmatrix} \succeq 0. \quad (34)$$

By $M_{\mathcal{I}} \succ 0$, (34) reduces to the following inequality (cf. [1, Lemma 4.2.1]):

$$\begin{aligned} &\sum_{i=1}^m \left(\alpha_i \beta_1^i + \sum_{l=1}^r \alpha_i^l \beta_{1,l}^i \right) + \sum_{j=1}^n \left(\lambda_j \beta_2^j + \sum_{s=1}^r \lambda_j^s \beta_{2,s}^j \right) - \sum_{i=1}^m \alpha_i \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}) \\ &\geq \frac{1}{4} v_{\mathcal{I}}^\top M_{\mathcal{I}}^{-1} v_{\mathcal{I}} = \sum_{k=1}^q t_k, \end{aligned}$$

which concludes that (29) holds.

Therefore, in all cases, we have shown that (6) under the current setting implies (29) and (30). The converse implication is similarly proceeded and so we omit it. □

3 Finding Pareto Solutions by Semidefinite Programming Relaxations

This section is devoted to showing how (weak) efficient solutions of our two-stage multiobjective program (RT) can be calculated by using associated *semidefinite programming* (SDP) or *second-order cone programming* (SOCP) relaxations.

Semidefinite Programming Relaxations. For each $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}_+^m \setminus \{0\}$, one considers a *robust two-stage* scalarized program of (UT) defined by

$$\begin{aligned} \min_{x \in \mathbb{R}^q, y^0 \in \mathbb{R}^p, Y \in \mathbb{R}^{p \times r}} &\left\{ \sum_{i=1}^m \alpha_i \max_{v \in V} \{ f_i(x, v) + (\theta_1^i)^\top y(v) \} \mid g_j(x, v) + (\theta_2^j)^\top y(v) \leq 0, \right. \quad (\text{P}_\alpha) \\ &\left. j = 1, \ldots, n, \quad y(v) = y^0 + Yv, \forall v \in V \right\}, \end{aligned}$$

where $f_i, i = 1, \ldots, m$ and $g_j, j = 1, \ldots, n$ are given as in (1), V is given as in (2), and $\theta_1^i, i = 1, \ldots, m$ and $\theta_2^j, j = 1, \ldots, n$ are given as in the definition of (UT).

We now address a semi-definite programming (SDP) relaxation program for (P_α) that is defined by

$$\min_{(z,\nu_0,\nu^l,Z,W_1^i,W_2^j)} \sum_{i=1}^{m} \alpha_i \big(\beta_1^i + (\xi_1^i)^\top z + (\theta_1^i)^\top \nu_0 + \mathrm{Tr}(ZQ_1^i) + \mathrm{Tr}(W_1^i A)\big) \tag{P_α^*}$$
$$\text{s.t.} \begin{pmatrix} 1 & z^\top \\ z & Z \end{pmatrix} \succeq 0, z \in \mathbb{R}^q, Z \in S^q, \nu_0 \in \mathbb{R}^p, \nu^l \in \mathbb{R}^p, l = 1, \ldots, r,$$
$$\beta_{1,l}^i + (\theta_1^i)^\top \nu^l + (\xi_{1,l}^i)^\top z + \mathrm{Tr}(W_1^i A_l) = 0, i = 1, \ldots, m, l = 1, \ldots, r,$$
$$\beta_2^j + (\xi_2^j)^\top z + (\theta_2^j)^\top \nu_0 + \mathrm{Tr}(ZQ_2^j) + \mathrm{Tr}(W_2^j A) \le 0, j = 1, \ldots, n,$$
$$\beta_{2,l}^j + (\theta_2^j)^\top \nu^l + (\xi_{2,l}^j)^\top z + \mathrm{Tr}(W_2^j A_l) = 0, j = 1, \ldots, n, l = 1, \ldots, r,$$
$$W_1^i \succeq 0, W_1^i \in S^{m_0}, W_2^j \succeq 0, W_2^j \in S^{m_0}, i = 1, \ldots, m, j = 1, \ldots, n.$$

The forthcoming theorem describes relationships between (weak) efficient solutions of the two-stage multiobjective problem (RT) and optimal solutions of the SDP problem (P_α^*), which is a relaxation of the two-stage scalarized problem (P_α). This provides a method to calculate (weak) efficient solutions of the robust two-stage multiobjective program (RT) by solving related (scalar) semidefinite programming relaxation problems (P_α^*) with $\alpha \in \mathbb{R}_+^m \setminus \{0\}$.

Theorem 3.1 (Calculating Solutions via Semidefinite Programming Relaxations) *For the problem* (RT)*, suppose that there is* $\widehat{v} := (\widehat{v}_1, \ldots, \widehat{v}_r) \in \mathbb{R}^r$ *such that*

$$A + \sum_{l=1}^{r} \widehat{v}_l A_l \succ 0. \tag{1}$$

Then, the following statements are valid.
(i) *Let the Slater qualification condition in* (3) *hold and assume that* $(\overline{x}, \overline{y}^0, \overline{Y})$ *is a weak efficient solution of* (RT). *Then one can find* $\alpha \in \mathbb{R}_+^m \setminus \{0\}$, $\overline{W}_1^i \in S^{m_0}, i = 1, \ldots, m$ *and* $\overline{W}_2^j \in S^{m_0}, j = 1, \ldots, n$ *such that* $(\overline{x}, \overline{y}^0, \overline{Y}_l, \overline{Z}, \overline{W}_1^i, \overline{W}_2^j)_{i=1,\ldots,m, j=1,\ldots,n, l=1,\ldots,r}$ *is a solution of* (P_α^*)*, where* $\overline{Z} := \overline{x}\overline{x}^\top$ *and* $\overline{Y}_l, l = 1, \ldots, r$ *are the columns of the matrix* $\overline{Y}$.
(ii) **(Finding weak efficient solutions)** *Suppose that the problem* (P_α) *admits a solution for* $\alpha \in \mathbb{R}_+^m \setminus \{0\}$ *and let* $(\overline{z}, \overline{\nu}_0, \overline{\nu}^l, \overline{Z}, \overline{W}_1^i, \overline{W}_2^j)_{i=1,\ldots,m, j=1,\ldots,n, l=1,\ldots,r}$ *be a solution of* (P_α^*). *Then* $(\overline{z}, \overline{\nu}_0, \overline{Y})$ *is a weak efficient solution of* (RT)*, where* $\overline{Y} := (\overline{\nu}^1, \ldots, \overline{\nu}^r)$ *is a matrix whose columns are those of* $\overline{\nu}^l, l = 1, \ldots, r$.
(iii) **(Finding efficient solutions)** *Suppose that the problem* (P_α) *admits a solution for* $\alpha \in \mathrm{int}\mathbb{R}_+^m$ *and let* $(\overline{z}, \overline{\nu}_0, \overline{\nu}^l, \overline{Z}, \overline{W}_1^i, \overline{W}_2^j)_{i=1,\ldots,m, j=1,\ldots,n, l=1,\ldots,r}$ *be a solution of* (P_α^*). *Then* $(\overline{z}, \overline{\nu}_0, \overline{Y})$ *is an efficient solution of* (RT)*, where* $\overline{Y} := (\overline{\nu}^1, \ldots, \overline{\nu}^r)$ *is a matrix whose columns are those of* $\overline{\nu}^l, l = 1, \ldots, r$.

Proof Let $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+ \setminus \{0\}$ be such that the two-stage scalarized problem (P_α) possesses a solution and assume that $(\overline{x}, \overline{y}^0, \overline{Y})$ is a solution of this problem. Let $\mathrm{val}(\mathrm{P}_\alpha)$ and $\mathrm{val}(\mathrm{P}^*_\alpha)$ denote the optimal values of problems (P_α) and (P^*_α), respectively. We justify that there exist $\overline{W}^i_1 \in S^{m_0}, i = 1, \ldots, m$ and $\overline{W}^j_2 \in S^{m_0}, j = 1, \ldots, n$ such that $(\overline{x}, \overline{y}^0, \overline{Y}_l, \overline{Z}, \overline{W}^i_1, \overline{W}^j_2)_{i=1,\ldots,m, j=1,\ldots,n, l=1,\ldots,r}$ is a solution of (P^*_α) and

$$\mathrm{val}(\mathrm{P}_\alpha) = \mathrm{val}(\mathrm{P}^*_\alpha) = \sum_{i=1}^m \alpha_i \big(\overline{x}^\top Q^i_1 \overline{x} + (\xi^i_1)^\top \overline{x} + \beta^i_1 + (\theta^i_1)^\top \overline{y}^0 + \mathrm{Tr}(\overline{W}^i_1 A)\big), \quad (2)$$

where $\overline{Z} := \overline{x}\overline{x}^\top$ and $\overline{Y}_l, l = 1, \ldots, r$ are the columns of the matrix $\overline{Y}$.

Since $(\overline{x}, \overline{y}^0, \overline{Y})$ is a solution of problem (P_α), we see that

$$\mathrm{val}(\mathrm{P}_\alpha) = \sum_{i=1}^m \alpha_i \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}), \quad (3)$$

where $\mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}) := \max_{v \in V}\{f_i(\overline{x}, v) + (\theta^i_1)^\top(\overline{y}^0 + \overline{Y}v)\}$ for $i = 1, \ldots, m$, and that

$$\max_{v \in V}\{g_j(\overline{x}, v) + (\theta^j_2)^\top(\overline{y}^0 + \overline{Y}v)\} \leq 0, \; j = 1, \ldots, n. \quad (4)$$

For each $i \in \{1, \ldots, m\}$, we rewrite $\mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}) = \max_{v \in V}\{f_i(\overline{x}, v) + (\theta^i_1)^\top(\overline{y}^0 + \overline{Y}v)\}$ as follows

$$\begin{aligned} &\max_{v \in \mathbb{R}^r}\Big\{\sum_{l=1}^r v_l\big((\xi^i_{1,l})^\top \overline{x} + \beta^i_{1,l} + (\theta^i_1)^\top \overline{Y}_l\big) \mid A + \sum_{l=1}^r v_l A_l \succeq 0\Big\} \\ &= \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}) - \overline{\lambda}^i_1 - (\theta^i_1)^\top \overline{y}^0, \end{aligned} \quad (5)$$

where $\overline{\lambda}^i_1 := \overline{x}^\top Q^i_1 \overline{x} + (\xi^i_1)^\top \overline{x} + \beta^i_1$. The condition (1) means that a regularity condition is valid for the semidefinite programming (SDP) problem in (5). So we can employ a strong duality in SDP (see e.g., [7, Theorem 2.15]) to find $\overline{W}^i_1 \in S^{m_0}, \overline{W}^i_1 \succeq 0$ such that

$$\begin{aligned} &\overline{\lambda}^i_1 + (\theta^i_1)^\top \overline{y}^0 + \mathrm{Tr}(\overline{W}^i_1 A) = \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}), \\ &\beta^i_{1,l} + (\theta^i_1)^\top \overline{Y}_l + (\xi^i_{1,l})^\top \overline{x} + \mathrm{Tr}(\overline{W}^i_1 A_l) = 0, l = 1, \ldots, r. \end{aligned} \quad (6)$$

