Journal of Global Optimization
https://doi.org/10.1007/s10898-025-01465-w

Conic relaxations for conic minimax convex polynomial programs with extensions and applications

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Received: 9 October 2023 / Accepted: 8 January 2025

Abstract
In this paper, we analyze conic minimax convex polynomial optimization problems. Under a suitable regularity condition, an exact conic programming relaxation is established based on a positivity characterization of a max function over a conic convex system. Further, we consider a general conic minimax ρ-convex polynomial optimization problem, which is defined by appropriately extending the notion of conic convexity of a vector-valued mapping. For this problem, it is shown that a Karush-Kuhn-Tucker condition at a global minimizer is necessary and sufficient for ensuring an exact relaxation with attainment of the conic programming relaxation. The exact conic programming relaxations are applied to SOS-convex polynomial programs, where appropriate choices of the data allow the associated conic programming relaxation to be reformulated as a semidefinite programming problem. In this way, we can further elaborate the obtained results for other special settings including conic robust SOS-convex polynomial problems and difference of SOS-convex polynomial programs.

Keywords Conic programming · Polynomial optimization · Minimax programs · Relaxations · Duality

1 Introduction

In this paper, we propose a *parametric conic minimax* polynomial problem (PCMP) that is defined as follows. Let $\mathcal{U} \subset \mathbb{R}^s$ be an index set that is a nonempty compact convex set, and $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ be a bifunction that is given by

$$f(x, u) := f_0(x) + \sum_{i=1}^{s} u_i f_i(x), \quad x \in \mathbb{R}^n, u := (u_1, \dots, u_s) \in \mathcal{U},$$

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Published online: 28 January 2025

where $f_0, f_1, \ldots, f_s$ are given polynomials on $\mathbb{R}^n$ such that $f(\cdot, u)$ is a *convex* polynomial for each $u \in \mathcal{U}$. For each f (regarded as a parameter), we consider a conic minimax polynomial problem of the form:

$$\inf_{x\in\mathbb{R}^n} \{\max_{u\in\mathcal{U}} f(x,u) : G(x) \in -K\}, \tag{P_f}$$

where $K \subset \mathbb{R}^m$ is a closed convex cone and $G : \mathbb{R}^n \to \mathbb{R}^m$ is a K-convex polynomial mapping (see its definition in Sect. 2). The problem of type (P_f) covers a broad spectrum of optimization models including common conic polynomial problems and robust polynomial optimization programs. For instance, if $f_0 := 0$, $K := \mathbb{R}^m_+$, $G := (g_1, \ldots, g_m)$ and $\mathcal{U}$ is a polytope given by $\mathcal{U} := \mathrm{conv}\{e^1, \ldots, e^s\}$, where e^k ($k \in \{1, \ldots, s\}$) is a unit vector in $\mathbb{R}^s$ whose k-th component is one and the others are zero, the problem (P_f) reduces to the following (classical) *minimax* polynomial problem

$$\inf_{x\in\mathbb{R}^n} \{\max_{j=1,\ldots,s} f_j(x) : g_i(x) \le 0, i = 1, \ldots, m\}, \tag{SOP_f}$$

which was studied in [22]. When $f_j := 0$, $j = 1, \ldots, s$ or $\mathcal{U} := \{0\}$, the problem (P_f) becomes a *cone convex polynomial* program studied in [11, 19]. The problem (P_f) also encompasses a class of *difference of SOS-convex* polynomial problems in [26] and other important *conic robust SOS-convex* polynomial programs examined in the last section by specifying the given data of $f, \mathcal{U}, K$ or G. It should be noted here that the problem (P_f) does not include the robust *difference-of-convex-max* optimization model in [17] as the latter is not necessarily a convex program.

Following the robust optimization approach (see, e.g., [5–8]), the problem (P_f) can be regarded as the *robust counterpart* of the following problem

$$\inf_{x\in\mathbb{R}^n} \{f(x,u) : G(x) \in -K\},$$

where $u \in \mathcal{U}$ is an *uncertain* parameter and $\mathcal{U} \subset \mathbb{R}^s$ is a nonempty compact uncertainty set. This model captures the uncertainty in the objective function of the problem $\inf_{x\in\mathbb{R}^n} \{f_0(x) : G(x) \in -K\}$ assuming affine parameterization.

It is worth mentioning that our newly-defined parametric conic minimax polynomial problem (PCMP) handles flexibly dynamic decision-making processes with data perturbations f by allowing decision variables to evolve with respect to other ambiguity parameters in $\mathcal{U}$. This is often seen in practice, for instance, the cost of producing a product is just an estimation within a prescribed range of unknown parameters until the product is actually made [23], and so a parametric optimization model enables the production decision maker the ability to update and possibly adjust the investment strategy according to the actual parameter of production cost. Furthermore, bearing the *conic* and *parametric* features, the generalized model of type (P_f) enhances the applicability of the obtained results to real-world scenarios. For instance, the *weighted Steiner* problem [7], which states that in a number of villages you want to place a telephone station for which the total cost of cables linking the station and the villages is as small as possible. This problem minimizes the weighted sum of its Euclidean distances to all villages and it is formulated as a *conic quadratic* program. If we consider additional constraints such as mountain blocks and river disruptions among the villages or replace the cables by different radio systems or by other means of telephone transmission, the Euclidean distance would be extended to a more general distance and in this case, the corresponding problem could be cast into our general framework.

The main aim of this work is to show that a broad class of conic minimax polynomial programming problems of type (P_f) exhibits new exact conic programming relaxations under suitable conditions. More precisely, we first establish an exact conic programming relaxation by using a positivity characterization of a max function over a conic convex system. This is done by employing some techniques from convex analysis, which have been used in the literature [13] including, for instance, a strict separation theorem (see, e.g., [32, Theorem 1.1.5]) and the classical minimax theorem (cf. [31, Theorem 4.2]) or the epigraph of the conjugate of an indicator function related to a system of infinite convex functions [15, Lemma 3.1].

We then consider a general conic minimax ρ-convex polynomial optimization problem and show that a Karush-Kuhn-Tucker condition at a global minimizer is necessary and sufficient for ensuring an exact relaxation with attainment of the conic programming problem. In addition, the exact conic programming relaxations are applied to the class of SOS-convex polynomial problems, where the associated conic programming relaxations can be reformulated and solved as a semidefinite programming problem. We further elaborate the obtained results for other special settings such as conic robust SOS-convex polynomial programs and difference of SOS-convex polynomial problems. We obtain these conic programming relaxations for the special frameworks by exploiting sum of squares characterizations of SOS-convex polynomials from [2, 14] or [22, Corollary 2.1]. In this way, the elaborations allow us to recover or develop some existing results in [12, 20, 22] or [26].

The outline of the paper is as follows. Section 2 introduces the definitions, notation and basic results needed ahead. We provide here the corresponding extension of generalized conic convexity to a vector-valued mapping, and prove dual characterizations of positivity and non-negativity of a max function over a conic convex system. Section 3 presents exact conic programming relaxation results for conic minimax convex polynomial programs. Section 4 provides further exact conic programming relaxation results for conic minimax generalized convex polynomial programs and shows applications of the obtained results with a numerical example. Finally, Sect. 5 is devoted to providing concluding remarks.

2 Preliminaries and positivity conic representations

We begin this section by providing notation and definitions of convex sets, functions and polynomials. Throughout this paper, $\mathbb{R}^n$ denotes the Euclidean space with dimension $n \in \mathbb{N} := \{1, 2, \ldots\}$. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^\top y$ for all $x, y \in \mathbb{R}^n$. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, \ldots, n\}$. We denote by $\Delta^n := \{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$ the simplex in $\mathbb{R}^n$. Moreover, the Euclidean norm on $\mathbb{R}^n$ is denoted by $\|\cdot\|$. The positive semidefiniteness of an $n \times n$ matrix B, denoted by $B \succeq 0$, is defined by $\langle x, Bx\rangle \geq 0$ for all $x \in \mathbb{R}^n$. If $\langle x, Bx\rangle > 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$, then B is called positive definite, denoted by $B \succ 0$. Let $S^n_+ \subset S^n$ denote the cone of $n \times n$ positive semidefinite matrices, where S^n stands for the space of symmetric $n \times n$ matrices. For $A, B \in S^n$, the inner product in S^n is given by $\langle A, B\rangle := \mathrm{Tr}(AB)$, where $\mathrm{Tr}(\cdot)$ refers to the trace operation. In particular, $\mathrm{Tr}(\lambda G)(x) := \mathrm{Tr}(\lambda G(x))$ for $x \in \mathbb{R}^n$, $\lambda \in S^m$ and $G : \mathbb{R}^n \to S^m$. As usual, $\mathrm{cl}\,\Gamma$ and $\mathrm{conv}\,\Gamma$ denote the closure and the convex hull of a nonempty set Γ, respectively.

For a closed convex subset $\Gamma \subset \mathbb{R}^n$, its indicator function $\iota_\Gamma : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is defined as $\iota_\Gamma(x) := 0$ if $x \in \Gamma$ and $\iota_\Gamma(x) := +\infty$ if $x \notin \Gamma$. For an extended real-valued function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$, we set its effective domain and its epigraph as

$$\mathrm{dom}\,\varphi := \{x \in \mathbb{R}^n : \varphi(x) < +\infty\}, \quad \mathrm{epi}\,\varphi := \{(x, \mu) \in \mathbb{R}^n \times \mathbb{R} : \varphi(x) \leq \mu\}.$$

The *conjugate function* of φ, $\varphi^*: \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined by

$$\varphi^*(w) := \sup\{\langle w, x\rangle - \varphi(x) : x \in \operatorname{dom} \varphi\}, \quad w \in \mathbb{R}^n.$$

An extended real-valued function is *proper* if it does not take the value $-\infty$ and its domain is nonempty. Let g, h and $g_i, i \in I$ (where I is an arbitrary index set) be proper lower semicontinuous convex functions. It is well known (see, e.g., [15, 32]) that, if $\operatorname{dom} g \cap \operatorname{dom} h \neq \emptyset$, $\sup_{i\in I} g_i$ is proper, and $\lambda > 0$, then

$$\operatorname{epi}(\lambda g)^* = \lambda \operatorname{epi} g^*, \tag{1}$$

$$\operatorname{epi}(g+h)^* = \operatorname{cl}\left(\operatorname{epi} g^* + \operatorname{epi} h^*\right), \tag{2}$$

$$\operatorname{epi}\left(\sup_{i\in I} g_i\right)^* = \operatorname{cl}\operatorname{conv}\left(\bigcup_{i\in I} \operatorname{epi} g_i^*\right). \tag{3}$$

The closure in (2) is superfluous if one of g and h is continuous at some $x_0 \in \operatorname{dom} g \cap \operatorname{dom} h$.

