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Solution existence for a class of nonsmooth robust optimization problems

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Abstract
The main purpose of this paper is to investigate the existence of global optimal solutions for nonsmooth and nonconvex robust optimization problems. To do this, we first introduce a concept called extended tangency variety and show how a robust optimization problem can be transformed into a minimizing problem of the corresponding tangency variety. We utilize this concept together with a constraint qualification condition and the boundedness of the objective function to provide relationships among the concepts of robust properness, robust M-tamesness and robust Palais-Smale condition related to the considered problem. The obtained results are also employed to derive necessary and sufficient conditions for the existence of global optimal solutions to the underlying robust optimization problem.

Keywords Mordukhovich/limiting subdifferential · Robust optimization · Extended tangency variety · Solution existence · Constraint qualification · Palais-Smale condition

1 Introduction

The existence of optimal solutions is an important topic in optimization theory and has recently received remarkable attention from the researchers; see, e.g. [2, 8, 9, 12, 15, 26] and other related references therein. Notably, we refer the reader to a recent work [16] for the study of the existence of solutions to a multiobjective optimization problem involving locally Lipschitz functions. As a matter of fact, there are vast real-world problems whose data are often uncertain or fluctuate. So, robust optimization has emerged as efficient techniques and

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effective frameworks for studying problems with uncertain data; see, [2, 3, 5–8, 13, 14, 17, 18, 27, 31] for more details.

In this paper, we consider an uncertain optimization problem that is defined by

$$\min_{x\in\mathbb{R}^n} \left\{ f(x,\tau) \mid x \in \Omega,\ h_i(x,u_i) \le 0,\ i = 1, ..., m \right\}, \tag{U}$$

where x is a decision variable, τ and u_i, $i = 1, \ldots, m$, are uncertain parameters, which reside in the uncertainty sets $\mathcal{T}$ and $\mathcal{V}_i$, respectively, $\Omega \subset \mathbb{R}^n$ is a nonempty closed set, $\mathcal{T} \subset \mathbb{R}^k$, $\mathcal{V}_i \subset \mathbb{R}^{n_i}$, $i = 1, \ldots, m$, are nonempty compact sets, and $f : \mathbb{R}^n \times \mathcal{T} \to \mathbb{R}$ and $h_i : \mathbb{R}^n \times \mathcal{V}_i \to \mathbb{R}$, $i = 1, ..., m$, are functions.

To deal with the uncertainty problem (U), we associate it with the following robust counterpart:

$$\min_{x\in\mathbb{R}^n} \left\{ \max_{\tau\in\mathcal{T}} f(x,\tau) \mid x \in \Omega,\ h_i(x,u_i) \le 0,\ \forall u_i \in \mathcal{V}_i,\ i = 1, ..., m \right\}. \tag{P}$$

In what follows, we assume that the feasible set of problem (P) is nonempty and it is denoted by S, i.e.,

$$S := \{x \in \Omega \mid h_i(x,u_i) \le 0, \quad \forall u_i \in \mathcal{V}_i,\ i = 1, ..., m\}. \tag{1.1}$$

The set S is also called the robust feasible set of problem (U).

The problem of type (P) covers an important class of robust optimization programs and has been intensively investigated in the literature. For instance, when $\Omega := \mathbb{R}^n$, the related functions are SOS-convex polynomials and the uncertainty sets $\mathcal{V}_i$, $i = 1, ..., m$, are boundedly intersection ellipsoidal, Chuong and Jeyakumar in [8] presented semidefinite programming relaxations for the corresponding problem under the well-posed regularity, where there is no uncertainty in the objective function. In [14], Jeyakumar et al. provided necessary and sufficient conditions for the robust optimality to convex optimization problems under the robust Slater constraint qualification. Some results of [14] have been improved and extended by Li and Wang in [19] using the robust Farkas-Minkowski constraint qualification. Recently, Sisarat et al. in [27] characterized the optimal solution set of an uncertain convex optimization problem by using convex subdifferentials and Lagrangian multipliers for the robust convex feasible set described by locally Lipschitz constraints. It should be noted further that checking whether a problem of polynomial with only degree 4 admits its solution is strongly NP-hard [1]. The interested reader is referred to [15, 16, 28] for some recent results on the existence of solutions to vector/polynomial optimization problems.

The main aim of this paper to investigate a question: When and under which conditions does the robust optimization problem (P) admit an optimal solution? Answering this question is generally challenging because the related functions of problem (P) are not only nonsmooth and nonconvex but also depend on uncertainties. In a special setting, where the problem does not have uncertainty, f is a quadratic polynomial bounded from below on S and $h_i, i = 1, ..., m$, are affine functions, Frank and Wolfe in [12] showed that an optimal solution of the underlying program exists. More generally, if a robust SOS-convex polynomial program is well-posed and its objective is bounded from below on S, then the considered problem attains its optimal solutions (cf. [8, Proposition 2.2]). However, an answer to the above question for a general robust optimization problem like (P) is currently unavailable.

In this work, we answer the above question by presenting new results on the existence of optimal solutions to the robust optimization problem (P). To this end, we define an extended tangency variety and prove that the robust optimization problem (P) can be reformulated as an equivalent problem in terms of the extended tangency variety. Consequently, the information for the existence of optimal solutions to the robust optimization problem (P) is revealed.

Moreover, under suitable hypotheses on the related functions of problem (P) and a constraint qualification, we present relationships among the concepts of robust M-tameness, robust properness and robust Palais-Smale condition related to the uncertain problem (U) or its robust version (P). In this way, we derive necessary and sufficient conditions for the existence of optimal solutions to the robust optimization problem (P).

The rest of the paper is organized as follows. Section 2 gives some basic concepts and calculus rules needed for proving our main results. In Sect. 3, we introduce the concept of extended tangency variety and use this concept to turn the problem (P) into a new equivalent problem. In Sect. 4, we examine links among the concepts of robust properness, robust Palai-Smale condition and robust M-tameness under certain assumptions. Section 5 provides conditions for the problem (P) to have an optimal solution. The last section summarizes the obtained results.

2 Preliminaries

Throughout the paper, let $\mathbb{R}^n$ be a finite-dimentional space with the usual scalar product $\langle\cdot,\cdot\rangle$ and the Euclidean norm $\|\cdot\|$, where $n \in \mathbb{N} := \{1, 2, ...\}$. For $n = 1$, let $\mathbb{R}^1 := \mathbb{R}$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. We use $I\!B_n$ and $\mathbb{R}^n_+$ to denote the closed unit ball and the nonnegative orthant of $\mathbb{R}^n$, respectively. For a nonempty subset $\Omega \subset \mathbb{R}^n$, the closure and covex hull of Ω are denoted by $\mathrm{cl}\Omega$ and $\mathrm{co}\,\Omega$, respectively. The notation $x \xrightarrow{\Omega} \overline{x}$ means that $x \to \overline{x}$ and $x \in \Omega$.

Let $F : X \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a multi-valued function/set-valued map. F is said to be closed at $\overline{x} \in X$ if for any sequence $\{x^k\} \subset X,\ x^k \to \overline{x}$ and any sequence $\{y^k\} \subset \mathbb{R}^m,\ y^k \in F(x^k)$, $y^k \to \overline{y}$ as $k \to \infty$, we have $\overline{y} \in F(\overline{x})$.

Let us recall some concepts and calculus rules from Variational Analysis (see e.g., [21, 22]). Given a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we denote by

$$\operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \overline{x}} F(x) := \left\{\vartheta \in \mathbb{R}^n \mid \exists \text{ sequences } x^k \xrightarrow{\Omega} \overline{x} \text{ and } \vartheta^k \to \vartheta \text{ with } \vartheta^k \in F(x^k) \text{ for all } k \in \mathbb{N}\right\}$$

the sequential Painlevé–Kuratowski upper/outer limit of F as $x \to \overline{x}$.