Similarly, for each $j \in \{1, \ldots, n\}$, we derive by (4) that there exist $\overline{W}^j_2 \in S^{m_0}, \overline{W}^j_2 \succeq 0$ such that

$$\begin{aligned} &\overline{\lambda}^j_2 + (\theta^j_2)^\top \overline{y}^0 + \mathrm{Tr}(\overline{W}^j_2 A) \leq 0, \\ &\beta^j_{2,s} + (\theta^j_2)^\top \overline{Y}_s + (\xi^j_{2,s})^\top \overline{x} + \mathrm{Tr}(\overline{W}^j_2 A_s) = 0, s = 1, \ldots, r, \end{aligned}$$

where $\overline{\lambda}_2^j := \overline{x}^\top Q_2^j \overline{x} + (\xi_2^j)^\top \overline{x} + \beta_2^j$.

Since $\overline{Z} := \overline{x}\overline{x}^\top$, we see that $(\overline{x}, \overline{y}^0, \overline{Y}_l, \overline{Z}, \overline{W}_1^i, \overline{W}_2^j)_{i=1,\ldots,m, j=1,\ldots,n, l=1,\ldots,r}$ is a feasible point of problem (P_α^*). This in turn implies that

$$\mathrm{val}(\mathrm{P}_\alpha^*)(\mathrm{P}_\alpha^*) \leq \sum_{i=1}^m \alpha_i \big(\overline{\lambda}_1^i + (\theta_1^i)^\top \overline{y}^0 + \mathrm{Tr}(\overline{W}_1^i A)\big) = \sum_{i=1}^m \alpha_i \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}) = \mathrm{val}(\mathrm{P}_\alpha), \tag{7}$$

where the first equality holds by (6) and the second one holds by (3).

Let us now justify that $\mathrm{val}(\mathrm{P}_\alpha) \leq \mathrm{val}(\mathrm{P}_\alpha^*)$. Assume that $(z, \nu_0, \nu^l, Z, W_1^i, W_2^j)_{i=1,\ldots,m, j=1,\ldots,n, l=1,\ldots,r}$ is feasible for (P_α^*). Then, $z \in \mathbb{R}^q$, $\nu_0 \in \mathbb{R}^p$, $\nu^l \in \mathbb{R}^p$, $Z \in S^q$, $W_1^i \in S^{m_0}$, $W_1^i \succeq 0$, $W_2^j \in S^{m_0}$, $W_2^j \succeq 0, i = 1, \ldots, m, j = 1, \ldots, n$, $l = 1, \ldots, r$ and

$$\begin{pmatrix} 1 & z^\top \\ z & Z \end{pmatrix} \succeq 0, \tag{8}$$

$$\beta_{1,l}^i + (\theta_1^i)^\top \nu^l + (\xi_{1,l}^i)^\top z + \mathrm{Tr}(W_1^i A_l) = 0, i = 1, \ldots, m, l = 1, \ldots, r, \tag{9}$$

$$\beta_2^j + (\xi_2^j)^\top z + (\theta_2^j)^\top \nu_0 + \mathrm{Tr}(Z Q_2^j) + \mathrm{Tr}(W_2^j A) \leq 0, j = 1, \ldots, n, \tag{10}$$

$$\beta_{2,l}^j + (\theta_2^j)^\top \nu^l + (\xi_{2,l}^j)^\top z + \mathrm{Tr}(W_2^j A_l) = 0, l = 1, \ldots, r, j = 1, ..., n. \tag{11}$$

Note that for each $j \in \{1, \ldots, n\}$ and any $v := (v_1, \ldots, v_r) \in V$, $\mathrm{Tr}\big[W_2^j (A + \sum_{l=1}^r v_l A_l)\big] \geq 0$ due to $W_2^j \succeq 0$ and $A + \sum_{l=1}^r v_l A_l \succeq 0$ and thus, $\mathrm{Tr}(W_2^j A) \geq -\sum_{l=1}^r v_l \mathrm{Tr}(W_2^j A_l)$. Denote $Y_0 := (\nu^1, \ldots, \nu^r)$ as a matrix whose columns are those of $\nu^l, l = 1, \ldots, r$. Then, for any $v \in V$, we derive from (11) that

$$\begin{aligned} g_j(z, v) + (\theta_2^j)^\top(\nu_0 + Y_0 v) &= \beta_2^j + (\xi_2^j)^\top z + \mathrm{Tr}(zz^\top Q_2^j) + (\theta_2^j)^\top \nu_0 - \sum_{l=1}^r v_l \mathrm{Tr}(W_2^j A_l) \\ &\leq \beta_2^j + (\xi_2^j)^\top z + (\theta_2^j)^\top \nu_0 + \mathrm{Tr}(Z Q_2^j) \\ &\quad + \mathrm{Tr}(W_2^j A), \quad j = 1, \ldots, n, \end{aligned} \tag{12}$$

where we note that $\mathrm{Tr}(Z Q_2^j) \geq \mathrm{Tr}(zz^\top Q_2^j)$ due to $Q_2^j \succeq 0$ and, by (8), $Z - zz^\top \succeq 0$. Now, we conclude by (10) and (12) that

$$\max_{v \in V}\{g_j(z, v) + (\theta_2^j)^\top(\nu_0 + Y_0 v)\} \leq 0, \ \ j = 1, \ldots, n,$$

which means that (z, ν_0, Y_0) is a feasible point of problem (P_α). Hence, it holds that

$$\mathrm{val}(\mathrm{P}_\alpha) \leq \sum_{i=1}^m \alpha_i \mathcal{F}_i(z, \nu_0, Y_0), \tag{13}$$

where $\mathcal{F}_i(z, \nu_0, Y_0) := \max_{v \in V}\{f_i(z, v) + (\theta_1^i)^\top(\nu_0 + Y_0 v)\}, i = 1, \ldots, m$.

Similarly, for any $v \in V$, we can derive from (8) and (9) that

$$f_i(z, v) + (\theta_1^i)^\top(\nu_0 + Y_0 v) \le \beta_1^i + (\xi_1^i)^\top z + (\theta_1^i)^\top \nu_0 + \mathrm{Tr}(Z Q_1^i) + \mathrm{Tr}(W_1^i A), \quad i = 1, \ldots, m,$$

which guarantees that

$$\mathcal{F}_i(z, \nu_0, Y_0) \le \beta_1^i + (\xi_1^i)^\top z + (\theta_1^i)^\top \nu_0 + \mathrm{Tr}(Z Q_1^i) + \mathrm{Tr}(W_1^i A), \; i = 1, \ldots, m.$$

This together with (13) entails that

$$\mathrm{val}(\mathrm{P}_\alpha) \le \sum_{i=1}^m \alpha_i \big(\beta_1^i + (\xi_1^i)^\top z + (\theta_1^i)^\top \nu_0 + \mathrm{Tr}(Z Q_1^i) + \mathrm{Tr}(W_1^i A)\big),$$

and consequently, $\mathrm{val}(\mathrm{P}_\alpha) \le \mathrm{val}(\mathrm{P}_\alpha^*)$ because $(z, \nu_0, \nu^l, Z, W_1^i, W_2^j)_{i=1,\ldots,m, j=1,\ldots,n, l=1,\ldots,r}$ was arbitrarily taken.

Now, in view of (7), we conclude that

$$\mathrm{val}(\mathrm{P}_\alpha) = \mathrm{val}(\mathrm{P}_\alpha^*) = \sum_{i=1}^m \alpha_i \big(\overline{\lambda}_1^i + (\theta_1^i)^\top \overline{y}^0 + \mathrm{Tr}(\overline{W}_1^i A)\big),$$

which also confirms that $(\overline{x}, \overline{y}^0, \overline{Y}_l, \overline{Z}, \overline{W}_1^i, \overline{W}_2^j)_{i=1,\ldots,m, j=1,\ldots,n, l=1,\ldots,r}$ is a solution of (P_α^*). Namely, the assertion in (2) has been justified.

(i) Assume that $(\overline{x}, \overline{y}^0, \overline{Y})$ is a weak efficient solution of (RT). Since the Slater qualification condition (3) is satisfied, we apply Theorem 2.1(i) to conclude that there exist $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}_+^m \setminus \{0\}$, $\alpha_i^s \in \mathbb{R}, i = 1, \ldots, m, s = 1, \ldots, r$ and $\lambda_j \ge 0, \lambda_j^s \in \mathbb{R}, j = 1, \ldots, n, s = 1, \ldots, r$ such that the linear matrix inequality optimality conditions in (4)–(6) are valid.

Recall $\mathcal{F}_i(x, y^0, Y) := \max_{v \in V}\{f_i(x, v) + (\theta_1^i)^\top(y^0 + Y v)\}, i = 1, \ldots, m$ for $(x, y^0, Y) \in \mathbb{R}^{q+p+p\times r}$ and $\mathcal{C}$ the set of feasible points of problem (RT). Note that $\mathcal{C}$ is also the set of feasible points of problem (P_α). Following similar arguments as in the proof of Theorem 2.1(ii), we employ (4)–(6) to arrive at

$$\sum_{i=1}^m \alpha_i \mathcal{F}_i(\widehat{x}, \widehat{y}^0, \widehat{Y}) \ge \sum_{i=1}^m \alpha_i \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}) \text{ for all } (\widehat{x}, \widehat{y}^0, \widehat{Y}) \in \mathcal{C},$$

which means that the triple $(\overline{x}, \overline{y}^0, \overline{Y})$ is a solution of (P_α). This, as shown above, guarantees that there exist $\overline{W}_1^i \in S^{m_0}, i = 1, \ldots, m$ and $\overline{W}_2^j \in S^{m_0}, j = 1, \ldots, n$ such that $(\overline{x}, \overline{y}^0, \overline{Y}_l, \overline{Z}, \overline{W}_1^i, \overline{W}_2^j)_{i=1,\ldots,m, j=1,\ldots,n, l=1,\ldots,r}$ is a solution of (P_α^*), where $\overline{Y}_l, l = 1, \ldots, r$ are the columns of $\overline{Y}$ and $\overline{Z} := \overline{x}\overline{x}^\top$.

