Quadratic Filtering for Linear Stochastic Systems With Dynamical Bias Under Amplify-and-Forward Relays: Dealing With Non-Gaussian Noises ⋆

Shaoying Wang [a,b], Zidong Wang [c], Qinyuan Liu [d,e], Hongli Dong [b,f,∗], Hongjian Liu [g,h]

[a] *College of Science, Shandong University of Aeronautics, Binzhou 256603, China*

[b] *Artificial Intelligence Energy Research Institute, Northeast Petroleum University, Daqing 163318, China*

[c] *Department of Computer Science, Brunel University London, Uxbridge, Middlesex, UB8 3PH, United Kingdom*

[d] *Department of Computer Science and Technology, Tongji University, Shanghai 201804, China*

[e] *Key Laboratory of Embedded System and Service Computing, Ministry of Education, Tongji University, Shanghai 201804, China*

[f] *Heilongjiang Provincial Key Laboratory of Networking and Intelligent Control, Northeast Petroleum University, Daqing 163318, China*

[g] *Key Laboratory of Advanced Perception and Intelligent Control of High-end Equipment, Ministry of Education, Anhui Polytechnic University, Wuhu 241000, China*

[h] *School of Mathematics and Physics, Anhui Polytechnic University, Wuhu 241000, China*

Abstract

In this paper, the recursive quadratic filtering problem is investigated for a class of linear non-Gaussian systems with dynamical bias and amplify-and-forward relays. The stochastic bias, characterized by a dynamical process with certain non-Gaussian noises, is incorporated into the system state equation. An amplify-and-forward relay is utilized in the sensor-to-filter network channel to enhance signal transmission performance. The transmission powers of the sensor and relay are governed by two sets of random variables. Particular attention is given to the design of a quadratic filter in the presence of the dynamical bias, the amplify-and-forward relay, and non-Gaussian noises. For this purpose, an augmented system is constructed by aggregating the augmented state (comprising the original state and the associated bias) and its second-order Kronecker power. Consequently, the addressed quadratic issue for the underlying non-Gaussian system is reformulated as a linear filtering problem for the augmented system. Using difference equations, the filtering error covariance is derived and subsequently minimized through the design of an appropriate gain matrix. Moreover, sufficient conditions are established to ascertain the existence of the lower and upper bounds on the filtering error covariance. Finally, the effectiveness of the designed quadratic filtering algorithm is demonstrated through a numerical example.

Key words: Quadratic filtering; Amplify-and-forward relays; Dynamical bias; Non-Gaussian systems; Boundedness analysis.

1 Introduction

State estimation or filtering, which is a fundamental topic in signal processing and control theory, has recently attracted increasing research attention due to its promising applications in areas like target tracking [3], aerospace [14], environmental monitoring [43, 44], and vehicle control [42], among others. At its core, state estimation/filtering endeavors to reconstruct the system state from available measurements, which may be compromised by noise. To address various system requirements and achieve desired performance, a myriad of sophisticated filters have been introduced in the literature, and notable examples include the Kalman filter [8, 35], the extended Kalman filter [7], the unscented Kalman filter [48], the H_∞ filter [11, 17, 25, 37, 40], set-membership filter [22, 23], and the particle filter [26, 45].

While many existing algorithms make the assumption of Gaussian noises due to its simplicity and the ease of al-

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∗ Corresponding author.

Email addresses: `shaoying2004@163.com` (Shaoying Wang), `Zidong.Wang@brunel.ac.uk` (Zidong Wang), `liuqy@tongji.edu.cn` (Qinyuan Liu), `shiningdhl@vip.126.com` (Hongli Dong), `hjliu1980@gmail.com` (Hongjian Liu).

gorithmic derivation, this assumption often falls short in practical engineering scenarios. As such, there has been significant effort dedicated to developing filtering methods suitable for non-Gaussian noises [1, 9, 32, 41, 46]. Notably, the quadratic filtering approach has emerged as a rather popular method whose aim is to deduce system states by harnessing more comprehensive information from state/measurement vectors and leveraging higher-order statistics of non-Gaussian noises. For instance, a feedback quadratic filter tailored for a non-Gaussian and unstable system has been introduced in [4]. Similarly, a suboptimal quadratic filter has been proposed in [24] for linear uncertain systems equipped with uniform quantization and non-Gaussian noises. When juxtaposed with the recursive linear filtering technique, the quadratic filtering approach showcases a superior ability to enhance filtering accuracy while maintaining ease of implementation [2, 3, 5, 6, 47].

In practical scenarios, inevitable perturbations in system dynamics can arise from various factors including unmodeled dynamics and unforeseen external excitations. These factors can complicate system modeling and analysis [10, 18, 19]. Stochastic bias, a specific type of unknown perturbation, is commonly represented by a dynamic equation with a particular white noise. The associated filtering challenges posed by this bias have garnered significant interest in recent research [13, 15, 33]. For instance, for multi-rate systems affected by dynamical bias, a recursive filter has been introduced in [33] to estimate both the system state and any faults concurrently. More recently, a novel distributed filter has been crafted for complex networks in [13]. This filter, rooted in the delay-prediction-compensation method, has addressed dynamics marred by bias, communication delays, and fading observations, where an upper bound has been derived for the filtering error covariance, and the monotonicity relationship between fading probability and filtering accuracy has also been elucidated. However, to date, scant research has been undertaken on the quadratic filtering problem in the presence of dynamical bias, and such a gap serves as the primary impetus for our current research endeavors.

A foundational presumption in the previously discussed filtering algorithms is that sensors can transmit signals across vast distances without hindrance. Unfortunately, this assumption often proves unrealistic in communication systems, especially when considering the limited transmission capabilities of sensors. These limitations can arise from technical or physical constraints, particularly with budget-friendly sensors. To address these transmission challenges, the integration of relays has recently seen burgeoning interest with aim to enhance the performance of long-range signal transmissions. Common relay models encompass amplify-and-forward relays, compute-and-forward relays, compress-and-forward relays, and filter-and-forward relays, see [12, 28, 30, 31, 34, 38] for more details. Recent explorations have ventured into filter design within the context of amplify-and-forward relays [12, 36, 39]. For instance, the study in [20] tackled the Kalman filtering problem for linear time-varying systems operating under the assumption of deterministic transmission power in the amplify-and-forward relay.

As energy harvesting technology gains traction, a challenge that emerges is the inherent unpredictability of transmission power in energy-harvested sensors or relays, which is because power sourced from the environment tends to exhibit natural variability [29]. To model this phenomenon, stochastic variables, which adhere to specific probability distributions, have been employed, see some recent literature illuminating insights into scenarios involving random transmission powers [16, 26, 27, 36]. For example, the centralized/distributed auxiliary particle filter has been introduced in [26] for multi-sensor systems, which accounts for both amplify-and-forward and decode-and-forward relay mechanisms in transmission links. Concurrently, innovative filtering techniques have been proposed, such as a recursive filter for complex networks [16] and a distributed H_∞ fusion filter for nonlinear systems [27]. However, it is noteworthy that research on the quadratic filtering challenge remains fragmented, especially when incorporating concerns related to amplify-and-forward relays.

Given the aforementioned analysis, our objective is to tackle the recursive quadratic filtering problem for a class of linear non-Gaussian systems that incorporate dynamical bias and amplify-and-forward relays. Unlike the constant and undetermined bias, we introduce a dynamical equation with non-Gaussian noises to depict the stochastic dynamical bias. An amplify-and-forward relay, equipped with random transmission power, is utilized in the sensor-to-filter network channel so as to ensure the quality of signal transmission. We propose a unique yet easy-to-implement quadratic filter for an augmented system, which leverages second-order Kronecker products of the augmented state, and encompasses both the original state and the dynamical bias, as well as its measurements. Then, the filtering error covariance is ascertained and subsequently minimized by choosing the appropriate gain matrix. We also provide sufficient conditions to guarantee the boundedness of the filtering error covariance. The challenges we foresee include: 1) analyzing the statistical properties of the random variables, which are influenced by transmission powers and the enhanced process/measurement noises; 2) constructing a recursive quadratic filter in the context of the dynamical bias, the amplify-and-forward relay, and non-Gaussian noises; and 3) examining the boundedness of the filtering error covariance. Given these challenges, we perceive the quadratic filter design problem to be notably intricate.

The core contributions of this paper can be encapsulated as follows: 1) the statistical properties of random variables (induced by transmission powers) and the expanded process/measurement noises are examined; 2) a novel recursive quadratic filtering paradigm is developed for linear non-Gaussian systems in the presence of the amplify-and-forward relay with random transmission powers, which is coupled with a thorough analysis of the filtering error covariance; and 3) sufficient conditions, grounded in system parameter information, are provisioned to ensure the boundedness of the filtering error covariance.

The structure of this paper is organized as follows. Section 2 delves into the quadratic filtering problem for linear systems by taking into account the dynamical bias, the amplify-and-forward relay, and non-Gaussian noises. In Section 3, we explore the stochastic properties of the augmented noises and discuss the quadratic filter design problem. Section 4 presents a simulation example to demonstrate the efficacy of the proposed quadratic filter. Section 5 concludes the paper with key findings and insights. Lastly, the Appendix provides a proof concerning the stochastic properties of augmented noises.

Notation. $\mathbb{R}^l$ is the l-dimensional Euclidean space. $\mathbb{P}\{\star\}$ stands for the occurring probability of the event $\star$. $\mathbb{E}\{\star\}$ denotes the expectation of random variable $\star$. Sym$\{\star\}$ means $\star + \star^T$. diag$\{\star\}$ is the block diagonal matrix. $\Re^T$ is the transpose of the matrix $\Re$. vec$(\Re)$ represents the vectorization of the matrix $\Re$. st$(\Re)$ denotes the operation that transfers vec$(\Re)$ to $\Re$. $\otimes$ is the Kronecker product. $\phi_z^{(m)}$ means the mth-order moment of z. $\tilde{S}_r(u \otimes v)$ denotes $u \otimes v + v \otimes u$ with $u \in \mathbb{R}^r$ and $v \in \mathbb{R}^r$.

2 Problem Formulation

Consider the following discrete linear time-varying systems with the dynamical bias and non-Gaussian noises:

$$x_{t+1} = A_t x_t + B_t w_t + E_t \varepsilon_t \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the system state, $w_t \in \mathbb{R}^{n_w}$ is the non-Gaussian process noise, and $\varepsilon_t \in \mathbb{R}^{n_\varepsilon}$ represents the random bias which satisfies the following dynamics:

$$\varepsilon_{t+1} = F_t \varepsilon_t + \xi_t. \tag{2}$$

Here, $\varepsilon_0 \in \mathbb{R}^{n_\varepsilon}$ and $\xi_t \in \mathbb{R}^{n_\varepsilon}$ denote non-Gaussian random sequences, and A_t, B_t, E_t, and F_t are given time-varying matrices with compatible dimensions.

The measurement model is described as

$$y_t = C_t x_t + D_t v_t \tag{3}$$

where $y_t \in \mathbb{R}^s$ is the measurement vector, $v_t \in \mathbb{R}^{n_v}$ is the non-Gaussian measurement noise, and C_t and D_t denote known matrices of suitable dimensions.