The Fréchet normal cone (regular normal cone) $\widehat{N}(\overline{x}; \Omega)$ to Ω at $\overline{x} \in \Omega$ is defined by

$$\widehat{N}(\overline{x}; \Omega) := \left\{\vartheta \in \mathbb{R}^n \mid \limsup_{x \xrightarrow{\Omega} \overline{x}} \frac{\langle \vartheta, x - \overline{x}\rangle}{\|x - \overline{x}\|} \le 0\right\}. \tag{2.1}$$

The *limiting normal cone* (basic/Mordukhovich normal cone) $N(\overline{x}; \Omega)$ to Ω at $\overline{x} \in \Omega$ is given by

$$N(\overline{x}; \Omega) := \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \overline{x}} \widehat{N}(x; \Omega). \tag{2.2}$$

For any $\overline{x} \in \mathbb{R}^n \setminus \Omega$, we put $\widehat{N}(\overline{x}; \Omega) := \emptyset$ and $N(\overline{x}; \Omega) := \emptyset$.

The limiting/Mordukhovich subdifferential of $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$ at $\overline{x} \in \mathbb{R}^n$ with $|\psi(\overline{x})| < \infty$ is defined by

$$\partial\psi(\overline{x}) := \left\{\vartheta \in \mathbb{R}^n \mid (\vartheta, -1) \in N((\overline{x}; \psi(\overline{x})); \mathrm{epi}\,\psi)\right\},$$

where

$$\mathrm{epi}\psi := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid \psi(x) \le y\}.$$

If $|\psi(\overline{x})| = \infty$, then one puts $\partial\psi(\overline{x}) := \emptyset$.

Remark that the above-defined normal cones and subdifferentials reduce to the corresponding concepts of normal cone and subdifferential in convex analysis when sets and functions are convex. Moreover, if ψ is strictly differentiable at $\overline{x}$ then $\partial\psi(\overline{x}) = \{\nabla\psi(\overline{x})\}$, where $\nabla\psi(\overline{x})$ stands for the derivative of ψ at $\overline{x}$.

The function ψ is locally Lipschitz at $\overline{x} \in \mathbb{R}^n$ if there exist a real number $L > 0$ and a neighborhood U of $\overline{x}$ such that

$$|\psi(z) - \psi(w)| \leq L\,\|z - w\| \quad \forall z,\ w \in U.$$

Furthermore, we obtain by [21, Corollary 1.81] that $\|\vartheta\| \leq L$ for any $\vartheta \in \partial\psi(\overline{x})$, and if $\overline{x}$ is a local minimizer for ψ, then we get, see [21, Proporsition 1.114],

$$0 \in \partial\psi(\overline{x}). \tag{2.3}$$

The following lemma presents a formula in convex analysis.

Lemma 2.1 *For $\overline{x} \in \mathbb{R}^n$, one has*

$$\partial(\|\cdot - \overline{x}\|)(x) = \begin{cases} I\!B_n & if\ x = \overline{x}, \\ \dfrac{x - \overline{x}}{\|x - \overline{x}\|} & if\ x \neq \overline{x}. \end{cases}$$

A sum rule for the limiting subdifferential is as follows.

Lemma 2.2 *(See [21, Theorem 3.36]). Let the functions $\psi_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ for $i = 1, ..., m,\ m \geq 2$, be lower semicontinuous around $\overline{x} \in \mathbb{R}^n$, and let all but one of these be Lipschitz continuous around $\overline{x}$. Then it holds*

$$\partial(\psi_1 + \psi_2 + \cdots + \psi_m)(\overline{x}) \subset \partial\psi_1(\overline{x}) + \partial\psi_2(\overline{x}) + \cdots + \partial\psi_m(\overline{x}). \tag{2.4}$$

Finally in this section, we recall a necessary optimality condition for a scalar nonsmooth optimization problem.

Lemma 2.3 (See [22, Corollary 6.6]). *Let the functions $\psi_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, ..., m + p$, be Lipschitz continuous around $\overline{x} \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is locally closed around this point. If ψ_0 attains its infimum value at $\overline{x}$ on the set*

$$\{x \in \Omega \mid \psi_i(x) \leq 0,\ i = 1, ..., m,\ \psi_i(x) = 0,\ i = m + 1, ..., m + p\},$$

then, one can find $(\mu_0, ..., \mu_{m+p}) \in \mathbb{R}^{m+p+1}\backslash\{0\}$ such that $\mu_i \geq 0, i = 0, ..., m$ and

$$\begin{aligned} &0 \in \sum_{i=0}^{m} \mu_i \partial\psi_i(\overline{x}) + \sum_{i=m+1}^{m+p} \mu_i \big[\partial\psi_i(\overline{x}) \cup \big(-\partial(-\psi_i)(\overline{x})\big)\big] + N(\overline{x}; \Omega), \\ &\mu_i \psi_i(\overline{x}) = 0,\ i = 1, ..., m. \end{aligned}$$

3 Extended tangency variety in robust optimization

In this section, we introduce an extended tangency variety for the feasible point set S of the robust optimization problem (P) defined by (1.1). Then, we show that the robust optimization problem (P) can be transformed into an equivalent problem in terms of this extended tangency variety.

In the rest of the paper, for a fixed $\overline{x} \in \mathbb{R}^n$, we denote

$$F(\overline{x}) := \max_{\tau \in \mathcal{T}} f(\overline{x}, \tau), \quad H_i(\overline{x}) := \max_{u_i \in \mathcal{V}_i} h_i(\overline{x}, u_i),\ i = 1, ..., m \tag{3.1}$$

and

$$\mathcal{T}(\overline{x}) := \{\tau \in \mathcal{T} \mid f(\overline{x}, \tau) = F(\overline{x})\}, \quad \mathcal{V}_i(\overline{x}) := \{u_i \in \mathcal{V}_i \mid h_i(\overline{x}, u_i) = H_i(\overline{x})\}. \tag{3.2}$$

Furthermore, we assume that the function f together with the constraint functions $h_1, ..., h_m$ of the problem (P) satisfy the following assumptions:

(A) For a fixed $\overline{x} \in \mathbb{R}^n$, there exist neighborhoods U_i, $i = 0, ..., m$, of $\overline{x}$ such that the functions $\tau \in \mathcal{T} \mapsto f(x, \tau)$, $x \in U_0$, and $u_i \in \mathcal{V}_i \mapsto h_i(x, u_i)$, $x \in U_i$, $i = 1 \ldots, m$ are upper semicontinuous and the functions f and h_i are partially uniformly Lipschitz of ranks $L_0 > 0$ and $L_i > 0$ on U_0 and U_i, respectively, i.e.,

$$|f(z, \tau) - f(w, \tau)| \le L_0 \|z - w\| \quad \forall z,\ w \in U_0,\ \forall \tau \in \mathcal{T},$$
$$|h_i(z, u_i) - h_i(w, u_i)| \le L_i \|z - w\| \quad \forall z,\ w \in U_i,\ \forall u_i \in \mathcal{V}_i.$$

(B) For the above $\overline{x} \in \mathbb{R}^n$, the multi-valued function $(x, \tau) \in U_0 \times \mathcal{T} \rightrightarrows \partial_x f(x, \tau) \subset \mathbb{R}^n$ is closed at $(\overline{x}, \overline{\tau})$ for each $\overline{\tau} \in \mathcal{T}(\overline{x})$ and the multi-valued function $(x, u_i) \in U_i \times \mathcal{V}_i \rightrightarrows \partial_x h_i(x, u_i) \subset \mathbb{R}^n$ is closed at $(\overline{x}, \overline{u}_i)$ for each $\overline{u}_i \in \mathcal{V}_i(\overline{x})$, where ∂_x stands for the limiting subdifferential with respect to the first variable x.

