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New control variates for pricing basket options

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Accurate pricing of basket options, which are financial derivatives on multiple underlying assets, is a challenging and practically important task for financial institutions. We propose several new control variates for accurate, fast and efficient pricing of basket options. The first approach to deriving new control variates is the use of Hermite polynomial approximation of appropriate function of the underlying asset prices, which leads to a Black–Scholes-like analytic solution. This approach is new in the option pricing context and opens up new possibilities in derivative pricing. Further control variates are analytically derived using Jensen's inequality in one case, and distributional properties of multivariate Wiener processes in other cases. All the newly proposed control variates are shown to lead to excellent variance reduction in numerical experiments based on realistic data. The proposed methods are novel, computationally simple and have a strong potential to replace more conventional methods, such as the geometric lower bound in simulation-based pricing of basket options and similar products used in financial risk management.

Keywords: finance; simulation; stochastic processes.

1. Introduction

A basket option is an option whose underlying is a group of assets and whose payoff depends on the weighted average of these assets at the maturity date. Basket options are popular over-the-counter financial instruments for cost-effective hedging of multiple underlying positions. Pricing baskets accurately can be challenging since the weighted average of the assets do not follow a closed-form distribution in general, even when the individual prices have a closed-form (e.g. lognormal) distribution. There is a significant amount of research over the past few decades on deriving approximate closed-form prices for baskets. Kemna & Vorst (1990) and Curran (1994) approximated the price of a basket option using the geometric mean of the prices of assets in the basket. Besides being a lower bound on the arithmetic mean, the geometric mean of lognormal random variables is also lognormal and leads to a closed-form estimate of basket option price. However, Gentle (1993) found that using a superposition of the strike and the expected difference of the average value of the basket and its geometric average yielded better approximations of the basket option price. In a different approach, Milevsky & Posner (1998) obtained closed-form estimates for the basket option price, by approximating the distribution of the sum of the lognormal distribution by reciprocal gamma distribution. The only drawback is it was found to under-price out-of-the-money call options when compared with the Monte Carlo prices. Other closed-form approximations are based on moment matching and numerical approximations.

Methods of moment matching were used by Brigo *et al.* (2001) and Henriksen (2008) to approximate the price of a basket option. This approach generally involves using a lognormal random variable and matching its first and second moments with those of the basket. Ju (2002) used Taylor series expansion to price basket and Asian options.

Despite several analytical approximations to pricing a basket option, Monte Carlo simulation remains the most accurate way of valuing such options. The biggest downsides of this approach are a high variance of the resulting estimates and a high computational cost. This has led to the use of various variance reduction methods, including antithetic variates and control variates based on closed-form bounds or approximations. De Luigi & Maire (2010) used adaptive numerical techniques to price low-dimensional basket options and found that this technique served as good control variates in pricing high-dimensional baskets. Dingeç& Hörmann (2013) used a conditional Monte Carlo method along with geometric average as control variate to price basket options. Korn & Zeytun (2013) proposed the use of a limiting approximation of the arithmetic mean by the geometric mean, to obtain closed-form estimates as well as low variance Monte Carlo estimates for the basket option price. Lai *et al.* (2015) used control variate to price basket option under jump diffusion models. This was extended to stochastic volatility models with jumps using relevant asymptotic expansions as control variates (Shiraya & Takahashi, 2017). This method of asymptotic expansion was also previously used by Xu & Zheng (2010) with Forward Partial Integro-Differential Equation to approximate the basket option price.

Dingeç (2019) also proposed new control variate models using time-changed Brownian motions for pricing and sensitivity analysis of Basket options. Shiraya *et al.* (2020) proposed a class of control variates for pricing basket options driven by Lévy processes with the use of subordinated Wiener processes. Zhang *et al.* (2019) extended this to exponential subordinated Wiener processes. Kreckel *et al.* (2004) provide a systematic numerical comparison of different basket option pricing methods. In the context of the existing work described above, the contribution of the paper can be summarized as follows:

1. For a basket of assets (which includes a practically important class of spread options), we introduce a new control variate for pricing basket options. This is done using a first-order Hermite polynomial approximation on the logarithm of a derived lower bound of the value of the basket, which is lognormally distributed. This leads to closed-form option price on the lower bound with modified strike, which is the logarithm of the strike of the basket.

2. We suggest the use of a new closed-form upper bound in terms of the Jensen's inequality as a control variate. To our knowledge, the use of this bound has not been explored in published literature. Using a fictitious basket option with volatilities and correlations of real-world indices, we demonstrate that the use of this bound as a control variate dramatically improves the variance of a Monte Carlo estimate.

3. We also propose two new distributional closed-form bounds on the basket option price: a lower bound using the minimum of Brownian motions and an upper bound using the maximum of Brownian motions provided certain integrability conditions are imposed. These bounds are new and contribute to the literature on stochastics. We demonstrate that these bounds can be used for variance reduction, with the distributional lower bound giving excellent variance reduction.

All our numerical experiments are benchmarked against a standard control variate, *viz* the geometric mean approximation as a lower bound on basket option price.

Note that the focus of this paper is variance reduction using novel control variates, which perform better than the benchmark method and lead to explicit analytic expressions. They also havethe potential to be useful for other multivariate Monte Carlo applications related to financial derivative pricing. Other variance reduction methods, such as importance sampling, hybrid methods (Sun & Chenglong, 2018) and quasi-Monte Carlo methods are important in their own right, but are beyond the scope of this paper.

The rest of the paper is organized as follows. Section 2 covers the modelling of the problem. Section 3 comprises numerical experiments which compare our new algorithms with the benchmark algorithms on realistic data sets. Section 4 concludes the paper.

2. Modelling approach

We model the economy with a filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{Q})$, where Ω represents the sample space, $\{\mathscr{F}_t\}_{t\geq 0}$ represents the filtration generated by n independent Brownian motions $W_1(t), ..., W_n(t)$ and $\mathbb{Q}$ is the risk neutral measure. We assume that the asset prices $S_i(t)$ are $\mathscr{F}_t$-measurable and follow a geometric Brownian motion (GBM) model given by

$$S_i(t) = S_i(0) \exp\left[\left(r - \frac{1}{2}\sigma_i^2\right) t + \sum_{j=1}^{n} \sigma_{ij} W_j(t)\right], \ \forall i = 1, ..., n, \tag{1}$$

i.e. each $S_i(t)$ satisfies the stochastic differential equation,

$$\frac{dS_i(t)}{S_i(t)} = rdt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t). \tag{2}$$

Here r is the risk-free rate and σ_i is the volatility of the asset i such that $\sigma_i^2 = \sum_{j=1}^{n} \sigma_{ij}^2$.

The value of a basket of n assets at a time t which satisfies (1)–(2) is given by

$$S(t) = \sum_{i=1}^{n} \omega_i S_i(t), \tag{3}$$

where $\omega_i \geq 0$ such that $\sum_{i=1}^{n} \omega_i = 1$.

The price $C(0, T, K)$ of a basket call option at a time 0, with a strike K and maturing at time T, is

$$C(0, T, K) = e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[(S(T) - K)^+\right], \tag{4}$$

where $\mathbb{E}^{\mathbb{Q}}[-]$ denotes the expectation under risk-neutral measure. In the subsequent subsections, we derive the first-order Hermite polynomial control variate for pricing a basket option, which will later be used in Sections 3 and 4.