In this paper, the measurement y_t is sent to the remote filter via an amplify-and-forward relay, and the corresponding measurement obtained by this relay can be given by

$$\tilde{y}_t = \sqrt{l_t}\Phi_t y_t + \varpi_t \tag{4}$$

where $\Phi_t \triangleq \mathrm{diag}\{\Phi_{1,t}, \Phi_{2,t}, \cdots, \Phi_{s,t}\}$ with $\Phi_{i,t}$ being the ith channel coefficient, ϖ_t is the non-Gaussian noise in the sensor-to-relay channel, and l_t denotes the transmission power of the sensor obeying certain probability distribution

$$\mathbb{P}\{l_t = l_{i,t}\} = \bar{l}_{i,t}, \ \ i = 1, 2, \cdots, p \tag{5}$$

where $l_{i,t} \geq 0$, $0 \leq \bar{l}_{i,t} \leq 1$ and $\sum_{i=1}^{p} \bar{l}_{i,t} = 1$.

Considering that the amplify-and-forward relay can amplify the received $\tilde{y}_t$ and further forward it to the remote filter, the real measurement acquired by the filter is described by

$$z_t = \theta_t \sqrt{m_t}\Psi_t \tilde{y}_t + \varsigma_t \tag{6}$$

where θ_t means the amplification factor, m_t is the transmission power, $\Psi_t \triangleq \mathrm{diag}\{\Psi_{1,t}, \Psi_{2,t}, \cdots, \Psi_{s,t}\}$ with $\Psi_{i,t}$ being the ith relay-to-filter channel coefficient, and ς_t denotes the non-Gaussian noise in the relay-to-filter channel. Moreover, the probability distribution of m_t satisfies

$$\mathbb{P}\{m_t = m_{i,t}\} = \bar{m}_{i,t}, \ \ i = 1, 2, \cdots, q \tag{7}$$

where $m_{i,t} \geq 0$, and $\bar{m}_{i,t}$ are known scalars with $0 \leq \bar{m}_{i,t} \leq 1$ and $\sum_{i=1}^{q} \bar{m}_{i,t} = 1$.

Before proceeding further, the following assumption is made in this paper.

Assumption 1 *1) The random sequences x_0, w_t, ε_t, ξ_t, v_t, ϖ_t, ς_t, l_t and m_t are mutually independent. 2) x_0, w_t, ε_t, ξ_t, v_t, ϖ_t and ς_t are zero-mean white sequences, and their second-, third- and fourth-order moments are known.*

For ease of notation, we introduce

$$\begin{aligned}
\bar{l}_t^{(1)} &\triangleq \mathbb{E}\{l_t^{\frac{1}{2}}\}, & \bar{m}_t^{(1)} &\triangleq \mathbb{E}\{m_t^{\frac{1}{2}}\}, \\
\bar{l}_t^{(2)} &\triangleq \mathbb{E}\{l_t\}, & \bar{m}_t^{(2)} &\triangleq \mathbb{E}\{m_t\}, \\
\bar{l}_t^{(3)} &\triangleq \mathbb{E}\{l_t^{\frac{3}{2}}\}, & \bar{m}_t^{(3)} &\triangleq \mathbb{E}\{m_t^{\frac{3}{2}}\}, \\
\bar{l}_t^{(4)} &\triangleq \mathbb{E}\{l_t^{2}\}, & \bar{m}_t^{(4)} &\triangleq \mathbb{E}\{m_t^{2}\},
\end{aligned} \tag{8}$$

and, accordingly, z_t in (6) can be rewritten as

$$z_t = \theta_t \bar{m}_t^{(1)} \bar{l}_t^{(1)} \Psi_t \Phi_t C_t x_t + \vec{v}_t \tag{9}$$

where

$$\begin{aligned}
\vec{v}_t \triangleq &\theta_t(\sqrt{m_t}\sqrt{l_t} - \bar{m}_t^{(1)}\bar{l}_t^{(1)})\Psi_t\Phi_t C_t x_t \\
&+ \theta_t\sqrt{m}_t\sqrt{l}_t\Psi_t\Phi_t D_t v_t + \theta_t\sqrt{m}_t\Psi_t\varpi_t + \varsigma_t.
\end{aligned}$$

Remark 1 *The randomness of the dynamical bias, attributed to factors such as random frictions, wind resistance, and/or electromagnetic interferences, is deemed worthy of consideration in the model. In comparison to the dynamical bias influenced by Gaussian white noise, non-Gaussian random sequences ε_0 and ξ_t are introduced in (2) to characterize the dynamics of the random bias.*

Remark 2 *Contrary to the deterministic transmission power model, in this paper, an amplify-and-forward relay influenced by the random transmission powers of the sensor and relay is introduced in (4) and (6). The randomness of these transmission powers arises primarily because the energy harvested from the environment is inherently random. As a result, the random variables l_t and m_t, which follow specific probability distributions, are employed to characterize these random power levels. The* *stochastic properties of these variables will be elaborated in a subsequent lemma. Furthermore, in (6), the parameter θ_t is used to represent the amplification factor, and ς_t is introduced to denote the non-Gaussian transmission noise.*

Defining

$$\vec{x}_t \triangleq \begin{bmatrix} x_t \\ \varepsilon_t \end{bmatrix}, \vec{w}_t \triangleq \begin{bmatrix} w_t \\ \xi_t \end{bmatrix},$$

the system (1) and (9) can be reformulated as

$$\begin{cases} \vec{x}_{t+1} = \vec{A}_t \vec{x}_t + \vec{B}_t \vec{w}_t \\ z_t = \vec{C}_t \vec{x}_t + \vec{v}_t \end{cases} \tag{10}$$

where

$$\vec{A}_t \triangleq \begin{bmatrix} A_t & E_t \\ 0 & F_t \end{bmatrix}, \vec{B}_t \triangleq \begin{bmatrix} B_t & 0 \\ 0 & I \end{bmatrix},$$
$$\vec{C}_t \triangleq \begin{bmatrix} \theta_t \bar{m}_t^{(1)} \bar{l}_t^{(1)} \Psi_t \Phi_t C_t & 0 \end{bmatrix}.$$

Based on the definition of $\vec{x}_t^{[2]} = \vec{x}_t \otimes \vec{x}_t$, we immediately have

$$\vec{x}_{t+1}^{[2]} = \vec{A}_t^{[2]} \vec{x}_t^{[2]} + \vec{B}_t^{[2]} \phi_{\vec{w}_t}^{(2)} + \tilde{w}_t \tag{11}$$

where

$$\tilde{w}_t \triangleq \tilde{S}_{n+n_\varepsilon}(\vec{A}_t \vec{x}_t \otimes \vec{B}_t \vec{w}_t) + \vec{B}_t^{[2]}(\vec{w}_t^{[2]} - \phi_{\vec{w}_t}^{(2)}).$$

Similarly, the sequence $z_t^{[2]}$ is given by

$$z_t^{[2]} = \vec{C}_t^{[2]} \vec{x}_t^{[2]} + \phi_{\vec{v}_t}^{(2)} + \tilde{v}_t \tag{12}$$

where $\tilde{v}_t \triangleq \tilde{S}_s(\vec{C}_t \vec{x}_t \otimes \vec{v}_t) + \vec{v}_t^{[2]} - \phi_{\vec{v}_t}^{(2)}$.

In light of (10)-(12), we construct the augmented state and measurement vectors as follows:

$$\mathcal{X}_t \triangleq \begin{bmatrix} \vec{x}_t \\ \vec{x}_t^{[2]} \end{bmatrix}, \quad \mathcal{Z}_t \triangleq \begin{bmatrix} z_t \\ z_t^{[2]} \end{bmatrix},$$

and the system (10) can further be converted into

$$\begin{cases} \mathcal{X}_{t+1} = \mathcal{A}_t \mathcal{X}_t + \mathcal{M}_t + W_t \\ \mathcal{Z}_t = \mathcal{C}_t \mathcal{X}_t + \mathcal{N}_t + V_t \end{cases} \tag{13}$$

where

$$\mathcal{A}_t \triangleq \begin{bmatrix} \vec{A}_t & 0 \\ 0 & \vec{A}_t^{[2]} \end{bmatrix}, \mathcal{M}_t \triangleq \begin{bmatrix} 0 \\ \vec{B}_t^{[2]} \phi_{\vec{w}_t}^{(2)} \end{bmatrix}, W_t \triangleq \begin{bmatrix} \vec{B}_t \vec{w}_t \\ \tilde{w}_t \end{bmatrix},$$
$$\mathcal{C}_t \triangleq \begin{bmatrix} \vec{C}_t & 0 \\ 0 & \vec{C}_t^{[2]} \end{bmatrix}, \mathcal{N}_t \triangleq \begin{bmatrix} 0 \\ \phi_{\vec{v}_t}^{(2)} \end{bmatrix}, V_t \triangleq \begin{bmatrix} \vec{v}_t \\ \tilde{v}_t \end{bmatrix}.$$

In this paper, a recursive quadratic filter is designed for the augmented system (13) as follows:

$$\begin{cases} \hat{\mathcal{X}}_{t+1|t} = \mathcal{A}_t \hat{\mathcal{X}}_{t|t} + \mathcal{M}_t \\ \hat{\mathcal{X}}_{t+1|t+1} = \hat{\mathcal{X}}_{t+1|t} + \mathcal{J}_{t+1}(\mathcal{Z}_{t+1} - \mathcal{C}_{t+1} \hat{\mathcal{X}}_{t+1|t} - \mathcal{N}_{t+1}) \end{cases} \tag{14}$$

where $\hat{\mathcal{X}}_{t+1|t}$ denotes the one-step prediction, $\hat{\mathcal{X}}_{t+1|t+1}$ is the filtered state, and $\mathcal{J}_{t+1}$ serves as the gain parameter.

From (13) and (14), the prediction error

$$\aleph_{t+1|t} \triangleq \mathcal{X}_{t+1} - \hat{\mathcal{X}}_{t+1|t}$$

and the filtering error

$$\aleph_{t+1|t+1} \triangleq \mathcal{X}_{t+1} - \hat{\mathcal{X}}_{t+1|t+1}$$

are, respectively, expressed by

$$\aleph_{t+1|t} = \mathcal{A}_t \aleph_{t|t} + W_t \tag{15}$$

and

$$\aleph_{t+1|t+1} = (I - \mathcal{J}_{t+1} \mathcal{C}_{t+1}) \aleph_{t+1|t} - \mathcal{J}_{t+1} V_{t+1}. \tag{16}$$

Furthermore, the associated error covariance matrices are defined as

$$\Re_{t+1} \triangleq \mathbb{E}\{\aleph_{t+1|t} \aleph_{t+1|t}^T\} \tag{17}$$

and

$$\Re_{t+1|t+1} \triangleq \mathbb{E}\{\aleph_{t+1|t+1} \aleph_{t+1|t+1}^T\}. \tag{18}$$

Our goal is to devise a recursive quadratic filter with the structure (14) such that, in the presence of the dynamical bias, the amplify-and-forward relay as well as the non-Gaussian noises, the filtering error covariance $\Re_{t+1|t+1}$ exists and is subsequently minimized by designing the appropriate gain $\mathcal{J}_{t+1}$.

Remark 3 *It should be noted that the proposal of the quadratic filtering algorithm for such a comprehensive system is made for the first time in this study, distinguishing it considerably from existing filtering schemes under amplify-and-forward relays. The distinctive features of the constructed filter in (14) include: 1) the simultaneous incorporation of information from the dynamical bias, the amplify-and-forward relay, and the non-Gaussian noises (e.g., high-order moments of random variables) within a unified framework; 2) the expression of the designed quadratic filter in a recursive form that is both computationally feasible and easy to implement; and 3) the simultaneous estimation of the involved dynamical bias, as part of the augmented state vector, alongside the system state.*

3 Main Results

In this section, we aim to investigate the recursive quadratic filtering problem under the dynamical bias, the amplify-and-forward relay, and non-Gaussian noises. More specifically, the filtering error covariance $\Re_{t+1|t+1}$ is obtained recursively and, furthermore, the boundedness of $\Re_{t+1|t+1}$ is analyzed.