(ii) Assume that the problem (P_α) possesses a solution for $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+ \setminus \{0\}$. Then, as shown by (2), we have

$$\mathrm{val}(\mathrm{P}_\alpha) = \mathrm{val}(\mathrm{P}^*_\alpha). \tag{14}$$

Let $(\overline{z}, \overline{v}_0, \overline{v}^l, \overline{Z}, \overline{W}^i_1, \overline{W}^j_2)_{i=1,\ldots,m,\, j=1,\ldots,n, l=1,\ldots,r}$ be a solution of (P^*_α). Then, $\overline{z} \in \mathbb{R}^q, \overline{v}_0 \in \mathbb{R}^p, \overline{v}^l \in \mathbb{R}^p, \overline{Z} \in S^q, \overline{W}^i_1 \in S^{m_0}, \overline{W}^i_1 \succeq 0, \overline{W}^j_2 \in S^{m_0}, \overline{W}^j_2 \succeq 0$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $l = 1, \ldots, r$ and

$$\mathrm{val}(\mathrm{P}^*_\alpha) = \sum_{i=1}^m \alpha_i \big(\beta^i_1 + (\xi^i_1)^\top \overline{z} + (\theta^i_1)^\top \overline{v}_0 + \mathrm{Tr}(\overline{Z} Q^i_1) + \mathrm{Tr}(\overline{W}^i_1 A)\big), \tag{15}$$

$$\begin{pmatrix} 1 & \overline{z}^\top \\ \overline{z} & \overline{Z} \end{pmatrix} \succeq 0, \tag{16}$$

$$\beta^i_{1,l} + (\theta^i_1)^\top \overline{v}^l + (\xi^i_{1,l})^\top \overline{z} + \mathrm{Tr}(\overline{W}^i_1 A_l) = 0, i = 1, \ldots, m, l = 1, \ldots, r, \tag{17}$$

$$\beta^j_2 + (\xi^j_2)^\top \overline{z} + (\theta^j_2)^\top \overline{v}_0 + \mathrm{Tr}(\overline{Z} Q^j_2) + \mathrm{Tr}(\overline{W}^j_2 A) \leq 0, j = 1, \ldots, n, \tag{18}$$

$$\beta^j_{2,l} + (\theta^j_2)^\top \overline{v}^l + (\xi^j_{2,l})^\top \overline{z} + \mathrm{Tr}(\overline{W}^j_2 A_l) = 0, j = 1, \ldots, n, l = 1, \ldots, r. \tag{19}$$

Denote by $\overline{Y} := (\overline{v}^1, \ldots \overline{v}^r)$ a matrix with columns of $\overline{v}^l$, $l = 1, \ldots, r$. Proceeding as above, we derive from (16), (18) and (19) that $(\overline{z}, \overline{v}_0, \overline{Y})$ is feasible for (P_α). This also shows that $(\overline{z}, \overline{v}_0, \overline{Y})$ is feasible for (RT). Similarly, we can derive from (16) and (17) that

$$\mathcal{F}_i(\overline{z}, \overline{v}_0, \overline{Y}) \leq \beta^i_1 + (\xi^i_1)^\top \overline{z} + (\theta^i_1)^\top \overline{v}_0 + \mathrm{Tr}(\overline{Z} Q^i_1) + \mathrm{Tr}(\overline{W}^i_1 A),\ i = 1, \ldots, m. \tag{20}$$

We assert that the triple $(\overline{z}, \overline{v}_0, \overline{Y})$ is a weak efficient solution of (RT). Otherwise, we would find $(\widehat{x}, \widehat{y}^0, \widehat{Y}) \in \mathcal{C}$ such that

$$\mathcal{F}_i(\widehat{x}, \widehat{y}^0, \widehat{Y}) < \mathcal{F}_i(\overline{z}, \overline{v}_0, \overline{Y}),\ i = 1, \ldots, m.$$

It is worth noting here that $(\widehat{x}, \widehat{y}^0, \widehat{Y})$ is also a feasible point of problem (P_α). In view of (20) and (15), we conclude that

$$\mathrm{val}(\mathrm{P}_\alpha) \leq \sum_{i=1}^m \alpha_i \mathcal{F}_i(\widehat{x}, \widehat{y}^0, \widehat{Y}) < \sum_{i=1}^m \alpha_i \mathcal{F}_i(\overline{z}, \overline{v}_0, \overline{Y}) \leq \mathrm{val}(\mathrm{P}^*_\alpha),$$

which is absurd by virtue of (14). In conclusion, the triple $(\overline{z}, \overline{v}_0, \overline{Y})$ is a weak efficient solution of (RT).

(iii) Assume that the problem (P_α) admits a solution for $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathrm{int}\mathbb{R}^m_+$. Let $(\overline{z}, \overline{v}_0, \overline{v}^l, \overline{Z}, \overline{W}^i_1, \overline{W}^j_2)_{i=1,\ldots,m,\, j=1,\ldots,n, l=1,\ldots,r}$ be a solution of (P^*_α). Then, (14)–(20) are valid for this setting. Proceeding similarly as in the proof of (ii), we come to an assertion that there does not exist $(\widehat{x}, \widehat{y}^0, \widehat{Y}) \in \mathcal{C}$ such that

$$\mathcal{F}_i(\widehat{x}, \widehat{y}^0, \widehat{Y}) \leq \mathcal{F}_i(\overline{z}, \overline{\nu}_0, \overline{Y}), \ i = 1, \ldots, m$$
$$\text{and } \mathcal{F}_i(\widehat{x}, \widehat{y}^0, \widehat{Y}) < \mathcal{F}_i(\overline{z}, \overline{\nu}_0, \overline{Y}) \text{ for some } i \in \{1, \ldots, m\},$$

which concludes that the triple $(\overline{z}, \overline{\nu}_0, \overline{Y})$ is an efficient solution of (RT). □

Second-Order Cone Programming Relaxations. We now consider the special robust two-stage multiobjective optimization problem (RT), where the objectives $f_i, i = 1, \ldots, m$ and the constraints $g_j, j = 1, \ldots, n$ are *separable* quadratic functions in the first-stage variable given by (25) and the uncertainty set V is an *ellipsoid* given by (26). It should be noted here that the two-stage scalarized problem (P_α) is considered under the current setting.

In this framework, we address a relaxation problem in terms of second-order cone programming (SOCP) for (P_α) as follows:

$$\min_{(z,\nu_0,\nu^l,\eta_k,\gamma_1^i,\gamma_2^j,z_1^i,z_2^j)} \sum_{i=1}^m \alpha_i \big(\beta_1^i + (\xi_1^i)^\top z + (\theta_1^i)^\top \nu_0 + \sum_{k=1}^q \omega_{1,k}^i \eta_k + \gamma_1^i\big) \tag{S_α^*}$$
$$\text{s.t. } \|(1-\eta_k, 2z_k)\| \leq 1 + \eta_k, z := (z_1, \ldots, z_q) \in \mathbb{R}^q, \eta_k \in \mathbb{R}, k = 1, \ldots, q,$$
$$\beta_{1,l}^i + (\theta_1^i)^\top \nu^l + (\xi_{1,l}^i)^\top z + (E_l^d)^\top z_1^i = 0, i = 1, \ldots, m, l = 1, \ldots, r,$$
$$\beta_2^j + (\xi_2^j)^\top z + (\theta_2^j)^\top \nu_0 + \sum_{k=1}^n \omega_{2,k}^j \eta_k + \gamma_2^j \leq 0, \nu_0 \in \mathbb{R}^p, j = 1, \ldots, n,$$
$$\beta_{2,l}^j + (\theta_2^j)^\top \nu^l + (\xi_{2,l}^j)^\top z + (E_l^d)^\top z_2^j = 0, j = 1, \ldots, n, l = 1, \ldots, r,$$
$$\|z_1^i\| \leq \gamma_1^i, \|z_2^j\| \leq \gamma_2^j, z_1^i \in \mathbb{R}^r, \gamma_1^i \in \mathbb{R}, z_2^j \in \mathbb{R}^r, \gamma_2^j \in \mathbb{R}, i = 1, \ldots, m, j = 1, \ldots, n,$$

where $E_l^d, l = 1, \ldots, r$ are the columns of the matrix E^d given as in (26).

In the forthcoming theorem, we present links between (weak) efficient solutions of the two-stage multiobjective problem (RT) under the current setting and optimal solutions of (scalar) second-order cone programming problem (S_α^*), which is a relaxation of the two-stage scalarized problem (P_α).

Theorem 3.2 (Calculating Solutions via second-order cone programming relaxations) *Consider the problem* (RT) *and the problem* (P_α) *with* $f_i, i = 1, \ldots, m$ *and* $g_j, j = 1, \ldots, n$ *given in* (25)*, and* V *given in* (26)*. Then, the following assertions are valid.*
(i) *Let the Slater qualification condition in* (3) *hold and let* $(\overline{x}, \overline{y}^0, \overline{Y})$ *be a weak efficient solution of* (RT)*. Then, we can find* $\alpha \in \mathbb{R}_+^m \backslash \{0\}$, $\overline{z}_1^i \in \mathbb{R}^r$, $\overline{\gamma}_1^i \in \mathbb{R}, i = 1, \ldots, m$ *and* $\overline{z}_2^j \in \mathbb{R}^r$, $\overline{\gamma}_2^j \in \mathbb{R}, j = 1, \ldots, n$ *such that* $(\overline{x}, \overline{y}^0, \overline{Y}_l, \overline{\eta}_k, \overline{\gamma}_1^i, \overline{\gamma}_2^j, \overline{z}_1^i, \overline{z}_2^j)_{k=1,\ldots,q,i=1,\ldots,m,j=1,\ldots,n,l=1,\ldots,r}$ *is a solution of* (S_α^*)*, where* $\overline{\eta}_k := \overline{x}_k^2, k = 1, \ldots, q$ *and* $\overline{Y}_l, l = 1, \ldots, r$ *are the columns of the matrix* $\overline{Y}$.
(ii) **(Finding weak efficient solutions)** *Suppose that the problem* (P_α) *admits a solution for* $\alpha \in \mathbb{R}_+^m \setminus \{0\}$ *and let* $(\overline{z}, \overline{\nu}_0, \overline{\nu}^l, \overline{\eta}_k, \overline{\gamma}_1^i, \overline{\gamma}_2^j, \overline{z}_1^i, \overline{z}_2^j)_{k=1,\ldots,q,i=1,\ldots,m,j=1,\ldots,n,l=1,\ldots,r}$

be a solution of (S_α^*)*. Then* $(\overline{z}, \overline{v}_0, \overline{Y})$ *is a weak efficient solution for* (RT)*, where* $\overline{Y} := (\overline{v}^1, \ldots, \overline{v}^r)$ *is a matrix whose columns are those of* $\overline{v}^l$, $l = 1, \ldots, r$.