2.1. *Lognormal approach using first-order hermite polynomials*

For $n = 1$, we know that $S(t)$ is lognormal and $C(0, T, K)$ is available in closed-form in terms of Black–Scholes formula. The main idea is to derive some lognormal random variable whose behaviour mimics that of the basket and which has a closed-form solution. If we can approximate the summation $\sum_{i=1}^{n} \omega_i S_i(t)$ by a lognormal random variable with a known finite variance in closed-form, we can obtain a Black–Scholes-type approximate solution for the price of the basket option. Instead of trying to approximate $S(t)$ directly e.g. using moments as in Leccadito *et al.* (2016), we construct a linear approximation of $y(t) := \ln(S(t))$ in terms of $\ln(S_i(t))$, using a multivariate Hermite polynomial basis. As each $\ln(S_i(t))$ is normal, so is our approximation to $y(t)$. Consequently, $e^{y(t)}$ is lognormal and this allows us to use Black–Scholes-like formula for the basket option price, in terms of the parameters of our linear approximation. Evaluating the option price under GBM assumption always involves an accurate evaluation of integrals with respect to the Gaussian measure, which motivates our choice of Hermite polynomials as a basis for linear approximation. Given the function φ, which is the log of the terminal value of the basket, we can rewrite φ as a function of standard normal variables such that

$$\varphi = \ln\big(S(T)\big), \tag{5}$$

$$= \ln\left(\sum_{i=1}^{n}\omega_i S_i(0)\exp\left(\left(r-\frac{1}{2}\sigma_i^2\right)T+\sum_{j=1}^{n}\sigma_{ij}W_j(T)\right)\right),$$

$$= \ln\left(\sum_{i=1}^{n}\omega_i S_i(0)\exp\left(\left(r-\frac{1}{2}\sigma_i^2\right)T+\sqrt{T}\sum_{j=1}^{n}\sigma_{ij}u_j\right)\right), \tag{6}$$

where $u_j \sim \mathcal{N}(0,1)\ \ \forall j$.

A non-linear function of Gaussian random variables can be approximated by a linear combination of standard normally distributed random variables using Hermite polynomial basis as follows. Consider a class of functions $\mathscr{Y}$ such that

$$\mathscr{Y} = \left\{\varphi(u) : \int_{-\infty}^{\infty}\phi(u;0,I)\Phi^2(u)du < \infty, \forall j\right\},$$

where $\phi(u;0,I)$ is the density of a standard normal vector $u = (u_1,...,u_n)$ with covariance matrix I.

We define the first-order Hermite polynomials $\{h_j^{(1)}(u)\}_{j=1}^{n}$ as

$$h_j^{(1)}(u) = (-1)\frac{\partial\phi(u;0,I)}{\partial u_j}\phi^{-1}(u;,0,I), \tag{7}$$

where $h_j^{(0)}(u) = 1\,\forall j$. These polynomials satisfy the orthogonality condition with respect to the Gaussian density:

$$\int_{-\infty}^{\infty} h_j^{(1)}(u)h_k^{(1)}(u)\phi(u;0,I)du = \delta_{jk},$$

where $\delta_{jk} = \begin{cases}1, & \text{if } j=k\\ 0, & \text{otherwise.}\end{cases}$

Since $\varphi \in \mathscr{Y}$, we can approximate $\varphi(u)$ as

$$\varphi(u) \approx \hat{\varphi}(u) = \sum_{j=0}^{n} b_j h_j^{(1)}(u),$$

$$= b_0 + \sum_{j=1}^{n} b_j u_j. \tag{8}$$

The coefficients of the orthogonal expansion are obtained as

$$b_0 = \int_{-\infty}^{\infty} \varphi(u)\phi(u)du \quad \text{and,} \tag{9}$$

$$b_j = \int_{-\infty}^{\infty} u_j\varphi(u)\phi(u)du,\ 1 \le j \le n. \tag{10}$$

We can now approximate the basket option price under the assumption that the basket of assets in lognormally distributed.

Proposition 1. The approximate price $\hat{C}(0,T,K)$ of a basket option at a time 0 at an earlier time T with non-negative strike K is given by

$$\hat{C}(0,T,K) = e^{-rT}\left[\exp\left(\frac{1}{2}\left(V+2b_0\right)\right)\Phi\left(\frac{b_0+V-\ln K}{\sqrt{V}}\right) - K\Phi\left(\frac{b_0-\ln K}{\sqrt{V}}\right)\right], \tag{11}$$

where $V = \sum_{j=1}^{n} b_j^2$.

Proof.

$$\hat{C}(0,T,K) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(e^{\hat{\Phi}(u)} - K\right)^+\right]. \tag{12}$$

We define the density $\rho_\Phi(y)$ of $\hat{\Phi}(u)$ as

$$\rho_\Phi(y) = \frac{1}{\sqrt{V}}\phi\left(\frac{y-b_0}{\sqrt{V}}\right). \tag{13}$$

The approximate terminal payoff is given by

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\left(e^{\hat{\Phi}(u)} - K\right)^+\right] &= \int_{-\infty}^{\infty}\left(e^y - K\right)^+\rho_\Phi(y)dy,\\ &= \exp\left(\frac{1}{2y}\left[(b_0+V)^2 - b_0^2\right]\right)\Phi\left(\frac{b_0+V-\ln K}{\sqrt{V}}\right) - K\Phi\left(\frac{b_0-\ln K}{\sqrt{V}}\right).\end{aligned} \tag{14}$$

The approximated price $\hat{C}(0,T,K)$ of the basket option, using a linear Hermite polynomial approximation, is thus given by

$$\hat{C}(0,T,K) = e^{-rT}\left[\exp\left(\frac{1}{2}\left(V+2b_0\right)\right)\Phi\left(\frac{b_0+V-\ln K}{\sqrt{V}}\right) - K\Phi\left(\frac{b_0-\ln K}{\sqrt{V}}\right)\right], \tag{15}$$

which completes the proof. □

The closed-form approximation in equation (15) for a basket option price is analogous to Black–Scholes representation for the price of a single asset.

In practice, we estimate b_j's using a third-order Taylor series expansion of Φ about zero in equations (9) and (10), which leads to the integral in question being approximated by a simple linear function of moments of a Gaussian variable. Details are straightforward and are omitted for brevity. An alternative would be to use Gaussian quadrature to evaluate the said one-dimensional integrals. Also, this closed-form estimate can be used as a control variate for pricing basket options. However, the computational complexity of calculating the basket option price increases as the number of assets in the basket increases. To overcome this, we suggest an adaptation to the previously mentioned method to allow for its use as a control variate for pricing basket options with sufficiently large assets in the basket.

In general,

$$\ln(1 + \overline{S_i(T)}) \leq \overline{S_i(T)}, \tag{16}$$

where $\overline{S_i(T)} = \omega_i S_i(T)$. Taking sums of (16) over i, we can deduce the following inequality:

$$\ln\left(\prod_{i=1}^{n} \overline{S_i(T)}\right) < \sum_{i=1}^{n} \ln\left(1 + \overline{S_i(T)}\right) < \sum_{i=1}^{n} \overline{S_i(T)} = S(T). \tag{17}$$

We can simplify the weighted products of assets as

$$\prod_{i=1}^{n} \overline{S_i(T)} = \left(\prod_{i=1}^{n} \overline{S_i(0)}\right) \exp\left[\left(nr - \frac{1}{2}\sum_{i=1}^{n} \sigma_i^2\right) T\right] \exp\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} W_j(T)\right), \tag{18}$$

and its logarithm becomes

$$\ln\left(\prod_{i=1}^{n} \overline{S_i(T)}\right) = \ln\left(\prod_{i=1}^{n} \overline{S_i(0)}\right) + \left(nr - \frac{1}{2}\sum_{i=1}^{n} \sigma_i^2\right) T + \sum_{i=1}^{n}\sum_{j=1}^{d} \sigma_{ij} W_j(T), \tag{19}$$

$$= \gamma + \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} W_j(T), \tag{20}$$

where $\gamma = \ln\left(\prod_{i=1}^{n} \overline{S_i(0)}\right) + \left(nr - \frac{1}{2}\sum_{i=1}^{n} \sigma_i^2\right) T$.

We define a new function φ' by simply replacing $S(T)$ in (5) with the logarithm of (20) and the strike of the basket with $\ln K$ to obtain

$$\varphi' = \ln\left(\ln\left(\prod_{i=1}^{n} S(T)\right)\right), \tag{21}$$

$$= \ln\left(\gamma + \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} W_j(T)\right),$$

$$= \ln\left(\gamma + \sqrt{T}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} u_j\right), \tag{22}$$

where $u_j \sim \Phi(0,1) \ \ \forall j$.