3.1 Preliminary Lemmas

For convenience of subsequent analysis, some lemmas are first introduced for the quadratic filter design.

Lemma 1 ([21]) *Let $\mathcal{L}, \mathcal{U}, \mathcal{N}$ and $\mathcal{V}$ be known matrices with $\mathcal{L}$ and $\mathcal{N}$ being invertible. Then, one has*

$$(\mathcal{L}+\mathcal{UNV})^{-1} =\mathcal{L}^{-1} - \mathcal{L}^{-1}\mathcal{U} \times (\mathcal{N}^{-1}+\mathcal{V}\mathcal{L}^{-1}\mathcal{U})^{-1}\mathcal{V}\mathcal{L}^{-1}. \quad (19)$$

Lemma 2 *$\bar{l}_t^{(i)}$ and $\bar{m}_t^{(i)}$ $(i = 1, 2, 3, 4)$ appearing in (8) are computed by*

$$\begin{aligned}
\bar{l}_t^{(1)} &= \sum_{i=1}^{p} l_{i,t}^{\frac{1}{2}}\bar{l}_{i,t}, \quad \bar{m}_t^{(1)} = \sum_{i=1}^{q} m_{i,t}^{\frac{1}{2}}\bar{m}_{i,t},\\
\bar{l}_t^{(2)} &= \sum_{i=1}^{p} l_{i,t}\bar{l}_{i,t}, \quad \bar{m}_t^{(2)} = \sum_{i=1}^{q} m_{i,t}\bar{m}_{i,t},\\
\bar{l}_t^{(3)} &= \sum_{i=1}^{p} l_{i,t}^{\frac{3}{2}}\bar{l}_{i,t}, \quad \bar{m}_t^{(3)} = \sum_{i=1}^{q} m_{i,t}^{\frac{3}{2}}\bar{m}_{i,t},\\
\bar{l}_t^{(4)} &= \sum_{i=1}^{p} l_{i,t}^{2}\bar{l}_{i,t}, \quad \bar{m}_t^{(4)} = \sum_{i=1}^{q} m_{i,t}^{2}\bar{m}_{i,t}. \quad (20)
\end{aligned}$$

Proof: The proof, readily derived from the definition of mathematical expectation, is omitted for brevity. ■

Lemma 3 *The recursion of the state covariance matrix $\Upsilon_{t+1} \triangleq \mathbb{E}\{\mathcal{X}_{t+1}\mathcal{X}_{t+1}^T\}$ for system (13) satisfies*

$$\begin{aligned}
\Upsilon_{t+1} =&\mathcal{A}_t\Upsilon_t\mathcal{A}_t^T + \mathcal{M}_t\mathcal{M}_t^T + \mathcal{R}_{W_t}\\
&+ \mathcal{A}_t\bar{\mathcal{X}}_t\mathcal{M}_t^T + \mathcal{M}_t\bar{\mathcal{X}}_t^T\mathcal{A}_t^T \quad (21)
\end{aligned}$$

where $\mathcal{R}_{W_t}$ is to be given in Lemma 4 and

$$\bar{\mathcal{X}}_t \triangleq \mathbb{E}\{\mathcal{X}_t\} = \begin{bmatrix} 0 \\ \phi_{\vec{x}_t}^{(2)} \end{bmatrix}.$$

Proof: From (13), it is easy to obtain that

$$\begin{aligned}
\Upsilon_{t+1} =&\mathcal{A}_t\Upsilon_t\mathcal{A}_t^T + \mathcal{M}_t\mathcal{M}_t^T + \mathbb{E}\{W_tW_t^T\}\\
&+ \mathrm{Sym}\Big\{\mathbb{E}\{\mathcal{A}_t\mathcal{X}_t\mathcal{M}_t^T\} + \mathbb{E}\{\mathcal{A}_t\mathcal{X}_tW_t^T\}\\
&+ \mathbb{E}\{\mathcal{M}_tW_t^T\}\Big\}. \quad (22)
\end{aligned}$$

Bearing in mind that x_0, w_t, ε_t and ξ_t are zero-mean random sequences, we know $\mathbb{E}\{\vec{x}_t\} = 0$. Taking the definition of $\vec{x}_t^{[2]}$ into account, we further have

$$\mathbb{E}\{\mathcal{A}_t\mathcal{X}_t\mathcal{M}_t^T\} = \mathcal{A}_t\bar{\mathcal{X}}_t\mathcal{M}_t^T. \quad (23)$$

In light of Assumption 1, we can verify that

$$\mathbb{E}\{\mathcal{A}_t\mathcal{X}_tW_t^T\} = 0, \;\; \mathbb{E}\{\mathcal{M}_tW_t^T\} = 0 \quad (24)$$

which yields (21), and the proof is complete. ■

On the other hand, recalling the definition of Υ_{t+1}, we know that

$$\begin{aligned}
\Upsilon_{t+1} &= \begin{bmatrix} \mathbb{E}\{\vec{x}_{t+1}\vec{x}_{t+1}^T\} & \mathbb{E}\{\vec{x}_{t+1}(\vec{x}_{t+1}^{[2]})^T\} \\ \mathbb{E}\{\vec{x}_{t+1}^{[2]}\vec{x}_{t+1}^T\} & \mathbb{E}\{\vec{x}_{t+1}^{[2]}(\vec{x}_{t+1}^{[2]})^T\} \end{bmatrix}\\
&= \begin{bmatrix} \mathrm{st}(\phi_{\vec{x}_{t+1}}^{(2)}) & \mathrm{st}(\phi_{\vec{x}_{t+1}}^{(3)}) \\ (\mathrm{st}(\phi_{\vec{x}_{t+1}}^{(3)}))^T & \mathrm{st}(\phi_{\vec{x}_{t+1}}^{(4)}) \end{bmatrix}, \quad (25)
\end{aligned}$$

that is, $\Upsilon_{11,t+1} = \mathrm{st}(\phi_{\vec{x}_{t+1}}^{(2)})$, $\Upsilon_{12,t+1} = \mathrm{st}(\phi_{\vec{x}_{t+1}}^{(3)})$ and $\Upsilon_{22,t+1} = \mathrm{st}(\phi_{\vec{x}_{t+1}}^{(4)})$.

Let $\vec{\mathcal{L}} \triangleq [I_n, 0_{n\times n_\varepsilon}]$, then we have $x_{t+1} = \vec{\mathcal{L}}\vec{x}_{t+1}$, which further implies that

$$\begin{aligned}
\mathrm{st}(\phi_{x_{t+1}}^{(2)}) =&\vec{\mathcal{L}}\mathrm{st}(\phi_{\vec{x}_{t+1}}^{(2)})\vec{\mathcal{L}}^T,\\
\mathrm{st}(\phi_{x_{t+1}}^{(3)}) =&\vec{\mathcal{L}}\mathrm{st}(\phi_{\vec{x}_{t+1}}^{(3)})(\vec{\mathcal{L}}^{[2]})^T, \quad (26)\\
\mathrm{st}(\phi_{x_{t+1}}^{(4)}) =&\vec{\mathcal{L}}^{[2]}\mathrm{st}(\phi_{\vec{x}_{t+1}}^{(4)})(\vec{\mathcal{L}}^{[2]})^T.
\end{aligned}$$

Similarly, defining

$$\tilde{\mathcal{L}} \triangleq [I_n, 0_{n\times(n_\varepsilon+(n+n_\varepsilon)^2)}]$$

and

$$\Re_{11,t+1|t+1} \triangleq \mathbb{E}\{(x_{t+1}-\hat{x}_{t+1|t+1})(x_{t+1}-\hat{x}_{t+1|t+1})^T\},$$

we obtain

$$\begin{aligned}
\hat{x}_{t+1|t+1} =&\tilde{\mathcal{L}}\hat{\mathcal{X}}_{t+1|t+1},\\
\Re_{11,t+1|t+1} =&\tilde{\mathcal{L}}\Re_{t+1|t+1}\tilde{\mathcal{L}}^T. \quad (27)
\end{aligned}$$

In what follows, we will concentrate on the stochastic properties of the augmented noises W_t and V_t. In other words, we need to calculate $\mathcal{R}_{W_t} \triangleq \mathbb{E}\{W_tW_t^T\}$ and $\mathcal{R}_{V_t} \triangleq \mathbb{E}\{V_tV_t^T\}$.

Lemma 4 *The variances of the augmented noises W_t and V_t satisfy*

$$\mathcal{R}_{W_t} = \begin{bmatrix} \vec{B}_t\mathrm{st}(\phi_{\vec{w}_t}^{(2)})\vec{B}_t^T & \vec{B}_t\mathrm{st}(\phi_{\vec{w}_t}^{(3)})(\vec{B}_t^{[2]})^T \\ \vec{B}_t^{[2]}(\mathrm{st}(\phi_{\vec{w}_t}^{(3)}))^T\vec{B}_t^T & \mathcal{R}_{W_{22,t}} \end{bmatrix} \quad (28)$$

$$\mathcal{R}_{V_t} = \begin{bmatrix} \mathcal{R}_{V_{11,t}} & \mathcal{R}_{V_{12,t}} \\ \mathcal{R}_{V_{12,t}}^T & \mathcal{R}_{V_{22,t}} \end{bmatrix} \quad (29)$$

where

$$\begin{aligned}
\mathcal{R}_{W_{22,t}} \triangleq&\tilde{S}_{n+n_\varepsilon}\Big(\vec{A}_t\mathrm{st}(\phi_{\vec{x}_t}^{(2)})\vec{A}_t^T \otimes \vec{B}_t\mathrm{st}(\phi_{\vec{w}_t}^{(2)})\vec{B}_t^T\Big)\tilde{S}_{n+n_\varepsilon}^T\\
&+ \vec{B}_t^{[2]}\Big(\mathrm{st}(\phi_{\vec{w}_t}^{(4)}) - \phi_{\vec{w}_t}^{(2)}(\phi_{\vec{w}_t}^{(2)})^T\Big)(\vec{B}_t^{[2]})^T,\\
\mathcal{R}_{V11,t} \triangleq&\theta_t^2\chi_{1,t}\Psi_t\Phi_tC_t\mathrm{st}(\phi_{x_t}^{(2)})C_t^T\Phi_t^T\Psi_t^T\\
&+ \theta_t^2\bar{m}_t^{(2)}\bar{l}_t^{(2)}\Psi_t\Phi_tD_t\mathrm{st}(\phi_{v_t}^{(2)})D_t^T\Phi_t^T\Psi_t^T\\
&+ \theta_t^2\bar{m}_t^{(2)}\Psi_t\mathrm{st}(\phi_{\varpi_t}^{(2)})\Psi_t^T + \mathrm{st}(\phi_{\varsigma_t}^{(2)}),\\
\mathcal{R}_{V12,t} \triangleq&\theta_t^3\bar{m}_t^{(1)}\bar{l}_t^{(1)}\chi_{1,t}\Psi_t\Phi_tC_t\mathrm{st}(\phi_{x_t}^{(3)})(\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]})^T\tilde{S}_s^T\\
&+ \theta_t^3\chi_{2,t}\Psi_t\Phi_tC_t\mathrm{st}(\phi_{x_t}^{(3)})(\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]})^T\\
&+ \theta_t^3\bar{m}_t^{(3)}\bar{l}_t^{(3)}\Psi_t\Phi_tD_t\mathrm{st}(\phi_{v_t}^{(3)})(\Psi_t^{[2]}\Phi_t^{[2]}D_t^{[2]})^T\\
&+ \theta_t^3\bar{m}_t^{(3)}\Psi_t\mathrm{st}(\phi_{\varpi_t}^{(3)})(\Psi_t^{[2]})^T + \mathrm{st}(\phi_{\varsigma_t}^{(3)}),\\
\mathcal{R}_{V22,t} \triangleq&\tilde{S}_s\Big[\theta_t^4\chi_{1,t}(\bar{m}_t^{(1)}\bar{l}_t^{(1)})^2\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]}\mathrm{st}(\phi_{x_t}^{(4)})(\Psi_t^{[2]}\Phi_t^{[2]}\\
&\times C_t^{[2]})^T + \theta_t^2(\bar{m}_t^{(1)}\bar{l}_t^{(1)})^2(\Psi_t\Phi_tC_t\mathrm{st}(\phi_{x_t}^{(2)})C_t^T
\end{aligned}$$