We want to emphasize that the above two assumptions often appear in studying robust optimization problems or in nonsmooth analysis such as calculating the nonsmooth subdifferentials/subgradients of max or supremum functions over an infinite set. More precisely, the hypothesis (A) ensures that the functions F and H_i, $i = 1, \ldots, m$ are well-defined, and furthermore, it entails that the functions F, H_i are locally Lipschitz of ranks L_0, L_i, $i = 1, ..., m$, respectively. The hypothesis (B) can be considered as a relaxation of subdifferentials for the class of convex functions, and actually, this assumption is still valid for a broader class of regular functions, including subsmooth and continuously prox-regularity functions whenever (A) holds. The reader is referred to [5, 7] and the references therein for a detailed review.

We are now ready to introduce the concept of extended tangency variety for the feasible set S of problem (P).

Definition 3.1 The extended tangency variety for the feasible set S of problem (P) is defined by

$$\begin{aligned}\Lambda(\mathrm{P}) := \Big\{ x \in S \mid\ & \exists \mu := (\mu_0, ..., \mu_m) \in \mathbb{R}^{m+1}_+,\ \exists \lambda \in \mathbb{R},\ (\mu, \lambda) \neq 0,\\ & 0 \in \mu_0 \mathrm{co}\{\partial_x f(x, \tau) \mid \tau \in \mathcal{T}(x)\}\\ &\quad + \sum_{i=1}^{m} \mu_i \mathrm{co}\{\partial_x h_i(x, u_i) \mid u_i \in \mathcal{V}_i(x)\} + \lambda x + N(x; \Omega),\\ & \mu_i \max_{u_i \in \mathcal{V}_i} h_i(x, u_i) = 0,\ i = 1, ..., m \Big\}.\end{aligned} \tag{3.3}$$

Consider a special setting, where there is no uncertainty and there are no constraints (i.e., $S = \mathbb{R}^n$) in the considered optimization problem. In this case, if the objective function f is a non-constant polynomial, then the concept in Definition 3.1 reduces to the tangency variety of f (see [29]), which is given by

$$\Gamma(f) := \left\{ x \in \mathbb{R}^n \mid \operatorname{rank} \begin{pmatrix} \nabla f(x) \\ x \end{pmatrix} \le 1 \right\}.$$

More particularly, if f is a non-constant polynomial on $\mathbb{R}^2$, then the above-defined concept agrees with the curve of tangency (see [10]) defined by

$$\Gamma(f) := \{x := (x_1, x_2) \in \mathbb{R}^2 \mid x_2 \frac{\partial f}{\partial x_1} - x_1 \frac{\partial f}{\partial x_2} = 0\},$$

where $\frac{\partial f}{\partial x_i}$ stands for the partial derivative of f with respect to the variable x_i for $i = 1, 2$.

The first theorem in this section gives a sufficient condition for the existence of the extended tangency variety for the feasible set S of problem (P).

Theorem 3.1 *Let the assumptions (A) and (B) hold for the problem* (P)*. If S is unbounded, then Λ(P) is nonempty and unbounded.*

Proof Let S be unbounded. Fixing any $k \in \mathbb{N}$, there exists $x_k \in S$ such that $||x_k|| > k$. Put

$$S_k := \{x \in \Omega \mid \|x\|^2 = \|x_k\|^2,\ H_i(x) \leq 0,\ i = 1, ..., m\},$$

where $H_i(x) := \max_{u_i \in \mathcal{V}_i} h_i(x, u_i)$ for $i = 1, ..., m$. Note that $S_k \neq \emptyset$ due to $x_k \in S_k$. We consider an auxiliary optimization problem (P_k) as follows:

$$\min\{F(x) \mid x \in S_k\}, \tag{P_k}$$

where $F(x) := \max_{\tau \in \mathcal{T}} f(x, \tau)$. Since S_k is a nonempty compact set, the problem (P_k) exists an optimal solution, denoted by x_k^*. According to Lemma 2.3, one can find the real numbers $\mu_i \geq 0$ for $i = 0, ..., m$ and $\mu_{m+1} \in \mathbb{R}$, not all zero, such that

$$0 \in \mu_0 \partial F(x_k^*) + \sum_{i=1}^{m} \mu_i \partial H_i(x_k^*) + 2\mu_{m+1} x_k^* + N(x_k^*; \Omega), \tag{3.4}$$

$$\mu_i H_i(x_k^*) = 0,\ i = 1, ..., m. \tag{3.5}$$

Under the assumptions (A) and (B), we use some simliar arguments of [5, Theorem 3.3] to obtain that

$$\begin{aligned}
&\partial F(x_k^*) \subset \operatorname{co}\{\partial_x f(x_k^*, \tau) \mid \tau \in \mathcal{T}(x_k^*)\},\\
&\partial H_i(x_k^*) \subset \operatorname{co}\{\partial_x h_i(x_k^*, u_i) \mid u_i \in \mathcal{V}_i(x_k^*)\},\ i = 1, ..., m,
\end{aligned}$$

where $\mathcal{T}(x_k^*)$ and $\mathcal{V}_i(x_k^*)$ are defined as in (3.2). Then, we get from (3.4) and (3.5) that

$$\begin{aligned}
&0 \in \mu_0 \operatorname{co}\{\partial_x f(x_k^*, \tau) \mid \tau \in \mathcal{T}(x_k^*)\}\\
&\quad + \sum_{i=1}^{m} \mu_i \operatorname{co}\{\partial h_i(x_k^*, u_i) \mid u_i \in \mathcal{V}_i(x_k^*)\} + 2\mu_{m+1} x_k^* + N(x_k^*; \Omega),\\
&\mu_i \max_{u_i \in \mathcal{V}_i} h_i(x_k^*, u_i) = 0,\ i = 1, ..., m.
\end{aligned}$$

This shows that $x_k^* \in \Lambda$(P) for all $k \in \mathbb{N}$, and so Λ(P) is a nonempty set. Moreover, it is obvious that $\|x_k^*\| = \|x_k\| \to \infty$ as $k \to \infty$, which means that Λ(P) is unbounded. The proof of the theorem is complete. □

We now show that the optimal value of problem (P), denoted by inf (P), can be found by minimizing of its robust objective over the extended tangency variety.

Theorem 3.2 *Let the assumptions (A) and (B) hold for the problem* (P). *Then, we have*

$$\inf\,(\mathrm{P}) = \inf\{F(x) \mid x \in \Lambda(\mathrm{P})\},$$

where $F(x) := \max_{\tau \in \mathcal{T}} f(x, \tau)$ *for* $x \in \mathbb{R}^n$.