Since $\varphi' \in \mathscr{Y}$, we can approximate $\varphi'(u)$ as

$$\varphi'(u) = b_0' + \sum_{j=1}^{n} b_j' u_j, \tag{23}$$

where the parameters b_j''s are estimated as b_j's in (9) and (10) but are estimated by replacing φ with φ' for all $0 \leq j \leq n$.

We can estimate the parameters of b_j''s in (23) using a third-order Taylor series approximation of φ' of u_j's about 0 given by

$$\varphi'(u) = \varphi'(0) + \sum_{j=1}^{n} \frac{\partial \varphi'}{\partial u_j}\bigg|_{u_j=0} u_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 \varphi'}{\partial u_j u_k}\bigg|_{u_j=u_k=0} u_j u_k$$
$$+ \frac{1}{6}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \frac{\partial^3 \varphi'}{\partial u_j u_k u_l}\bigg|_{u_j=u_k=u_l=0} u_j u_k u_l. \tag{24}$$

Thus, the coefficients of the parameters of Φ' are

$$b_0' = \ln\gamma - \frac{1}{2}\frac{T}{\rho^2}\sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{n} \sigma_{ij}\sigma_{kj}, \tag{25}$$

$$= \ln\gamma - \frac{1}{2}\frac{T}{\rho^2}\sum_{k=1}^{n}\sum_{i=1}^{n} A_{ik}, \tag{26}$$

where A is the volatility matrix, $\rho = \gamma + \sqrt{T}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} \sigma_{ij} u_j$ and

$$b_j' = \frac{\sqrt{T}}{\gamma}\sum_{i=1}^{n}\sigma_{ij} + \left(\frac{\sqrt{T}}{\gamma}\right)^3 \sum_{i=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n}\sigma_{ij}\sigma_{lj}\sigma_{mj} + \left(\frac{\sqrt{T}}{\gamma}\right)^3 \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n}\sigma_{ij}\sigma_{mk}\sigma_{lj}\delta_{jk}, \tag{27}$$

for all $1 \leq j \leq n$.

We denote the price at time 0 of an option on the logarithm of the product of weighted assets in the basket with a strike $\ln K$, maturing at time T as $C'(0, T, \ln K)$ is given by

$$C'(0, T, \ln K) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(\ln\left(\prod_{i=1}^{n} S(T)\right) - \ln K\right)^{+}\right]. \tag{28}$$

Using the lognormal approximation for $\ln\left(\prod_{i=1}^{n} \overline{S(T)}\right)$ in (23), we can obtain the option price $C'(0, T, \ln K)$ in a lognormal framework. Thus, the option price $C'(0, T, \ln K)$ is given by

$$C'(0, T, \ln K) = e^{-rT}\left[\exp\left(\frac{1}{2}\left(\bar{V} + 2b_0'\right)\right)\Phi\left(\frac{b_0' + V - \ln \bar{K}}{\sqrt{\bar{V}}}\right) - \bar{K}\Phi\left(\frac{b_0' - \ln \bar{K}}{\sqrt{\bar{V}}}\right)\right], \tag{29}$$

where $\bar{K} = \ln K$ and $\bar{V} = \sum_{j=1}^{n} b_j'^2$.

Given that we have the price of the option $C'(0, T, \ln K)$ in closed-form, we have all the essential ingredients necessary to use this method for pricing basket options, using the first-order Hermite polynomial as a control variate for a large number of underlying assets.

2.2. *Direct upper bound*

As a second control variate, we look at an upper bound on the option price. As mentioned earlier, a lot of analytical work was done in estimating bounds on its price (Rogers & Shi (1995), Xu & Zheng (2010)). However, we can obtain an easy and direct upper bound on the price of a basket option by a direct application of the Jensen's inequality due to the convexity of the payoff function of the basket. To our knowledge, this bound has not been used in the published literature and can be explained as follows.

$$\mathbb{E}\left(S(T) - K\right)^{+} = \mathbb{E}\left(\sum_{i=1}^{n} \omega_i S_i(T) - K\right)^{+} \leq \sum_{i=1}^{n} \omega_i \mathbb{E}\left(S_i(T) - K\right)^{+}. \tag{30}$$

So that,

$$e^{-rT}\mathbb{E}\left(S(T) - K\right)^{+} \leq e^{-rT}\sum_{i=1}^{n} \omega_i \mathbb{E}\left(S_i(T) - K\right)^{+} =: U_B. \tag{31}$$

The upper bound U_B on the price of a basket option can be seen as the same as holding n options of different assets with the same strike K. The price U_B of such a fictitious portfolio is given by

$$U_B = e^{-rT}\sum_{i=1}^{n}\left[S_i(0)e^{rT}\Phi(h_i^{+}) - K\Phi(h_i^{-})\right], \tag{32}$$

where $h_i \pm = \dfrac{\ln\left(\frac{S_i(0)e^{rT}}{K}\right) \pm \frac{1}{2}\sigma_i^2 T}{\sigma_i\sqrt{T}}$.

While an upper bound similar to the one presented in this paper has also been discussed in Yu *et al.* (2022), there is a conceptual difference in how the bound is arrived at: we average over individual options with the same strike, whereas Yu *et al.* (2022) average over the geometric mean of the basket.

2.3. *Distributional bounds*

Some of the research into finding suitable bounds on the price of a basket option involve using the properties of its payoff function such as in the Rogers–Shi lower bound (Rogers & Shi, 1995). In this section, we derive new upper and lower bounds on a basket of assets and their corresponding option price using the distributional properties of Brownian motions.

2.3.1. *Lower and upper distributional bounds of a basket option.*

We can obtain an upper bound on the basket of assets in (3) and the price of a basket option in (4) by replacing the independent Wiener processes with their joint maximum. Similarly, we can also obtain lower bounds on the basket and its corresponding option price by replacing the independent Wiener processes with their joint minimum.

PROPOSITION 2. The value $S(t)$ of the basket of assets at any time t is bounded above by

$$S_u(t) = \sum_{i=1}^{n} \omega_i S_i(0) \exp\Big((r - \frac{1}{2}\sigma_i^2)t\Big) \exp\Big(M_n(t) \sum_{j=1}^{n} \sigma_{ij}\Big), \tag{33}$$

and bounded below by

$$S_l(t) = \sum_{i=1}^{n} \omega_i S_i(0) \exp\Big((r - \frac{1}{2}\sigma_i^2)t\Big) \exp\Big(m_n(t) \sum_{j=1}^{n} \sigma_{ij}\Big), \tag{34}$$

where $M_n(t) = \max_{1\le j\le n} W_j(t)$, $m_n(t) = \min_{1\le j\le n} W_j(t)$ and provided $\sum_{j=1}^{n} \sigma_{ij}$ is non-negative for $1 \le i \le n$.

Proof. Given that the assets in the basket follow a GBM model, we set up the following inequalities:

$$m_n(t) \sum_{j=1}^{n} \sigma_{ij} \le \sum_{j=1}^{n} \sigma_{ij} W_j(t) \le M_n(t) \sum_{j=1}^{n} \sigma_{ij}, \tag{35}$$

where $M_n(t) = \max_{1\le j\le n} W_j(t))$ and $m_n(t) = \min_{1\le j\le n} W_j(t)$,
so that the price of the basket at time t is bounded by

$$S_l(t) = \sum_{i=1}^{n} Y_i \exp\Big(m(t)\sigma_i^*\Big) \le S(t) \le \sum_{i=1}^{n} Y_i \exp\Big(M(t)\sigma_i^*\Big) = S_u(t), \tag{36}$$

where $\sigma_i^* = \sum_{j=1}^{n} \sigma_{ij}$, $\sigma_i^2 = \sum_{j=1}^{n} \sigma_{ij}^2$ and $Y_i = \omega_i S_i(0) \exp\left[\left(r - \frac{1}{2}\sigma_i^2\right) T\right]$. □

These bounds on the value of the basket given by $S_u(t)$ and $S_l(t)$ are analytically intractible and are of a similar problem-type as the the basket of assets. To this end, we can estimate options on $S_u(t)$ and

$S_l(t)$ using their respective geometric means as suggested by Gentle (1993). Given the representations of the distributional upper and lower bound on the value of a basket of n assets at a time T given by $S_u(T)$ and $S_l(T)$, respectively, in proposition 2. Let $G_u(T)$ denote the geometric mean of (33), which has the following representation:

$$G_u(T) = \prod_{i=1}^{n}\left[S_i(0)\exp\left[\left(r - \frac{1}{2}\sigma_i^2\right)T\right]\exp\left(M_n(T)\sum_{j=1}^{n}\sigma_{ij}\right)\right]^{\frac{1}{n}}. \tag{37}$$

Given this representation of $G_u(T)$, we can use it to estimate the price of an option on $S_u(T)$, which is an the upper bound on the basket option price with the same strike.