$$
\begin{aligned}
&\times \Phi_t^T\Psi_t^T) \otimes \big(\theta_t^2\bar{m}_t^{(2)}\bar{l}_t^{(2)}\Psi_t\Phi_t D_t \mathrm{st}(\phi_{v_t}^{(2)})D_t^T\Phi_t^T\Psi_t^T \\
&+ \theta_t^2\bar{m}_t^{(2)}\Psi_t \mathrm{st}(\phi_{\varpi_t}^{(2)})\Psi_t^T + \mathrm{st}(\phi_{\varsigma_t}^{(2)})\big)\Big]\tilde{S}_s^T \\
&+ \theta_t^4\chi_{3,t}\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]}\mathrm{st}(\phi_{x_t}^{(4)})(C_t^{[2]})^T(\Phi_t^{[2]})^T(\Psi_t^{[2]})^T \\
&+ \mathrm{st}(\phi_{\varsigma_t}^{(4)}) + \theta_t^4\bar{m}_t^{(4)}\Psi_t^{[2]}\mathrm{st}(\phi_{\varpi_t}^{(4)})(\Psi_t^{[2]})^T - \phi_{\vec{v}_t}^{(2)}(\phi_{\vec{v}_t}^{(2)})^T \\
&+ \theta_t^4\bar{m}_t^{(4)}\bar{l}_t^{(4)}\Psi_t^{[2]}\Phi_t^{[2]}D_t^{[2]}\mathrm{st}(\phi_{v_t}^{(4)})(\Psi_t^{[2]}\Phi_t^{[2]}D_t^{[2]})^T \\
&+ \tilde{S}_s\Big\{(\theta_t^4\chi_{4,t}\Psi_t\Phi_t C_t\mathrm{st}(\phi_{x_t}^{(2)})C_t^T\Phi_t^T\Psi_t^T) \\
&\otimes(\Psi_t\Phi_t D_t\mathrm{st}(\phi_{v_t}^{(2)})D_t^T\Phi_t^T\Psi_t^T) + (\theta_t^2\chi_{1,t}\Psi_t \\
&\times \Phi_t C_t\mathrm{st}(\phi_{x_t}^{(2)})C_t^T\Phi_t^T\Psi_t^T)\otimes\mathrm{st}(\phi_{\varsigma_t}^{(2)}) \\
&+ (\theta_t^2\bar{m}_t^{(2)}\Psi_t\mathrm{st}(\phi_{\varpi_t}^{(2)})\Psi_t^T)\otimes\mathrm{st}(\phi_{\varsigma_t}^{(2)}) \\
&+ (\theta_t^4\chi_{5,t}\Psi_t\Phi_t C_t\mathrm{st}(\phi_{x_t}^{(2)})C_t^T\Phi_t^T\Psi_t^T) \\
&\otimes(\Psi_t\mathrm{st}(\phi_{\varpi_t}^{(2)})\Psi_t^T) + \theta_t^2\bar{m}_t^{(2)}\bar{l}_t^{(2)}\Psi_t \\
&\times\Phi_t D_t\mathrm{st}(\phi_{v_t}^{(2)})D_t^T\Phi_t^T\Psi_t^T\otimes\mathrm{st}(\phi_{\varsigma_t}^{(2)}) \\
&+ (\theta_t^4\bar{m}_t^{(4)}\bar{l}_t^{(2)}\Psi_t\Phi_t D_t\mathrm{st}(\phi_{v_t}^{(2)})D_t^T\Phi_t^T\Psi_t^T) \\
&\otimes(\Psi_t\mathrm{st}(\phi_{\varpi_t}^{(2)})\Psi_t^T)\Big\}\tilde{S}_s^T + \mathrm{Sym}\Big\{\theta_t^2\chi_{1,t}\Psi_t^{[2]} \\
&\times\Phi_t^{[2]}C_t^{[2]}\phi_{x_t}^{(2)}(\phi_{\varsigma_t}^{(2)})^T + \theta_t^2\bar{m}_t^{(2)}\Psi_t^{[2]}\phi_{\varpi_t}^{(2)}(\phi_{\varsigma_t}^{(2)})^T \\
&+ \theta_t^4\chi_{4,t}\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]}\phi_{x_t}^{(2)}(\phi_{v_t}^{(2)})^T(\Psi_t^{[2]}\Phi_t^{[2]}D_t^{[2]})^T \\
&+ \theta_t^4\chi_{5,t}\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]}\phi_{x_t}^{(2)}(\phi_{\varpi_t}^{(2)})^T(\Psi_t^{[2]})^T \\
&+ \theta_t^4\bar{m}_t^{(4)}\bar{l}_t^{(2)}\Psi_t^{[2]}\Phi_t^{[2]}D_t^{[2]}\phi_{v_t}^{(2)}(\phi_{\varpi_t}^{(2)})^T(\Psi_t^{[2]})^T \\
&+ \theta_t^2\bar{m}_t^{(2)}\bar{l}_t^{(2)}\Psi_t^{[2]}\Phi_t^{[2]}D_t^{[2]}\phi_{v_t}^{(2)}(\phi_{\varsigma_t}^{(2)})^T \\
&+ \tilde{S}_s\Big[\theta_t^4\chi_{2,t}\bar{m}_t^{(1)}\bar{l}_t^{(1)}\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]}\mathrm{st}(\phi_{x_t}^{(4)}) \\
&\times(\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]})^T + \theta_t^4\chi_{6,t}\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]}\phi_{x_t}^{(2)}(\phi_{v_t}^{(2)})^T \\
&\times(\Psi_t^{[2]}\Phi_t^{[2]}D_t^{[2]})^T + \theta_t^4\chi_{7,t}\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]}\phi_{x_t}^{(2)}(\phi_{\varpi_t}^{(2)})^T \\
&\times(\Psi_t^{[2]})^T + \theta_t^4\chi_{6,t}\big[(\Psi_t\Phi_t C_t\mathrm{st}(\phi_{x_t}^{(2)})C_t^T\Phi_t^T\Psi_t^T) \\
&\otimes(\Psi_t\Phi_t D_t\mathrm{st}(\phi_{v_t}^{(2)})D_t^T\Phi_t^T\Psi_t^T)\tilde{S}_s^T\big] + \theta_t^4\chi_{7,t}\big[(\Psi_t\Phi_t \\
&\times C_t\mathrm{st}(\phi_{x_t}^{(2)})C_t^T\Phi_t^T\Psi_t^T)\otimes(\Psi_t\mathrm{st}(\phi_{\varpi_t}^{(2)})\Psi_t^T)\tilde{S}_s^T\big]\Big]\Big\}
\end{aligned}
$$

with

$$
\begin{aligned}
\chi_{1,t} \triangleq& \bar{m}_t^{(2)}\bar{l}_t^{(2)} - (\bar{m}_t^{(1)}\bar{l}_t^{(1)})^2, \\
\chi_{2,t} \triangleq& \bar{m}_t^{(3)}\bar{l}_t^{(3)} - 3\bar{m}_t^{(1)}\bar{l}_t^{(1)}\bar{m}_t^{(2)}\bar{l}_t^{(2)} + 2(\bar{m}_t^{(1)}\bar{l}_t^{(1)})^3, \\
\chi_{3,t} \triangleq& \bar{m}_t^{(4)}\bar{l}_t^{(4)} - 4\bar{m}_t^{(1)}\bar{l}_t^{(1)}\bar{m}_t^{(3)}\bar{l}_t^{(3)} \\
&+ 6(\bar{m}_t^{(1)}\bar{l}_t^{(1)})^2\bar{m}_t^{(2)}\bar{l}_t^{(2)} - 3(\bar{m}_t^{(1)}\bar{l}_t^{(1)})^4, \\
\chi_{4,t} \triangleq& \bar{m}_t^{(4)}\bar{l}_t^{(4)} - 2\bar{m}_t^{(1)}\bar{l}_t^{(1)}\bar{m}_t^{(3)}\bar{l}_t^{(3)} + \bar{m}_t^{(2)}\bar{l}_t^{(2)}(\bar{m}_t^{(1)}\bar{l}_t^{(1)})^2, \\
\chi_{5,t} \triangleq& \bar{m}_t^{(4)}\bar{l}_t^{(2)} - 2\bar{m}_t^{(1)}(\bar{l}_t^{(1)})^2\bar{m}_t^{(3)} + \bar{m}_t^{(2)}(\bar{m}_t^{(1)}\bar{l}_t^{(1)})^2, \\
\chi_{6,t} =& \bar{m}_t^{(1)}\bar{l}_t^{(1)}(\bar{m}_t^{(3)}\bar{l}_t^{(3)} - \bar{m}_t^{(1)}\bar{l}_t^{(1)}\bar{m}_t^{(2)}\bar{l}_t^{(2)}), \\
\chi_{7,t} =& \bar{m}_t^{(1)}\bar{l}_t^{(1)}(\bar{m}_t^{(3)}\bar{l}_t^{(1)} - \bar{m}_t^{(1)}\bar{l}_t^{(1)}\bar{m}_t^{(2)}).
\end{aligned}
$$

Proof: See Appendix. ∎

Next, we focus our attention on the derivation of the filtering error covariance $\Re_{t+1|t+1}$ and the gain matrix $\mathcal{J}_{t+1}$.

3.2 Design of Quadratic Filter

In this subsection, we aim to give the one-step prediction error covariance and filtering error covariance in Theorem 1, which is subsequently followed by selecting a gain parameter $\mathcal{J}_{t+1}$ in Theorem 2.