The space of all real polynomials on $\mathbb{R}^n$ is denoted by $\mathbb{R}[x]$ and the set of all $n \times r$ matrix polynomials is denoted by $\mathbb{R}[x]^{n\times r}$. We say that $f \in \mathbb{R}[x]$ is sum-of-squares (see, e.g., [3, 24, 25]) if there exist $f_j \in \mathbb{R}[x]$, $j = 1, \ldots, r$, such that $f = \sum_{j=1}^r f_j^2$. The set consisting of all sum-of-squares polynomials is denoted by Σ^2, which is a subset of $\mathcal{P}$, the set of all nonnegative polynomials. Moreover, the set consisting of all sum-of-squares (respectively, nonnegative) polynomials with degree at most d is denoted by Σ_d^2 (respectively, $\mathcal{P}_d$). We say that $F \in \mathbb{R}[x]^{n\times n}$ is an SOS matrix polynomial if $F(x) = H(x)H(x)^\top$, where $H(x) \in \mathbb{R}[x]^{n\times r}$ is a matrix polynomial for some $r \in \mathbb{N}$. A real polynomial f on $\mathbb{R}^n$ is called *SOS-convex* if the Hessian matrix function $F : x \mapsto \nabla^2 f(x)$ is an SOS matrix polynomial [14]. Clearly, an SOS-convex polynomial is convex, but the converse is not true in general [1]. It is well-known that any convex quadratic function and any convex separable polynomial is SOS-convex (see, e.g., [22]).

For a differentiable mapping $G : \mathbb{R}^n \to \mathbb{R}^m$, we use $DG(x)$ to denote the derivative of G at $x \in \mathbb{R}^n$, and $DG(x)^* : \mathbb{R}^m \to \mathbb{R}^n$ to denote the *adjoint* of $DG(x)$, which is characterized by the following property: $\langle DG(x)v, w\rangle = \langle v, DG(x)^* w\rangle$ for every $v \in \mathbb{R}^n$, $w \in \mathbb{R}^m$. For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, we often use $\nabla f(x)$ to denote the transpose of $Df(x)$.

Let $K \subset \mathbb{R}^m$ be a closed convex cone. We denote by $K^\oplus$ the dual cone of K, that is, $K^\oplus := \{y \in \mathbb{R}^m : \langle y, x\rangle \geq 0, \forall x \in K\}$. A polynomial vector-valued mapping $G : \mathbb{R}^n \to \mathbb{R}^m$, that is, $G(x) = (g_1(x), \ldots, g_m(x))$ where each g_i is a polynomial on $\mathbb{R}^n$, is said to be *K-convex* if for all $x, y \in \mathbb{R}^n$ and for all $\gamma \in [0, 1]$,

$$(1-\gamma)G(x) + \gamma G(y) - G((1-\gamma)x + \gamma y) \in K.$$

Equivalently, G is a K-convex polynomial if and only if for any $\gamma \in [0, 1]$ and for any $\lambda \in K^\oplus$, $\langle \lambda, G\rangle$ is a convex polynomial on $\mathbb{R}^n$. Analogously, $G : \mathbb{R}^n \to \mathbb{R}^m$ is said to be a *K-SOS-convex polynomial* [19] if, for any $\alpha \in [0, 1]$ and for any $\lambda \in K^\oplus$,

$$h(x, y) = \langle \lambda, \alpha G(x) + (1-\alpha)G(y) - G(\alpha x + (1-\alpha)y)\rangle$$

is a sum-of-squares polynomial on $\mathbb{R}^n \times \mathbb{R}^n$. Equivalently, in virtue of [19, Lemma 2.2] and [2, Theorem 3.1], G is a K-SOS-convex polynomial if and only if, for any $\lambda \in K^\oplus$, $\langle \lambda, G\rangle$ is an SOS-convex polynomial on $\mathbb{R}^n$. The degree of the polynomial mapping G is defined by $\deg G = \max\{\deg g_j, j = 1, \ldots, m\}$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be ρ-convex [34, 35] if there exists some real number ρ such that for all $\lambda \in [0, 1]$ and all $x, y \in \mathbb{R}^n$,

$$(1-\lambda)f(x) + \lambda f(y) - f((1-\lambda)x + \lambda y) - \rho\lambda(1-\lambda)\|x - y\|^2 \geq 0. \tag{4}$$

If $\rho = 0$ then f is convex, if $\rho > 0$ then f is called strong convex, and if $\rho < 0$ then f is said to be weak convex. It is well-known that f is ρ-convex if and only if $f(x) - \rho\|x\|^2$ is convex.

A numerically tractable relaxation of the ρ-convexity for real polynomials is given as follows. Given $\rho \in \mathbb{R}$, a real polynomial f on $\mathbb{R}^n$ is said to be ρ-SOS-*convex* [16] if for all $x, y \in \mathbb{R}^n$ and for all $\lambda \in [0, 1]$, the expression of the left-hand side in (4) is a sum-of-squares polynomial in $\mathbb{R}[x; y]$. If $\rho > 0$, then f is called a strong SOS-convex polynomial with modulus ρ. If $\rho = 0$, then f is an SOS-convex polynomial. If $\rho < 0$, then f is called a weak SOS-convex polynomial with modulus ρ. Clearly, it follows by definition that the strong SOS-convexity implies the SOS-convexity which, in turn, implies the weak SOS-convexity. Moreover, a ρ-SOS-convex polynomial is SOS-convex if and only if $\rho \geq 0$. According to [16, Theorem 2.2], a given polynomial f is a ρ-SOS-convex polynomial if and only if $f(x) - \rho\|x\|^2$ is an SOS-convex polynomial.

We now extend the notion of ρ-convexity to a vector-valued mapping.

Definition 2.1 (Conic ϱ-convexity) Let K be a closed and convex cone in $\mathbb{R}^m$. Let $G : \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued mapping and $\varrho \in \mathbb{R}^m$. We say that G is ϱ-K-*convex* if for all $x, y \in \mathbb{R}^n$ and for all $\alpha \in [0, 1]$,

$$(1-\alpha)G(x) + \alpha G(y) - G((1-\alpha)x + \alpha y) - \varrho\alpha(1-\alpha)\|x - y\|^2 \in K.$$

As a straightforward consequence of this definition, it follows that G is ϱ-K-convex if and only if, for every $\lambda \in K^{\oplus}$, $\langle\lambda, G\rangle$ is $\langle\lambda, \varrho\rangle$-convex. Analogously, we say that G is ϱ-K-*SOS-convex* if, for any $\alpha \in [0, 1]$ and for any $\lambda \in K^{\oplus}$,

$$h(x, y) = \langle\lambda, \alpha G(x) + (1-\alpha)G(y) - G(\alpha x + (1-\alpha)y) - \varrho\alpha(1-\alpha)\|x - y\|^2\rangle$$

is a sum-of-squares polynomial on $\mathbb{R}^n \times \mathbb{R}^n$. Thus, G is ϱ-K-SOS-convex if and only if for every $\lambda \in K^{\oplus}$, $\langle\lambda, G\rangle$ is $\langle\lambda, \varrho\rangle$-SOS-convex.

Throughout this paper, we assume that the feasible set of problem (P_f) is nonempty (i.e., $\mathcal{F} := \{x \in \mathbb{R}^n : G(x) \in -K\} \neq \emptyset$), and denote by d the smallest even number satisfying $d \geq \max\{\deg f_i, i = 0, 1, \ldots, s, \deg G\}$. We also put

$$F(x) := \max\{f(x, u) : u \in \mathcal{U}\}, \quad x \in \mathbb{R}^n. \tag{5}$$

Let us first provide a characterization of positivity of a max function in the form of F over a conic polynomial convex system under the condition that the convex set

$$\Omega := \{(y, z) \in \mathbb{R} \times \mathbb{R}^m : \exists x \in \mathbb{R}^n \text{ s.t. } F(x) \leq y, z \in G(x) + K\} \tag{6}$$

is closed, where F is given by (5) and G and K are given as in the definition of problem (P_f). The set Ω in (6) is closed if F is coercive on $\mathbb{R}^n$; i.e., $\lim_{||x|| \to \infty} F(x) = +\infty$. Besides, the closedness of Ω automatically holds whenever $\mathcal{U}$ is a polytope and K is a polyhedral cone as we will see in Sect. 3.

Proposition 2.1 (Positivity conic characterization) *Consider the problem* (P_f) *and assume that the set* Ω *in* (6) *is closed. Then, the following statements are equivalent:*

(i) $[x \in \mathbb{R}^n, G(x) \in -K] \Rightarrow F(x) > 0.$
(ii) $\exists \overline{u} \in \mathcal{U}, \overline{\lambda} \in K^{\oplus}, \overline{\delta} > 0$ *such that* $f(\cdot, \overline{u}) + \langle\overline{\lambda}, G\rangle - \overline{\delta} \in \mathcal{P}_d$.