PROPOSITION 3. The price $C_{G_u}(0,T)$ at time 0 of an option on the geometric mean $G_u(T)$, maturing at time T with non-negative strike K, is given by

$$C_{G_u}(0,T) = \alpha_1\beta_1 e^{-rT}\int_{\tilde{K}}^{\infty} e^{\gamma_1 y}\frac{n}{\sqrt{T}}\phi\left(\frac{y}{\sqrt{T}}\right)\left(\Phi\left(\frac{y}{\sqrt{T}}\right)\right)^{n-1}dy - Ke^{-rT}\left[1-\left(\Phi\left(\frac{\tilde{K}}{\sqrt{T}}\right)\right)^{n}\right],$$

where $\alpha_1 = \prod_{i=1}^{n} S_i(0)^{\frac{1}{n}}$, $\beta_1 = \exp\left[\left(r - \frac{1}{2}\sum_{i=1}^{n}\frac{\sigma_i^2}{n}\right)T\right]$, $\gamma_1 = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}$ and $\tilde{K} = \frac{1}{\gamma_1}\ln\left(\frac{K}{\alpha_1\beta_1}\right)$.

Proof. We can simplify the expression for $G_u(T)$ in (37) to become

$$G_u(T) = \left(\prod_{i=1}^{n}\left(S_i(0)\right)^{\frac{1}{n}}\right)\exp\left[\left(r - \frac{1}{2}\sum_{i=1}^{n}\frac{\sigma_i^2}{n}\right)T\right]\exp\left(M_n(T)\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}\right). \tag{38}$$

The price $C_{G_u}(0,T)$ of the option on $G_u(T)$ at a time 0 is given by

$$C_{G_u}(0,T) = e^{-rT}\mathbb{E}\left[\left(G_u(T) - K\right)^{+}\right], \tag{39}$$

$$= e^{-rT}\int_{\tilde{K}}^{\infty}\left(\alpha_1\beta_1 e^{\gamma_1 y} - K\right)\frac{n}{\sqrt{T}}\phi\left(\frac{y}{\sqrt{T}}\right)\left(\Phi\left(\frac{y}{\sqrt{T}}\right)\right)^{n-1}dy, \tag{40}$$

$$= \alpha_1\beta_1 e^{-rT}\int_{\tilde{K}}^{\infty} e^{\gamma_1 y}\frac{n}{\sqrt{T}}\phi\left(\frac{y}{\sqrt{T}}\right)\left(\Phi\left(\frac{y}{\sqrt{T}}\right)\right)^{n-1}dy - Ke^{-rT}\left[1-\left(\Phi\left(\frac{\tilde{K}}{\sqrt{T}}\right)\right)^{n}\right]. \tag{41}$$

This completes the proof. □

Similarly, given the geometric mean of $S_l(T)$ in (34), we can also derive the price of an option on it with the same strike K. We can define the geometric mean $G_l(T)$ of $S_l(T)$ as

$$G_l(T) = \prod_{i=1}^{n} \left[S_i(0) \exp\left[\left(r - \frac{1}{2}\sigma_i^2 \right) T \right] \exp\left(m_n(T) \sum_{j=1}^{n} \sigma_{ij} \right) \right]^{\frac{1}{n}}. \tag{42}$$

PROPOSITION 4. The price $C_{G_l}(0,T)$ of an option on $G_l(T)$, maturing at T at a time 0 with non-negative strike K, is given by

$$C_{G_l}(0,T) = e^{-rT} \int_{\tilde{K}}^{\infty} e^{\gamma_1 y} \frac{n}{\sqrt{T}} \phi\left(\frac{y}{\sqrt{T}} \right) \left(\Phi\left(-\frac{y}{\sqrt{T}} \right) \right)^{n-1} dy + K e^{-rT} \left(\Phi\left(-\frac{\tilde{K}}{T} \right) \right)^{n},$$

where $\alpha_1 = \prod_{i=1}^{n} S_i(0)^{\frac{1}{n}}$, $\beta_1 = \exp\left[\left(r - \frac{1}{2} \sum_{i=1}^{n} \frac{\sigma_i^2}{n} \right) T \right]$, $\gamma_1 = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}$ and $\tilde{K} = \frac{1}{\gamma_1} \ln\left(\frac{K}{\alpha_1 \beta_1} \right)$.

Hence, we are able to use the bounds (33) and (34) on the value of the basket to obtain closed-form bounds on the price of a basket option using their respective geometric means. Alternatively, we can obtain more accurate distributional bounds on the price of a basket option. This can be achieved by conditioning the option price of the distributional bounds on the value of the basket with the same strike K on its geometric mean as suggested by Curran (1994). We demonstrate this in the next two propositions.

PROPOSITION 5. The estimated option price $C_B^u(0,T)$ on $S_u(T)$ with strike K, at a time 0 prior to its maturity T using Curran's conditioning arguments, is given by

$$C_B^u(0,T) = e^{-rT} \mathbb{E}\left[S_u(T) \mathbb{1}\{G_u(T) \geq K\} \right] + K e^{-rT} \left[\left(\Phi\left(\frac{1}{\gamma_1} \ln\left(\frac{K}{\alpha_1 \beta_1} \right) \right) \right)^{n} - 1 \right], \tag{43}$$

where $S_u(T)$ is as defined is (33), $\alpha_1 = \prod_{i=1}^{n} S_i(0)^{\frac{1}{n}}$, $\beta_1 = \exp\left[\left(r - \frac{1}{2} \sum_{i=1}^{n} \frac{\sigma_i^2}{n} \right) T \right]$, $\gamma_1 = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}$.

Proof. The price at time 0 an option on $S_u(T)$ prior to the maturity T is given by

$$C_B^u(0,T) = e^{-rT}\mathbb{E}\left[\left(S_u(T)-K\right)^+\right], \tag{44}$$

$$= e^{-rT}\mathbb{E}\left[\mathbb{E}\left[\left(S_u(T)-K\right)^+ \middle| G_u(T)=y\right]\right], \tag{45}$$

$$= e^{-rT}\int_0^K \mathbb{E}\left[\left(S_u(T)-K\right)^+ \middle| G_u(T)=y\right]\mathbb{Q}\left(G_u(T)\in dy\right),$$

$$+ e^{-rT}\int_K^\infty \mathbb{E}\left[\left(S_u(T)-K\right)^+ \middle| G_u(T)=y\right]\mathbb{Q}\left(G_u(T)\in dy\right). \tag{46}$$

We use the fact that

$$\int_0^K \mathbb{E}\left[\left(S_u(T)-K\right)^+ \middle| G_u(T)=y\right]\mathbb{Q}\left(G_u(T)\in dy\right) \approx 0. \tag{47}$$

Substituting (47) in (46) to obtain

$$C_B^u(0,T) = e^{-rT}\int_K^\infty \mathbb{E}\left[\left(S_u(T)-K\right)^+ \middle| G_u(T)=y\right]\mathbb{Q}\left(G_u(T)\in dy\right), \tag{48}$$

$$= e^{-rT}\int_K^\infty \mathbb{E}\left[S_u(T)-K \middle| G_u(T)=y\right]\mathbb{Q}\left(G_u(T)\in dy\right), \tag{49}$$

$$= e^{-rT}\mathbb{E}\left[S_u(T)\mathbb{1}\{G_u(T)\geq K\}\right] - Ke^{-rT}\mathbb{Q}\left(G_u(T)\geq K\right), \tag{50}$$

$$= e^{-rT}\mathbb{E}\left[S_u(T)\mathbb{1}\{G_u(T)\geq K\}\right] - Ke^{-rT}\left[1-\left(\Phi\left(\frac{1}{\gamma_1}\ln\left(\frac{K}{\alpha_1\beta_1}\right)\right)\right)^n\right]. \tag{51}$$

Thus, we are able to obtain the required results. □

The option price given by $C_B^u(0,T)$ is an upper bound on the basket option price in (4). Next, we shall proceed to work out the lower bound on the basket option price using similar conditioning arguments.