(iii) **(Finding efficient solutions)** *Suppose that the problem* (P_α) *admits a solution for* $\alpha \in \operatorname{int}\mathbb{R}^m_+$ *and let* $(\overline{z}, \overline{v}_0, \overline{v}^l, \overline{\eta}_k, \overline{\gamma}_1^i, \overline{\gamma}_2^j, \overline{z}_1^i, \overline{z}_2^j)_{k=1,\ldots,q, i=1,\ldots,m, j=1,\ldots,n, l=1,\ldots,r}$ *be a solution of* (S_α^*)*. Then* $(\overline{z}, \overline{v}_0, \overline{Y})$ *is an efficient solution for* (RT)*, where* $\overline{Y} := (\overline{v}^1, \ldots, \overline{v}^r)$ *is a matrix whose columns are those of* $\overline{v}^l$, $l = 1, \ldots, r$.

Proof Let $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+ \setminus \{0\}$ be such that the problem (P_α) admits a solution and assume that $(\overline{x}, \overline{y}^0, \overline{Y})$ is a solution of (P_α) under the current setting. Let val(P_α) denote the optimal value of (P_α) and let val(S_α^*) denote the optimal value of (S_α^*). We justify that there exist $\overline{z}_1^i \in \mathbb{R}^r$, $\overline{\gamma}_1^i \in \mathbb{R}$, $i = 1, \ldots, m$ and $\overline{z}_2^j \in \mathbb{R}^r$, $\overline{\gamma}_2^j \in \mathbb{R}$, $j = 1, \ldots, n$ such that $(\overline{x}, \overline{y}^0, \overline{Y}_l, \overline{\eta}_k, \overline{\gamma}_1^i, \overline{\gamma}_2^j, \overline{z}_1^i, \overline{z}_2^j)_{k=1,\ldots,q, i=1,\ldots,m, j=1,\ldots,n, l=1,\ldots,r}$ is a solution of (S_α^*) and

$$\operatorname{val}(P_\alpha) = \operatorname{val}(S_\alpha^*) = \sum_{i=1}^m \alpha_i \big(\beta_1^i + (\xi_1^i)^\top \overline{x} + (\theta_1^i)^\top \overline{y}^0 + \sum_{k=1}^q \omega_{1,k}^i \overline{\eta}_k + \overline{\gamma}_1^i\big), \tag{21}$$

where $\overline{x} := (\overline{x}_1, \ldots, \overline{x}_q)$, $\overline{\eta}_k := \overline{x}_k^2$, $k = 1, \ldots, q$ and $\overline{Y}_l$, $l = 1, \ldots, r$ are the columns of the matrix $\overline{Y}$.

Since $(\overline{x}, \overline{y}^0, \overline{Y})$ is a solution of (P_α), we see that

$$\operatorname{val}(P_\alpha) = \sum_{i=1}^m \alpha_i \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}), \tag{22}$$

where $\mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}) := \max_{v \in V}\{f_i(\overline{x}, v) + (\theta_1^i)^\top(\overline{y}^0 + \overline{Y}v)\}$ for $i = 1, \ldots, m$, and that

$$\max_{v \in V}\{g_j(\overline{x}, v) + (\theta_2^j)^\top(\overline{y}^0 + \overline{Y}v)\} \le 0, \; j = 1, \ldots, n. \tag{23}$$

For each $i \in \{1, \ldots, m\}$, we rewrite $\mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}) = \max_{v \in V}\{f_i(\overline{x}, v) + (\theta_1^i)^\top(\overline{y}^0 + \overline{Y}v)\}$ as follows

$$\min_{v \in \mathbb{R}^r}\Big\{ -\sum_{l=1}^r v_l\big((\xi_{1,l}^i)^\top \overline{x} + \beta_{1,l}^i + (\theta_1^i)^\top \overline{Y}_l\big) \mid \|E^d v\| \le 1\Big\} = \sum_{k=1}^q \omega_{1,k}^i \overline{x}_k^2 + (\xi_1^i)^\top \overline{x} + \beta_1^i + (\theta_1^i)^\top \overline{y}^0 - \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}). \tag{24}$$

Letting $\widehat{v} := 0_r \in \mathbb{R}^r$, it holds that $\|E^d \widehat{v}\| < 1$, i.e., the strict feasibility condition holds for the second-order cone programming problem in (24). Thus, we can invoke a strong duality result in second-order cone programming (see e.g., [1, Page 81]) to find $\overline{z}_1^i \in \mathbb{R}^r, \overline{\gamma}_1^i \in \mathbb{R}, \|\overline{z}_1^i\| \le \overline{\gamma}_1^i$ such that

$$\beta_1^i + (\xi_1^i)^\top \overline{x} + (\theta_1^i)^\top \overline{y}^0 + \sum_{k=1}^{q} \omega_{1,k}^i \overline{x}_k^2 + \overline{\gamma}_1^i = \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}),$$
$$\beta_{1,l}^i + (\theta_1^i)^\top \overline{Y}_l + (\xi_{1,l}^i)^\top \overline{x} + (E_l^d)^\top \overline{z}_1^i = 0, l = 1, \dots, r. \tag{25}$$

Similarly, for each $j \in \{1, \dots, n\}$, we derive by (23) that there exist $\overline{z}_2^j \in \mathbb{R}^r, \overline{\gamma}_2^j \in \mathbb{R}$, $\|\overline{z}_2^j\| \le \overline{\gamma}_2^j$ such that

$$\beta_2^j + (\xi_2^j)^\top \overline{x} + (\theta_2^j)^\top \overline{y}^0 + \sum_{k=1}^{q} \omega_{2,k}^j \overline{x}_k^2 + \overline{\gamma}_2^j \le 0,$$
$$\beta_{2,l}^j + (\theta_2^j)^\top \overline{Y}_l + (\xi_{2,l}^j)^\top \overline{x} + (E_l^d)^\top \overline{z}_2^j = 0, l = 1, \dots, r.$$

So, it holds that $(\overline{x}, \overline{y}^0, \overline{Y}_l, \overline{\eta}_k, \overline{\gamma}_1^i, \overline{\gamma}_2^j, \overline{z}_1^i, \overline{z}_2^j)_{k=1,\dots,q,i=1,\dots,m,j=1,\dots,n,l=1,\dots,r}$ is a feasible point of problem (S_α^*). This in turn implies that

$$\begin{aligned} \mathrm{val}(S_\alpha^*) &\le \sum_{i=1}^{m} \alpha_i \big(\beta_1^i + (\xi_1^i)^\top \overline{x} + (\theta_1^i)^\top \overline{y}^0 + \sum_{k=1}^{q} \omega_{1,k}^i \overline{\eta}_k + \overline{\gamma}_1^i\big) \\ &= \sum_{i=1}^{m} \alpha_i \mathcal{F}_i(\overline{x}, \overline{y}^0, \overline{Y}) = \mathrm{val}(P_\alpha), \end{aligned} \tag{26}$$

where the first equality holds by (25) and the second one holds by (22). Let us now justify that val(P_α)$\le$ val(S_α^*). Assume that $(z, \nu_0, \nu^l, \eta_k, \gamma_1^i, \gamma_2^j, z_1^i, z_2^j)_{k=1,\dots,q,i=1,\dots,m,j=1,\dots,n,l=1,\dots,r}$ is feasible for (S_α^*). Then, $z := (z_1, \dots, z_q) \in \mathbb{R}^q, \nu_0 \in \mathbb{R}^p, \nu^l \in \mathbb{R}^p, \eta_k \in \mathbb{R}, \gamma_1^i \in \mathbb{R}, z_1^i \in \mathbb{R}^r, \|z_1^i\| \le \gamma_1^i, \gamma_2^j \in \mathbb{R}, z_2^j \in \mathbb{R}^r, \|z_2^j\| \le \gamma_2^j, k = 1, \dots, q, i = 1, \dots, m, j = 1, \dots, n, l = 1, \dots, r$ and

$$\|(1 - \eta_k, 2z_k)\| \le 1 + \eta_k, k = 1, \dots, q, \tag{27}$$
$$\beta_{1,l}^i + (\theta_1^i)^\top \nu^l + (\xi_{1,l}^i)^\top z + (E_l^d)^\top z_1^i = 0, i = 1, \dots, m, l = 1, \dots, r, \tag{28}$$
$$\beta_2^j + (\xi_2^j)^\top z + (\theta_2^j)^\top \nu_0 + \sum_{k=1}^{q} \omega_{2,k}^j \eta_k + \gamma_2^j \le 0, j = 1, \dots, n, \tag{29}$$
$$\beta_{2,l}^j + (\theta_2^j)^\top \nu^l + (\xi_{2,l}^j)^\top z + (E_l^d)^\top z_2^j = 0, j = 1, ..., n, l = 1, \dots, r. \tag{30}$$

Note that for each $j \in \{1, \dots, n\}$ and any $v := (v_1, \dots, v_r) \in V$, $\|E^d v\| \le 1$. Then, we get by $\|z_2^j\| \le \gamma_2^j$ and the Cauchy-Schwarz inequality that

$$\gamma_2^j \ge \|z_2^j\| \|E^d v\| \ge -(z_2^j)^\top (E^d v) = -\sum_{l=1}^{r} v_l (E_l^d)^\top z_2^j.$$