Proof It is clear that $\Lambda(\mathrm{P}) \subset S$, and one has

$$\inf\,(\mathrm{P}) = \inf\{F(x) \mid x \in S\} \leq \inf\{F(x) \mid x \in \Lambda(\mathrm{P})\}.$$

In the rest of the proof, we show that

$$\inf\,(\mathrm{P}) \geq \inf\{F(x) \mid x \in \Lambda(\mathrm{P})\}. \tag{3.6}$$

Denote $\alpha^* := \inf\,(\mathrm{P})$. Then, one finds a sequence $\{x_k\}_{k\in\mathbb{N}} \subset S$ such that $F(x_k) \to \alpha^*$ as $k \to \infty$. For each $k \in \mathbb{N}$, let

$$S_k := \{x \in \Omega \mid \|x\|^2 = \|x_k\|^2,\ H_i(x) \leq 0,\ i = 1, ..., m\},$$

and consider the optimization problem (P_k) as in the proof of Theorem 3.1. Similarly as in the proof of Theorem 3.1, for $k \in \mathbb{N}$, we find an optimal solution of problem (P_k), say x_k^*, such that $x_k^* \in \Lambda(\mathrm{P})$. Then, for each $k \in \mathbb{N}$, we get

$$\inf\{F(x) \mid x \in \Lambda(\mathrm{P})\} \leq F(x_k^*) = \min\{F(x) \mid x \in S_k\} \leq F(x_k),$$

where the last inequality holds due to $x_k \in S_k$. Hence,

$$\inf\{F(x) \mid x \in \Lambda(\mathrm{P})\} \leq F(x_k) \text{ for all } k \in \mathbb{N}. \tag{3.7}$$

Letting $k \to \infty$ in (3.7), we obtain that

$$\inf\{F(x) \mid x \in \Lambda(\mathrm{P})\} \leq \alpha^*,$$

which shows that (3.6) is valid and so the proof of the theorem is complete. □

The following example illustrates Theorem 3.2.

Example 3.1 Let $f : \mathbb{R}^2 \times \mathcal{T} \to \mathbb{R}$ and $h_1 : \mathbb{R}^2 \times \mathcal{V}_1 \to \mathbb{R}$ be given respectively by

$$f(x, \tau) := |x_1| - |x_2| + \tau - 3, \quad h_1(x, u_1) := -x_1 - x_2 - u_1^2,$$

where $x := (x_1, x_2) \in \mathbb{R}^2$, $\tau \in \mathcal{T} := [-5, 3]$ and $u_1 \in \mathcal{V}_1 := [-4, -2] \cup [-1, 3]$. Consider the following robust optimization problem

$$\min_{x\in\mathbb{R}^2} \{\max_{\tau\in\mathcal{T}} f(x, \tau) \mid x \in \Omega,\ h_1(x, u_1) \leq 0,\ \forall u_1 \in \mathcal{V}_1\}, \tag{EP1}$$

where the set Ω is define by

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 - x_2 \geq 0\}.$$

The problem (EP1) is in the form of (P), and we can verify that both assumptions (A) and (B) are satisfied for this setting.

On the one hand, we can check that the feasible set of problem (EP1) is Ω (i.e., $S = \Omega$, see Fig. 1a) and moreover, it holds that

$$\inf\,(\mathrm{EP1}) = 0. \tag{3.8}$$

One the other hand, by direct calculation, we see that, for $x \in S$,

$$\partial_x f(x, \tau) = \{(\nu, -1) \mid -1 \leq \nu \leq 1\} \cup \{(\nu, 1) \mid -1 \leq \nu \leq 1\} \text{ for } x = (0, 0),$$

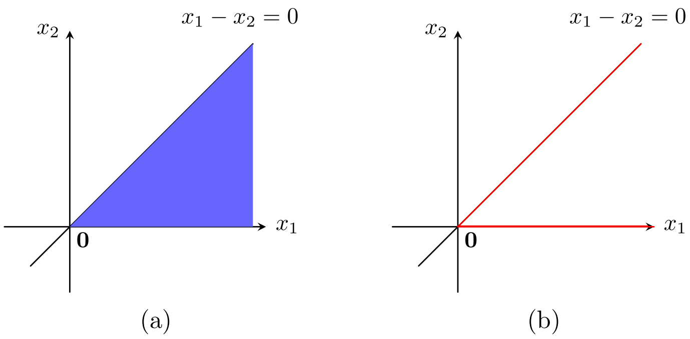


Fig. 1 (a) The feasible set S is shaded in blue, (b) The extended tangency set Λ(EP1) is in red

$$
\begin{aligned}
&\partial_x f(x,\tau) = \{(1,-1)\} \text{ for } x = (x_1, x_2) \text{ with } x_1 \geq x_2 > 0,\\
&\partial_x f(x,\tau) = \{(1,1),(1,-1)\} \text{ for } x = (x_1, 0) \text{ with } x_1 > 0,\\
&\partial_x h_1(x,u_1) = \{(-1,-1)\},\ \max_{u_1 \in \mathcal{V}_1} h_1(x,u_1) = -x_1 - x_2,\\
&N(x;\Omega) = \{(a,b) \in \mathbb{R}^2 \mid a \leq 0,\ a+b \leq 0\} \text{ for } x = (0,0),\\
&N(x;\Omega) = \{(-a,a) \in \mathbb{R}^2 \mid a \geq 0\} \text{ for } x = (x_1,x_2) \text{ with } x_1 = x_2 > 0,\\
&N(x;\Omega) = \{(0,a) \in \mathbb{R}^2 \mid a \leq 0\} \text{ for } x = (x_1,0) \text{ with } x_1 > 0,\\
&N(x;\Omega) = \{(0,0)\} \text{ for } x = (x_1,x_2) \in \operatorname{int} S,
\end{aligned}
$$

where $\tau \in \mathcal{T}(x) = \{3\}$ and $u_1 \in \mathcal{V}_1(x) = \{0\}$. Then, the extended tangency variety set (see Fig. 1b) is given by

$$
\begin{aligned}
\Lambda(\text{EP1}) = \Big\{ x \in S \mid\ & \exists(\mu_0,\mu_1) \in \mathbb{R}^2_+,\ \exists\lambda \in \mathbb{R},\ (\mu_0,\mu_1,\lambda) \neq 0,\ 0 \in \mu_0 \text{co}\{\partial_x f(x,\tau) \mid \tau \in \mathcal{T}(x)\}\\
& + \mu_1 \text{co}\{\partial_x h_1(x,u_1) \mid u_1 \in \mathcal{V}_1(x)\} + \lambda x + N(x;\Omega),\ \mu_1 \max_{u_1 \in \mathcal{V}_1} h_1(x,u_1) = 0 \Big\}\\
= \Big\{ & (x_1,x_1) \in \mathbb{R}^2 \mid x_1 \geq 0 \Big\} \cup \Big\{ (x_1,0) \in \mathbb{R}^2 \mid x_1 > 0 \Big\},
\end{aligned}
$$

and so

$$
\inf\{F(x) \mid x \in \Lambda(\text{EP1})\} = 0, \tag{3.9}
$$

where $F(x) := \max_{\tau \in \mathcal{T}} f(x,\tau) = |x_1| - |x_2|$ for $x := (x_1,x_2) \in \mathbb{R}^2$. From (3.8) and (3.9), we see that the conclusion of Theorem 3.2 holds for this setting. In fact, we can verify further that any $x^* := (a,a)$ with $a \geq 0$ is an optimal solution of problem (EP1) and the problem in the left-hand side of (3.9).

4 Asymptotic conditions for robust optimization problems

This section is devoted to introducing and establishing relations between the notions of asymptotic robust conditions including robust properness, robust M-tameness and robust Palais-Smale condition for the uncertain optimization problem (U). These asymptotic robust conditions provide sufficient criteria that guarantee the solution existence of the robust optimization problem (P) studied in the next section.

Definition 4.1 Consider the uncertain optimization problem (U) with its robust feasible set S defined by (1.1).

(i) The problem (U) is called robust proper at a sublevel $\overline{y} \in \overline{\mathbb{R}}$ if

$$\left[\forall \{x_k\}_{k\in\mathbb{N}} \subset S,\ \|x_k\| \to \infty,\ F(x_k) \leq \overline{y},\ k \in \mathbb{N}\right] \Rightarrow \left[\,|F(x_k)| \to \infty \text{ as } k \to \infty\right],$$

where $F(x) := \max_{\tau\in\mathcal{T}} f(x,\tau)$ for $x \in \mathbb{R}^n$.