Proof *((i) ⇒ (ii))* Let (i) hold, and consider Ω in (6). Then, we see that $0 \notin \Omega$. A strict separation theorem (see, e.g., [32, Theorem 1.1.5]) shows that there exist $(\mu, \lambda) \in (\mathbb{R} \times \mathbb{R}^m)\backslash\{0\}$, $\alpha \in \mathbb{R}$ and $\delta > 0$ such that

$$0 \leq \alpha < \alpha + \delta \leq \mu y + \langle\lambda, z\rangle \quad \forall (y, z) \in \Omega. \tag{7}$$

As $\Omega+(\mathbb{R}_+\times K)\subset\Omega$, we derive by (7) that $(\mu,\lambda)\in\mathbb{R}_+\times K^{\oplus}$. Observe that $\big(F(x),G(x)\big)\in\Omega$ for every $x\in\mathbb{R}^n$, and so it follows by (7) that

$$\delta\leq\mu F(x)+\langle\lambda,G(x)\rangle\quad\forall x\in\mathbb{R}^n.$$

As $\mathcal{F}\neq\emptyset$, we claim that $\mu\neq 0$. Otherwise, if $\mu=0$, then we get $\langle\lambda,G(x)\rangle>0$ for any $x\in\mathcal{F}$. However, $G(x)\in -K$ and $\lambda\in K^{\oplus}$, then $\langle\lambda,G(x)\rangle\leq 0$, which is a contradiction. Thus, $\mu>0$, and we obtain that

$$0\leq F(x)+\langle\overline{\lambda},G(x)\rangle-\overline{\delta}\quad\forall x\in\mathbb{R}^n,$$

where $\overline{\delta}:=\mu^{-1}\delta$ and $\overline{\lambda}:=\mu^{-1}\lambda$. Then, it entails that

$$\inf_{x\in\mathbb{R}^n}\Big\{\max_{u\in\mathcal{U}}\{f(x,u)+\langle\overline{\lambda},G(x)\rangle-\overline{\delta}\}\Big\}\geq 0 \tag{8}$$

by (5). Let $H:\mathbb{R}^n\times\mathbb{R}^s\to\mathbb{R}$ be defined by $H(x,u):=f(x,u)+\langle\overline{\lambda},G(x)\rangle-\overline{\delta}$ for $x\in\mathbb{R}^n$ and $u\in\mathbb{R}^s$. Then, H is an affine function in variable u and a convex function in variable x and so, we invoke the classical minimax theorem (see, e.g., [31, Theorem 4.2]) and (8) to claim that

$$\max_{u\in\mathcal{U}}\inf_{x\in\mathbb{R}^n}H(x,u)=\inf_{x\in\mathbb{R}^n}\max_{u\in\mathcal{U}}H(x,u)\geq 0.$$

Therefore, we can find $\overline{u}\in\mathcal{U}$ such that $\inf\limits_{x\in\mathbb{R}^n}H(x,\overline{u})\geq 0$, which means that

$$f(x,\overline{u})+\langle\overline{\lambda},G(x)\rangle-\overline{\delta}\geq 0\quad\forall x\in\mathbb{R}^n,$$

and so

$$f(\cdot,\overline{u})+\langle\overline{\lambda},G\rangle-\overline{\delta}\in\mathcal{P}_d.$$

$[(ii)\Rightarrow(i)]$ Let (ii) hold. This means there exist $p_0\in\mathcal{P}_d$, $\overline{u}\in\mathcal{U}$, $\overline{\lambda}\in K^{\oplus}$ and $\overline{\delta}>0$ such that $f(\cdot,\overline{u})+\langle\overline{\lambda},G\rangle-\overline{\delta}=p_0$. Then, for $x\in\mathbb{R}^n$ with $G(x)\in -K$ one has

$$F(x)\geq f(x,\overline{u})=p_0(x)-\langle\overline{\lambda},G(x)\rangle+\overline{\delta}\geq\overline{\delta}>0.$$

The proof of the proposition is complete. □

The following result stems from the above positivity conic charaterization that can alternatively be used to derive exact conic relaxations in the sequel.

Corollary 2.2 (Nonnegativity conic characterization) *Consider the problem* (P_f) *and assume that the set* Ω *in* (6) *is closed. Then, the following statements are equivalent:*

(i) $[x\in\mathbb{R}^n, G(x)\in -K]\ \Rightarrow\ F(x)\geq 0.$
(ii) $\forall\varepsilon>0$, $\exists\,\overline{u}\in\mathcal{U}$, $\overline{\lambda}\in K^{\oplus}$ *such that* $f(\cdot,\overline{u})+\langle\overline{\lambda},G\rangle+\varepsilon\in\mathcal{P}_d$.

Proof *$((i)\Rightarrow(ii))$* Assume that (i) holds. Then, for any $\varepsilon>0$, $F+\varepsilon>0$ is positive on $\mathcal{F}$. So, Proposition 2.1 implies that there exist $p_0\in\mathcal{P}_d$, $\overline{u}\in\mathcal{U}$, $\overline{\lambda}\in K^{\oplus}$ and $\overline{\delta}>0$ such that, for all $x\in\mathbb{R}^n$, $f(x,\overline{u})+\langle\overline{\lambda},G(x)\rangle+\varepsilon=p_0(x)+\overline{\delta}$. So $f(\cdot,\overline{u})+\langle\overline{\lambda},G\rangle+\varepsilon\in\mathcal{P}_d$.

$[(ii)\Rightarrow(i)]$ Suppose that for each $\varepsilon>0$ there exist $p_0\in\mathcal{P}_d$, $\overline{u}\in\mathcal{U}$ and $\overline{\lambda}\in K^{\oplus}$ such that $f(\cdot,\overline{u})+\varepsilon=p_0-\langle\overline{\lambda},G\rangle$. Then, for each $x\in\mathbb{R}^n$ with $G(x)\in -K$ one has

$$F(x)+\varepsilon\geq f(x,\overline{u})+\varepsilon=p_0(x)-\langle\overline{\lambda},G(x)\rangle\geq 0.$$

Letting $\varepsilon\to 0$, we see that $F(x)\geq 0$ for all $x\in\mathbb{R}^n$ with $G(x)\in -K$. Hence, the conclusion follows. □

3 Exact conic relaxations for conic minimax convex polynomial programs

In this section, we employ the certificates of conic positivity/nonnegativity to provide (exact) conic relaxations for the conic minimax convex polynomial program defined by (P_f). The conic programming relaxation of (P_f) is given by

$$\sup_{(u,\lambda,\mu)} \{\mu : f_0 + \sum_{i=1}^{s} u_i f_i + \langle \lambda, G \rangle - \mu \in \mathcal{P}_d, \quad u = (u_1, \ldots, u_s) \in \mathcal{U}, \lambda \in K^{\oplus}, \mu \in \mathbb{R}\}. \tag{R_f}$$

The first result in this section is an exact conic programming relaxation that is expressed in the form of a *zero duality* gap (i.e., the solution attainment of the conic programming relaxation is not guaranteed).

Theorem 3.1 (Exact conic relaxation) *Consider the problem* (P_f) *and assume that the set* Ω *in* (6) *is closed. Then, we have*

$$\inf\,(\mathrm{P}_f) = \sup\,(\mathrm{R}_f)$$

Proof Recall that $\mathcal{F} := \{x \in \mathbb{R}^n : G(x) \in -K\}$. Clearly, for any $x \in \mathcal{F}$, $u \in \mathcal{U}$, $\lambda \in K^{\oplus}$ and $\mu \in \mathbb{R}$, with $f_0 + \sum_{i=1}^{s} u_i f_i + \langle \lambda, G \rangle - \mu \in \mathcal{P}_d$,

$$F(x) - \mu \geq f(x, u) - \mu \geq f(x, u) + \langle \lambda, G(x) \rangle - \mu \geq 0.$$

Hence, $\inf\,(\mathrm{P}_f) \geq \sup\,(\mathrm{R}_f)$. To see the reverse inequality, we assume without loss of generality that $\inf\,(\mathrm{P}_f) > -\infty$; otherwise, the conclusion trivially holds. As $\mathcal{F} \neq \emptyset$, we have $r := \inf\,(\mathrm{P}_f) \in \mathbb{R}$. Then, for any $\varepsilon > 0$, $F - r + \varepsilon$ is positive on $\mathcal{F}$. So, our Proposition 2.1 implies that there exist $p_0 \in \mathcal{P}_d$, $\overline{u} \in \mathcal{U}$, $\overline{\lambda} \in K^{\oplus}$ and $\overline{\delta} > 0$ such that

$$f(\cdot, \overline{u}) + \langle \overline{\lambda}, G \rangle - (r - \varepsilon) = p_0 + \overline{\delta} \in \mathcal{P}_d.$$

This shows that, for any $\varepsilon > 0$, $\sup\,(\mathrm{R}_f) \geq r - \varepsilon$. Letting $\varepsilon \to 0$, we see that $\sup\,(\mathrm{R}_f) \geq r$, and so the conclusion follows. □

In the following corollary, we show that the closedness of the set Ω in Theorem 3.1 automatically holds for the setting, where the index set $\mathcal{U}$ is a polytope and the cone K is a polyhedral cone. So we obtain an exact conic relaxation for the corresponding conic minimax convex polynomial program.

Corollary 3.2 (Qualification-free exact conic relaxation) *Consider the problem* (P_f)*, where* $\mathcal{U} = \mathrm{conv}\{\overline{u}^1, \ldots, \overline{u}^r\}$ *with* $\overline{u}^\ell \in \mathbb{R}^s, \ell = 1, \ldots, r$, $K = \{x \in \mathbb{R}^m : \langle a^j, x \rangle \geq 0, j = 1, \ldots, q\}$ *with* $a^j \in \mathbb{R}^m$, $j = 1, \ldots, q$, *and* $G := (g_1, \ldots, g_m)$. *Then, we have*

$$\inf\,(\mathrm{P}_f) = \sup_{(\delta,\beta,\mu)} \{\mu : f_0 + \sum_{i=1}^{s} \sum_{\ell=1}^{r} \delta_\ell \overline{u}_i^\ell f_i + \sum_{i=1}^{m} \sum_{j=1}^{q} \beta_j a_i^j g_i - \mu \in \mathcal{P}_d, \quad \delta = (\delta_1, \ldots, \delta_r) \in \Delta^r, \beta := (\beta_1, \ldots, \beta_q) \in \mathbb{R}_+^q, \mu \in \mathbb{R}\}.$$

Proof Firstly, we show that the set Ω in (6) is closed under the given assumptions. In this setting, the set Ω in (6) reduces to the following simpler form:

$$\Omega = \{ (y, z) \in \mathbb{R} \times \mathbb{R}^m : \exists\, x \in \mathbb{R}^n \text{ s.t. } f_0(x) + \sum_{i=1}^{s} \overline{u}_i^{\ell} f_i(x) \leq y,\, \ell = 1, \ldots, r,$$
$$\sum_{i=1}^{m} a_i^j (g_i(x) - z_i) \leq 0,\, j = 1, \ldots, q \}.$$

Let $\{(y^k, z^k)\}_{k\in\mathbb{N}} \subset \Omega$ be such that $(y^k, z^k) \to (\overline{y}, \overline{z})$ as $k \to \infty$. Then, for each $k \in \mathbb{N}$, there exists $x^k \in \mathbb{R}^n$ such that $f_0(x^k) + \sum_{i=1}^{s} \overline{u}_i^{\ell} f_i(x^k) \leq y^k$, $\ell = 1, \ldots, r$, and $\sum_{i=1}^{m} a_i^j (g_i(x^k) - z_i^k) \leq 0$, $j = 1, \ldots, q$. Now, consider the optimization problem

$$\begin{aligned} \inf_{(x,\gamma,\zeta)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^m} \quad & (\gamma - \overline{y})^2 + \sum_{i=1}^{m} (\zeta_i - \overline{z}_i)^2 \\ \text{s.t.} \quad & f_0(x) + \sum_{i=1}^{s} \overline{u}_i^{\ell} f_i(x) \leq \gamma,\, \ell = 1, \ldots, r, \\ & \sum_{i=1}^{m} a_i^j (g_i(x) - \zeta_i) \leq 0,\, j = 1, \ldots, q. \end{aligned} \tag{AP}$$

It is easy to see that (AP) is a convex polynomial optimization problem. Furthermore, one has

$$0 \leq \inf\,(\text{AP}) \leq (y^k - \overline{y})^2 + \sum_{i=1}^{m} (z_i^k - \overline{z}_i)^2 \to 0.$$

Then, by virtue of [4, Theorem 3], inf (AP) is attained, and so there exists $\overline{x} \in \mathbb{R}^n$ such that $f_0(\overline{x}) + \sum_{i=1}^{s} \overline{u}_i^{\ell} f_i(\overline{x}) \leq \overline{y}$, $\ell = 1, \ldots, r$, and $\sum_{i=1}^{m} a_i^j (g_i(\overline{x}) - \overline{z}_i) \leq 0$, $j = 1, \ldots, q$, which shows that $(\overline{y}, \overline{z}) \in \Omega$. Therefore, Ω is closed.