Denote $Y_0 := (\nu^1, \ldots, \nu^r)$ as a matrix whose columns are those of $\nu^l, l = 1, \ldots, r$. Then, for any $v \in V$, we derive from (30) that

$$g_j(z, v) + (\theta_2^j)^\top(\nu_0 + Y_0 v) = \beta_2^j + (\xi_2^j)^\top z + \sum_{k=1}^{q} \omega_{2,k}^j z_k^2 + (\theta_2^j)^\top \nu_0 - \sum_{l=1}^{r} v_l (E_l^d)^\top z_2^j$$
$$\leq \beta_2^j + (\xi_2^j)^\top z + (\theta_2^j)^\top \nu_0 + \sum_{k=1}^{q} \omega_{2,k}^j \eta_k + \gamma_2^j, \quad (31)$$

where we note that $z_k^2 \leq \eta_k, k = 1, \ldots, q$ by virtue of (27). Now, we conclude by (29) and (31) that

$$\max_{v \in V} \{g_j(z, v) + (\theta_2^j)^\top(\nu_0 + Y_0 v)\} \leq 0, \;\; j = 1, \ldots, n,$$

which means that (z, ν_0, Y_0) is feasible for (P_α). Hence, we obtain that

$$\mathrm{val}(\mathrm{P}_\alpha) \leq \sum_{i=1}^{m} \alpha_i \mathcal{F}_i(z, \nu_0, Y_0), \quad (32)$$

where $\mathcal{F}_i(z, \nu_0, Y_0) := \max_{v \in V} \{f_i(z, v) + (\theta_1^i)^\top(\nu_0 + Y_0 v)\}, i = 1, \ldots, m$. Similarly, for any $v \in V$, we derive from (27) and (28) that

$$\mathcal{F}_i(z, \nu_0, Y_0) \leq \beta_1^i + (\xi_1^i)^\top z + (\theta_1^i)^\top \nu_0 + \sum_{k=1}^{q} \omega_{1,k}^i \eta_k + \gamma_1^i, \;\; i = 1, \ldots, m.$$

This together with (32) entails that

$$\mathrm{val}(\mathrm{P}_\alpha) \leq \sum_{i=1}^{m} \alpha_i \big(\beta_1^i + (\xi_1^i)^\top z + (\theta_1^i)^\top \nu_0 + \sum_{k=1}^{q} \omega_{1,k}^i \eta_k + \gamma_1^i\big),$$

and consequently, $\mathrm{val}(\mathrm{P}_\alpha) \leq \mathrm{val}(\mathrm{S}_\alpha^*)$ because $(z, \nu_0, \nu^l, \eta_k, \gamma_1^i, \gamma_2^j, z_1^i, z_2^j)_{k=1,\ldots,q, i=1,\ldots,m, j=1,\ldots,n, l=1,\ldots,r}$ was arbitrarily taken.

Now, in view of (26), we conclude that

$$\mathrm{val}(\mathrm{P}_\alpha) = \mathrm{val}(\mathrm{S}_\alpha^*) = \sum_{i=1}^{m} \alpha_i \big(\beta_1^i + (\xi_1^i)^\top \overline{x} + (\theta_1^i)^\top \overline{y}^0 + \sum_{k=1}^{q} \omega_{1,k}^i \overline{\eta}_k + \overline{\gamma}_1^i\big),$$

which also confirms that $(\overline{x}, \overline{y}^0, \overline{Y}_l, \overline{\eta}_k, \overline{\gamma}_1^i, \overline{\gamma}_2^j, \overline{z}_1^i, \overline{z}_2^j)_{k=1,\ldots,q, i=1,\ldots,m, j=1,\ldots,n, l=1,\ldots,r}$ is a solution of (S_α^*). Namely, the assertion in (21) has been justified.

(i) Let $(\overline{x}, \overline{y}^0, \overline{Y})$ be a weak efficient solution of (RT). Since the Slater qualification condition (3) holds, we apply Corollary 2.1(i) to assert that there exist $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}_+^m \setminus \{0\}$, $\alpha_i^s \in \mathbb{R}, i = 1, \ldots, m, s = 1, \ldots, r, \lambda_j \geq 0, \lambda_j^s \in$

$\mathbb{R}$, $j = 1, \ldots, n$, $s = 1, \ldots, r$ and $t_k \geq 0$, $k = 1, \ldots, q$ such that the second-order cone conditions in (27)–(30) are valid. Similarly as in the proof of Theorem 3.1, we can show that $(\overline{x}, \overline{y}^0, \overline{Y})$ is a solution of (P_α). This, as shown above, guarantees that there exist $\overline{z}_1^i \in \mathbb{R}^r$, $\overline{\gamma}_1^i \in \mathbb{R}$, $i = 1, \ldots, m$ and $\overline{z}_2^j \in \mathbb{R}^r$, $\overline{\gamma}_2^j \in \mathbb{R}$, $j = 1, ..., n$ such that $(\overline{x}, \overline{y}^0, \overline{Y}_l, \overline{\eta}_k, \overline{\gamma}_1^i, \overline{\gamma}_2^j, \overline{z}_1^i, \overline{z}_2^j)_{k=1,...,q, i=1,...,m, j=1,...,n, l=1,...,r}$ is a solution of (S_α^*), where $\overline{\eta}_k := \overline{x}_k^2$, $k = 1, \ldots, q$ and $\overline{Y}_l$, $l = 1, \ldots, r$ are the columns of the matrix $\overline{Y}$.

The proofs of (ii) and (iii) are similar to the corresponding ones in Theorem 3.1 specified for this setting and so, they are omitted. □

4 Solving Examples Numerically via Relaxations

In this section, we present numerical examples to illustrate that one can employ the proposed semidefinite programming (SDP) or second-order cone programming (SOCP) relaxations to find (weak) efficient solutions for the considered two-stage multiobjective problems including those arisen from practical applications.

4.1 A Numerical Example

Consider an *uncertain two-stage* multiobjective program:

$$\begin{aligned} \min_{x,y} & \left(f_1(x, v) + (\theta_1^1)^\top y(v),\, f_2(x, v) + (\theta_1^2)^\top y(v),\, f_3(x, v) + (\theta_1^3)^\top y(v) \right) \\ \text{s.t. } & g_j(x, v) + (\theta_2^j)^\top y(v) \leq 0,\;\; j = 1, 2, 3, \end{aligned} \tag{E2}$$

where $\theta_1^i \in \mathbb{R}^3$, $i = 1, 2, 3$ and $\theta_2^j \in \mathbb{R}^3$, $j = 1, 2, 3$ are fixed parameters, v is an *uncertain* parameter, which is residing in the uncertainty set $V \subset \mathbb{R}^2$, $x \in \mathbb{R}^3$ is the first-stage decision variable, $y : V \to \mathbb{R}^3$ is the second-stage decision variable, and $f_i : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2, 3$, $g_j : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}$, $j = 1, 2, 3$ are functions, defined respectively by, for $x := (x_1, x_2, x_3) \in \mathbb{R}^3$, $v := (v_1, v_2) \in V$,

$$\begin{aligned} f_1(x, v) &:= 2x_1^2 - (v_1 + 4)x_1 + (1 - v_2)x_2 + x_3 + v_1 - 1, \\ g_1(x, v) &:= -x_2 - x_3 + v_1, \\ f_2(x, v) &:= x_1^2 + (v_2 - 2)x_1 - v_1 x_2 + x_3 - v_2 + 1, \\ g_2(x, v) &:= 2x_1^2 + (v_1 - 4)x_1 + v_2 x_2 - v_1 + 2, \\ f_3(x, v) &:= x_1^2 - (v_1 + 2)x_1 + (1 - v_2)x_2 + v_1 + 2, \\ g_3(x, v) &:= x_1^2 - (v_2 + 2)x_1 + v_1 x_2 + v_2 + 1. \end{aligned}$$

Here, the uncertainty set V is given by

$$V := \{v := (v_1, v_2) \in \mathbb{R}^2 \mid v_1^2 + \frac{1}{2} v_2^2 \leq 1\}$$

and the second-stage decision variable y is given by

$$y(v) := y^0 + Yv,\ \ v \in V,$$

where $y^0 \in \mathbb{R}^3$ and $Y \in \mathbb{R}^{3\times 2}$ are nonadjustable variables.

We consider a *robust two-stage* multiobjective optimization problem that is a robust counterpart of (E2) defined by

$$\begin{aligned}\min_{x\in\mathbb{R}^3, y^0\in\mathbb{R}^3, Y\in\mathbb{R}^{3\times 2}} \Big(& \max_{v\in V}\{f_1(x,v) + (\theta_1^1)^\top y(v)\}, \\ & \max_{v\in V}\{f_2(x,v) + (\theta_1^2)^\top y(v)\}, \max_{v\in V}\{f_3(x,v) + (\theta_1^3)^\top y(v)\}\Big) \\ & \text{s.t. } g_j(x,v) + (\theta_2^j)^\top y(v) \le 0,\ j=1,2,3,\ y(v) = y^0 + Yv, \forall v \in V.\end{aligned} \tag{R2}$$

It is easy to see that the problem (R2) is of the form of (RT), where the functions $f_i, i = 1,2,3$ and $g_j,\ j = 1,2,3$ are defined by $Q_1^1 := Q_2^2 := \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $Q_2^1 := 0_{3\times 3}$, $Q_1^2 := Q_1^3 := Q_2^3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\xi_1^1 := (-4,1,1), \xi_1^2 := (-2,0,1)$, $\xi_1^3 := (-2,1,0)$, $\xi_{1,1}^1 := \xi_{1,1}^3 := (-1,0,0)$, $\xi_{1,2}^1 := \xi_{1,1}^2 := \xi_{1,2}^3 := (0,-1,0)$, $\xi_{1,2}^2 := (1,0,0)$, $\beta_1^1 := -1$, $\beta_1^2 := 1$, $\beta_1^3 := 2$, $\beta_{1,1}^1 := 1$, $\beta_{1,2}^1 := \beta_{1,2}^3 := 0$, $\beta_{1,1}^3 := 1$, $\beta_{1,1}^2 := 0$, $\beta_{1,2}^2 := -1$, $\xi_2^1 := (0,-1,-1)$, $\xi_2^3 := (-2,0,0)$, $\xi_2^2 := (-4,0,0)$, $\xi_{2,1}^1 := \xi_{2,2}^1 := 0_3$, $\xi_{2,1}^2 := (1,0,0)$, $\xi_{2,1}^3 := \xi_{2,2}^2 := (0,1,0)$, $\xi_{2,2}^3 := (-1,0,0)$, $\beta_2^1 := 0$, $\beta_2^3 := 1$, $\beta_2^2 := 2$, $\beta_{2,1}^1 := 1$, $\beta_{2,2}^1 := \beta_{2,2}^2 := \beta_{2,1}^3 := 0$, $\beta_{2,1}^2 := -1$, $\beta_{2,2}^3 := 1$, and the uncertainty set V is described by

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A_1 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, A_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Case I: Two-stage multiobjective programs. Let $\theta_1^1 := \theta_1^3 := (1,1,-1)$, $\theta_2^1 := 0_3$, $\theta_1^2 := (-1,1,-1)$, $\theta_2^2 := (-1,-1,1)$ and $\theta_2^3 := (1,-1,1)$. We employ the obtained relaxations from Theorem 3.1 to find (weak) efficient solutions of problem (R2). By choosing $\widehat{v} := 0_2 \in \mathbb{R}^2$, it holds that $A + \sum_{l=1}^{2} \widehat{v}_l A_l \succ 0$, which means that the condition (1) holds for this setting.