Theorem 1 *The one-step prediction error covariance $\Re_{t+1|t}$ and the filtering error covariance $\Re_{t+1|t+1}$ satisfy the following difference equations*

$$
\begin{aligned}
\Re_{t+1|t} =& \mathcal{A}_t\Re_{t|t}\mathcal{A}_t^T + \mathcal{R}_{W_t} &(30)\\
\Re_{t+1|t+1} =& (I - \mathcal{J}_{t+1}\mathcal{C}_{t+1})\Re_{t+1|t}(I - \mathcal{J}_{t+1}\mathcal{C}_{t+1})^T \\
&+ \mathcal{J}_{t+1}\mathcal{R}_{V_{t+1}}\mathcal{J}_{t+1}^T &(31)
\end{aligned}
$$

Proof: Substituting (15) into $\Re_{t+1|t}$, we have

$$
\begin{aligned}
\Re_{t+1|t} =& \mathbb{E}\{\mathcal{A}_t\aleph_{t|t}\aleph_{t|t}^T\mathcal{A}_t^T\} + \mathcal{A}_t\mathbb{E}\{\aleph_{t|t}W_t^T\} \\
&\mathbb{E}\{W_t\aleph_{t|t}^T\}\mathcal{A}_t^T + \mathbb{E}\{W_tW_t^T\}. &(32)
\end{aligned}
$$

Noticing $\mathbb{E}\{\mathcal{X}_tW_t^T\} = 0$ and $\mathbb{E}\{\hat{\mathcal{X}}_{t|t}W_t^T\} = 0$, it is easy to verify that $\mathbb{E}\{\aleph_{t|t}W_t^T\} = 0$. Therefore, the dynamics of the one-step prediction error covariance satisfies (30). In view of the expression of $\aleph_{t+1|t+1}$ and the definition of $\Re_{t+1|t+1}$, we obtain

$$
\begin{aligned}
\Re_{t+1|t+1} =& (I - \mathcal{J}_{t+1}\mathcal{C}_{t+1})\mathbb{E}\{\aleph_{t+1|t}\aleph_{t+1|t}^T\}(I - \mathcal{J}_{t+1}\mathcal{C}_{t+1})^T \\
&- \mathbb{E}\{(I - \mathcal{J}_{t+1}\mathcal{C}_{t+1})\aleph_{t+1|t}V_{t+1}^T\}\mathcal{J}_{t+1}^T \\
&- \mathcal{J}_{t+1}\mathbb{E}\{V_{t+1}\aleph_{t+1|t}^T(I - \mathcal{J}_{t+1}\mathcal{C}_{t+1})\}^T \\
&+ \mathcal{J}_{t+1}\mathbb{E}\{V_{t+1}V_{t+1}^T\}\mathcal{J}_{t+1}^T. &(33)
\end{aligned}
$$

Since the noises ε_{t+1}, ϖ_{t+1} and ς_{t+1} are independent of $\aleph_{t+1|t}$, we derive $\mathbb{E}\{\aleph_{t+1|t}\vec{v}_{t+1}^T\} = 0$ and $\mathbb{E}\{\aleph_{t+1|t}\tilde{v}_{t+1}^T\} = 0$, from which we further obtain $\mathbb{E}\{\aleph_{t+1|t}V_{t+1}^T\} = 0$. Accordingly, the evolution of the filtering error covariance's dynamics is given as

$$
\begin{aligned}
\Re_{t+1|t+1} =& (I - \mathcal{J}_{t+1}\mathcal{C}_{t+1})\mathbb{E}\{\aleph_{t+1|t}\aleph_{t+1|t}^T\}(I - \mathcal{J}_{t+1}\mathcal{C}_{t+1})^T \\
&+ \mathcal{J}_{t+1}\mathbb{E}\{V_{t+1}V_{t+1}^T\}\mathcal{J}_{t+1}^T &(34)
\end{aligned}
$$

which yields (31). This proof is complete. ∎

Theorem 2 *The filtering error covariance $\Re_{t+1|t+1}$ in (31) is minimized by designing the gain matrix as follows:*

$$
\mathcal{J}_{t+1} = \Re_{t+1|t}\mathcal{C}_{t+1}^T(\mathcal{C}_{t+1}\Re_{t+1|t}\mathcal{C}_{t+1}^T + \mathcal{R}_{V_{t+1}})^{-1} \quad (35)
$$

and the corresponding minimal filtering error covariance $\Re_{t+1|t+1}$ is computed recursively as

$$
\Re_{t+1|t+1} = \Re_{t+1|t} - \mathcal{J}_{t+1}(\mathcal{C}_{t+1}\Re_{t+1|t}\mathcal{C}_{t+1}^T + \mathcal{R}_{V_{t+1}})\mathcal{J}_{t+1}^T. \quad (36)
$$

Proof: By means of the completing-the-square method, $\Re_{t+1|t+1}$ in (31) can be calculated as follows:

$$
\begin{aligned}
\Re_{t+1|t+1} =& \Re_{t+1|t} + \Big[\mathcal{J}_{t+1} - \Re_{t+1|t}\mathcal{C}_{t+1}^T(\mathcal{C}_{t+1}\Re_{t+1|t}\mathcal{C}_{t+1}^T \\
&+ \mathcal{R}_{V_{t+1}})^{-1}\Big](\mathcal{C}_{t+1}\Re_{t+1|t}\mathcal{C}_{t+1}^T + \mathcal{R}_{V_{t+1}})\Big[\mathcal{J}_{t+1}
\end{aligned}
$$

$$
\begin{aligned}
&- \Re_{t+1|t}\mathcal{C}_{t+1}^T(\mathcal{C}_{t+1}\Re_{t+1|t}\mathcal{C}_{t+1}^T + \mathcal{R}_{V_{t+1}})^{-1}\Big]^T \\
&- \Re_{t+1|t}\mathcal{C}_{t+1}^T(\mathcal{C}_{t+1}\Re_{t+1|t}\mathcal{C}_{t+1}^T + \mathcal{R}_{V_{t+1}})^{-1} \\
&\times \mathcal{C}_{t+1}\Re_{t+1|t}. \qquad (37)
\end{aligned}
$$

Consequently, it is straightforward to see that $\Re_{t+1|t+1}$ is minimized by choosing the following gain matrix

$$
\mathcal{J}_{t+1} = \Re_{t+1|t}\mathcal{C}_{t+1}^T(\mathcal{C}_{t+1}\Re_{t+1|t}\mathcal{C}_{t+1}^T + \mathcal{R}_{V_{t+1}})^{-1}
$$

which, together with (37), leads to (36). The proof is now complete. ■

Remark 4 *The one-step prediction error covariance and filtering error covariance have been established in (30) and (31), and the gain matrix $\mathcal{J}_{t+1}$ has been obtained in (35). It is observed that the filtering error covariance relies on $\mathcal{A}_t$, $\mathcal{C}_t$, $\mathcal{R}_{W_t}$ and $\mathcal{R}_{V_t}$, which indicate that the influences from the dynamical bias, the amplify-and-forward relay, and the non-Gaussian noises have all been reflected in the design of the quadratic filter. To be specific, F_t in $\vec{A}_t$ accounts for the effect from the dynamical bias, θ_t and $\chi_{i,t}(i = 1, 2, \cdots, 7)$ reflect the influence of the amplify-and-forward relay, and $\phi^{(j)}_{\vec{w}_t}$, $\phi^{(j)}_{v_t}$, $\phi^{(j)}_{\varpi_t}$ and $\phi^{(j)}_{\varsigma_t}$ $(j = 2, 3, 4)$ in $\mathcal{R}_{W_t}$ and $\mathcal{R}_{V_t}$ cater for the impact of high-order moments of non-Gaussian noises on $\Re_{t+1|t+1}$.*

3.3 Boundedness Analysis

In this subsection, we examine the boundedness of the filtering error covariance $\Re_{t+1|t+1}$.

Assumption 2 *There exist positive scalars $\bar{q}_a$, $\underline{q}_a$, $\bar{q}_c$, $\underline{q}_v$, $\underline{q}_w$ and $\bar{q}_w$ satisfying*

$$
\begin{aligned}
&\mathcal{A}_t\mathcal{A}_t^T \le \bar{q}_a I, \mathcal{A}_t^T\mathcal{A}_t \ge \underline{q}_a I, \\
&\mathcal{C}_t^T\mathcal{C}_t \le \bar{q}_c I, \mathcal{R}_{V_t} \ge \underline{q}_v I, \\
&\underline{q}_w I \le \mathcal{R}_{W_t} \le \bar{q}_w I.
\end{aligned}
$$

Theorem 3 *Under Assumption 2, there exists a positive scalar $\underline{\beta}$ such that*

$$
\Re_{t+1|t+1} \ge \underline{\beta} I \qquad (38)
$$

for every $t > 0$, where

$$
\underline{\beta} \triangleq (\underline{q}_w^{-1} + \bar{q}_c\underline{q}_v^{-1})^{-1}.
$$

Proof: By means of Lemma 1, $\Re_{t+1|t+1}$ can be reorganized as

$$
\Re_{t+1|t+1} = \left(\Re_{t+1|t}^{-1} + \mathcal{C}_{t+1}^T\mathcal{R}_{V_{t+1}}^{-1}\mathcal{C}_{t+1}\right)^{-1}. \qquad (39)
$$

Noting

$$
\Re_{t+1|t} \ge \mathcal{R}_{W_t} \ge \underline{q}_w I \qquad (40)
$$

we have

$$
\begin{aligned}
\Re_{t+1|t+1}^{-1} =&\Re_{t+1|t}^{-1} + \mathcal{C}_{t+1}^T\mathcal{R}_{V_{t+1}}^{-1}\mathcal{C}_{t+1} \\
\le&(\underline{q}_w^{-1} + \bar{q}_c\underline{q}_v^{-1})I \qquad (41)
\end{aligned}
$$

which implies that

$$
\Re_{t+1|t+1} \ge (\underline{q}_w^{-1} + \bar{q}_c\underline{q}_v^{-1})^{-1} I \triangleq \underline{\beta} I \qquad (42)
$$

for every $t > 0$. ■

Theorem 4 *Under Assumption 2, there exists a positive scalar β_{t+1} such that*

$$
\Re_{t+1|t+1} \le \beta_{t+1} I \qquad (43)
$$

where

$$
\beta_0 \triangleq \lambda_{\max}(\Re_{0|0}), \quad \beta_{t+1} \triangleq \beta_0\bar{q}_a^{t+1} + \bar{q}_w\sum_{i=0}^{t}\bar{q}_a^i.
$$

Proof: This theorem is proved by the mathematical induction method. It is obvious that $\Re_{0|0} \le \lambda_{\max}(\Re_{0|0})I$. Suppose that $\Re_{t|t} \le \beta_t I$ holds, we need to prove $\Re_{t+1|t+1} \le \beta_{t+1} I$.

It follows from (36) that

$$
\begin{aligned}
\Re_{t+1|t+1} \le&\Re_{t+1|t} = \mathcal{A}_t\Re_{t|t}\mathcal{A}_t^T + \mathcal{R}_{W_t} \\
\le&\beta_t\bar{q}_a I + \bar{q}_w I \\
\le&(\beta_0\bar{q}_a^{t+1} + \bar{q}_w\sum_{i=0}^{t}\bar{q}_a^i)I \\
\triangleq&\beta_{t+1} I \qquad (44)
\end{aligned}
$$

which ends the proof. ■

We notice from Theorem 4 that, if $\bar{q}_a < 1$, then the matrix $\Re_{t+1|t+1}$ also satisfies

$$
\Re_{t+1|t+1} \le (\beta_0\bar{q}_a + \frac{\bar{q}_w}{1 - \bar{q}_a})I.
$$

Furthermore, if $\bar{q}_a > 1$, we are interested in establishing the corresponding sufficient condition on the upper bound of $\Re_{t+1|t+1}$.