As $\mathcal{U} = \operatorname{conv}\{\overline{u}^1, \ldots, \overline{u}^r\}$ is a polytope and $K = \{x \in \mathbb{R}^m : \langle a^j, x\rangle \geq 0,\, j = 1, \ldots, q\}$ is a polyhedral cone, it holds that

$$\mathcal{U} = \left\{ \sum_{\ell=1}^{r} \delta_\ell \overline{u}^\ell : \delta := (\delta_1, \ldots, \delta_r) \in \Delta^r \right\} \quad \text{and} \quad K^{\oplus} = \left\{ \sum_{j=1}^{q} \beta_j a^j : \beta := (\beta_1, \ldots, \beta_q) \in \mathbb{R}_+^q \right\}.$$

Hence, the existence of $(u, \lambda, \mu) \in \mathcal{U} \times K^{\oplus} \times \mathbb{R}$ with $f_0 + \sum_{i=1}^{s} u_i f_i + \langle \lambda, G\rangle - \mu \in \mathcal{P}_d$ is equivalent to the existence of $(\delta, \beta, \mu) \in \Delta^r \times \mathbb{R}_+^q \times \mathbb{R}$ with

$$f_0 + \sum_{i=1}^{s}\sum_{\ell=1}^{r} \delta_\ell \overline{u}_i^{\ell} f_i + \sum_{i=1}^{m}\sum_{j=1}^{q} \beta_j a_i^j g_i - \mu \in \mathcal{P}_d.$$

Consequently,

$$\sup\,(\text{R}_f) = \sup_{\delta\in\Delta^r,\beta\in\mathbb{R}_+^q,\mu\in\mathbb{R}} \{ \mu : f_0 + \sum_{i=1}^{s}\sum_{\ell=1}^{r} \delta_\ell \overline{u}_i^{\ell} f_i + \sum_{i=1}^{m}\sum_{j=1}^{q} \beta_j a_i^j g_i - \mu \in \mathcal{P}_d \},$$

and the conclusion follows by Theorem 3.1. □

Under a strict feasibility (Slater) condition, the following theorem establishes an exact conic relaxation that is expressed in the form of a *strong duality* relation (i.e., there is the solution attainment of the conic relaxation problem).

Theorem 3.3 (Exact conic relaxation with attainment) *Consider the problem* (P_f)*, where* K *is a closed convex cone with nonempty interior, and let* $x_0 \in \mathbb{R}^n$ *be such that* $G(x_0) \in -\operatorname{int} K$*. Assume that* $\inf(\mathrm{P}_f) > -\infty$*. Then, we have*

$$\inf(\mathrm{P}_f) = \max(\mathrm{R}_f).$$

Proof In this setting, it holds that $r := \inf(\mathrm{P}_f) \in \mathbb{R}$. Observe first as in the proof of Theorem 3.1 that $\sup(\mathrm{R}_f) \leq \inf(\mathrm{P}_f)$ (weak duality).

Now, let

$$D := \{(y, z) \in \mathbb{R} \times \mathbb{R}^m : \exists\, x \in \mathbb{R}^n \text{ s.t. } F(x) - r < y, z \in G(x) + K\}. \tag{9}$$

We see that D is a convex set and $0 \notin D$. Using a separation theorem (see, e.g., [28, Theorem 2.5]), we find $(\nu, \lambda) \in (\mathbb{R} \times \mathbb{R}^m)\backslash\{0\}$ such that

$$0 \leq \nu y + \langle \lambda, z\rangle \quad \forall (y, z) \in D. \tag{10}$$

Arguing as in the proof of Proposition 2.1, we get by (10) that $(\nu, \lambda) \in \mathbb{R}_+ \times K^{\oplus}$. Since $\big(F(x) - r + \varepsilon, G(x)\big) \in D$ for every $x \in \mathbb{R}^n$ and $\varepsilon > 0$, we conclude by (10) that

$$\nu\big(F(x) - r + \varepsilon\big) + \langle \lambda, G(x)\rangle \geq 0 \qquad \forall x \in \mathbb{R}^n.$$

Letting $\varepsilon \to 0$, we obtain

$$\nu\big(F(x) - r\big) + \langle \lambda, G(x)\rangle \geq 0 \qquad \forall x \in \mathbb{R}^n.$$

If $\nu = 0$, then $\lambda \neq 0$ and $\langle \lambda, G(x)\rangle \geq 0$ for all $x \in \mathbb{R}^n$, which is a contradiction with the hypothesis that $G\,(x_0) \in -\operatorname{int} K$ for some $x_0 \in \mathbb{R}^n$, and then $\langle \lambda, G(x_0)\rangle < 0$. Hence, $\nu > 0$, and so we arrive at

$$F(x) - r + \langle \overline{\lambda}, G(x)\rangle \geq 0 \qquad \forall x \in \mathbb{R}^n, \tag{11}$$

where $\overline{\lambda} := \nu^{-1}\lambda$. Following a similar argument as in the proof of Proposition 2.1, we can find $\overline{u} := (\overline{u}_1, \ldots, \overline{u}_s) \in \mathcal{U}$ such that

$$f(x, \overline{u}) - r + \langle \overline{\lambda}, G(x)\rangle \geq 0 \quad \forall x \in \mathbb{R}^n,$$

and so

$$f_0 + \sum_{i=1}^{s} \overline{u}_i f_i + \langle \overline{\lambda}, G\rangle - r \in \mathcal{P}_d.$$

It shows that $(\overline{u}, \overline{\lambda}, r)$ is a feasible point of problem (R_f), and so $\sup(\mathrm{R}_f) \geq r = \inf(\mathrm{P}_f)$. This, together with the above weak duality, entails that $\sup\,(\mathrm{R}_f) = \inf\,(\mathrm{P}_f)$ and the program (R_f) attains its optimal value as r. Consequently, $\inf\,(\mathrm{P}_f) = \max\,(\mathrm{R}_f)$. □

The next theorem establishes a characterization for *stable* exact conic relaxations of the parametric conic minimax convex polynomial problem (PCMP) under the following constraint qualification:

$$\mathcal{H} := \bigcup_{\lambda \in K^{\oplus}} \operatorname{epi}\langle \lambda, G\rangle^* \;\text{ is closed.} \tag{CQ}$$

It is worth observing here that $\mathcal{H}$ is a convex cone [21, Lemma 6.1] and the constraint qualification (CQ) is guaranteed by the Slater condition [21, Proposition 6.1]. In our setting, *stable* means that the exact conic relaxation continues to hold when the objective function of the primal problem is perturbed with any affine function.

Theorem 3.4 (Stable exact conic relaxation) *Consider the parametric problem* (PCMP). *Then, the following statements are equivalent:*

(i) (CQ) *holds.*
(ii) *For any function f in the form of* (PCMP) *with* $\inf(\mathrm{P}_f) > -\infty$, *one has* $\inf(\mathrm{P}_f) = \max(\mathrm{R}_f)$.

Proof *$((i) \Rightarrow (ii))$* Consider a given function f in the form of (PCMP) and let $\alpha := \inf(\mathrm{P}_f) > -\infty$. Since $\mathcal{F} \neq \emptyset$, $\alpha < +\infty$, and so α is finite. Observe first as in the proof of Theorem 3.1 that $\sup(\mathrm{R}_f) \leq \inf(\mathrm{P}_f)$ (weak duality). Since $\alpha := \inf\{F(x) \mid x \in \mathcal{F}\}$, it holds that

$$F(x) + \iota_{\mathcal{F}}(x) \geq \alpha \text{ for all } x \in \mathbb{R}^n.$$

This, by the notion of conjugate function of $F + \iota_{\mathcal{F}}$, means that

$$(0, -\alpha) \in \mathrm{epi}(F + \iota_{\mathcal{F}})^*. \tag{12}$$

Since F is a continuous function, one has

$$\mathrm{epi}(F + \iota_{\mathcal{F}})^* = \mathrm{epi}\, F^* + \mathrm{epi}\, \iota_{\mathcal{F}}^* = \mathrm{epi}\, F^* + \mathrm{cl}\,\mathcal{H} = \mathrm{epi}\, F^* + \mathcal{H}, \tag{13}$$

where the first equality holds by (2), the second equality holds by virtue of [15, Lemma 3.1] and the last one holds under the assumption (i).

We now show that the set $\mathcal{E} := \bigcup_{u \in \mathcal{U}} \mathrm{epi}\, f(\cdot, u)^*$ is convex and closed. To see this, let $(x, w) \in \mathrm{conv}(\mathcal{E})$. Then, there exist $\{u_j\} \subset \mathcal{U}$, $\{(x_j, w_j)\} \subset \mathcal{E}$ and $\{\gamma_j\} \subset \mathbb{R}_+$ for $j = 1, \ldots, k$, such that $(x, w) = \sum_{j=1}^k \gamma_j (x_j, w_j)$, $\sum_{j=1}^k \gamma_j = 1$ and $(x_j, w_j) \in \mathrm{epi}\, f(\cdot, u_j)^*$ for $j = 1, \ldots, k$. Without loss of generality, we can assume that $\gamma_j > 0$ for all $j = 1, \ldots, k$, and by the Carathéodory theorem that $k \leq n + 2$. Then

$$\begin{aligned}(x, w) \in \sum_{j=1}^k \gamma_j\, \mathrm{epi}\, f(\cdot, u_j)^* &\overset{(1)}{=} \sum_{j=1}^k \mathrm{epi}(\gamma_j f(\cdot, u_j))^* \\ &\overset{(2)}{=} \mathrm{epi}\left(\sum_{j=1}^k \gamma_j f(\cdot, u_j)\right)^* = \mathrm{epi}\, f(\cdot, \overline{u})^* \subset \mathcal{E},\end{aligned}$$

where $\overline{u} := \sum_{j=1}^k \gamma_j u_j \in \mathcal{U}$. Thus $\mathrm{conv}(\mathcal{E}) \subset \mathcal{E}$, which entails that $\mathcal{E}$ is convex.