(ii) The problem (U) is called robust proper if

$$\left[\forall \{x_k\}_{k\in\mathbb{N}} \subset S,\ \|x_k\| \to \infty\right] \Rightarrow \left[\,|F(x_k)| \to \infty \text{ as } k \to \infty\right].$$

(iii) The problem (U) is called robust coercive if

$$\left[\forall \{x_k\}_{k\in\mathbb{N}} \subset S,\ \|x_k\| \to \infty\right] \Rightarrow \left[\,F(x_k) \to +\infty \text{ as } k \to \infty\right].$$

Remark 4.1 (i) The above asymptotic robust concepts can be viewed as extensions from the coercivity/properness of functions on a set, which provide sufficient conditions for an unconstrained optimization problem to attain its optimal value, see e.g., [24, 25, 30].

(ii) If the problem (U) is robust coercive, then it is robust proper, and these properties are the same whenever the optimal value of problem (P) is finite.

The following simple example illustrates a difference.

Example 4.1 Let $f : \mathbb{R} \times \mathcal{T} \to \mathbb{R}$ and $h_1 : \mathbb{R} \times \mathcal{V}_1 \to \mathbb{R}$ be given respectively by

$$f(x,\tau) := x^3 + \tau,\ h_1(x,u_1) := -x^2 - u_1^2,\ x \in \mathbb{R},\ \tau \in \mathcal{T},\ u_1 \in \mathcal{V}_1,$$

where $\mathcal{T} := [-3, 5]$ and $\mathcal{V}_1 := [-5, 5]$. Let $\Omega := \mathbb{R}$ and consider an uncertain optimization problem of the form (U) as

$$\min_{x\in\mathbb{R}}\{f(x,\tau) \mid x \in \Omega,\ h_1(x,u_1) \leq 0\}, \tag{EU2}$$

where $\tau \in \mathcal{T}$ and $u_1 \in \mathcal{V}_1$ are uncertain. In this setting, we have $F(x) := \max_{\tau\in\mathcal{T}} f(x,\tau) = x^3 + 5$ for $x \in \mathbb{R}$ and the robust feasible set of (EU2) is $\mathbb{R}$, i.e., $S := \{x \in \Omega \mid h_1(x,u_1) \leq 0,\ \forall u_1 \in \mathcal{V}_1\} = \mathbb{R}$. Taking any sequence $\{x_k\}_{k\in\mathbb{N}} \subset S$ satisfying $|x_k| \to \infty$, one has $|F(x_k)| \to \infty$ as $k \to \infty$. This means that the problem (EU2) is robust proper. However, it is not robust coercive as $F(x_k) \to -\infty$ with $x_k := -k$ when $k \to \infty$.

To introduce asymptotic robust properties for the uncertain optimization problem (U), we consider an extended Rabier function $\mathcal{R} : S \to \overline{\mathbb{R}}$ for the robust problem (P) by

$$\mathcal{R}(x) := \inf\Big\{\|x^*\| \mid x^* \in \operatorname{co}\{\partial_x f(x,\tau) \mid \tau \in \mathcal{T}(x)\} + \sum_{i=1}^{m} \mu_i \operatorname{co}\{\partial_x h_i(x,u_i) \mid u_i \in \mathcal{V}_i(x)\}$$
$$+ N(x;\Omega),\ \mu_i \geq 0,\ \mu_i \max_{u_i\in\mathcal{V}_i} h_i(x,u_i) = 0,\ i = 1,\dots,m\Big\},\ x \in S, \tag{4.1}$$

and define sets of asymptotic values at a sublevel $\overline{y} \in \overline{\mathbb{R}}$ for the robust problem (P) as follows:

$$\widetilde{K}_{\overline{y}}^{\infty}(\text{P}) := \{y \in \mathbb{R} \mid \exists \{x_k\}_{k\in\mathbb{N}} \subset S,\ \|x_k\| \to \infty,\ F(x_k) \leq \overline{y},\ k \in \mathbb{N},$$
$$F(x_k) \to y,\ \mathcal{R}(x_k) \to 0 \text{ as } k \to \infty\},$$
$$K_{\overline{y}}^{\infty}(\text{P}) := \{y \in \mathbb{R} \mid \exists \{x_k\}_{k\in\mathbb{N}} \subset S,\ \|x_k\| \to \infty,\ F(x_k) \leq \overline{y},\ k \in \mathbb{N},$$
$$F(x_k) \to y,\ \|x_k\|\mathcal{R}(x_k) \to 0 \text{ as } k \to \infty\},$$
$$T_{\overline{y}}^{\infty}(\text{P}) := \{y \in \mathbb{R} \mid \exists \{x_k\}_{k\in\mathbb{N}} \subset \Lambda(\text{P}),\ \|x_k\| \to \infty,\ F(x_k) \leq \overline{y},\ k \in \mathbb{N},$$

$$F(x_k) \to y \text{ as } k \to \infty\},$$

where Λ(P) is defined by (3.3). If $\overline{y} = +\infty$, the above notations can be simply denoted respectively by $\widetilde{K}$(P), K(P) and T(P).

Definition 4.2 Consider the uncertain optimization problem (U) with its robust feasible set S defined by (1.1).

(i) The problem (U) is called to satisfy the robust Palais-Smale condition at a sublevel $\overline{y} \in \overline{\mathbb{R}}$ if $\widetilde{K}^{\infty}_{\overline{y}}$(P) $= \emptyset$.
(ii) The problem (U) is called to satisfy the weak robust Palais-Smale condition at a sublevel $\overline{y} \in \overline{\mathbb{R}}$ if $K^{\infty}_{\overline{y}}$(P) $= \emptyset$.
(iii) The problem (U) is called to satisfy the robust M-tame condition at a sublevel $\overline{y} \in \overline{\mathbb{R}}$ if $T^{\infty}_{\overline{y}}$(P) $= \emptyset$.

Remark 4.2 (i) The above-defined concepts can be regarded as variants and extensions of a so-called compactness condition/Palais-Smale condition introduced in [23] (see also, [20]) for the smooth setting, which is stated that, for a differentiable real-valued function $\psi : X \to \mathbb{R}$, if for a sequence $\{x_k\}_{k\in\mathbb{N}} \subset X$, $\psi(x_k)$ is bounded and $\|\nabla\psi(x_k)\| \to 0$ as $k \to \infty$, then $\{x_k\}_{k\in\mathbb{N}}$ contains a convergent subsequence.

(ii) It is clear that $K^{\infty}_{\overline{y}}$(P) $\subset \widetilde{K}^{\infty}_{\overline{y}}$(P) for $\overline{y} \in \overline{\mathbb{R}}$. For a special class of polynomial problems, where there are no uncertainty and constraint functions, i.e., $S = \Omega = \mathbb{R}^n$, we have inclusions

$$T^{\infty}_{\overline{y}}(\text{P}) \subset K^{\infty}_{\overline{y}}(\text{P}) \subset \widetilde{K}^{\infty}_{\overline{y}}(\text{P}), \tag{4.2}$$

and moreover these inclusions may be strict, see [15] for more details. However, the first inclusion in (4.2) (i.e., $T^{\infty}_{\overline{y}}$(P) $\subset K^{\infty}_{\overline{y}}$(P)) might not be valid for the robust problem (P) in general as the following example shows.