PROPOSITION 6. The estimated option price $C_B^l(0,T)$ on $S_l(T)$, with strike K at a time 0 prior to its maturity T using Curran's conditioning arguments, is given by

$$C_B^l(0,T) = e^{-rT}\mathbb{E}\left[S_l(T)\mathbb{1}\{G_l(T)\geq K\}\right] + Ke^{-rT}\left[\left(\Phi\left(-\frac{1}{\gamma_1}\ln\left(\frac{K}{\alpha_1\beta_1}\right)\right)\right)^n\right], \tag{52}$$

where $S_l(T)$ is as defined is (34), $\alpha_1 = \prod_{i=1}^n S_i(0)^{\frac{1}{n}}$, $\beta_1 = \exp\left[\left(r-\frac{1}{2}\sum_{i=1}^n \frac{\sigma_i^2}{n}\right)T\right]$, $\gamma_1 = \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n \sigma_{ij}$.

Proof. The price at time 0 an option on $S_l(T)$ prior to the maturity T is given by

$$C_B^l(0,T) = e^{-rT}\mathbb{E}\left[\left(S_l(T)-K\right)^+\right], \tag{53}$$

$$= e^{-rT}\mathbb{E}\left[\mathbb{E}\left[\left(S_l(T)-K\right)^+ \middle| G_l(T)=y\right]\right], \tag{54}$$

$$= e^{-rT}\int_0^K \mathbb{E}\left[\left(S_l(T)-K\right)^+ \middle| G_l(T)=y\right]\mathbb{Q}\left(G_l(T)\in dy\right),$$

$$+ e^{-rT}\int_K^\infty \mathbb{E}\left[\left(S_l(T)-K\right)^+ \middle| G_l(T)=y\right]\mathbb{Q}\left(G_l(T)\in dy\right). \tag{55}$$

We use the fact that

$$\int_0^K \mathbb{E}\left[\left(S_l(T)-K\right)^+ \middle| G_l(T)=y\right]\mathbb{Q}\left(G_l(T)\in dy\right) \approx 0. \tag{56}$$

Substituting (56) in (55) to obtain

$$C_B^l(0,T) = e^{-rT}\int_K^\infty \mathbb{E}\left[\left(S_l(T)-K\right)^+ \middle| G_l(T)=y\right]\mathbb{Q}\left(G_l(T)\in dy\right), \tag{57}$$

$$= e^{-rT}\int_K^\infty \mathbb{E}\left[S_l(T)-K \middle| G_l(T)=y\right]\mathbb{Q}\left(G_l(T)\in dy\right), \tag{58}$$

$$= e^{-rT}\mathbb{E}\left[S_l(T)\mathbb{1}\{G_l(T)\geq K\}\right] - Ke^{-rT}\mathbb{Q}\left(G_l(T)\geq K\right), \tag{59}$$

$$= e^{-rT}\mathbb{E}\left[S_l(T)\mathbb{1}\{G_l(T)\geq K\}\right] + Ke^{-rT}\left[\left(\Phi\left(-\frac{1}{\gamma_1}\ln\left(\frac{K}{\alpha_1\beta_1}\right)\right)\right)^n\right]. \tag{60}$$

Thus, we are able to obtain the required results. □

Given the analytic intractibility of the bounds $S_l(t)$ and $S_u(t)$ on the value of the basket at any time t, we can impose integrability conditions on the the volatility parameters, allowing for closed-form evaluation of options on these bounds. These integrability conditions lead to bounds on the value of the basket given by

$$\bar{S}_l(t) = \sum_{i=1}^n Y_i \exp\left(m_n(t)\sigma_m\right) \leq S(t) \leq \sum_{i=1}^n Y_i \exp\left(M_n(t)\sigma_M\right) = \bar{S}_u(t), \tag{61}$$

where $\sigma_i^2 = \sum_{j=1}^n \sigma_{ij}^2$, $\sigma_M = \max_{1\leq i\leq n} \sigma_i^*$ and $\sigma_m = \min_{1\leq i\leq n} \sigma_i^*$ and $Y_i = \omega_i S_i(0)\exp\left[\left(r-\frac{1}{2}\sigma_i^2\right)T\right]$.

Proposition 7. Given the lower bound $\bar{S}_l(t)$ on the value of a basket $S(t)$ at a time t, its density and distribution

$$\mathbb{Q}\left(\bar{S}_l(t)\in dy\right) = \frac{1}{y}\frac{1}{\sigma_m\sqrt{T}}n\phi\left(\frac{1}{\sigma_m\sqrt{T}}\ln\left(\frac{y}{\vartheta}\right)\right)\left[1-\Phi\left(\frac{1}{\sigma_m\sqrt{T}}\ln\left(\frac{y}{\vartheta}\right)\right)\right]^{n-1} dy, \tag{62}$$

are given by the following and

$$\mathbb{Q}\left(\bar{S}_l(t) \leq y\right) = 1 - \left[1 - \Phi\left(\frac{1}{\sigma_m\sqrt{T}} \ln\left(\frac{y}{\vartheta}\right)\right)\right]^n, \tag{63}$$

respectively, where $\vartheta = \sum_{i=1}^{n} \omega_i S_i(0) \exp\left[(r - \frac{1}{2}\sigma_i^2)T\right]$.

Proposition 8. Given the upper bound $\bar{S}_u(t)$ on the value of a basket $S_u(t)$, its density and distribution

$$\mathbb{Q}\left(\bar{S}_u(t) \in dy\right) = \frac{1}{y}\frac{1}{\sigma_M\sqrt{T}} n\phi\left(\frac{1}{\sigma_M\sqrt{T}} \ln\left(\frac{y}{\vartheta}\right)\right)\left(\Phi\left(\frac{1}{\sigma_M\sqrt{T}} \ln\left(\frac{y}{\alpha}\right)\right)\right)^{n-1} dy, \tag{64}$$

are given by the following and

$$\mathbb{Q}\left(\bar{S}_u(t) \leq y\right) = \left(\Phi\left(\frac{1}{\sigma_M\sqrt{T}} \ln\left(\frac{y}{\vartheta}\right)\right)\right)^n, \tag{65}$$

respectively, where $\vartheta = \sum_{i=1}^{n} \omega_i S_i(0) \exp\left[(r - \frac{1}{2}\sigma_i^2)T\right]$.