To show that $\mathcal{E}$ is closed, we pick $(y, v) \in \mathrm{cl}\,\mathcal{E}$. Then, there exists $\{(y^k, v^k)\}_{k \in \mathbb{N}} \subset \mathcal{E}$ such that $(y^k, v^k) \to (y, v)$ as $k \to \infty$. Consequently, there exists $\{u^k\}_{k \in \mathbb{N}} \subset \mathcal{U}$ such that $(y^k, v^k) \in \mathrm{epi}\, f(\cdot, u^k)^*$, and so

$$\langle y^k, x\rangle - f(x, u^k) \leq v^k \qquad \forall x \in \mathbb{R}^n, k \in \mathbb{N}. \tag{14}$$

Since $\mathcal{U}$ is a compact set, we may assume that $\overline{u} := \lim_{k \to \infty} u^k \in \mathcal{U}$. Note further that $f(x, \cdot)$ is a continuous function. By letting $k \to \infty$ in (14) we get $\langle y, x\rangle - f(x, \overline{u}) \leq v$ for all $x \in \mathbb{R}^n$. This shows that $f(\cdot, \overline{u})^*(y) = \sup\{\langle y, x\rangle - f(x, \overline{u}) : x \in \mathbb{R}^n\} \leq v$ and so $(y, v) \in \mathrm{epi}\, f(\cdot, \overline{u})^* \subset \mathcal{E}$. Thus $\mathrm{cl}\,\mathcal{E} \subset \mathcal{E}$, which shows that $\mathcal{E}$ is closed.

Now, taking the above assertion, the definition of F and the properties of the conjugate, we obtain that

$$\mathrm{epi}\, F^* \overset{(3)}{=} \mathrm{cl}\,\mathrm{conv}\left(\bigcup_{u \in \mathcal{U}} \mathrm{epi}\, f(\cdot, u)^*\right) = \mathcal{E}.$$

This together with (12) and (13) entails that $(0, -\alpha) \in \mathcal{E} + \mathcal{H}$. Then, there exists $\overline{u} \in \mathcal{U}$, $(a, b) \in \operatorname{epi} f(\cdot, \overline{u})^*$, $\overline{\lambda} \in K^{\oplus}$ and $(c, d) \in \operatorname{epi}\langle \overline{\lambda}, G\rangle^*$, such that $a+c = 0$ and $b+d+\alpha = 0$. Now, by the definition of conjugate function, one has $\langle a, x\rangle - f(x, \overline{u}) \leq b$ for all $x \in \mathbb{R}^n$ and $\langle c, x\rangle - \langle \overline{\lambda}, G(x)\rangle \leq d$ for all $x \in \mathbb{R}^n$. Therefore, we have

$$f(x, \overline{u}) + \langle \overline{\lambda}, G(x)\rangle - \alpha \geq \langle a + c, x\rangle - (b + d + \alpha) = 0 \qquad \forall x \in \mathbb{R}^n,$$

and so

$$f_0 + \sum_{i=1}^{s} \overline{u}_i f_i + \langle \overline{\lambda}, G\rangle - \alpha \in \mathcal{P}_d.$$

This means that $(\overline{u}, \overline{\lambda}, \alpha)$ is a feasible point of problem (R_f), and so sup (R_f) $\geq \alpha =$ inf (P_f). This, together with the above weak duality, entails that sup (R_f) = inf (P_f) and the program (R_f) attains its optimal value as α, and so inf (P_f) = max (R_f). Thus, (ii) holds.

[$(ii) \Rightarrow (i)$] Suppose that (ii) holds. Let $(v, r) \in \operatorname{cl}\mathcal{H}$. Then, there exist $\{\lambda^k\}_{k\in\mathbb{N}} \subset K^{\oplus}$ and $\{(v^k, r^k)\}_{k\in\mathbb{N}} \subset \mathbb{R}^n \times \mathbb{R}$ such that $(v^k, r^k) \in \operatorname{epi}\langle \lambda^k, G\rangle^*$ and $(v^k, r^k) \to (v, r)$ as $k \to \infty$. Hence, for $x \in \mathbb{R}^n$ such that $G(x) \in -K$ one has

$$\langle v^k, x\rangle \leq \langle v^k, x\rangle - \langle \lambda^k, G(x)\rangle \leq r^k.$$

Letting $k \to \infty$, we obtain $\langle v, x\rangle \leq r$ for all $x \in \mathcal{F}$, and so $\alpha := \inf\{-\langle v, x\rangle : G(x) \in -K\} \geq -r$. Putting $\tilde{f} := \tilde{f}_0 + \sum_{i=1}^{s} u_i \tilde{f}_i$ for $\tilde{f}_0 := -\langle v, \cdot\rangle$, $\tilde{f}_i := 0, i = 1, \ldots, s$ and $u \in \mathcal{U}$, we see that $\tilde{f}$ is in the form of (PCMP) with inf ($\mathrm{P}_{\tilde{f}}$) $= \alpha > -\infty$. Then, it follows from (ii) that there exist $\overline{u} \in \mathcal{U}$ and $\overline{\lambda} \in K^{\oplus}$ such that

$$\tilde{f}_0(x) + \sum_{i=1}^{s} \overline{u}_i \tilde{f}_i(x) + \langle \overline{\lambda}, G(x)\rangle - \alpha \geq 0 \qquad \forall x \in \mathbb{R}^n.$$

Thus, $-\langle v, x\rangle + \langle \overline{\lambda}, G(x)\rangle \geq -r$ for all $x \in \mathbb{R}^n$, which shows that

$$\langle \overline{\lambda}, G\rangle^*(v) = \sup\{\langle v, x\rangle - \langle \overline{\lambda}, G(x)\rangle : x \in \mathbb{R}^n\} \leq r.$$

Hence, $(v, r) \in \operatorname{epi}\langle \overline{\lambda}, G\rangle^* \subset \mathcal{H}$. Consequently, $\mathcal{H}$ is closed, and (i) holds. □

4 Extensions and applications of exact conic relaxations

This section is devoted to considering how to generalize our exact conic relaxations to a more general conic minimax ρ-convex polynomial optimization problem and derive exact semidefinite programming (SDP) relaxations for some tractable classes of parametric conic mathematical models such as the class of conic robust SOS-convex polynomial problems, the class of second-order conic robust SOS-convex polynomial problems and the class of difference of SOS-convex polynomial programs.

4.1 Exact conic relaxations for conic minimax ρ-Convex polynomial programs

In this subsection, we consider a (parametric) conic minimax ρ-convex polynomial problem of the form

$$\inf_{x\in\mathbb{R}^n} \{\max_{u\in\mathcal{U}} f(x, u) : G(x) \in -K\}, \tag{GP$_f$}$$

where $\mathcal{U} \subset \mathbb{R}^s$ is a nonempty compact convex set, $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ is given by

$$f(x, u) := f_0(x) + \sum_{i=1}^{s} u_i f_i(x), \quad x \in \mathbb{R}^n, u := (u_1, \ldots, u_s) \in \mathcal{U}$$

for certain polynomials $f_0, f_1, \ldots, f_s$ on $\mathbb{R}^n$ such that for each $u \in \mathcal{U}$, $f(\cdot, u)$ is a $\rho(u)$-*convex* polynomial for some $\rho(u) \in \mathbb{R}$, $K \subset \mathbb{R}^m$ is a closed convex cone and $G : \mathbb{R}^n \to \mathbb{R}^m$ is a ϱ-K-*convex* polynomial mapping for some $\varrho \in \mathbb{R}^m$. We recall that $\mathcal{F} := \{x \in \mathbb{R}^n : G(x) \in -K\} \neq \emptyset$, F is defined as in (5) and the conic relaxation for the problem (GP_f) is given as in (R_f). Observe that the problem (P_f) is a particular case of problem (GP_f).

The forthcoming theorem provides necessary and sufficient conditions in terms of the *Karush-Kuhn-Tucker* optimality condition for exact conic relaxations of problem (GP_f).

Definition 4.1 We say that the *Karush-Kuhn-Tucker* (KKT) condition holds at $\overline{x} \in \mathcal{F}$ whenever there exist $\overline{u} \in \mathcal{U}$ and $\overline{\lambda} \in K^{\oplus}$ such that

$$\begin{aligned} \nabla_x f(\overline{x}, \overline{u}) + DG(\overline{x})^* \overline{\lambda} &= 0, \\ \langle \overline{\lambda}, G(\overline{x}) \rangle &= 0, \\ f(\overline{x}, \overline{u}) - F(\overline{x}) &= 0, \end{aligned} \tag{15}$$

where $\overline{\lambda}$ is called a *Lagrange multiplier* at $(\overline{x}, \overline{u})$.

It is worth mentioning here that when $\mathcal{U}$ is a singleton (equivalently, there is no index set $\mathcal{U}$), the condition of $f(\overline{x}, \overline{u}) - F(\overline{x}) = 0$ is superfluous, and so above-defined KKT condition collapses to the classical KKT one given, for example, in [9].

Theorem 4.1 (Exact conic relaxations via KKT) *Consider the problem* (GP_f) *and its conic relaxation* (R_f). *Then, the following statements hold:*

(i) If the KKT condition holds for (GP_f) *at* $\overline{x} \in \mathcal{F}$ *with a Lagrange multiplier* $\overline{\lambda}$ *at* $(\overline{x}, \overline{u})$ *satisfying* $\rho(\overline{u}) + \langle \overline{\lambda}, \varrho \rangle \geq 0$, *then* $\overline{x}$ *is a global minimizer and* min (GP_f) = max (R_f).

(ii) If min (GP_f) = max (R_f), *then the KKT condition holds at each global minimizer of* (GP_f).