Example 4.2 Let $f : \mathbb{R}^3 \times \mathcal{T} \to \mathbb{R}$ and $h_i : \mathbb{R}^3 \times \mathcal{V}_i \to \mathbb{R}$, $i = 1, 2$ be given respectively by

$$f(x, \tau) := \sin(x_1 + x_2 + x_3) + \tau - 4,$$
$$h_1(x, u_1) := u_1(x_2^2 + x_3^2),\ \ h_2(x, u_2) := x_3^3 - |u_2| + 3,$$

where $x := (x_1, x_2, x_3) \in \mathbb{R}^3$, $\tau \in \mathcal{T} := [-2, 4]$, $u_1 \in \mathcal{V}_1 := [1, 5]$ and $u_2 \in \mathcal{V}_2 := [-9, -7] \cup [6, 9]$. Consider a robust optimization problem in the form of (P) as

$$\min_{x\in\mathbb{R}^3}\{\max_{\tau\in\mathcal{T}} f(x, \tau) \mid x \in \Omega,\ h_i(x, u_1) \leq 0,\ \forall u_i \in \mathcal{V}_i,\ i = 1, 2\}, \tag{EP3}$$

where $\Omega := \mathbb{R}^3$. In this setting, the feasible set S of (EP3) is computed by

$$S := \{x \in \mathbb{R}^3 \mid x \in \Omega,\ h_i(x, u_i) \leq 0,\ \forall u_i \in \mathcal{V}_i,\ \ i = 1, 2\} = \{(x_1, 0, 0) \in \mathbb{R}^3 \ \mid\ x_1 \in \mathbb{R}\},$$

and for each $x := (x_1, 0, 0) \in S$, $\tau \in \mathcal{T}(x) = \{4\}$, $u_1 \in \mathcal{V}_1(x) = \{5\}$ and $u_2 \in \mathcal{V}_2(x) = \{6\}$, we have

$$\nabla_x f(x, \tau) = (\cos x_1, \cos x_1, \cos x_1),\ \ \nabla_x h_1(x, u_1) = (0, 0, 0),$$
$$\nabla_x h_2(x, u_2) = (0, 0, 0),\ \ N(x; \Omega) = \{(0, 0, 0)\},$$

where $\nabla_x h(\overline{x}, \overline{v})$ stands for the derivative of h with respect to the first variable x at a given point $(\overline{x}, \overline{v})$. The extended tangency variety set Λ(EP3) is given by

$$\Lambda(\text{EP3}) = \Big\{x \in S \ \mid\ \exists\mu := (\mu_0, \mu_1, \mu_2) \in \mathbb{R}^3_+,\ \exists\lambda \in \mathbb{R},\ (\mu, \lambda) \neq 0,\ \ 0 \in \mu_0\nabla_x f(x, \tau)$$

$$+\sum_{i=1}^{2}\mu_i\nabla_x h_i(x,u_i)+\lambda x+N(x;\Omega),\ \tau\in\mathcal{T}(x),\ u_i\in\mathcal{V}_i(x),\ i=1,2,$$
$$\mu_i\max_{u_i\in\mathcal{V}_i}h_i(x,u_i)=0,\ i=1,2\big\}=S,$$

and the extended Rabier function $\mathcal{R}:S\to\overline{\mathbb{R}}$ for the problem (EP3) is given by

$$\mathcal{R}(x):=\inf\big\{\|x^*\|\ \mid\ x^*\in\nabla_x f(x,\tau)+\sum_{i=1}^{2}\mu_i\nabla_x h_i(x,u_i)+N(x;\Omega),\ \tau\in\mathcal{T}(x),$$
$$u_i\in\mathcal{V}_i(x),\ \mu_i\geq 0,\ \mu_i\max_{u_i\in\mathcal{V}_i}h_i(x,u_i)=0,\ i=1,2\big\}=\sqrt{3\cos^2 x_1},\ x\in S.$$

Let $\overline{y}:=0\in\mathbb{R}$. Taking a sequence $\{x_k\}_{k\in\mathbb{N}}\subset\Lambda$(EP3) with $x_k:=(2k\pi,0,0)$ for $k\in\mathbb{N}$, we see that $F(x_k)=0\leq\overline{y}$ for $k\in\mathbb{N}$, where $F(x):=\max_{\tau\in\mathcal{T}}f(x,\tau)=\sin(x_1+x_2+x_3)$ and $\|x_k\|\to\infty$ and $F(x_k)\to 0$ as $k\to\infty$. Hence, $0\in T^{\infty}_{\overline{y}}$(EP3).

We now show that $0\notin K^{\infty}_{\overline{y}}$(EP3). Assume the contrary that $0\in K^{\infty}_{\overline{y}}$(EP3). Then, there exists a sequence $\{\widetilde{x}_k\}_{k\in\mathbb{N}}\subset S$, where $\widetilde{x}_k:=(\widetilde{x}_{1k},0,0)$, such that

$$\|\widetilde{x}_k\|\to\infty,\ F(\widetilde{x}_k)\leq 0,\ F(\widetilde{x}_k)=\sin\widetilde{x}_{1k}\to 0 \text{ and } \|\widetilde{x}_k\|\mathcal{R}(\widetilde{x}_k)\to 0 \text{ as } k\to\infty,$$

where $\mathcal{R}(\widetilde{x}_k)=\sqrt{3\cos^2\widetilde{x}_{1k}}$ for $k\in\mathbb{N}$. By taking a subsequence if necessary, we conclude from $\|\widetilde{x}_k\|\mathcal{R}(\widetilde{x}_k)\to 0$ that $\cos^2\widetilde{x}_{1k}\to 0$ as $k\to\infty$. Then, $1=\sin^2\widetilde{x}_{1k}+\cos^2\widetilde{x}_{1k}\to 0$ as $k\to\infty$, which is impossible. Consequently, $0\notin K^{\infty}_{\overline{y}}$(EP3). So we conclude that

$$T^{\infty}_{\overline{y}}(\text{EP3})\not\subset K^{\infty}_{\overline{y}}(\text{EP3}).$$

It is also worth mentioning here that if the problem (U) is robust proper, then we have

$$\widetilde{K}^{\infty}_{\overline{y}}(\text{P})=K^{\infty}_{\overline{y}}(\text{P})=T^{\infty}_{\overline{y}}(\text{P})=\emptyset \text{ for } \overline{y}\in\overline{\mathbb{R}}. \tag{4.3}$$

The following example shows that (4.3) is not a sufficient condition for an uncertain optimization problem to have the robust properness.

Example 4.3 Let $f:\mathbb{R}^2\times\mathcal{T}\to\mathbb{R}$ and $h_1:\mathbb{R}^2\times\mathcal{V}_1\to\mathbb{R}$ be given respectively by

$$f(x_1,x_2):=x_1-x_2+\tau,\ \ h_1(x,u_1):=u_1x_1^2x_2^2-1,\ \ x:=(x_1,x_2)\in\mathbb{R}^2,\ \tau\in\mathcal{T},\ u_1\in\mathcal{V}_1,$$

where $\mathcal{T}:=[-5,0]$ and $\mathcal{V}_1:=[-4,-1]$. Let $\Omega:=\mathbb{R}^2$ and consider an uncertain optimization problem in the form of (U) as follows:

$$\min_{x\in\mathbb{R}^2}\{f(x,\tau)\ \mid\ x\in\Omega,\ h_1(x,u_1)\leq 0\}, \tag{EU4}$$

where $\tau\in\mathcal{T}$ and $u_1\in\mathcal{V}_1$ are uncertain. The robust counterpart of (EU4) can be captured by

$$\min_{x\in\mathbb{R}^2}\{\max_{\tau\in\mathcal{T}}f(x,\tau)\ \mid\ x\in\Omega,\ h_1(x,u_1)\leq 0,\ \forall u_1\in\mathcal{V}_1\}. \tag{EP4}$$