For each $\alpha := (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3_+ \backslash \{0\}$, we examine a robust scalarized optimization problem of (R2) given by

$$\begin{aligned}\min_{x\in\mathbb{R}^2, y^0\in\mathbb{R}^3, Y\in\mathbb{R}^{3\times 2}} \Big\{ \sum_{i=1}^{3} \alpha_i \max_{v\in V}\{f_i(x,v) + (\theta_1^i)^\top y(v)\} \,|\, & g_j(x,v) + (\theta_2^j)^\top y(v) \le 0,\ j=1,2,3, \\ & y(v) = y^0 + Yv, \forall v \in V\Big\}.\end{aligned} \tag{E2$_\alpha$}$$

By considering the case of $\alpha := (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3_+ \backslash \{0\}$ with $\alpha_2 = \alpha_3$, we assert that this weighted sum problem admits an optimal solution. To see this, just take $\overline{x} := (1, 0, 1)$, $\overline{y}^0 := 0_3$ and $\overline{Y} := 0_{3\times 2}$. Then $(\overline{x}, \overline{y}^0, \overline{Y})$ is a solution of problem (E2_α) and the optimal value is val(E2_α)$= -2\alpha_1 + \alpha_2 + \alpha_3$.

The SDP relaxation problem of (E2_α) is given by

$$\min_{(z,\nu_0,\nu^l,Z,W_1^i,W_2^j)} \sum_{i=1}^{3} \alpha_i \big(\beta_1^i + (\xi_1^i)^\top z + (\theta_1^i)^\top \nu_0 + \text{Tr}(ZQ_1^i) + \text{Tr}(W_1^i A)\big) \qquad (\text{E2}^*_\alpha)$$

$$\text{s.t.} \begin{pmatrix} 1 & z^\top \\ z & Z \end{pmatrix} \succeq 0, z \in \mathbb{R}^3, Z \in S^3, \nu_0 \in \mathbb{R}^3, \nu^l \in \mathbb{R}^3, l = 1, 2,$$
$$\beta_{1,l}^i + (\theta_1^i)^\top \nu^l + (\xi_{1,l}^i)^\top z + \text{Tr}(W_1^i A_l) = 0, i = 1, 2, 3, l = 1, 2,$$
$$\beta_2^j + (\xi_2^j)^\top z + (\theta_2^j)^\top \nu_0 + \text{Tr}(ZQ_2^j) + \text{Tr}(W_2^j A) \leq 0, j = 1, 2, 3,$$
$$\beta_{2,l}^j + (\theta_2^j)^\top \nu^l + (\xi_{2,l}^j)^\top z + \text{Tr}(W_2^j A_l) = 0, j = 1, 2, 3, l = 1, 2,$$
$$W_1^i \succeq 0, W_1^i \in S^3, W_2^j \succeq 0, W_2^j \in S^3, i = 1, 2, 3, j = 1, 2, 3.$$

Consider (for instance) $\alpha := (1, 2, 2)$. We utilize the toolbox CVX (see e.g., [21]) to solve the problem (E2^*_α) and obtain the optimal value as 2.0000, which is nothing else but val(E2_α). Furthermore, the solver returns (optimal) variables as $(\overline{z}, \overline{\nu}_0, \overline{\nu}^l, \overline{Z}, \overline{W}_1^i, \overline{W}_2^j)_{i=1,2,3, j=1,2,3, l=1,2}$, where $\overline{z} = (1.0000, 0.4025, 0.5975)$, $\overline{\nu}_0 = (0.0000, 0.0000, 0.0000)$, $\overline{\nu}^1 = (-0.2012, 0.1006, -0.1006)$ and $\overline{\nu}^2 = (0.2012, 0.1006, -0.1006)$. By denoting $\overline{Y} := (\overline{\nu}^1, \overline{\nu}^2)$ as a matrix with the columns of $\overline{\nu}^1$ and $\overline{\nu}^2$, we assert from Theorem 3.1 that $(\overline{z}, \overline{\nu}_0, \overline{Y})$ is a weak/efficient solution of (R2).

Case II: Multiobjective programs. Let $\theta_1^1 := \theta_2^1 := \theta_1^2 := \theta_2^2 := \theta_1^3 := \theta_2^3 := 0_3$. Then, the problem (E2) collapses to an uncertain (single-stage) multiobjective problem, which lands in the form of (P). By fixing $\theta_1^1 := \theta_2^1 := \theta_1^2 := \theta_2^2 := \theta_1^3 := \theta_2^3 := 0_3$ and disregarding redundant variables in (R2), (E2_α) and (E2^*_α), we obtain corresponding problems for this case.

In this way, we can use other different vectors of $\alpha := (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3_+ \setminus \{0\}$ with $\alpha_2 = \alpha_3$ to locate and compare (weak) efficient solutions for the frameworks in Case I and Case II.

As we can see from Table 1 that, when scalarizing with a same set of weighted sum vectors, the two-stage multiobjective problem of Case I provides many different (first-stage) weak/efficient solutions, while the (standard/single-stage) multiobjective program of Case II just gives a unique weak/efficient solution.

4.2 An Example Inspired by Bidding Strategies in Electricity Markets

Consider an *uncertain two-stage* multiobjective program:

Table 1 Comparison of solutions of Case I and Case II

Vectors: α's	(First-stage) Solutions of Case I: x's	Solutions of Case II: x's
(0, 0.5, 0.5)	(1.0000, 0.4823, 0.5177)	(1.0000, 0.0000, 1.0000)
(0.1, 0.45, 0.45)	(1.0000, 0.4420, 0.5580)	(1.0000, 0.0000, 1.0000)
(0.2, 0.4, 0.4)	(1.0000, 0.2878, 0.7122)	(1.0000, 0.0000, 1.0000)
(0.3, 0.35, 0.35)	(1.0000, −0.0294, 1.0294)	(1.0000, 0.0000, 1.0000)
(0.4, 0.3, 0.3)	(1.0000, −0.0864, 1.0864)	(1.0000, 0.0000, 1.0000)
(0.5, 0.25, 0.25)	(1.0000, 0.2996, 0.7004)	(1.0000, 0.0000, 1.0000)
(0.6, 0.2, 0.2)	(1.0000, 0.8889, 0.1111)	(1.0000, 0.0000, 1.0000)
(0.7, 0.15, 0.15)	(1.0000, 1.0240, −0.0240)	(1.0000, 0.0000, 1.0000)
(0.8, 0.1, 0.1)	(1.0000, 0.9151, 0.0849)	(1.0000, 0.0000, 1.0000)
(0.9, 0.05, 0.05)	(1.0000, 0.8622, 0.1378)	(1.0000, 0.0000, 1.0000)
(1, 0, 0)	(1.0000, 0.8837, 0.1163)	(1.0000, 0.0000, 1.0000)

$$\min_{x,y} \left(f_1(x,v) + (\theta_1^1)^\top y(v),\ f_2(x,v) + (\theta_1^2)^\top y(v),\ f_3(x,v) + (\theta_1^3)^\top y(v) \right) \tag{E3}$$

$$\text{s.t. } g_j(x,v) + (\theta_2^j)^\top y(v) \le 0,\ \ j = 1,2,3, \tag{33}$$

$$0 \le x_l \le x_c^{max},\ l = 1, ..., T, \tag{34}$$

$$0 \le x_{l+T} \le x_p^{max},\ l = 1, ..., T,$$

$$x_l + x_{l+T} \ge \sigma_4 D_l,\ l = 1, ..., T, \tag{35}$$

where $\theta_1^i \in \mathbb{R}^{2T}, i = 1, 2, 3, \theta_2^j \in \mathbb{R}^{2T}, j = 1, 2, 3, D_l \in \mathbb{R}, l = 1, \ldots, T, \sigma_4 > 0$, x_c^{max} and x_p^{max} are fixed parameters, $v := (v_1, ..., v_{2T})$ is an *uncertain* parameter, which resides in the uncertainty set $V \subset \mathbb{R}^{2T}$, $x := (x_1, ..., x_{2T})$ is the first-stage variable and $y : V \to \mathbb{R}^{2T}$ is the second-stage variable. In this setting, the functions $f_i : \mathbb{R}^{2T} \times \mathbb{R}^{2T} \to \mathbb{R}, i = 1, 2, 3, g_j : \mathbb{R}^{2T} \times \mathbb{R}^{2T} \to \mathbb{R}, j = 1, 2, 3$ are defined, respectively by, for $x := (x_1, ..., x_{2T}) \in \mathbb{R}^{2T}$ and $v := (v_1, ..., v_{2T}) \in V$,

$$f_1(x,v) := \sum_{i=1}^{T} (\pi_i x_i^2 - v_i x_i) - M, \quad f_2(x,v) := \sum_{i=T+1}^{2T} (\pi_i x_i^2 - C v_i x_i), \tag{36}$$

$$f_3(x,v) := \sum_{i=1}^{T} \pi_i x_i + \sum_{i=T+1}^{2T} v_i \pi_i x_i, \quad g_1(x,v) := \sum_{i=1}^{T} (\sigma_1 D_i - v_i x_i), \tag{37}$$

$$g_2(x,v) := \sum_{i=T+1}^{2T} (\sigma_2 D_{i-T} - v_i x_i), \quad g_3(x,v) := \sum_{i=1}^{T} (\sigma_3 D_i - v_i x_i - v_{i+T} x_{i+T}), \tag{38}$$

where $\pi_i > 0, i = 1, \ldots, 2T, M \in \mathbb{R}, \sigma_l > 0, l = 1, 2, 3$ and $C \in \mathbb{R}$. The uncertainty set V is defined by $V := \prod_{j=1}^{2T} [\lambda_j, \gamma_j]$, where $\lambda_j \in \mathbb{R}, \gamma_j \in \mathbb{R}$ are fixed and $\lambda_j < \gamma_j$ for $j = 1, \ldots, 2T$, and the second-stage decision variable y is given by

$$y(v) := y^0 + \sum_{i=1}^{2T} v_i y^i, \ v := (v_1, \ldots, v_{2T}) \in V,$$

where $y^i \in \mathbb{R}^{2T}, i = 0, 1, \ldots, 2T$ are non-adjustable decision variables.