Proof (i) Observe first as in the proof of Theorem 3.1 that sup (R_f) $\leq$ inf (GP_f) (weak duality). Assume that the KKT condition holds for (GP_f) at $\overline{x} \in \mathcal{F}$ with a Lagrange multiplier $\overline{\lambda}$ at $(\overline{x}, \overline{u})$ satisfying $\rho(\overline{u}) + \langle \overline{\lambda}, \varrho \rangle \geq 0$. We note from the definition of ρ-convexity that $f(\cdot, \overline{u}) + \langle \overline{\lambda}, G \rangle$ is $(\rho(\overline{u}) + \langle \overline{\lambda}, \varrho \rangle)$-convex. Then, $f(\cdot, \overline{u}) + \langle \overline{\lambda}, G \rangle$ is convex if and only if $\rho(\overline{u}) + \langle \overline{\lambda}, \varrho \rangle \geq 0$. So, the condition of $\rho(\overline{u}) + \langle \overline{\lambda}, \varrho \rangle \geq 0$ (hence, the convexity of $f(\cdot, \overline{u}) + \langle \overline{\lambda}, G \rangle$) follows that

$$f(x, \overline{u}) + \langle \overline{\lambda}, G(x) \rangle - f(\overline{x}, \overline{u}) - \langle \overline{\lambda}, G(\overline{x}) \rangle - \langle \nabla_x f(\overline{x}, \overline{u}) + DG(\overline{x})^* \overline{\lambda}, x - \overline{x} \rangle \geq 0$$

for all $x \in \mathbb{R}^n$. Then, we get by the KKT condition at $\overline{x}$ in (15) that

$$f(\cdot, \overline{u}) + \langle \overline{\lambda}, G \rangle - F(\overline{x}) \in \mathcal{P}_d.$$

If $x \in \mathbb{R}^n$ is such that $G(x) \in -K$, since $0 \geq \langle \overline{\lambda}, G(x) \rangle$, then $F(x) \geq f(x, \overline{u}) \geq F(\overline{x})$, which shows that $\overline{x}$ is a global minimizer of (GP_f) and min (GP_f) = $F(\overline{x})$. Furthermore, we see that $(\overline{u}, \overline{\lambda}, F(\overline{x}))$ is a feasible point of (R_f), and so sup (R_f) $\geq F(\overline{x})$. This, together with the above weak duality, entails that sup (R_f) = min (GP_f) and the program (R_f) attains its optimal value; i.e., min (GP_f) = max (R_f).

(ii) Assume that $\min$ (GP$_f$) $=$ $\max$ (R$_f$), and let $\overline{x} \in \mathcal{F}$ be an arbitrary global minimizer of (GP$_f$). Then, $F(\overline{x}) = \max$ (R$_f$) and so, there exist $\overline{u} \in \mathcal{U}$ and $\overline{\lambda} \in K^{\oplus}$ such that

$$h(\cdot) := f(\cdot, \overline{u}) + \langle \overline{\lambda}, G \rangle - F(\overline{x}) \in \mathcal{P}_d$$

and $(\overline{u}, \overline{\lambda}, F(\overline{x}))$ is an optimal solution to (R$_f$). Now, we observe that $F(\overline{x}) = \max_{u \in \mathcal{U}} f(\overline{x}, u) = f(\overline{x}, \overline{u})$. Otherwise, if $F(\overline{x}) > f(\overline{x}, \overline{u})$, since $G(\overline{x}) \in -K$ implies $\langle \overline{\lambda}, G(\overline{x}) \rangle \leq 0$, then $h(\overline{x}) < 0$, a contradiction with $h \in \mathcal{P}_d$. Hence, $F(\overline{x}) = f(\overline{x}, \overline{u})$ and so, $h(\overline{x}) = \langle \overline{\lambda}, G(\overline{x}) \rangle \geq 0$. Consequently, we get that $\langle \overline{\lambda}, G(\overline{x}) \rangle = 0$. We thus get that $h(\overline{x}) = 0$, which means that $\overline{x}$ is a minimizer of h on $\mathbb{R}^n$ since for each $x \in \mathbb{R}^n$, $h(x) \geq 0 = h(\overline{x})$. Then, by the necessary condition for a minimizer of h, we obtain that $\nabla h(\overline{x}) = 0$, that is, $\nabla_x f(\overline{x}, \overline{u}) + DG(\overline{x})^* \overline{\lambda} = 0$. Consequently, the KKT condition holds at each global minimizer of (GP$_f$). □

Since the problem (P$_f$) is a particular case of problem (GP$_f$) and moreover, it is a convex program, we get by Theorem 4.1 a characterization for exact conic relaxations at each minimizer of problem (P$_f$) as follows:

Corollary 4.2 *Let $\overline{x} \in \mathcal{F}$ be a minimizer of problem* (P$_f$)*. Then, the KKT condition holds for* (P$_f$) *at $\overline{x} \in \mathcal{F}$ if and only if* $\min$ (P$_f$) $=$ $\max$ (R$_f$).

4.2 Applications to conic robust SOS-convex and difference of SOS-convex polynomial programs

In this subsection, we apply the exact conic relaxations obtained in the previous sections to derive stable exact semidefinite programming (SDP) relaxations for some tractable classes of parametric conic mathematical models including the class of conic robust SOS-convex polynomial problems, the class of second-order conic robust SOS-convex polynomial problems and the class of difference of SOS-convex polynomial programs. When restricting our problems into particular settings, the obtained results develop and recover some corresponding ones existing the literature such as those in [20, 22] and [26], and more importantly, all exact conic relaxations are *numerically tractable* in the sense that they can be reformulated and solved as SDP problems [33].

Parametric conic robust SOS-convex polynomial problems. Let us first consider a parametric *conic robust SOS-convex* polynomial problem (PCRP) that is defined as follows: Let $\mathcal{U} \subset \mathbb{R}^s$ be a nonempty uncertainty set that is a *spectrahedron* (cf. [30, 36]) defined by

$$\mathcal{U} := \{u := (u_1, \ldots, u_s) \in \mathbb{R}^s : A_0 + \sum_{i=1}^{s} u_i A_i \succeq 0\} \tag{16}$$

for some given symmetric matrices $A_i, i = 0, 1, \ldots, s$, and $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ be a bifunction that is given by

$$f(x, u) := f_0(x) + \sum_{i=1}^{s} u_i f_i(x), \quad x \in \mathbb{R}^n, u := (u_1, \ldots, u_s) \in \mathcal{U} \tag{17}$$

for certain polynomials $f_0, f_1, \ldots, f_s$ on $\mathbb{R}^n$ such that $f(\cdot, u)$ is an *SOS-convex* polynomial for each $u \in \mathcal{U}$, where u is an *uncertain* vector. For each f (regarded as a parameter), one considers a *conic robust SOS-convex* polynomial problem of the form:

$$\inf_{x \in \mathbb{R}^n} \{\max_{u \in \mathcal{U}} f(x, u) : G(x) \in -S^p_+\}, \tag{RP$_f$}$$

where $G : \mathbb{R}^n \to S^p$ is an S^p_+-*SOS-convex* polynomial mapping. As previously, we assume that the uncertainty set $\mathcal{U}$ in (16) is bounded and the feasible set of (RP_f) is non-empty, and denote by d the smallest even number satisfying $d \geq \max\{\deg f_i, i = 0, 1, \ldots, s, \deg G\}$.

The following theorem provides a characterization of exact conic relaxations for the parametric conic robust SOS-convex polynomial problem when f varies in the class of SOS-convex polynomials of the form (17).

Theorem 4.3 (Stable exact SDP relaxation of PCRP) *Consider the parametric conic robust SOS-convex polynomial problem* (PCRP). *Then, the following statements are equivalent:*

(i) $\bigcup_{\lambda \in S^p_+} \textit{epi}\operatorname{Tr}(\lambda G)^*$ *is closed.*

(ii) For any function f in the form of (PCRP) *with* $\inf(\text{RP}_f) > -\infty$, *one has*

$$\inf(\text{RP}_f) = \max_{(\mu,\lambda,w)} \{\mu : f_0 + \sum_{i=1}^{s} w_i f_i + \operatorname{Tr}(\lambda G) - \mu \in \Sigma^2_d,\ A_0 + \sum_{i=1}^{s} w_i A_i \succeq 0,$$
$$\mu \in \mathbb{R}, \lambda \in S^p_+, w = (w_1, \ldots, w_s) \in \mathbb{R}^s\}. \tag{18}$$

Proof Let $m := \frac{p(p+1)}{2}$. Then, it holds that the dimension of S^p is the same as the dimension of $\mathbb{R}^m$, and so there exists an invertible linear map $\mathcal{T} : S^p \to \mathbb{R}^m$ such that

$$\mathcal{T}(A)^\top \mathcal{T}(B) = \operatorname{Tr}(AB) \text{ for all } A, B \in S^p.$$

Thus, one can identify S^p equipped with the trace inner product as $\mathbb{R}^m$ and the Euclidean inner product by associating each symmetric matrix $A \in S^p$ to $\mathcal{T}(A) \in \mathbb{R}^m$. Now, for any function f in the form of (PCRP) with $\inf(\text{RP}_f) > -\infty$, we see that the problem (RP_f) can be viewed as a particular case of problem (P_f) with $K := S^p_+$ and $\mathcal{U}$ given by (16). Invoking Theorem 3.4, we conclude that the closedness of $\bigcup_{\lambda \in K^\oplus} \operatorname{epi}\langle \lambda, G\rangle^*$ is equivalent to the assertion that for any function f in the form of (PCRP) with $\inf(\text{RP}_f) > -\infty$, one has

$$\inf(\text{RP}_f) = \max_{(u,\lambda,\mu)} \{\mu : f_0 + \sum_{i=1}^{s} u_i f_i + \langle \lambda, G\rangle - \mu \in \mathcal{P}_d,$$
$$u = (u_1, \ldots, u_s) \in \mathcal{U}, \lambda \in K^\oplus, \mu \in \mathbb{R}\}. \tag{19}$$

In this setting, for $u := (u_1, \ldots, u_s) \in \mathcal{U}$, $\lambda \in K^\oplus$ and $\mu \in \mathbb{R}$, the function $f_0 + \sum_{i=1}^s u_i f_i + \langle \lambda, G\rangle - \mu$ is an SOS-convex polynomial and so $f_0 + \sum_{i=1}^s u_i f_i + \langle \lambda, G\rangle - \mu \in \mathcal{P}_d$ is equivalent to $f_0 + \sum_{i=1}^s u_i f_i + \langle \lambda, G\rangle - \mu \in \Sigma^2_d$ (see e.g., [22, Corollary 2.1]). Moreover, since $K^\oplus = S^p_+$, (18) is nothing else but (19). So the proof is complete. □

Parametric robust SOS-convex polynomial problems. We now consider a parametric *robust SOS-convex* polynomial problem (PRP) that is defined as follows: Let $\mathcal{U} \subset \mathbb{R}^s$ be a nonempty *uncertainty set* given as in (16), and $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ be a bifunction given as in (17). For each f (regarded as a parameter), one considers a *robust SOS-convex* polynomial problem of the form

$$\inf_{x \in \mathbb{R}^n} \{\max_{u \in \mathcal{U}} f(x, u) : g_i(x) \leq 0,\ i = 1, \ldots, m\}, \tag{SP_f}$$

where $g_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$ are *SOS-convex* polynomials. As previously, we assume that the uncertainty set $\mathcal{U}$ in (16) is bounded and the feasible set of (SP_f) is non-empty, and denote by d the smallest even number satisfying $d \geq \max\{\deg f_j, j = 0, 1, \ldots, s, \deg g_i, i = 1, \ldots, m\}$.