Proposition 9. Given a basket of assets, which has the lower and upper bounds $\bar{S}_l(t)$ and $\bar{S}_u(t)$, respectively, as specified in (61), such a basket has the following bounds on the price of the basket option with maturity T and strike K at a time 0 given by

$$UB_1^n = ne^{-rT}\frac{\sum_{i=1}^{n} Y_i}{\sqrt{T}} \int_{\xi}^{\infty} e^{\sigma_M y}\phi\left(\frac{y}{\sqrt{T}}\right)\left(\Phi\left(\frac{y}{\sqrt{T}}\right)\right)^{n-1} dy - Ke^{-rT}\left[1 - \left(\Phi\left(\frac{\xi}{\sqrt{T}}\right)\right)^n\right], \tag{66}$$

and

$$LB_1^n = ne^{-rT}\frac{\sum_{i=1}^{n} Y_i}{\sqrt{T}} \int_{\tau}^{\infty} e^{\sigma_m y}\phi\left(\frac{y}{\sqrt{T}}\right)\left(1 - \Phi\left(\frac{y}{\sqrt{T}}\right)\right)^{n-1} dy + Ke^{-rT}\left[1 - \Phi\left(\frac{\tau}{\sqrt{T}}\right)\right]^n, \tag{67}$$

where $Y_i = \omega_i S_i(0) \exp\left[\left(r - \frac{1}{2}\sigma_i^2\right)T\right]$, $\xi = \frac{1}{\sigma_M} \ln \frac{K}{\sum_{i=1}^{n} Y_i}$ and $\tau = \frac{1}{\sigma_m} \ln \frac{K}{\sum_{i=1}^{n} Y_i}$. UB_1^n and LB_1^n denote the respective option price of $S_u(t)$ and $S_l(t)$.

The integral expressions for UB_1^n and LB_1^n are difficult to solve analytically and can be simplified using suitable approximations.

PROPOSITION 10. The upper bound UB_1^n on the basket option price of, as defined in equation (66), can be bounded from above and below as follows:

$$UB_1^n \in (LUB_1^n, UUB_1^n), \text{ where}$$
$$LUB_1^n = ne^{-rT}\left[\alpha e^{\frac{1}{2}\sigma_M^2 T}\Phi(\eta) - K\Phi\left(-\frac{\xi}{\sqrt{T}}\right)\right] \tag{68}$$

and

$$UUB_1^n = n\alpha e^{-rT}e^{\frac{1}{2}\sigma_M^2 T}\int_{\xi}^{\infty}\phi\left(\frac{y-\sigma_M T}{\sqrt{T}}\right)\Phi\left(\frac{y}{\sqrt{T}}\right)dy - \frac{nKe^{-rT}}{2}\left[1-\Phi^2\left(\frac{\xi}{\sqrt{T}}\right)\right]. \tag{69}$$

where, $\sigma_M = \max_{1\le i\le n}\sum_{j=1}^n \sigma_{ij}$ as before, $\alpha = \sum_{i=1}^n \omega_i S_i(0)\exp\left[\left(r-\frac{1}{2}\sigma_i^2\right)T\right]$, $\xi = \frac{1}{\sigma_M}\ln\left(\frac{K}{\alpha}\right)$, $\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ and $\eta = \frac{\xi-\sigma_M T}{\sqrt{T}}$.

Proof. The proof is available as online-only material and omitted from the text for brevity. □

PROPOSITION 11. The lower bound LB_1^n on the basket option price of n assets, as defined in (67), can be bounded from above and below as follows:

$$LB_1^n \in (LLB_1^n, ULB_1^n) \text{ where}$$
$$LLB_1^n = ne^{-rT}\left(\Phi\left(-\frac{|\zeta|}{\sqrt{T}}\right)\right)^{n-1}\left[\alpha e^{\frac{1}{2}\sigma_m^2 T}\Phi(-\mu) - K\Phi\left(-\frac{\zeta}{\sqrt{T}}\right)\right], \tag{70}$$
$$\text{and } ULB_1^n = ne^{-rT}\left[\alpha e^{\frac{1}{2}\sigma_m^2 T}\Phi\left(\frac{\sigma_m T-\zeta}{\sqrt{T}}\right) - K\Phi\left(-\frac{\zeta}{\sqrt{T}}\right)\right], \tag{71}$$

where $\zeta_m = \frac{1}{\sigma_m}\ln\left(\frac{K}{\alpha}\right)$, $\sigma_m = \min_{1\le i\le n}\sum_{j=1}^n \sigma_{ij}$ and $\mu = \frac{\zeta-\sigma_m T}{\sqrt{T}}$. α and $\phi(z)$ are as defined in proposition 10.

Proof. The proof is available as online-only material and omitted from the text for brevity. □

In the discussion on control variates in subsequent sections, we will use a numerical approximation of UB_1^n and LB_1^n as control variates. Furthermore, the bounds derived and the analysis presented in this section are new and may have wider applications beyond the ones in this paper.

2.4. *Control variates*

The control variate approach is a method of variance reduction for Monte Carlo estimates. While this method is standard, we outline it here for completeness and to establish the notation which we will use. Consider a random variable Y which is a function $h(X)$ whose distribution is not known, but the distribution of X is known. We can estimate the value of Y using a random variable $Z = h^*(X)$ whose distribution is known using the random variable Ψ such that

$$\Psi = Y - \lambda(Z - \mathbb{E}(Z)), \tag{72}$$

where the optimal value of the parameter λ which minimizes the variance of Ψ is given by

$$\lambda = \frac{Cov(Y, Z)}{Var(Z)}. \tag{73}$$

This shows that Ψ is an estimator of Y. The algorithm (1) in the appendix shows the implementation of the control variate methodology.

For the purpose of this research we have used $\lambda = 1$ for the direct upper bound and geometric lower bound control variates, because their empirical values for λ are close to one, which was also observed by Dingeç & Hörmann (2013).

3. Numerical implementation and findings

To conduct the numerical experiments for the control variate analysis, we estimated the sample covariance matrix using 522 observations of daily prices from 1 January 2018 to 31 December 2019, for five market indices namely FTSE 100, FTSE 250, S&P 500, NIKKEI 225 and IMOEX (from Thomson Reuters Datastream), using log return of the prices.[1] The prices of two-asset and five-asset baskets were simulated using these covariance estimates. The control variates used for this experiment are the geometric lower bound of the basket option price, the modified geometric lower bound, the direct upper bound, the distributional upper and lower bound on the basket option price. For the distributional upper and lower bounds we use (69) and (70), respectively, as the closed-form estimates for the respective control variates. The modified geometric lower bound is simply using an option on the geometric lower bound on the final value of the basket with a modified strike $\hat{K}$. This strike $\hat{K}$ is obtained by equating the expected differences of the final value of the basket from the strike K, and that of the geometric estimate of the basket's final value from the modified strike $\hat{K}$ as suggested by Gentle (1993). This relation can be simplified to obtain

$$\hat{K} = K + \mathbb{E}[G(0, T)] - F(0, T), \tag{74}$$

where $F(0, T) = S(0)e^{rT}$ and $G(0, T)$ is the geometric average of the final value of the basket. We will use the geometric lower bound as well as the modified geometric lower bound as benchmark control variates in our numerical experiments.

3.1. *Control variate analysis of basket option price*

In this section, we simulate the price of two- and five-asset basket option using 10^6 simulations, with initial value of the assets in the basket being $S_i(0) = 80$ and weightings $\omega_i = \frac{1}{n}$, for $i = 1, 2, ...n$. Also, the simulations are carried out for a variety of strikes $K = 60, 80, 100$ and for different maturities, $T = 0.5, 1, 2$. We assume a constant risk-free rate r of 1%. For the two-asset case, the simulation are

[1] Data availability: the results described in this paper are fully reproducible based on the covariance matrix presented and do not need any external data. The data were used simply to produce a realistic enough covariance matrix.