Motivation by Bidding Strategies of Electricity Markets. The study of problem (E3) is inspired by modeling *coal-fired* power plants and *photovoltaic* (PV) systems in the electricity market. In this scenario, the objective function f_1 in (36) is to minimize cost for the coal-fired power stations, where π_i is a day-ahead electricity price at time slot i. M is the fixed operation cost for coal-fired power plants. x_i is the reserved power generation amount at time slot i in the day-ahead market. The objective function f_2 in (36) is to minimize cost for the PV systems, where C is the generation cost for the PV systems. The objective function f_3 in (37) is to minimize power purchasing costs for the customers. In the objective f_3, the first term denotes the total cost of purchasing power from the coal-fired power stations, while the second term indicates the cost of purchasing power from the PV systems. The first-stage constraints g_1 in (37) and g_2, g_3 in (38) explain the total power generation from the coal-fired power stations and the PV systems should satisfy the power demand at every time slot, where D_i is the amount of power demand at time slot i. $\sigma_l, l = 1, 2, 3, 4$ is scale parameters for the power demand. The constraints (34) to (35) explain the lower bound and upper bound of power generation amount for the coal-fired power plants and the PV systems, respectively, where x_c^{max} is the maximal power output from the coal-fired power plants and x_p^{max} is the maximal power output from the PV systems. v_i is an uncertain extra electricity price that is applied based on the actual scenarios such as higher or lower demands of power customers or affected weather conditions on the PV power generation over the prescribed time periods, while y is an adjustable rule of reserved power generation amounts that could be implemented and controlled by θ_1^i and θ_2^j.

Semidefinite Programming Relaxations. For $x := (x_1, ..., x_{2T}) \in \mathbb{R}^{2T}$ and $v := (v_1, ..., v_{2T}) \in V$, put

$$g_j(x, v) := \begin{cases} -x_{j-3}, & j = 4, \ldots, 3 + T, \\ x_{j-3-T} - x_c^{max}, & j = 4 + T, \ldots, 3 + 2T, \\ -x_{j-3-T}, & j = 4 + 2T, \ldots, 3 + 3T, \\ x_{j-3-2T} - x_p^{max}, & j = 4 + 3T, \ldots, 3 + 4T, \\ \sigma_4 D_{j-3-4T} - x_{j-3-4T} - x_{j-3-3T}, & j = 4 + 4T, \ldots, 3 + 5T, \end{cases}$$

and denote by $Y := (y^1, \ldots y^{2T})$ a matrix whose columns are those of $y^i, i = 1, \ldots, 2T$. Then the uncertain two-stage problem (E3) can be relabeled as the following one:

$$\min_{x,y} \left(f_1(x,v) + (\theta_1^1)^\top y(v),\ f_2(x,v) + (\theta_1^2)^\top y(v),\ f_3(x,v) + (\theta_1^3)^\top y(v)\right) \quad \text{(A3)}$$
$$\text{s.t. } g_j(x,v) + (\theta_2^j)^\top y(v) \le 0,\ \ j = 1,\dots,3+5T,$$

where $x \in \mathbb{R}^{2T}$, $y(v) = y^0 + Yv$ for $v \in V$ and $\theta_2^j := 0_{2T}$ for all $j = 4,\dots,3+5T$.

Now, consider a *robust two-stage* multiobjective problem, which is the robust counterpart of (A3) defined by

$$\min_{(x,y^0,Y)} \left(\max_{v\in V}\{f_1(x,v) + (\theta_1^1)^\top y(v)\}, \max_{v\in V}\{f_2(x,v) + (\theta_1^2)^\top y(v)\}, \max_{v\in V}\{f_3(x,v) + (\theta_1^3)^\top y(v)\}\right) \quad \text{(R3)}$$
$$\text{s.t. } g_j(x,v) + (\theta_2^j)^\top y(v) \le 0,\ \ j = 1,\dots,3+5T,$$
$$x \in \mathbb{R}^{2T}, y^0 \in \mathbb{R}^{2T}, Y \in \mathbb{R}^{2T\times 2T}, y(v) = y^0 + Yv, \forall v \in V.$$

Note further that the box $V := \prod_{j=1}^{2T} [\lambda_j, \gamma_j]$ can be written as the following spectrahedron:

$$V = \{v := (v_1,\dots,v_{2T}) \in \mathbb{R}^{2T} \mid A + \sum_{l=1}^{2T} v_l A_l \succeq 0\}, \quad (39)$$

where $A := \operatorname{diag}(-\lambda_1,\dots,-\lambda_{2T},\gamma_1,\dots,\gamma_{2T})$ and $A_l := \operatorname{diag}\begin{pmatrix} e_l^{2T} \\ -e_l^{2T}\end{pmatrix}, l = 1,\dots,2T$. Therefore, the problem (R3) lands in the form of (RT), where the functions $f_i, i = 1,2,3$ and $g_j, j = 1,\dots,3+5T$ are defined by $Q_1^1 := \operatorname{diag}\left(\sum_{l=1}^{T} \pi_l e_l^{2T}\right)$, $Q_1^2 := \operatorname{diag}\left(\sum_{l=T+1}^{2T} \pi_l e_l^{2T}\right)$, $Q_1^3 := 0_{2T\times 2T}$, $\xi_1^1 := \xi_1^2 := 0_{2T}$, $\xi_1^3 := \sum_{l=1}^{T} \pi_l e_l^{2T}$, $\beta_1^1 := -M$, $\beta_1^2 := \beta_1^3 := 0$, $\xi_{1,l}^1 := \begin{cases} -e_l^{2T}, & l = 1,\dots,T, \\ 0_{2T}, & l = T+1,\dots,2T,\end{cases}$ $\xi_{1,l}^2 := \begin{cases} 0_{2T}, & l = 1,\dots,T, \\ -Ce_l^{2T}, & l = T+1,\dots,2T,\end{cases}$ $\xi_{1,l}^3 := \begin{cases} 0_{2T}, & l = 1,\dots,T, \\ \pi_l e_l^{2T}, & l = T+1,\dots,2T,\end{cases}$ $\beta_{1,l}^i := 0, l = 1,\dots,2T, i = 1,2,3$, $Q_2^j := 0_{2T\times 2T}, j = 1,\dots 3+5T$, $\beta_{2,l}^j := 0, l = 1,\dots,2T, j = 1,\dots,3+5T$, $\xi_{2,l}^1 := \begin{cases} -e_l^{2T}, & l = 1,\dots,T, \\ 0_{2T}, & l = T+1,\dots,2T,\end{cases}$ $\xi_{2,l}^2 := \begin{cases} 0_{2T}, & l = 1,\dots,T, \\ -e_l^{2T}, & l = T+1,\dots,2T,\end{cases}$ $\xi_{2,l}^3 := -e_l^{2T}, l = 1,\dots,2T$, $\xi_{2,l}^j := 0_{2T}, l = 1,\dots,2T, j = 4,\dots,3+5T$,

$$\xi_2^j := \begin{cases} 0_{2T}, & j = 1, 2, 3, \\ -e_{j-3}^{2T}, & j = 4, \ldots, 3+T, \\ e_{j-3-T}^{2T} & j = 4+T, \ldots, 3+2T, \\ -e_{j-3-T}^{2T}, & j = 4+2T, \ldots, 3+3T, \\ e_{j-3-2T}^{2T}, & j = 4+3T, \ldots, 3+4T, \\ -e_{j-3-4T}^{2T} - e_{j-3-3T}^{2T}, & j = 4+4T, \ldots, 3+5T, \end{cases}$$

$$\beta_2^j := \begin{cases} \sum_{l=1}^{T} \sigma_1 D_l, & j = 1, \\ \sum_{l=1}^{T} \sigma_2 D_l, & j = 2, \\ \sum_{l=1}^{T} \sigma_3 D_l, & j = 3, \\ 0, & j = 4, \ldots, 3+T, \\ -x_c^{max} & j = 4+T, \ldots, 3+2T, \\ 0, & j = 4+2T, \ldots, 3+3T, \\ -x_p^{max}, & j = 4+3T, \ldots, 3+4T, \\ \sigma_4 D_{j-3-4T}, & j = 4+4T, \ldots, 3+5T. \end{cases}$$

Let us employ the obtained relaxations from Theorem 3.1 to calculate (weak) efficient solutions of (R3). By choosing $\widehat{v} := (\frac{\lambda_1+\gamma_1}{2}, \ldots, \frac{\lambda_{2T}+\gamma_{2T}}{2}) \in \mathbb{R}^{2T}$, it holds that $A + \sum_{l=1}^{2T} \widehat{v}_l A_l \succ 0$, which means that the condition (1) holds for this setting.