In this case, we see that the feasible set S of (EP4) is defined by

$$S:=\{x\in\mathbb{R}^2\mid x\in\Omega,\ h_1(x,u_1)\leq 0,\ \ \forall u_1\in\mathcal{V}_1\}=\mathbb{R}^2$$

and the extended tangency variety Λ(EP4) is computed by

$$\Lambda(\text{EU4}):=\big\{x\in S\ \mid\ \exists(\mu_0,\mu_1)\in\mathbb{R}^2_+,\ \exists\lambda\in\mathbb{R},\ (\mu_0,\mu_1,\lambda)\neq 0,\ 0\in\mu_0\nabla_x f(x,\tau)$$
$$+\mu_1\mathrm{co}\{\nabla_x h_1(x,u_1)\mid u_1\in\mathcal{V}_1(x)\}+\lambda x+N(x;\Omega),\ \ \tau\in\mathcal{T}(x),\ \mu_1\max_{u_1\in\mathcal{V}_1}h_1(x,u_1)=0\big\}$$

$= \{(x_1, -x_1) \mid x_1 \in \mathbb{R}\}.$

Similarly, we can calculate the extended Rabier function $\mathcal{R} : S \to \overline{\mathbb{R}}$ for the problem (EP4), which is given by $\mathcal{R}(x) = \sqrt{2}$ for $x \in S$.

Now, take any $\overline{y} \in \overline{\mathbb{R}}$. For any sequence $\{x_k\}_{k\in\mathbb{N}} \subset S$, it holds that $\mathcal{R}(x_k) = \sqrt{2}$ for all $k \in \mathbb{N}$, and so $\widetilde{K}^{\infty}_{\overline{y}}$(EP4) $= K^{\infty}_{\overline{y}}$(EP4) $= \emptyset$. Choosing any sequence $\{x_k\}_{k\in\mathbb{N}} \subset \Lambda$(EP4) satisfying $\|x_k\| \to \infty$ as $k \to \infty$, it holds that $x_k := (x_{1k}, -x_{1k})$ with $x_{1k} \in \mathbb{R}$ for all $k \in \mathbb{N}$ and $|x_{1k}| \to \infty$ as $k \to \infty$. This entails that the sequence $F(x_{1k}) = 2x_{1k}$ does not converge to some $y \in \mathbb{R}$, where $F(x) := \max_{\tau\in\mathcal{T}} f(x, \tau) = x_1 - x_2$ for $x := (x_1, x_2) \in \mathbb{R}^2$. This means that $T^{\infty}_{\overline{y}}$(EP4) $= \emptyset$, and consequently,

$$\widetilde{K}^{\infty}_{\overline{y}}(\text{EP4}) = K^{\infty}_{\overline{y}}(\text{EP4}) = T^{\infty}_{\overline{y}}(\text{EP4}) = \emptyset.$$

However, the problem (EU4) is not robust proper. To see this, just take a sequence $\{x_k\}_{k\in\mathbb{N}} \subset S$ with $x_k := (k, k)$ for $k \in \mathbb{N}$. It is clear that $\|x_k\| \to \infty$ as $k \to \infty$ and $F(x_k) = 0$ for all $k \in \mathbb{N}$.

To proceed, we state the robust qualification conditions for the uncertain optimization problem (U).

Definition 4.3 Consider the uncertain optimization problem (U) with its robust feasible set S defined by (1.1).

(i) The problem (U) is called to satisfy the robust qualification (RQ) at $x \in S$ if there does not exist $(\mu_1, ..., \mu_m) \in \mathbb{R}^m_+ \setminus \{0\}$ such that

$$0 \in \sum_{i=1}^{m} \mu_i \text{co}\{\partial_x h_i(x, u_i) \mid u_i \in \mathcal{V}_i(x)\} + N(x; \Omega). \tag{4.4}$$

(ii) The problem (U) is called to satisfy the robust qualification (RQ) if it satisfies the (RQ) at every $x \in S$.

(iii) The problem (U) is called to satisfy the robust qualification at infinity $(\text{RQ})_\infty$ if there exists a real number $r > 0$ such that (U) satisfies the (RQ) at any $x \in S$ with $\|x\| \geq r$.

Remark 4.3 (i) In the above definition, the concept of (RQ) can be viewed as a development of the classical Mangasarian-Fromovitz constraint qualification in the smooth setting (see, e.g. [4, 5, 21, 22] for more details), while the concept of $(\text{RQ})_\infty$ reduces to a so-called regularity at infinity $(\text{CQ})_\infty$ in [30], which was stated for a setting of polynomial functions.

(ii) Obviously, the (RQ) implies the $(\text{RQ})_\infty$. However, the inverted statement is not true in general as the following example illustrates.

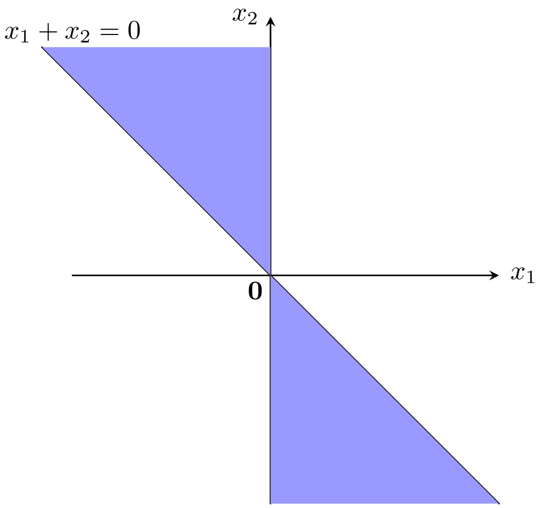
Example 4.4 Let $h_i : \mathbb{R}^2 \times \mathcal{V}_i \to \mathbb{R},\ i = 1, 2$ be given respectively by

$$h_1(x, u_1) := 2(x_1^2 - x_2^2) + u_1,\ h_2(x, u_2) := x_1x_2 + u_2 - 3,$$
$$x := (x_1, x_2) \in \mathbb{R}^2,\ u_1 \in \mathcal{V}_1,\ u_2 \in \mathcal{V}_2,$$

where $\mathcal{V}_1 := [-5, 0]$ and $\mathcal{V}_2 := [-3, 3]$. Let $f : \mathbb{R}^2 \times \mathcal{T} \to \mathbb{R}$ be a function and $\mathcal{T} \subset \mathbb{R}$ be a nonempty compact set. We consider an uncertain optimization problem of the form (U) as

$$\min_{x\in\mathbb{R}^2} \{f(x, \tau) \mid x \in \Omega,\ h_i(x, u_i) \leq 0,\ i = 1, 2\}, \tag{EU5}$$

Fig. 2 The robust feasible set S of (EU5) is in blue



where $\Omega := \mathbb{R}^2$ and $\tau \in \mathcal{T}$ and $u_i \in \mathcal{V}_i$, $i = 1, 2$ are uncertain parameters.

In this setting, we see that the robust feasible set S of (EU5) is computed by

$$S := \{x \in \Omega \mid h_i(x, u_i) \le 0, \ \forall u_i \in \mathcal{V}_i, \ i = 1, 2\} = \{x \in \mathbb{R}^2 \mid |x_1| \le |x_2|, \ x_1 x_2 \le 0\},$$

which is depicted in Fig. 2.

Moreover, it can be verified that the problem (EU5) does not satisfy the (RQ) as it does not satisfy the (RQ) at $\overline{x} := (0, 0) \in S$. However, for any $r > 0$, the problem (EU5) satisfies the (RQ) at any $x \in S$ with $\|x\| \ge r$. This means that (EU5) satisfies the $(\mathrm{RQ})_\infty$.