TABLE 1 *Daily covariance estimates of market indices ($\times 10^{-5}$)*

	FTSE 100	**FTSE 250**	**S&P 500**	**NIKKEI 225**	**IMOEX**
FTSE 100	7.18	6.90	3.48	1.86	4.40
FTSE 250	6.90	9.29	3.67	2.58	3.75
S&P 500	3.48	3.67	8.60	7.00	3.89
NIKKEI 225	1.86	2.58	7.00	9.57	1.57
IMOEX	4.40	3.75	3.89	1.57	16.31

TABLE 2 *Hermite polynomial approximation for a two-asset basket option*

T	**K**	**Price**	**Error**
0.5	60	22.285	1.927
	80	7.534	2.901
	100	1.560	1.243
1	60	21.743	0.742
	80	7.230	0.533
	100	1.4681	0.209
2	60	20.685	-1.795
	80	6.652	-2.954
	100	1.298	-1.987

carried out using only the volatility estimates from FTSE 100 and FTSE 250, while for the five-asset case we use volatility estimates from FTSE 100, FTSE 250, S&P 500, NIKKEI 225 AND IMOEX. Table 2 shows the estimated prices of a two-asset basket option using first-order Hermite polynomial approximation and the corresponding error (E), which is given by

$$E = HPPrice - MCPrice, \tag{75}$$

where *HPPrice* is the first-order Hermite polynomial approximation of basket option price calculated in (15) and *MCPrice* is the standard Monte Carlo price of the basket option.

The results show a good approximation for the two-asset basket option price, when the strike price K of the basket is 60 and 80, but is significantly different when the basket is out-of-the money. This observation is consistent with findings in Milevsky & Posner (1998): that lognormal models tend to over-value out-of-the-money call option prices.

For control variate analysis, we simulate Monte Carlo option price with control variates using 10^6 simulations, for the two-asset and five-asset basket, for different maturities and a variety of strikes[2]. We use a 95% confidence interval for our results and the general idea is that, for the same number of

[2]The implementation of these control variates is reproducible using the control variate algorithm and implementation chart given in the appendix.

TABLE 3 *Basket option prices for a two-asset basket, $T = 0.5$*

K	Method	Price	Variance	CI Lower	CI Upper	Time
60	MC	20.3633	1.0000	20.3415	20.3850	0.0567
	LB	20.3615	0.0001	20.3612	20.3617	0.2320
	MLB	20.3616	0.0001	20.3613	20.3618	0.1548
	UB	20.3618	0.0002	20.3614	20.3620	0.0733
	HP	20.3609	0.0225	20.3586	20.3631	0.0068
	UBM	20.3581	0.4601	20.3435	20.3626	0.4486
	LBM	20.3094	0.0596	20.3042	20.3147	0.0469
80	MC	4.6431	1.0000	4.6285	4.6567	0.0152
	LB	4.6387	0.0001	4.6385	4.6389	0.2362
	MLB	4.6387	0.0002	4.6381	4.6396	0.1497
	UB	4.6392	0.0047	4.6382	4.6402	0.0769
	HP	4.6392	0.0285	4.6373	4.6409	0.0869
	UBM	4.6374	0.4327	4.6289	4.6403	0.2613
	LBM	4.6464	0.0784	4.6424	4.6504	0.0865
100	MC	0.3122	1.0000	0.3085	0.3159	0.0541
	LB	0.3145	0.0004	0.3144	0.3146	0.2355
	MLB	0.3142	0.0007	0.3139	0.3149	0.1889
	UB	0.3147	0.0454	0.3139	0.3155	0.0782
	HP	0.3144	0.1912	0.3127	0.3161	0.0875
	UBM	0.3138	0.5194	0.3111	0.3154	0.2853
	LBM	0.3127	0.1781	0.3112	0.3142	0.0867

simulations, we can obtain tighter confidence intervals with faster times (given that standard Monte Carlo needs significantly more simulations to obtain better confidence intervals).

In Tables 3– 8, we compare the price, (normalized) variance and confidence intervals of our Monte Carlo estimates for a two- and five-asset basket option with those obtained using different control variates, for different maturities and strikes. The normalized variances are simply the estimated variances, normalized with respect to the Monte Carlo variance for a fixed maturity and strike. The Monte Carlo, first-order Hermite polynomial, geometric lower bound, modified geometric lower bound, direct upper bound, distributional upper bound and distributional lower bounds are abbreviated as MC, HP, LB, MLB, UB, UBM and LBM, respectively, in all of the tables. Furthermore, CI Lower and CI Upper are the respective lower and upper confidence intervals of our estimates. Our numerical experiments show that for the same number of simulations, we obtain significant variance reduction, and tighter confidence intervals compared with standard Monte Carlo results for both the two-asset and five-asset case. In Tables 3– 5, we find that our control variates (with the exception of the UBM) for the two-asset basket option have over 80% variance reduction, but were on average outperformed by the reference control variate. In terms of computation times, we find that LBM and HP control variate outperforms other control variates, while the UBM recorded the slowest computation times. The results for the five-asset basket option are presented in Tables 6– 8. The results show significant variance reduction for all control variates, with the UB and LB having the highest variance reduction. The computation times are found to be largely similar for all the control variates besides the LBM.

TABLE 4 *Basket option prices for a two-asset basket, $T = 1$*

K	Method	Price	Variance	CI Lower	CI Upper	Time
60	MC	20.9871	1.0000	20.9569	21.0179	0.0335
	LB	20.9849	0.0002	20.9845	20.9853	0.2036
	MLB	20.9845	0.0002	20.9840	20.9852	0.1505
	UB	20.9844	0.0005	20.9837	20.9852	0.0341
	HP	20.9847	0.0194	20.9813	20.9881	0.0435
	UBM	20.9844	0.4363	20.9701	20.9997	0.2325
	LBM	20.9791	0.0607	20.9716	20.9865	0.0495
80	MC	6.6758	1.0000	6.6549	6.6966	0.0501
	LB	6.6595	0.0003	6.6592	6.6599	0.2364
	MLB	6.6595	0.0003	6.6580	6.6612	0.1511
	UB	6.6599	0.0007	6.6585	6.6613	0.0751
	HP	6.6609	0.0185	6.6581	6.6637	0.0857
	UBM	6.6631	0.4791	6.6501	6.6771	0.2541
	LBM	6.6174	0.0771	6.6115	6.6233	0.0872
100	MC	1.2449	1.0000	1.2356	1.2542	0.0500
	LB	1.2487	0.0006	1.2484	1.2489	0.2356
	MLB	1.2487	0.0007	1.2484	1.2497	0.1505
	UB	1.2483	0.0221	1.2469	1.2497	0.0771
	HP	1.2486	0.0361	1.2469	1.2500	0.0868
	UBM	1.2322	0.5101	1.2256	1.2338	0.2620
	LBM	1.2359	0.1186	1.2325	1.2391	0.0869

TABLE 5 *Basket option prices for a two-asset basket, $T = 2$*

K	Method	Price	Variance	CI Lower	CI Upper	Time
60	MC	22.4924	1.0000	22.4509	22.5238	0.0545
	LB	22.4834	0.0004	20.4819	22.4845	0.2401
	MLB	22.4836	0.0004	20.4816	22.4850	0.1555
	UB	22.4832	0.0009	22.4819	22.4845	0.0810
	HP	22.4839	0.0311	22.4770	22.4883	0.0906
	UBM	22.2479	0.5278	22.2210	22.2747	0.3034
	LBM	22.2396	0.0645	22.2291	22.2502	0.0965
80	MC	9.5914	1.0000	9.5601	9.6226	0.0574
	LB	9.6002	0.0006	9.5994	9.6009	0.2423
	MLB	9.6002	0.0009	9.5974	9.6003	0.1582
	UB	9.6019	0.0037	9.5596	9.6039	0.0814
	HP	9.5974	0.0421	9.5930	9.6017	0.0928
	UBM	9.4249	0.4424	9.4044	9.4374	0.1918
	LBM	9.4150	0.0777	9.4062	9.4238	0.0566
100	MC	3.3437	1.0000	3.3242	3.3632	0.0375
	LB	3.3377	0.0008	3.3371	3.3383	0.2108
	MLB	3.3479	0.0011	3.3456	3.3493	0.1564
	UB	3.3467	0.0119	3.3445	3.3488	0.0396
	HP	3.3531	0.0718	3.3479	3.3583	0.0525
	UBM	3.3406	0.4914	3.3374	3.3563	0.2172
	LBM	3.2906	0.1152	3.2841	3.2971	0.0601