Theorem 5 *Under Assumption 2, if there exist positive scalars $\underline{q}_a$, $\bar{q}_w$, $\underline{\beta}$, $\bar{\beta}$ and r such that the following inequalities*

$$
\sum_{i=t-r+1}^{t+1}\zeta^{i-t-1}\Omega^T(i, t+1)\mathcal{C}_i^T\mathcal{R}_{V_i}^{-1}\mathcal{C}_i\Omega(i, t+1) \ge \bar{\beta}^{-1}I, t \ge r \qquad (45)
$$

$$
\beta_{t+1} \le \bar{\beta}, \quad 0 \le t < r - 1 \qquad (46)
$$

hold for all $t \ge 0$, then the filtering error covariance $\Re_{t+1|t+1}$ satisfies

$$
\Re_{t+1|t+1} \le \bar{\beta} I \qquad (47)
$$

where

$$
\begin{aligned}
&\zeta \triangleq 1 + \bar{q}_w\underline{q}_a^{-1}\underline{\beta}^{-1}, \ \Omega(t+1, t+1) \triangleq I, \\
&\Omega(i, t+1) \triangleq \mathcal{A}_i^{-1}\mathcal{A}_{i+1}^{-1}\cdots\mathcal{A}_t^{-1}(i < t). \qquad (48)
\end{aligned}
$$

Proof: For $0 \le t < r-1$, we can conclude that $\Re_{t+1|t+1} \le \beta_{t+1} I \le \bar{\beta} I$.

Based on Assumption 2 and (30), we obtain that

$$\begin{aligned}\Re_{t+1|t} =& \mathcal{A}_t[\Re_{t|t} + \mathcal{A}_t^{-1}\mathcal{R}_{W_t}(\mathcal{A}_t^{-1})^T]\mathcal{A}_t^T \\ \le& \mathcal{A}_t[\Re_{t|t} + \bar{q}_w \underline{q}_a^{-1}\underline{\beta}^{-1}\Re_{t|t}]\mathcal{A}_t^T \\ =& (1+\bar{q}_w \underline{q}_a^{-1}\underline{\beta}^{-1})\mathcal{A}_t\Re_{t|t}\mathcal{A}_t^T. \end{aligned} \tag{49}$$

Substituting (49) into $\Re_{t+1|t}^{-1}$ leads to

$$\begin{aligned}\Re_{t+1|t+1}^{-1} =& \Re_{t+1|t}^{-1} + \mathcal{C}_{t+1}^T\mathcal{R}_{V_{t+1}}^{-1}\mathcal{C}_{t+1} \\ \ge& \zeta^{-1}\mathcal{A}_t^{-T}\Re_{t|t}^{-1}\mathcal{A}_t^{-1} + \mathcal{C}_{t+1}^T\mathcal{R}_{V_{t+1}}^{-1}\mathcal{C}_{t+1} \\ \ge& \zeta^{-2}\mathcal{A}_t^{-T}\mathcal{A}_{t-1}^{-T}\Re_{t-1|t-1}^{-1}\mathcal{A}_{t-1}^{-1}\mathcal{A}_t^{-1} \\ & + \zeta^{-1}\mathcal{A}_t^{-T}\mathcal{C}_t^T\mathcal{R}_{V_t}^{-1}\mathcal{C}_t\mathcal{A}_t^{-1} + \mathcal{C}_{t+1}^T\mathcal{R}_{V_{t+1}}^{-1}\mathcal{C}_{t+1} \\ \ge& \sum_{i=t-r+1}^{t+1}\zeta^{i-t-1}\Omega^T(i,t+1)\mathcal{C}_i^T R_{V_i}^{-1}\mathcal{C}_i\Omega(i,t+1) \\ \ge& \bar{\beta}^{-1}I. \end{aligned} \tag{50}$$

Therefore, we can derive that $\Re_{t+1|t+1} \le \bar{\beta} I$ when (45)-(46) hold. The proof is complete. ■

Remark 5 *The uniform lower bound $\underline{\beta} I$ and the upper bound $\bar{\beta} I$ of the filtering error covariance $\Re_{t+1|t+1}$ have been given, respectively, in Theorems 3-4, from which we can observe that the scalars $\underline{\beta}$ and $\bar{\beta}$ are closely related to the augmented system matrices and the augmented noises covariance matrices.*

Remark 6 *Thus far, the quadratic filtering issue has been addressed for linear non-Gaussian systems equipped with an amplify-and-forward relay possessing random transmission powers. Given the dynamical bias and the amplify-and-forward relay, both a recursive quadratic filter and its filtering error covariance have been derived. Moreover, the influence of system coefficients on the filtering performance has been elucidated. Contrasting with extant filtering outcomes related to the amplify-and-forward relay/dynamical bias, the filtering methodology presented in this study boasts several unique attributes: 1) the quadratic filtering challenge explored is pioneering, especially given the simultaneous presence of dynamical bias, non-Gaussian noises, and an amplify-and-forward relay influenced by random transmission powers; 2) the devised quadratic filtering method is innovative, delving into the stochastic nuances of the non-Gaussian noises/transmission powers while shedding light on the interplay between the filtering error covariance and the aforementioned determinants; and 3) the analysis result of boundedness is new that offers both the lower and upper boundaries for the filtering error covariance.*

4 An Illustrative Example

In this section, some experiment results are provided to demonstrate the validity of the presented quadratic filtering algorithm.

Table 1
The 2nd, 3rd and 4th-order moments of random variables

	$\mathbb{E}\{(\cdot)^2\}$	$\mathbb{E}\{(\cdot)^3\}$	$\mathbb{E}\{(\cdot)^4\}$
w_t	1	0	1
v_t	0.9100	−0.5460	1.1557
$\varepsilon_{1,t}$	0.9600	0.3840	1.0752
$\varepsilon_{2,t}$	0.1875	−0.0938	0.0820
$\xi_{1,t}$	0.7500	−0.7500	1.3125
$\xi_{2,t}$	0.3750	−0.0938	0.1641
ϖ_t	0.3600	−0.5760	1.0512
ς_t	0.2100	−0.0840	0.0777

Consider a linear discrete system with the following parameters:

$$A_t = \begin{bmatrix} 0.6 & 0.5 \\ -0.1\sin(0.2t) & 0.8 \end{bmatrix}, E_t = \begin{bmatrix} 0.1 & 0 \\ 0.03\cos(0.2t) & 0.15 \end{bmatrix},$$

$$B_t = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix}, C_t = \begin{bmatrix} 0.2 & 0.18 \end{bmatrix}, F_t = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.2 \end{bmatrix}.$$

Other parameters are set as $D_t = 0.01$, $\Phi_t = 1$, $\Psi_t = 1$ and $\theta_t = 3$. The probability distributions of the transmission power are given as follows:

$$\mathbb{P}\{l_t = 1\} = 0.1,\ \ \mathbb{P}\{l_t = 1.5\} = 0.3,\ \ \mathbb{P}\{l_t = 2\} = 0.6,$$
$$\mathbb{P}\{m_t = 1\} = 0.2,\ \mathbb{P}\{m_t = 1.1\} = 0.4,\ \mathbb{P}\{l_t = 1.2\} = 0.4.$$

The initial state x_0 is supposed to satisfy the Gaussian distribution with zero mean and covariance $0.01I_2$. The non-Gaussian random sequences, i.e., $w_t, v_t, \varepsilon_{i,t}, \xi_{i,t}$ ($i = 1, 2$), ϖ_t and ς_t, are adopted as follows:

$$\begin{aligned} & w_t = -\rho_{w_t} + (1-\rho_{w_t}), \\ & v_t = 0.7\rho_{v_t} - 1.3(1-\rho_{v_t}), \\ & \varepsilon_{1,t} = -0.8\rho_{\varepsilon_{1,t}} + 1.2(1-\rho_{\varepsilon_{1,t}}), \\ & \varepsilon_{2,t} = -0.75\rho_{\varepsilon_{2,t}} + 0.25(1-\rho_{\varepsilon_{2,t}}), \\ & \xi_{1,t} = -1.5\rho_{\xi_{1,t}} + 0.5(1-\rho_{\xi_{1,t}}), \\ & \xi_{2,t} = -0.75\rho_{\xi_{2,t}} + 0.5(1-\rho_{\xi_{2,t}}), \\ & \varpi_t = -1.8\rho_{\varpi_t} + 0.2(1-\rho_{\varpi_t}), \\ & \varsigma_t = -0.7\rho_{\varsigma_t} + 0.3(1-\rho_{\varsigma_t}), \end{aligned}$$

where the random variables $\rho_{w_t}, \rho_{v_t}, \rho_{\varepsilon_{i,t}}, \rho_{\xi_{i,t}}, \rho_{\varpi_t}$ and ρ_{ς_t} obey Bernoulli distributions

$$\begin{aligned} &\mathbb{P}\{\rho_{w_t} = 1\} = 0.5, && \mathbb{P}\{\rho_{v_t} = 1\} = 0.65, \\ &\mathbb{P}\{\rho_{\varepsilon_{1,t}} = 1\} = 0.6, && \mathbb{P}\{\rho_{\varepsilon_{2,t}} = 1\} = 0.25, \\ &\mathbb{P}\{\rho_{\xi_{1,t}} = 1\} = 0.25, && \mathbb{P}\{\rho_{\xi_{2,t}} = 1\} = 0.4, \\ &\mathbb{P}\{\rho_{\varpi_t} = 1\} = 0.1, && \mathbb{P}\{\rho_{\varsigma_t} = 1\} = 0.3. \end{aligned}$$

Moreover, the 2nd, 3rd and 4th-order moments of the involved random sequences are given in Table 1.

Based on the aforementioned parameters, (30)-(31) and (35), the quadratic filtering algorithm is employed to estimate x_t, and the associated simulation results are depicted in Figs. 1-6. Specifically, Figs. 1-2 illustrate the trajectories of $x_{1,t}$ and $x_{2,t}$, along with their estimates $\hat{x}_{1,t|t}$ and $\hat{x}_{2,t|t}$. From these figures, it can be observed that the proposed filter aligns closely with the actual states, from which we notice that the proposed filter can follow the actual states closely. To underscore the superiority of the devised quadratic filtering approach, comparisons of the filtering error covariances between the quadratic and linear filters (the latter utilizing only z_t) are presented in Figs. 3-4. Moreover, Figs. 5-6 showcase the Mean Squared Error (MSE) trajectories of both the quadratic and linear filters. It is clear from the graphs that the covariance/MSE trajectories of the linear filter consistently exceed those of the quadratic filter, further attesting to the enhanced filtering accuracy of the proposed quadratic method.