Corollary 4.4 (Stable exact SDP relaxation of PRP) *Consider the parametric robust SOS-convex polynomial problem* (PRP). *Then, the following statements are equivalent:*

(i) $\bigcup_{\lambda_1 \geq 0, \ldots, \lambda_m \geq 0} \operatorname{epi}\left(\sum_{i=1}^{m} \lambda_i g_i\right)^*$ *is closed.*

(ii) For any function f in the form of (PRP) *with* $\inf$ (SP$_f$) $> -\infty$*, one has*

$$\inf (\mathrm{SP}_f) = \max_{(\mu,\lambda,w)} \{\mu : f_0 + \sum_{i=1}^{s} w_i f_i + \sum_{i=1}^{m} \lambda_i g_i - \mu \in \Sigma_d^2, A_0 + \sum_{i=1}^{s} w_i A_i \succeq 0,$$
$$\mu \in \mathbb{R}, \lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}_+^m, w := (w_1, \ldots, w_s) \in \mathbb{R}^s\}.$$

Proof Letting $K := \mathbb{R}_+^m$ and $G := (g_1, \ldots, g_m)$, the problem (SP$_f$) is in the form of problem (P$_f$). In this setting, it holds $K^{\oplus} = \mathbb{R}_+^m$, and then $\langle \lambda, G \rangle = \sum_{i=1}^{m} \lambda_i g_i$ is an SOS-convex polynomial on $\mathbb{R}^n$ for any $\lambda := (\lambda_1, \ldots, \lambda_m) \in K^{\oplus}$. This ensures that G is a K-SOS-convex polynomial. So the desired result is now followed by invoking Theorem 3.4. □

Remark 4.1 By considering $f_0 := 0$ and $\mathcal{U} := \operatorname{conv}\{e^1, \ldots, e^s\}$, the problem (SP$_f$) collapses to the form of (SOP$_f$), and then we apply Corollary 4.4 to assert that if $\bigcup_{\lambda_1 \geq 0, \ldots, \lambda_m \geq 0} \operatorname{epi}\left(\sum_{i=1}^{m} \lambda_i g_i\right)^*$ is closed and $\inf$ (SOP$_f$) $> -\infty$, then one has

$$\inf (\mathrm{SOP}_f) = \max_{(\mu,\lambda,w)} \{\mu : \sum_{i=1}^{s} w_i f_i + \sum_{i=1}^{m} \lambda_i g_i - \mu \in \Sigma_d^2, \mu \in \mathbb{R},$$
$$\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}_+^m, w = (w_1, \ldots, w_s) \in \Delta^s\}.$$

This result was established in [22, Theorem 3.2] under the validation of the Slater condition, that is, there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0) < 0$ for all $i = 1, \ldots, m$.

The interested reader is also referred to [12, Theorem 2.2] for a characterization of stable exact SDP relaxations via the characteristic cone for another class of robust SOS-convex polynomial problems that involve uncertainty data in their constraints.

Parametric second-order conic robust SOS-convex polynomial problems. We now consider a parametric *second-order conic robust SOS-convex* polynomial problem (PSOC) that is defined as follows: Let $\mathcal{U} \subset \mathbb{R}^s$ be a nonempty *uncertainty set* that is an ellipsoid given by

$$\mathcal{U} := \left\{u := (u_1, \ldots, u_s) \in \mathbb{R}^s : \langle u, Eu \rangle \leq 1\right\} \tag{20}$$

for a given symmetric $s \times s$ matrix $E \succ 0$, and $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ be a bifunction given as in (17). For each f (regarded as a parameter), one considers a *second-order conic robust SOS-convex* polynomial problem of the form:

$$\inf_{x \in \mathbb{R}^n} \{\max_{u \in \mathcal{U}} f(x, u) : G(x) \in -\mathcal{L}_m, \}, \tag{SOC$_f$}$$

where $\mathcal{L}_m := \left\{(y_1, \ldots, y_m) \in \mathbb{R}^m : y_1 \geq \sqrt{\sum_{i=2}^{m} (y_i)^2}\right\}$ is the second-order cone or Lorentz cone in $\mathbb{R}^m$, and $G : \mathbb{R}^n \to \mathbb{R}^m$ is an $\mathcal{L}_m$-*SOS-convex* polynomial mapping. We assume that the feasible set of (SOC$_f$)) is non-empty, and denote by d the smallest even number satisfying $d \geq \max\{\deg f_i, i = 0, 1, \ldots, s, \deg G\}$. We also use the notation L to denote a decomposition factor of E, i.e., $E = L^\top L$.

Theorem 4.5 (Stable exact SDP relaxation of PSOC) *Consider the parametric second-order conic robust SOS-convex polynomial problem* (PSOC). *Then, the following statements are equivalent:*

(i) $\bigcup_{\lambda \in \mathcal{L}_m} \text{epi}\langle \lambda, G\rangle^*$ *is closed.*

(ii) For any function f in the form of (PSOC) *with* $\inf\,(\text{SOC}_f) > -\infty$, *one has*

$$\inf\,(\text{SOC}_f) = \max_{(\mu,\lambda,w)} \{\mu : f_0 + \sum_{i=1}^{s} w_i f_i + \langle \lambda, G\rangle - \mu \in \Sigma_d^2,\ w = (w_1, \ldots, w_s) \in \mathbb{R}^s,$$
$$||Lw|| \le 1, \mu \in \mathbb{R}, \lambda_1 \ge \sqrt{\sum_{i=2}^{m} (\lambda_i)^2}, \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m\}. \tag{21}$$

Proof Letting $K := \mathcal{L}_m$, the problem (SOC_f) is in the form of problem (P_f). Arguing as in the proof of Theorem 4.3, we invoke Theorem 3.4 to conclude that the closedness of $\bigcup_{\lambda \in K^{\oplus}} \text{epi}\langle \lambda, G\rangle^*$ is equivalent to the assertion that for any function f in the form of (PSOC) with $\inf\,(\text{SOC}_f)) > -\infty$, one has

$$\inf\,(\text{SOC}_f) = \max_{(u,\lambda,\mu)} \{\mu\ : f_0 + \sum_{i=1}^{s} u_i f_i + \langle \lambda, G\rangle - \mu \in \Sigma_d^2,$$
$$u = (u_1, \ldots, u_s) \in \mathcal{U}, \lambda \in K^{\oplus}, \mu \in \mathbb{R}\}. \tag{22}$$

In this setting, it holds that $K^{\oplus} = \mathcal{L}_m$. Moreover, for $u \in \mathcal{U}$, we have

$$\langle u, Eu\rangle \le 1 \Leftrightarrow ||Lu|| \le 1$$

as $E = L^\top L$. So (22) coincides with (21), which completes the proof. □

Parametric conic difference of SOS-convex polynomial problems. Let us now consider a parametric conic *difference of SOS-convex* polynomial problem (PCDC) that is defined as follows: Let $f_0 : \mathbb{R}^m \to \mathbb{R}$ be an *SOS-convex* polynomial and $h : \mathbb{R}^n \to \mathbb{R}$ be a support function that is given by

$$h(x) := \max\left\{\langle u, x\rangle : u \in \text{conv}\{\overline{u}^1, \ldots, \overline{u}^r\}\right\},\ x \in \mathbb{R}^n$$

for given $\overline{u}^1, \ldots, \overline{u}^r$ in $\mathbb{R}^n$. For each pair of (f_0, h) (regarded as a parameter), one considers a conic *difference of SOS-convex* polynomial problem (cf. [26]):

$$\inf_{x \in \mathbb{R}^n} \{f_0(x) - h(x) : G(x) \in -S_+^p\}, \tag{$\text{DC}_{f_0,h}$}$$

where $G : \mathbb{R}^n \to S^p$ is an S_+^p-*SOS-convex* polynomial mapping. As previously, we assume that the feasible set of ($\text{DC}_{f_0,h}$) is non-empty, and denote by d the smallest even number satisfying $d \ge \max\{\deg f_0, 1, \deg G\}$.

The next theorem presents a characterization of exact conic relaxations for the parametric conic difference of SOS-convex polynomial problem when the pair of (f_0, h) varies in the above class of difference of SOS-convex polynomials.

Theorem 4.6 (Stable exact SDP relaxation of PCDC) *Consider the parametric conic difference of SOS-convex polynomial problem* (PCDC). *Then, the following statements are equivalent:*

(i) $\bigcup_{\lambda \in S^p_+} \operatorname{epi}\langle \lambda, G\rangle^*$ *is closed.*

(ii) *For any pair of* (f_0, h) *in the form of* (PCDC) *with* $\inf\,(\mathrm{DC}_{f_0,h}) > -\infty$*, one has*

$$\inf\,(\mathrm{DC}_{f_0,h}) = \max_{(\mu,\lambda,\delta)} \{\mu : f_0 - \sum_{j=1}^{r} \delta_j \langle \overline{u}^j, \cdot\rangle + \operatorname{Tr}(\lambda G) - \mu \in \Sigma^2_d,$$
$$\delta = (\delta_1, \ldots, \delta_r) \in \Delta^r, \mu \in \mathbb{R}, \lambda \in S^p_+\}. \tag{23}$$

Proof For any pair of (f_0, h) in the form of (PCDC) with $\inf\,(\mathrm{DC}_{f_0,h}) > -\infty$, the problem $(\mathrm{DC}_{f_0,h})$ can be rewritten as

$$\inf_{x\in\mathbb{R}^n} \Big\{ \max_{u\in \operatorname{conv}\{\overline{u}^1,\ldots,\overline{u}^r\}} \{f_0(x) - \langle u, x\rangle\} : G(x) \in -S^p_+\Big\}. \tag{24}$$