For each $\alpha := (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}_+^3 \backslash \{0\}$, one considers a robust weighted sum optimization problem of (R3) given by

$$\min_{x\in\mathbb{R}^{2T}, y^0\in\mathbb{R}^{2T}, Y\in\mathbb{R}^{2T\times 2T}} \Big\{ \sum_{i=1}^{3} \alpha_i \max_{v\in V}\{f_i(x,v) + (\theta_1^i)^\top y(v)\} \mid g_j(x,v) + (\theta_2^j)^\top y(v) \le 0, \tag{E3$_\alpha$}$$

$$j = 1, \ldots, 3+5T, \, y(v) = y^0 + Yv, \forall v \in V \Big\}.$$

The SDP relaxation problem of (E3$_\alpha$) is given by

$$\begin{aligned} \min_{(z,\nu_0,\nu^l,Z,W_1^i,W_2^j)} \; & \sum_{i=1}^{3} \alpha_i \big(\beta_1^i + (\xi_1^i)^\top z + (\theta_1^i)^\top \nu_0 + \mathrm{Tr}(ZQ_1^i) + \mathrm{Tr}(W_1^i A)\big) && \text{(E3}^*_\alpha\text{)} \\ \text{s.t.} \; & \begin{pmatrix} 1 & z^\top \\ z & Z \end{pmatrix} \succeq 0, z \in \mathbb{R}^{2T}, Z \in S^{2T}, \nu_0 \in \mathbb{R}^{2T}, \nu^l \in \mathbb{R}^{2T}, l = 1, \ldots, 2T, \\ & \beta_{1,l}^i + (\theta_1^i)^\top \nu^l + (\xi_{1,l}^i)^\top z + \mathrm{Tr}(W_1^i A_l) = 0, i = 1, 2, 3, l = 1, \ldots, 2T, \\ & \beta_2^j + (\xi_2^j)^\top z + (\theta_2^j)^\top \nu_0 + \mathrm{Tr}(ZQ_2^j) + \mathrm{Tr}(W_2^j A) \le 0, j = 1, \ldots, 3+5T, \\ & \beta_{2,l}^j + (\theta_2^j)^\top \nu^l + (\xi_{2,l}^j)^\top z + \mathrm{Tr}(W_2^j A_l) = 0, j = 1, \ldots, 3+5T, l = 1, \ldots, 2T, \end{aligned}$$

$$W_1^i \succeq 0,\, W_1^i \in S^{4T},\, W_2^j \succeq 0,\, W_2^j \in S^{4T},\, i = 1, 2, 3,\, j = 1, \ldots, 3 + 5T.$$

According to the obtained relaxation schemes in Theorem 3.1, we assert that, for a given $\alpha \in \mathrm{int}\mathbb{R}^3$ (resp., $\alpha \in \mathbb{R}^3_+\backslash\{0\}$), if $(\overline{z}, \overline{\nu}_0, \overline{\nu}^l, \overline{Z}, \overline{W}_1^i, \overline{W}_2^j)_{i=1,2,3,j=1,\ldots,3+5T,l=1,\ldots,2T}$ is a solution of ($E3^*_\alpha$), then $(\overline{z}, \overline{\nu}_0, \overline{Y})$ is a (resp., weak) efficient solution of (R3), where $\overline{Y} := (\overline{\nu}^1, \ldots, \overline{\nu}^{2T})$ stands for a matrix with the columns of $\overline{\nu}^l, l = 1, \ldots, 2T$. So, $(\overline{z}, \overline{y})$ is a robust solution of (E3), where $\overline{y}(\upsilon) = \overline{\nu}^0 + \sum_{l=1}^{2T} \upsilon_l \overline{\nu}^l$ for $\upsilon := (\upsilon_1, \ldots, \upsilon_{2T}) \in V$.

Numerical Simulations. Our study utilizes a day-ahead dataset provided by the Australian Energy Market Operator (AEMO), which features information on electricity price and demand in the Victoria state of Australia. We extract data from 8:00 to 17:00 on March 2, 2022, as shown in Table 2. In our model, we set the maximal power output from the coal-fired power plants as $x_c^{max} = 71160$ MWh, while the maximal power output by the PV systems is $x_p^{max} = 1000$ MWh. Furthermore, we utilize a fixed operation cost of AUD \$50,000 for coal-fired power plants, and a generation cost of AUD \$40 per MWh for the PV systems. We random the values of θ_1^1, θ_1^2 and θ_1^3 as given in Table 3 and set $\theta_2^1 = -\theta_1^1, \theta_2^2 = -\theta_1^2$ and $\theta_2^3 = -\theta_1^3$.

Note that the weights of the objective functions satisfy $\alpha_1 + \alpha_2 + \alpha_3 = 1$, where the values of α_1, α_2 and α_3 range from 0.00001 to 1 with increments of 0.025 up to 0.05, and further increments of 0.075 and so on. To evaluate the effectiveness of the proposed two-stage model, we conducted simulations using two-stage multiobjective programs and (single-stage/standard) multiobjective programs. We simulated all possible combinations of weights and obtained the results presented in Fig. 1, which is an almost comprehensive set of (first-stage) efficient/Pareto solutions.

The figure presented in the analysis illustrates the varying costs associated with three different tasks, based on the different weights assigned to three objective functions. In the case of single-stage analysis, the efficient/Pareto solutions for the three objective functions are consolidated within a single range. However, in the two-stage analysis, the (first-stage) efficient/Pareto solutions for the three objective functions are distributed across multiple ranges, highlighting the greater flexibility afforded to the electricity market operator to adjust costs between different sources of electricity generation, such as power plant generation, PV generation and market purchasing. The deployment of a two-stage multiobjective approach presents several advantages, such as facilitating the increased integration of renewable energy sources into the grid and reducing reliance on fossil fuels. The flexibility provided by this approach enables the operator to adjust costs to incentivize the use of surplus electricity generated by renewable energy sources, and thus promote their utilization and facilitate their integration into the electricity system. This not only contributes to the achievement of environmental objectives by reducing greenhouse gas emissions but also supports the overall stability and reliability of the electricity grid. Consequently, the two-stage multiobjective approach offers more flexible and efficient means of adjusting electricity costs than a standard (single-stage) multiobjective approach that leads to far-reaching benefits for the electricity market and society at large.

Table 2 The day-ahead electricity price in wholesale market and PV market, and electricity demand on 02/03/2022

Time	8:00	9:00	10:00	11:00	12:00	13:00	14:00	15:00	16:00	17:00
π ($/MWh)	75.83	82.64	72.58	74.76	83.44	81.50	72.17	79.66	69.71	78.37
π_{PV} ($/MWh)	1.44	1.69	1.68	1.54	1.64	1.44	1.52	1.67	1.63	1.68
D_i (MWh)	59245	52208	49014	46753	44228	45289	45810	47687	53317	59404

Table 3 The values of θ_1^i

i	1	2	3	4	5	6	7	8	9	10
θ_1^1	0.547	0.296	0.745	0.189	0.687	0.184	0.368	0.626	0.780	0.081
θ_1^2	0.644	0.379	0.812	0.533	0.351	0.939	0.876	0.550	0.622	0.587
θ_1^3	0.311	0.923	0.430	0.185	0.905	0.980	0.439	0.111	0.258	0.409
i	11	12	13	14	15	16	17	18	19	20
θ_1^1	0.929	0.776	0.487	0.436	0.447	0.306	0.509	0.511	0.818	0.795
θ_1^2	0.208	0.301	0.471	0.230	0.844	0.195	0.226	0.171	0.228	0.436
θ_1^3	0.595	0.262	0.603	0.711	0.222	0.117	0.297	0.319	0.424	0.508

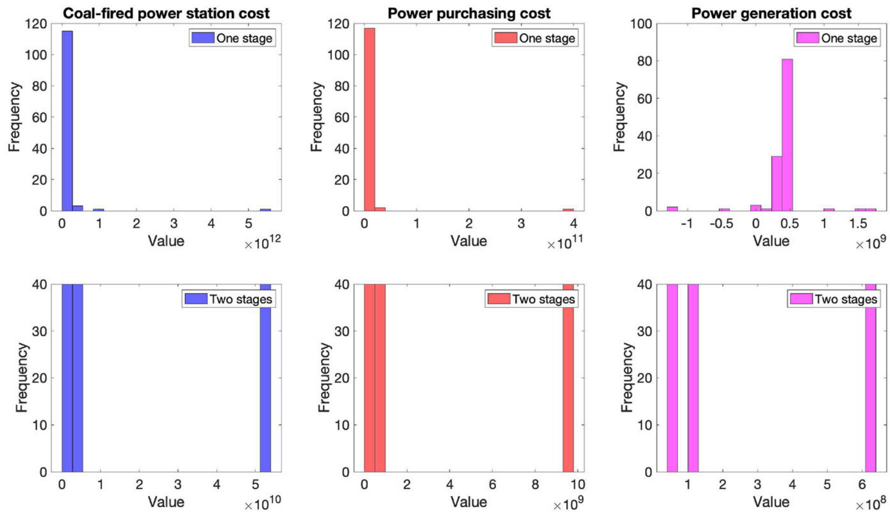

Fig. 1 Solutions of single-stage and two-stage bidding strategies

5 Conclusions with Study Perspectives

We provided verifiable necessary and sufficient optimality conditions for a class of robust two-stage multiobjective optimization problems. The obtained optimality conditions can be numerically validated by using a semidefinite programming problem as they are presented by way of linear matrix inequalities. These optimality criteria can be expressed further as second-order conic conditions for robust two-stage multiobjective optimization problems involving separable functions and ellipsoidal uncertainty sets. We addressed relaxation schemes that allow one to locate (weak) efficient solutions for a robust two-stage multiobjective optimization problem by solving related semidefinite programming or second-order cone programming relaxation problems.

Given numerical examples have exemplified that our two-stage multiobjective model is more flexible than a (single-stage) multiobjective version in terms of solution variety. In more detail, the former produces more (first-stage) efficient solutions than the latter does when solving them through a same set of weighted sum vectors. Furthermore, the simulation on an example arisen from practical applications has shown that the two-stage multiobjective approach offers various benefits such as reducing the dependence on fossil fuels, achieving environmental objectives and supporting the stability and reliability of the electricity grid. Consequently, this approach presents a more efficient and flexible mean of adjusting electricity costs than a standard (single-stage) multiobjective approach that leads to far-reaching benefits for the electricity market and society.

When dealing with a general two-stage multiobjective optimization problem in practice, where the problem data involve *nonconvex* functions, corresponding scalarized/weighted sum methods often lose their effectiveness in locating (weak) efficient/Pareto solutions. Therefore, a further investigation on how to make use of such relaxation schemes to solve nonconvex two-stage multiobjective problems would be of great interest from practical implementations. It is also worth looking at whether the current approach can be exploited to formulate and solve other real-world uncertain multiobjective models such as *integrated energy systems* with load uncertainties (see e.g., [32]), where the decision makers would deal with conflicting demands and benefits of the energy efficiency and the carbon emissions reduction under load budget uncertainty scenarios.

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Declarations

Conflict of interest: The authors declare that they have no Conflict of interest.

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