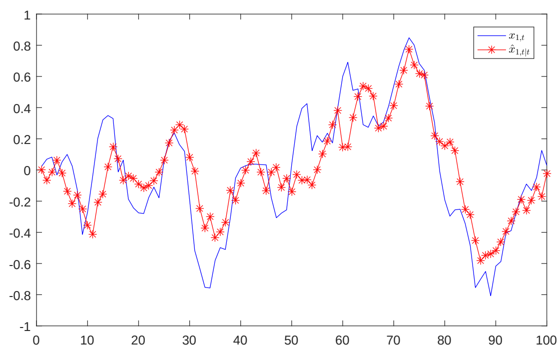

Fig. 1. $x_{1,t}$ and its estimation $\hat{x}_{1,t|t}$

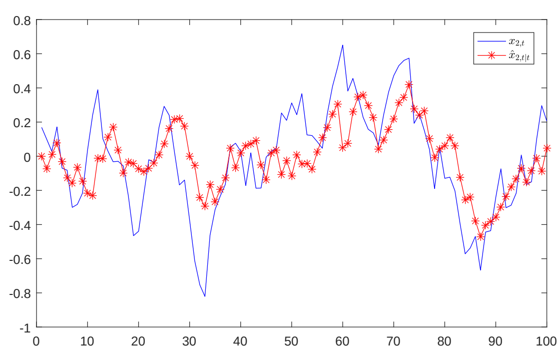

Fig. 2. $x_{2,t}$ and its estimation $\hat{x}_{2,t|t}$

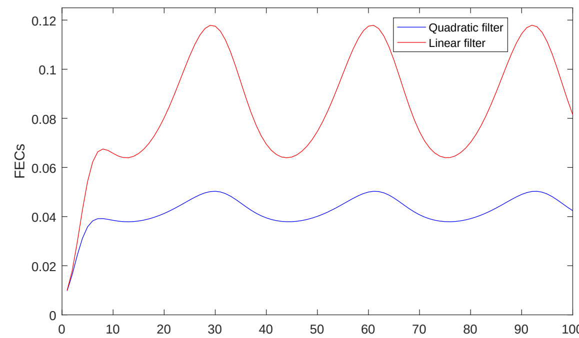

Fig. 3. Comparisons of filtering error covariances between the quadratic filter and the linear filter for $x_{1,t}$

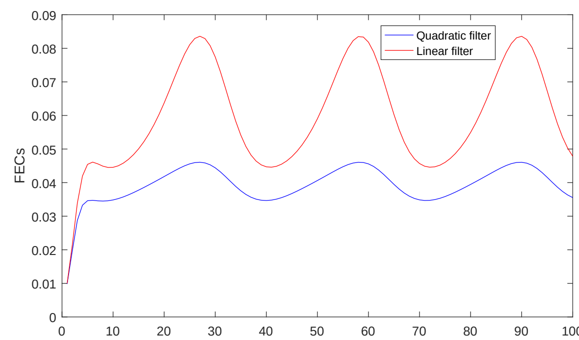

Fig. 4. Comparisons of filtering error covariances between the quadratic filter and the linear filter for $x_{2,t}$

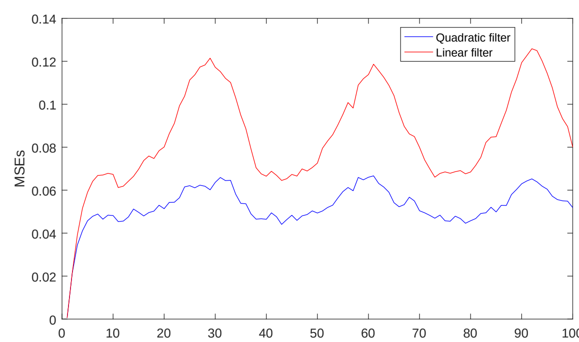

Fig. 5. Comparisons of MSEs between the quadratic filter and the linear filter for $x_{1,t}$

5 Conclusions

This paper has tackled the quadratic filtering challenge for linear stochastic systems influenced by the dynamical bias, an amplify-and-forward relay, and non-Gaussian noises. Random transmission powers of the amplify-and-forward relay have been represented using two series of random variables. Utilizing the state augmentation method, an augmented system has been constructed, encapsulating high-order moments of the implicated random variables. From this foundation, a recursive quadratic filter has been established, and the filtering error covariance has been derived and then minimized through designing the optimal gain matrix. In the subsequent sections, the filtering error covariance's lower and upper bounds have been computed. Concluding the study, numerical simulations have been presented to validate the efficacy of the formulated quadratic filtering approach. For upcoming research, the developed quadratic filtering results would be extended to more general systems, such as multi-sensor systems and complex networks.

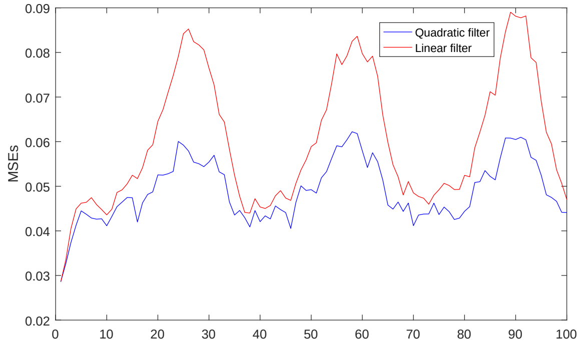

Fig. 6. Comparisons of MSEs between the quadratic filter and the linear filter for $x_{2,t}$

6 Appendix

6.1 Proof of Lemma 4

Proof: Utilizing the expression of W_t, we compute $\mathcal{R}_{W_t}$ as follows:

$$\mathcal{R}_{W_t}=\begin{bmatrix}\mathbb{E}\{\vec{B}_t\vec{w}_t\vec{w}_t^T\vec{B}_t^T\} & \mathbb{E}\{\vec{B}_t\vec{w}_t(\tilde{w}_t)^T\}\\ \mathbb{E}\{\tilde{w}_t\vec{w}_t^T\vec{B}_t^T\} & \mathbb{E}\{\tilde{w}_t\tilde{w}_t^T\}\end{bmatrix}. \tag{51}$$

For the first term $\mathbb{E}\{\vec{B}_t\vec{w}_t\vec{w}_t^T\vec{B}_t^T\}$, we have

$$\mathbb{E}\{\vec{B}_t\vec{w}_t\vec{w}_t^T\vec{B}_t^T\}=\vec{B}_t\begin{bmatrix}\mathrm{st}(\phi^{(2)}_{w_t}) & 0\\ 0 & \mathrm{st}(\phi^{(2)}_{\xi_t})\end{bmatrix}\vec{B}_t^T. \tag{52}$$

For the second term $\mathbb{E}\{\vec{B}_t\vec{w}_t\tilde{w}_t^T\}$, we derive

$$\begin{aligned}\mathbb{E}\{\vec{B}_t\vec{w}_t\tilde{w}_t^T\}=&\mathbb{E}\Big\{\vec{B}_t\vec{w}_t\Big[\tilde{S}_{n+n_\varepsilon}(\vec{A}_t\vec{x}_t\otimes\vec{B}_t\vec{w}_t)\\&+\vec{B}_t^{[2]}(\vec{w}_t^{[2]}-\phi^{(2)}_{\vec{w}_t})\Big]^T\Big\}\\=&\vec{B}_t\mathbb{E}\{\vec{w}_t(\vec{w}_t^{[2]})^T\}(\vec{B}_t^{[2]})^T\end{aligned} \tag{53}$$

where the conclusion that $\vec{x}_t$ is uncorrelated with $\vec{w}_t$ has been utilized.

Recalling the definition of $\tilde{w}_t$, we can easily see that

$$\begin{aligned}&\mathbb{E}\{\tilde{w}_t\tilde{w}_t^T\}\\=&\mathbb{E}\Big\{\tilde{S}_{n+n_\varepsilon}(\vec{A}_t\vec{x}_t\vec{x}_t^T\vec{A}_t^T\otimes\vec{B}_t\vec{w}_t\vec{w}_t^T\vec{B}_t^T)\tilde{S}^T_{n+n_\varepsilon}\\&+\vec{B}_t^{[2]}\vec{w}_t^{[2]}(\vec{w}_t^{[2]})^T(\vec{B}_t^{[2]})^T-\vec{B}_t^{[2]}\phi^{(2)}_{\vec{w}_t}(\phi^{(2)}_{\vec{w}_t})^T(\vec{B}_t^{[2]})^T\\&+\mathrm{Sym}\Big\{\tilde{S}_{n+n_\varepsilon}(\vec{A}_t\vec{x}_t\otimes\vec{B}_t\vec{w}_t)(\vec{w}_t^{[2]})^T(\vec{B}_t^{[2]})^T\\&-\tilde{S}_{n+n_\varepsilon}(\vec{A}_t\vec{x}_t\otimes\vec{B}_t\vec{w}_t)(\phi^{(2)}_{\vec{w}_t})^T(\vec{B}_t^{[2]})^T\Big\}\Big\}\end{aligned} \tag{54}$$

which, together with the properties of $\vec{x}_t$ and $\vec{w}_t$, leads to $\mathcal{R}_{W_{22,t}}$.

Similarly, we have

$$\mathcal{R}_{V_t}=\begin{bmatrix}\mathbb{E}\{\vec{v}_t\vec{v}_t^T\} & \mathbb{E}\{\vec{v}_t\tilde{v}_t^T\}\\ \mathbb{E}\{\tilde{v}_t\vec{v}_t^T\} & \mathbb{E}\{\tilde{v}_t\tilde{v}_t^T\}\end{bmatrix}. \tag{55}$$

For the sake of simplicity, we denote

$$\gamma_t\triangleq\sqrt{m_t}\sqrt{l_t}-\bar{m}_t^{(1)}\bar{l}_t^{(1)}.$$

In light of $\vec{v}_t$ and (8), it is not difficult to derive that

$$\begin{aligned}\mathbb{E}\{\vec{v}_t\vec{v}_t^T\}=&\theta_t^2\mathbb{E}\{\gamma_t^2\Psi_t\Phi_tC_tx_tx_t^TC_t^T\Phi_t^T\Psi_t^T\}\\&+\theta_t^2\bar{m}_t^{(2)}\bar{l}_t^{(2)}\Psi_t\Phi_tD_t\mathbb{E}\{v_tv_t^T\}D_t^T\Phi_t^T\Psi_t^T\\&+\theta_t^2\mathbb{E}\{m_t\Psi_t\varpi_t\varpi_t^T\Psi_t^T\}+\mathbb{E}\{\varsigma_t\varsigma_t^T\}.\end{aligned} \tag{56}$$

A tedious algebraic manipulation of (56) yields $\mathcal{R}_{V11,t}$.

Taking $\vec{v}_t$ and $\tilde{v}_t$ into account, the term $\mathbb{E}\{\vec{v}_t\tilde{v}_t^T\}$ can be expressed as

$$\begin{aligned}\mathbb{E}\{\vec{v}_t\tilde{v}_t^T\}=&\mathbb{E}\Big\{\vec{v}_t\Big[\tilde{S}_s(\vec{C}_t\vec{x}_t\otimes\vec{v}_t)+\vec{v}_t^{[2]}-\phi^{(2)}_{\vec{v}_t}\Big]^T\Big\}\\=&\mathbb{E}\{\vec{v}_t(\vec{C}_t\vec{x}_t\otimes\vec{v}_t)^T\tilde{S}_s^T\}+\mathbb{E}\{\vec{v}_t(\vec{v}_t^{[2]})^T\}\\&-\mathbb{E}\{\vec{v}_t(\phi^{(2)}_{\vec{v}_t})^T\}.\end{aligned} \tag{57}$$

Since $\vec{x}_t$ is uncorrelated with $\vec{v}_t$ and $\mathbb{E}\{\vec{v}_t\}=0$, (57) reduces to

$$\begin{aligned}\mathbb{E}\{\vec{v}_t\tilde{v}_t^T\}=&\theta_t^3\bar{m}_t^{(1)}\bar{l}_t^{(1)}\mathbb{E}\{\gamma^2\Psi_t\Phi_tC_tx_t(x_t^{[2]})^T\\&\times(\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]})^T\}\tilde{S}_s^T+\mathbb{E}\{\vec{v}_t(\vec{v}_t^{[2]})^T\}.\end{aligned} \tag{58}$$