Let $\mathcal{U} := \operatorname{conv}\{\overline{u}^1, \ldots, \overline{u}^r\}$ and $f_i(x) := -x_i, i = 1, \ldots, n$ for $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$. We see that the problem in (24) can be expressed further as the following one:

$$\inf_{x\in\mathbb{R}^n} \Big\{ \max_{u\in\mathcal{U}} \{f_0(x) + \sum_{i=1}^{n} u_i f_i(x)\} : G(x) \in -S^p_+\Big\}, \tag{25}$$

which is in the form of problem (P_f) with $K := S^p_+$ and $f(\cdot, u) := f_0 + \sum_{i=1}^n u_i f_i$, $u := (u_1, \ldots, u_n) \in \mathcal{U}$. Note that, in this setting, $f(\cdot, u)$ is an SOS-convex polynomial for all $u \in \mathcal{U}$. Arguing similarly as in the proof of Theorem 4.3, we invoke Theorem 3.4 to conclude that the closedness of $\bigcup_{\lambda\in S^p_+} \operatorname{epi}\langle\lambda, G\rangle^*$ is equivalent to the assertion that for any pair of (f_0, h) in the form of (PCDC) with $\inf\,(\mathrm{DC}_{f_0,h}) > -\infty$ one has

$$\inf\,(\mathrm{DC}_{f_0,h}) = \max_{(u,\lambda,\mu)} \{\mu : f_0 + \sum_{i=1}^{n} u_i f_i + \operatorname{Tr}(\lambda G) - \mu \in \Sigma^2_d,$$
$$u = (u_1, \ldots, u_n) \in \mathcal{U}, \lambda \in S^p_+, \mu \in \mathbb{R}\}. \tag{26}$$

Note further that $\mathcal{U} = \{\sum_{j=1}^r \delta_j \overline{u}^j : \delta = (\delta_1, \ldots, \delta_r) \in \Delta^r\}$. Then, (26) is equivalent to

$$\inf\,(\mathrm{DC}_{f_0,h}) = \max_{(\delta,\lambda,\mu)} \{\mu : f_0 + \sum_{i=1}^{n}\sum_{j=1}^{r} \delta_j \overline{u}^j_i f_i + \operatorname{Tr}(\lambda G) - \mu \in \Sigma^2_d,$$
$$\delta = (\delta_1, \ldots, \delta_r) \in \Delta^r, \lambda \in S^p_+, \mu \in \mathbb{R}\},$$

which amounts to (23), and so the proof is complete. ◻

The following corollary provides an exact SDP relaxation for a conic difference of SOS-convex polynomial problem under the Slater condition. This result was obtained in [26, Theorem 3] by using a different dual approach.

Corollary 4.7 (Exact SDP relaxation under the Slater condition) *Consider a conic difference of SOS-convex polynomial problem of the form* $(\mathrm{DC}_{f_0,h})$*. Let* $x_0 \in \mathbb{R}^n$ *be such that*

$$G(x_0) \in -\operatorname{int} S^p_+. \tag{27}$$

Assume that $\inf\,(\mathrm{DC}_{f_0,h}) > -\infty$*. Then, we have*

$$\inf\,(\mathrm{DC}_{f_0,h}) = \max_{(\mu,\lambda,\delta)} \{\mu : f_0 - \sum_{j=1}^{r} \delta_j \langle \overline{u}^j, \cdot\rangle + \operatorname{Tr}(\lambda G) - \mu \in \Sigma^2_d,$$

$$\delta = (\delta_1, \ldots, \delta_r) \in \Delta^r, \mu \in \mathbb{R}, \lambda \in S^p_+\}.$$

Proof The Slater condition in (27) entails that $\bigcup_{\lambda \in S^p_+} \operatorname{epi}\langle \lambda, G\rangle^*$ is closed (see, e.g., [21, Proposition 6.1]). The desired conclusion follows by Theorem 4.6. □

We close this section with an example which shows how we can employ the obtained exact SDP relaxations to find the robust optimal value of an uncertain optimization problem by solving its corresponding SDP relaxation problem.

Example 4.1 Consider an *uncertain* optimization problem of the form

$$\inf_{x:=(x_1,x_2)\in\mathbb{R}^2} \left\{x_1^4 + 2x_2^4 + (u_1+3)x_2^2 + (u_2+1)x_1^2 + 1 : 1 - x_1^4 - 3x_2^2 \geq \sqrt{x_1^8 + 9x_2^4}\right\}, \tag{EUP}$$

where $u := (u_1, u_2) \in \mathcal{U}$ is an *uncertain* vector and $\mathcal{U}$ is an uncertainty set given by $\mathcal{U} := \{u = (u_1, u_2) \in \mathbb{R}^2 : \frac{u_1^2}{9} + u_2^2 \leq 1\}$. To treat the problem (EUP), one considers its *robust* counterpart as follows:

$$\inf_{x:=(x_1,x_2)\in\mathbb{R}^2} \left\{\max_{u\in\mathcal{U}}\{x_1^4 + 2x_2^4 + (u_1+3)x_2^2 + (u_2+1)x_1^2 + 1\} : \; 1 - x_1^4 - 3x_2^2 \geq \sqrt{x_1^8 + 9x_2^4}\right\}. \tag{ERP}$$

Let $f_0(x) := x_1^4 + 2x_2^4 + 3x_2^2 + x_1^2 + 1$, $f_1(x) := x_2^2$, $f_2(x) := x_1^2$, and $G := (g_1, g_2, g_3)$, where $g_1(x) := x_1^4 + 3x_2^2 - 1$, $g_2(x) := x_1^4$, $g_3(x) := 3x_2^2$ for $x := (x_1, x_2) \in \mathbb{R}^2$. The problem (ERP) can be rewritten as the following second-order conic robust convex polynomial problem:

$$\inf_{x\in\mathbb{R}^2} \{\max_{u\in\mathcal{U}}\{f_0(x) + \sum_{i=1}^{2} u_i f_i(x)\} : G(x) \in -\mathcal{L}_3\}, \tag{SOCE}$$

where $\mathcal{L}_3 := \{y := (y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1 \geq \sqrt{y_2^2 + y_3^2}\}$, which is a second-oder cone in $\mathbb{R}^3$ depicted in Fig. 1.

Taking any $\lambda := (\lambda_1, \lambda_2, \lambda_3) \in \mathcal{L}_3$, we see that $\lambda_1 \geq \sqrt{\lambda_2^2 + \lambda_3^2} \geq \max\{|\lambda_2|, |\lambda_3|\}$, and then $g_\lambda := \langle \lambda, G\rangle$ is a convex function by the fact that $g_\lambda(x) = (\lambda_1+\lambda_2)x_1^4 + 3(\lambda_1+\lambda_3)x_2^2 - \lambda_1$ for $x \in \mathbb{R}^2$. Since g_λ is a separable polynomial, then $\langle \lambda, G\rangle$ is an SOS-convex polynomial. This in turn entails that G is an $\mathcal{L}_3$-SOS-convex polynomial mapping.

So the problem (SOCE) lands in the form of problem (SOC_f)) with $f := f_0 + \sum_{i=1}^{2} u_i f_i$ with $u := (u_1, u_2) \in \mathcal{U}$. It is easy to see that $\inf(SOCE) > -\infty$. Moreover, since the Slater condition holds for this problem, the set $\bigcup_{\lambda\in\mathcal{L}_3} \operatorname{epi}\langle \lambda, G\rangle^*$ is closed. We invoke the stable exact SDP relaxation in Theorem 4.5 to assert that

$$\inf(SOCE) = \max(SDPE), \tag{28}$$

where (SDPE) is the following SDP relaxation problem

$$\max_{(\mu,\lambda,w)} \{\mu : f_0 + \sum_{i=1}^{2} w_i f_i + \langle \lambda, G\rangle - \mu \in \Sigma^2_d, \; w := (w_1, w_2) \in \mathbb{R}^2, \tag{SDPE}$$

$$||Lw|| \leq 1, \mu \in \mathbb{R}, \lambda_1 \geq \sqrt{\lambda_2^2 + \lambda_3^2}, \lambda := (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3\}$$

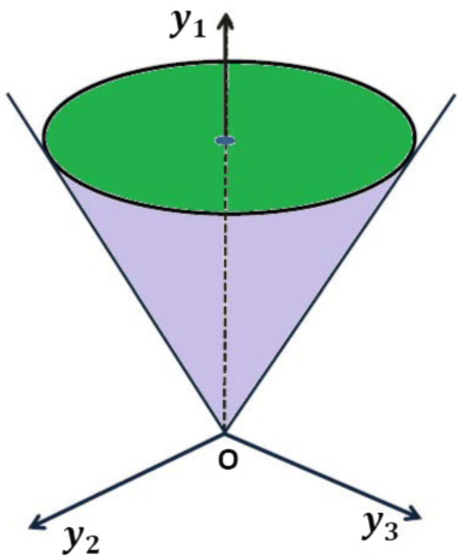

Fig. 1 The shaded set is the cone $\mathcal{L}_3$ in $\mathbb{R}^3$

with $L := \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$ and $d := 4$. Using the Matlab toolbox YALMIP [27], we solve the problem (SDPE) and the solver returns its optimal value as 1.000 (i.e., $\max(SDPE) = 1$). This together with (28) concludes that the optimal value of problem (SOCE) (thus the optimal value of our robust problem (ERP)) is 1 (i.e., $\inf$ (ERP) $= \inf(SOCE) = 1$).

Note that in this setting one can verify directly that $\bar{x} := (0, 0)$ is an optimal solution of problem (SOCE) with the corresponding optimal value $\min(SOCE) = f(\bar{x}) = 1$.

5 Concluding remarks

In this paper, we established that conic minimax convex polynomial programs exhibit exact conic programming relaxations under suitable regularity assumptions. We extended the notion of ρ-convexity of a real-valued polynomial to a more general notion of conic convexity for polynomial mappings. Then, we further proved that conic minimax generalized convex polynomial programs exhibit exact conic programming relaxations under the KKT condition. All these results can be applied to the framework of SOS-convexity related to f and G and so, the conic programming relaxations, for appropriate instances of $\mathcal{U}$ and K, can be reformulated as a semidefinite programming problem.

Consequently, the conic relaxation problems can be solved by using commonly used numerical methods such as interior point algorithms. Moreover, as conic programming models can be found in many applications of various disciplines [10], our results become significant to study classes of conic minimax generalized convex programs and their related applications. It would be of great interest to see how we can develop associated numerical methods/schemes to verify the exact conic relaxations for some specific convex polynomial problems and deploy applications to practical scenarios such as a generalization of the *weighted Steiner* problem in [7].

Acknowledgements The authors would like to thank the reviewers for their constructive comments and valuable suggestions that helped improve the final version of the paper.

Funding Research of J. Vicente-Pérez was partially supported by Grant PID2022-136399NB-C21 funded by MICIU/AEI/10.13039/501100011033 and by ERDF/EU, and by Grant AICO/2021/165 from the Generalitat Valenciana.

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