TABLE 6 *Basket option prices for a five-asset basket, $T = 0.5$*

K	Method	Price	Variance	CI Lower	CI Upper	Time
60	MC	20.3111	1.0000	20.2937	20.3285	0.1401
	LB	20.3107	0.0021	20.3099	20.3107	0.3675
	MLB	20.3108	0.0023	20.3101	20.3108	0.3674
	UB	20.3082	0.0035	20.3072	20.3092	0.2131
	HP	20.3109	0.3406	20.3007	20.3186	0.2016
	UBM	20.3027	0.5521	20.2897	20.3157	0.4189
	LBM	20.3107	0.2322	20.3024	20.3190	1.6879
80	MC	3.7339	1.0000	3.7228	3.7451	0.1352
	LB	3.7341	0.0051	3.7334	3.7342	0.3754
	MLB	3.7340	0.0051	20.3097	3.7341	0.3672
	UB	3.7237	0.0635	3.7208	3.7265	0.2188
	HP	3.7371	0.3438	3.7284	3.7404	0.2018
	UBM	3.7371	0.5701	3.7297	3.7414	0.3795
	LBM	3.7371	0.2844	3.7312	3.7431	1.6751
100	MC	0.0931	1.0000	0.0914	0.0948	0.1446
	LB	0.0941	0.0227	0.0939	0.0941	0.3624
	MLB	0.0940	0.0227	0.0938	0.0941	0.3625
	UB	0.0909	1.4135	0.0892	0.0917	0.2145
	HP	0.0945	0.3987	0.0929	0.0948	0.1972
	UBM	0.0945	0.6239	0.0931	0.0949	0.3864
	LBM	0.0946	0.3917	0.0934	0.0948	1.6667

TABLE 7 *Basket option prices for five-asset basket, $T = 1$*

K	Method	Price	Variance	CI Lower	CI Upper	Time
60	MC	20.7163	1.0000	20.6920	20.7407	0.1546
	LB	20.7216	0.0047	20.7199	20.7216	0.3869
	MLB	20.7219	0.0047	20.7202	20.7219	0.3871
	UB	20.7114	0.0086	20.7096	20.7136	0.2363
	HP	20.7188	0.3712	20.7051	20.7291	0.2222
	UBM	20.7141	0.5647	20.7057	20.7221	0.4474
	LBM	20.7217	0.2357	20.7101	20.7291	1.7071
80	MC	5.3851	1.0000	5.3688	5.4015	0.1434
	LB	5.3925	0.0103	5.3908	5.3925	0.3682
	MLB	5.3920	0.0104	5.3902	5.3921	0.3681
	UB	5.3764	0.0585	5.3724	5.3804	0.2163
	HP	5.3941	0.3191	5.3834	5.3991	0.2019
	UBM	5.3853	0.5441	5.3734	5.3911	0.3795
	LBM	5.3902	0.2764	5.3814	5.3963	1.7035
100	MC	0.5749	1.0000	0.5694	0.5893	0.1673
	LB	0.5698	0.0309	0.5689	0.5699	0.3891
	MLB	0.5695	0.0306	0.5686	0.5696	0.3893
	UB	0.5612	0.5094	0.5572	0.5651	0.6194
	HP	0.5656	0.4133	0.5609	0.5704	0.2286
	UBM	0.5654	0.7069	5609	0.5698	0.6194
	LBM	0.5669	0.3771	0.5635	0.5701	1.7383

TABLE 8 *Basket option price for a five-asset basket, $T = 2$*

K	Method	Price	Variance	CI Lower	CI Upper	Time
60	MC	22.7791	1.0000	21.7453	21.8128	0.1579
	LB	21.7702	0.0103	21.7667	21.7702	0.3845
	MLB	21.7707	0.0103	21.7673	21.7708	0.3845
	UB	21.7462	0.0126	21.7424	21.7501	0.2341
	HP	21.7692	0.2846	21.7505	21.7723	0.2189
	UBM	21.7814	0.5704	21.7907	22.0133	0.2948
	LBM	21.7698	0.2427	21.7532	21.7764	1.7223
80	MC	7.8575	1.0000	7.8320	7.8879	0.1574
	LB	7.8367	0.0212	7.8333	7.8367	0.3814
	MLB	7.8385	0.0215	7.8316	7.8351	0.3814
	UB	7.8232	0.0529	7.8177	7.8288	0.2303
	HP	7.8402	0.3803	7.8296	7.8504	0.2166
	UBM	7.8424	0.5132	7.8269	7.8593	0.3953
	LBM	7.8402	0.2925	7.8338	7.8596	1.7361
100	MC	1.9530	1.0000	1.9659	1.9560	0.0129
	LB	1.9419	0.0428	1.9393	1.9419	0.3791
	MLB	1.9421	0.0429	1.9398	1.9422	0.3793
	UB	1.9294	0.3107	1.9231	1.9357	0.2309
	HP	1.9454	0.4719	1.9351	1.9508	0.2153
	UBM	1.9394	0.6037	1.9293	1.9421	0.4001
	LBM	1.9454	0.3255	1.9351	1.9511	1.7026

All the computations in this paper have been carried out using an Apple M1 Pro MacBook Pro 2021 with 16GB of unified memory and 8-core CPU (with six performance cores and two efficiency cores). The matlab version used is MatlabR2022a.

4. Conclusion

In this paper, we have proposed several new control variate for efficient simulation-based pricing of basket options. The first method is a lognormal approximation of the basket (or logarithm of the product of weighted assets in the basket) using first-order Hermite polynomials.

The second control variate is a direct upper bound on the payoff of the basket option obtained by a direct application of the Jensen's inequality.

The third and fourth control variates are the distributional upper and lower bounds, which involve obtaining bounds on the price of a basket option, whose randomness is driven by the maximum or minimum of independent Brownian motions.

Our numerical results show that all our control variates achieve significant variance reduction compared with standard Monte Carlo. Also, the variance reductions obtained from the use of the distributional lower bound and first-order Hermite polynomial approximation control variates are comparable with the benchmark control variates (geometric lower bound without and with modified strike), but have significantly faster computation times.

Since basket options are commonly used by financial institutions for cost-effective hedging of multiple underlying positions, new methods to price them efficiently and accurately is a useful contribution in risk management. Further, the lower distributional bound derived in this paper is believed to be new, as is the derivation of Hermite polynomial-based approximation. Both these approaches will have applications beyond basket option pricing and beyond GBM set-up. Adaptation of these control variates to the pricing of Asian options and basket options under stochastic volatility are topics of current research.

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Data availability

The data underlying this article will be shared on reasonable request to the corresponding author.

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A. Appendix

A.1. *Control Variate Algorithm*

Algorithm 1 Algorithm for control variates

1. Pick a large number N for number of simulations.
2. Set $k = 1$.
3. For each k, generate $Y^{(k)}$ and $Z^{(k)}$ and evaluate the mean of Z in closed-form.
4. Calculate $\Psi^{(k)}$ given in (72) such that

$$\Psi^{(k)} = Y^{(k)} - \lambda(Z^{(k)} - \mathbb{E}(Z)),$$

   where λ is of the form given in (73).

5. Set k = k+1.
6. If $k < N$, go to step 3. Else, compute

$$\widehat{\Psi} = \frac{1}{N}\sum_{k=1}^{N} \Psi^{(k)}.$$

A.2. *Algorithm Implementation Table*

Table 9 gives pointers to computation of Z^k and $\mathbb{E}(Z)$ in the algorithm above, for each control variate.

Table 9 *Algorithm implementation chart for control variates*

Method	Eqn for $\mathbb{E}(Z)$	$Z^{(k)}$
UB	(32)	$e^{-rT}\sum_{i=1}^{n}\omega_i\left(S_i(T)-K\right)^+$
HP	(15)	$e^{-rT}\left(e^{\Phi(u)}-K\right)^+$
UBM	(69)	$e^{-rT}\left(\bar{S}_u(T)-K\right)^+$
LBM	(70)	$e^{rT}\left(\bar{S}_l(T)-K\right)^+$