A combination of $\vec{v}_t^{[2]}=\vec{v}_t\otimes\vec{v}_t$ and the properties of Kronecker algebra leads to

$$\begin{aligned}\vec{v}_t^{[2]}\triangleq&\theta_t^2\gamma_t^2\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]}x_t^{[2]}+\theta_t^2m_tl_t\Psi_t^{[2]}\Phi_t^{[2]}D_t^{[2]}v_t^{[2]}\\&+\theta_t^2m_t\Psi_t^{[2]}\varpi_t^{[2]}+\varsigma_t^{[2]}+\tilde{S}_s\Big\{(\theta_t\gamma_t\Psi_t\Phi_tC_tx_t)\\&\otimes(\theta_t\sqrt{m_t}\sqrt{l_t}\Psi_t\Phi_tD_tv_t)+\theta_t\sqrt{m_t}\Psi_t\varpi_t\otimes\varsigma_t\\&+(\theta_t\gamma_t\Psi_t\Phi_tC_tx_t)\otimes(\theta_t\sqrt{m_t}\Psi_t\varpi_t)\\&+(\theta_t\sqrt{m_t}\sqrt{l_t}\Psi_t\Phi_tD_tv_t)\otimes(\theta_t\sqrt{m_t}\Psi_t\varpi_t)\\&+(\theta_t\sqrt{m_t}\sqrt{l_t}\Psi_t\Phi_tD_tv_t)\otimes\varsigma_t\\&+(\theta_t\gamma_t\Psi_t\Phi_tC_tx_t)\otimes\varsigma_t\Big\}.\end{aligned} \tag{59}$$

Therefore, we have

$$\begin{aligned}\mathbb{E}\{\vec{v}_t(\vec{v}_t^{[2]})^T\}=&\mathbb{E}\{\theta_t^3\gamma_t^3\Psi_t\Phi_tC_tx_t(x_t^{[2]})^T(\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]})^T\}\\&+\mathbb{E}\{\theta_t^3m_t^{\frac{3}{2}}l_t^{\frac{3}{2}}\Psi_t\Phi_tD_tv_t(v_t^{[2]})^T(\Psi_t^{[2]}\Phi_t^{[2]}D_t^{[2]})^T\}\\&+\mathbb{E}\{\theta_t^3m_t^{\frac{3}{2}}\Psi_t\varpi_t(\varpi_t^{[2]})^T(\Psi_t^{[2]})^T\}\\&+\mathbb{E}\{\varsigma_t(\varsigma_t^{[2]})^T\}\end{aligned} \tag{60}$$

and reorganizing the above formula results in $\mathcal{R}_{V12,t}$.

On the other hand, $\mathbb{E}\{\tilde{v}_t\tilde{v}_t^T\}$ is calculated as

$$\begin{aligned}\mathbb{E}\{\tilde{v}_t\tilde{v}_t^T\}=&\tilde{S}_s\mathbb{E}\Big\{(\vec{C}_t\vec{x}_t\otimes\vec{v}_t)(\vec{C}_t\vec{x}_t\otimes\vec{v}_t)^T\Big\}\tilde{S}_s^T\\&+\mathbb{E}\{\vec{v}_t^{[2]}(\vec{v}_t^{[2]})^T\}-\phi^{(2)}_{\vec{v}_t}(\phi^{(2)}_{\vec{v}_t})^T\\&+\mathrm{Sym}\Big\{\mathbb{E}\{\tilde{S}_s(\vec{C}_t\vec{x}_t\otimes\vec{v}_t)(\vec{v}_t^{[2]})^T\}\\&-\mathbb{E}\{\tilde{S}_s(\vec{C}_t\vec{x}_t\otimes\vec{v}_t)(\phi^{(2)}_{\vec{v}_t})^T\}\Big\}\end{aligned}$$

$$
\begin{aligned}
&=\tilde{S}_s\mathbb{E}\Big\{(\vec{C}_t\vec{x}_t\vec{x}_t^T\vec{C}_t^T)\otimes(\vec{v}_t\vec{v}_t^T)\Big\}\tilde{S}_s^T \\
&\quad+\mathbb{E}\{\vec{v}_t^{[2]}(\vec{v}_t^{[2]})^T\}-\phi_{\vec{v}_t}^{(2)}(\phi_{\vec{v}_t}^{(2)})^T \\
&\quad+\mathrm{Sym}\Big\{\mathbb{E}\{\tilde{S}_s(\vec{C}_t\vec{x}_t\otimes\vec{v}_t)(\vec{v}_t^{[2]})^T\}\Big\}. \qquad (61)
\end{aligned}
$$

Furthermore, it is straightforward to derive that

$$
\begin{aligned}
&\mathbb{E}\Big\{(\vec{C}_t\vec{x}_t\vec{x}_t^T\vec{C}_t^T)\otimes(\vec{v}_t\vec{v}_t^T)\Big\} \\
&=\theta_t^4\chi_{1,t}(\bar{m}_t^{(1)}\bar{l}_t^{(1)})^2\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]}\mathrm{st}(\phi_{x_t}^{(4)})(\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]})^T \\
&\quad+\theta_t^2(\bar{m}_t^{(1)}\bar{l}_t^{(1)})^2(\Psi_t\Phi_tC_t\mathrm{st}(\phi_{x_t}^{(2)})C_t^T\Phi_t^T\Psi_t^T) \\
&\quad\otimes\big(\theta_t^2\bar{m}_t^{(2)}\bar{l}_t^{(2)}\Psi_t\Phi_tD_t\mathrm{st}(\phi_{v_t}^{(2)})D_t^T\Phi_t^T\Psi_t^T \\
&\quad+\theta_t^2\bar{m}_t^{(2)}\Psi_t\mathrm{st}(\phi_{\varpi_t}^{(2)})\Psi_t^T+\mathrm{st}(\phi_{\varsigma_t}^{(2)})\big), \qquad (62)\\
&\phi_{\vec{v}_t}^{(2)}=\theta_t^2\chi_{1,t}\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]}\phi_{x_t}^{(2)}+\theta_t^2\bar{m}_t^{(2)}\Psi_t^{[2]}\phi_{\varpi_t}^{(2)} \\
&\quad+\theta_t^2\bar{m}_t^{(2)}\bar{l}_t^{(2)}\Psi_t^{[2]}\Phi_t^{[2]}D_t^{[2]}\phi_{v_t}^{(2)}+\phi_{\varsigma_t}^{(2)}, \qquad (63)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\{(\vec{C}_t\vec{x}_t\otimes\vec{v}_t)(\vec{v}_t^{[2]})^T\} \\
&=\theta_t^4\chi_{2,t}\bar{m}_t^{(1)}\bar{l}_t^{(1)}\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]}\mathrm{st}(\phi_{x_t}^{(4)})(\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]})^T \\
&\quad+\theta_t^4\chi_{6,t}\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]}\phi_{x_t}^{(2)}(\phi_{v_t}^{(2)})^T(\Psi_t^{[2]}\Phi_t^{[2]}D_t^{[2]})^T \\
&\quad+\theta_t^4\chi_{7,t}\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]}\phi_{x_t}^{(2)}(\phi_{\varpi_t}^{(2)})^T(\Psi_t^{[2]})^T \\
&\quad+\theta_t^4\chi_{6,t}\big[(\Psi_t\Phi_tC_t\mathrm{st}(\phi_{x_t}^{(2)})C_t^T\Phi_t^T\Psi_t^T)\otimes(\Psi_t\Phi_tD_t\mathrm{st}(\phi_{v_t}^{(2)}) \\
&\quad\times D_t^T\Phi_t^T\Psi_t^T)\tilde{S}_s^T\big]+\theta_t^4\chi_{7,t}\big[(\Psi_t\Phi_tC_t\mathrm{st}(\phi_{x_t}^{(2)})C_t^T\Phi_t^T\Psi_t^T) \\
&\quad\otimes(\Psi_t\mathrm{st}(\phi_{\varpi_t}^{(2)})\Psi_t^T)\tilde{S}_s^T\big]. \qquad (64)
\end{aligned}
$$

For the term $\mathbb{E}\{\vec{v}_t^{[2]}(\vec{v}_t^{[2]})^T\}$, we have

$$
\begin{aligned}
&\mathbb{E}\{\vec{v}_t^{[2]}(\vec{v}_t^{[2]})^T\} \\
&=\mathbb{E}\Big\{\theta_t^4\gamma_t^4\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]}x_t^{[2]}(x_t^{[2]})^T(C_t^{[2]})^T(\Phi_t^{[2]})^T(\Psi_t^{[2]})^T \\
&\quad+\theta_t^4m_t^2l_t^2\Psi_t^{[2]}\Phi_t^{[2]}D_t^{[2]}v_t^{[2]}(v_t^{[2]})^T(D_t^{[2]})^T(\Phi_t^{[2]})^T(\Psi_t^{[2]})^T \\
&\quad+\theta_t^4m_t^2\Psi_t^{[2]}\varpi_t^{[2]}(\varpi_t^{[2]})^T(\Psi_t^{[2]})^T+\varsigma_t^{[2]}(\varsigma_t^{[2]})^T \\
&\quad+\tilde{S}_s\Big(\big(\theta_t^2\gamma_t^2\Psi_t\Phi_tC_tx_tx_t^TC_t^T\Phi_t^T\Psi_t^T\big)\otimes(\theta_t^2m_tl_t\Psi_t\Phi_tD_t \\
&\quad\times v_tv_t^TD_t^T\Phi_t^T\Psi_t^T)+\big(\theta_t^2\gamma_t^2\Psi_t\Phi_tC_tx_tx_t^TC_t^T\Phi_t^T\Psi_t^T\big) \\
&\quad\otimes(\theta_t^2m_t\Psi_t\varpi_t(\varpi_t)^T\Psi_t^T)+\big(\theta_t^2\gamma_t^2\Psi_t\Phi_tC_tx_tx_t^TC_t^T\Phi_t^T\Psi_t^T\big) \\
&\quad\otimes(\varsigma_t\varsigma_t^T)+(\theta_t^2m_tl_t\Psi_t\Phi_tD_tv_tv_t^TD_t^T\Phi_t^T\Psi_t^T)\otimes(\varsigma_t\varsigma_t^T) \\
&\quad+(\theta_t^2m_tl_t\Psi_t\Phi_tD_tv_tv_t^TD_t^T\Phi_t^T\Psi_t^T)\otimes(\theta_t^2m_t\Psi_t\varpi_t\varpi_t^T\Psi_t^T) \\
&\quad+(\theta_t^2m_t\Psi_t\varpi_t\varpi_t^T\Psi_t^T)\otimes(\varsigma_t\varsigma_t^T)\Big)\tilde{S}_s^T \\
&\quad+\mathrm{Sym}\Big\{\theta_t^2\gamma_t^2\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]}x_t^{[2]}(\theta_t^2m_tl_t\Psi_t^{[2]}\Phi_t^{[2]}D_t^{[2]}v_t^{[2]})^T \\
&\quad+\theta_t^2m_t\Psi_t^{[2]}\varpi_t^{[2]}(\varsigma_t^{[2]})^T+\theta_t^2\gamma_t^2\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]}x_t^{[2]}(\varsigma_t^{[2]})^T \\
&\quad+\theta_t^4m_t\gamma_t^2\Psi_t^{[2]}\Phi_t^{[2]}C_t^{[2]}x_t^{[2]}(\varpi_t^{[2]})^T(\Psi_t^{[2]})^T \\
&\quad+\theta_t^4m_t^2l_t\Psi_t^{[2]}\Phi_t^{[2]}D_t^{[2]}v_t^{[2]}(\varpi_t^{[2]})^T(\Psi_t^{[2]})^T \\
&\quad+\theta_t^2m_tl_t\Psi_t^{[2]}\Phi_t^{[2]}D_t^{[2]}v_t^{[2]}(\varsigma_t^{[2]})^T\Big\}\Big\}. \qquad (65)
\end{aligned}
$$

Substituting (62)-(65) into (61) yields $\mathcal{R}_{V22,t}$, and this ends the proof. ■

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