We are now ready to describe relationships among the asymptotic robust properties of the uncertain optimization problem (U).

Theorem 4.1 *Let the assumptions (A) and (B) hold for the problem* (P) *with* inf (P) $> -\infty$. *Assume that the problem* (U) *satisfies the* $(RQ)_\infty$. *Then, for any feasible point* $\overline{x}$ *of* (P)*, the following statements are equivalent to each other:*

- *(i) The problem* (U) *is robust proper at a sublevel* $F(\overline{x})$*, where* $F(x) := \max_{\tau \in \mathcal{T}} f(x, \tau)$ *for* $x \in \mathbb{R}^n$.
- *(ii) The problem* (U) *satisfies the robust Palais-Smale condition at a sublevel* $F(\overline{x})$.
- *(iii) The problem* (U) *satisfies the weak robust Palais-Smale condition at a sublevel* $F(\overline{x})$.
- *(iv) The problem* (U) *is robust M-tame at a sublevel* $F(\overline{x})$.

Proof We exploit some techniques from the proof of [16, Theorem 3.1]. Observe first by definition that the assertions (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) are straightforward. To finish the proof, it remains to show that (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (i): Let the problem (U) satisfy the weak robust Palais-Smale condition at $F(\overline{x})$, where $\overline{x}$ is a feasible point of problem (P), i.e., $\overline{x} \in S$. Suppose for the contradiction that (U) is not robust proper at the sublevel $F(\overline{x})$. This means that one finds a sequence $\{x_k\}_{k \in \mathbb{N}} \subset S$ satisfying

$$F(x_k) \le F(\overline{x}), \ \forall k \in \mathbb{N}, \ \|x_k\| \to \infty \text{ as } k \to \infty, \tag{4.5}$$

and the sequence $\{|F(x_k)|\}_{k\in\mathbb{N}}$ is bounded. Put

$$X := \{x \in \Omega \mid F(x) - F(\overline{x}) \le 0,\ H_i(x) \le 0,\ i = 1, \ldots, m\}, \tag{4.6}$$

where $F(x) := \max_{\tau\in\mathcal{T}} f(x,\tau)$ and $H_i(x) := \max_{u_i\in\mathcal{V}_i} h_i(x,u_i),\ i = 1, ..., m$ for $x \in \mathbb{R}^n$. Clearly, X is unbounded as X contains the sequence $\{x_k\}_{k\in\mathbb{N}}$ in (4.5).

Since inf (P) $> -\infty$, it follows that $\liminf_{x\in X,\ \|x\|\to\infty} F(x)$ exists and is finite, and so we denote

$$\liminf_{x\in X,\ \|x\|\to\infty} F(x) := \omega \in \mathbb{R}. \tag{4.7}$$

Consider a function $\theta : [0, +\infty) \to \mathbb{R}$ by

$$\theta(r) := \inf_{x\in X,\ \|x\|\ge r} F(x),\ \ r \in [0, +\infty).$$

We see that the function θ is nondecreasing and that $\theta(r) \to \omega$ as $r \to \infty$. This means that for each $k \in \mathbb{N}$, there exists $r_k > \|x_k\|$ such that

$$-\frac{1}{k} < \theta(r) - \omega < \frac{1}{k} \ \text{ for all } \ r \ge r_k. \tag{4.8}$$

Fix $k \in \mathbb{N}$. By the definition of $\theta(3r_k)$, there exists $\widetilde{x}_k \in X$ with $\|\widetilde{x}_k\| \ge 3r_k$ and $\theta(3r_k) > F(\widetilde{x}_k) - \frac{1}{k}$. This, together with (4.8), entails that

$$F(\widetilde{x}_k) < \theta(3r_k) + \frac{1}{k} \le \omega + \frac{2}{k} < \theta(r_k) + \frac{3}{k} \tag{4.9}$$

and so

$$F(\widetilde{x}_k) < \inf_{x\in S_k} F(x) + \epsilon_k,$$

where $S_k := \{x \in X \mid \|x\| \ge r_k\}$ and $\epsilon_k := \frac{3}{k}$. Note that $\widetilde{x}_k \in S_k$ and F is bounded from below on S_k. We apply the Ekeland variational principle (see [11] and also, [21, Theorem 2.26]) to F on S_k and arrive at an assertion that for $\lambda_k := \frac{\|\widetilde{x}_k\|}{2}$, there exists $v_k \in S_k$ such that the following conditions hold:

(a) $F(v_k) \le F(\widetilde{x}_k)$,

(b) $\|v_k - \widetilde{x}_k\| \le \lambda_k$,

(c) $F(v_k) < F(x) + \frac{\epsilon_k}{\lambda_k}\|x - v_k\|$ for all $x \in S_k \setminus \{v_k\}$.

Note that $\theta(r_k) \le F(v_k)$ because of $v_k \in S_k$. Granting this, we derive from (a) and (4.9) that

$$\theta(r_k) \le F(v_k) < \theta(r_k) + \epsilon_k,$$

which shows that $F(v_k) \to \omega$ as $k \to \infty$. Keeping in mind $\|\widetilde{x}_k\| \ge 3r_k$, we derive from (b) that

$$r_k < \frac{\|\widetilde{x}_k\|}{2} \le \|v_k\| \le \frac{3\|\widetilde{x}_k\|}{2},$$

where $r_k > \|x_k\|$. Since $\|x_k\| \to \infty$ as $k \to \infty$, it holds that $\|v_k\| \to \infty$ as $k \to \infty$.

From (c), it shows that v_k is a minimizer of $\psi := F + \frac{\epsilon_k}{\lambda_k}\|\cdot - v_k\|$ on the set S_k, and so v_k is a minimizer of ψ on the set E_k, where E_k is given by

$$E_k := \{x \in \Omega \mid r_k - \|x\| \leq 0,\ H_i(x) \leq 0,\ i = 1, \ldots, m\}.$$

By Lemma 2.3, there exists $(\mu_0, \ldots, \mu_{m+1}) \in \mathbb{R}^{m+2}_+ \setminus \{0\}$ such that

$$\begin{aligned}
&0 \in \mu_0 \partial\psi(v_k) + \sum_{i=1}^{m} \mu_i \partial H_i(v_k) + \mu_{m+1}\partial(r_k - \|\cdot\|)(v_k) + N(v_k; \Omega), \\
&\mu_i H_i(v_k) = 0,\ i = 1, \ldots, m,\ \mu_{m+1}(r_k - ||v_k||) = 0.
\end{aligned} \tag{4.10}$$

Note that $r_k < \|v_k\|$ and so we conclude by the last equation in (4.10) that $\mu_{m+1} = 0$. Moreover, by Lemma 2.1 and Lemma 2.2, it holds that

$$\partial\psi(v_k) \subset \partial F(v_k) + \frac{\epsilon_k}{\lambda_k} I\!B_n,$$

and under the hypotheses (A) and (B), we use similar arguments of [5, Theorem 3.3] to obtain that

$$\begin{aligned}
&\partial F(v_k) \subset \text{co}\{\partial_x f(v_k, \tau) \mid \tau \in \mathcal{T}(v_k)\}, \\
&\partial H_i(v_k) \subset \text{co}\{\partial_x h_i(v_k, u_i) \mid u_i \in \mathcal{V}_i(v_k)\} \text{ for } i = 1, \ldots, m.
\end{aligned}$$

Therefore, we arrive at

$$0 \in \mu_0 \text{co}\{\partial_x f(v_k, \tau) \mid \tau \in \mathcal{T}(v_k)\} + \sum_{i=1}^{m} \mu_i \text{co}\{\partial_x h_i(v_k, u_i) \mid u_i \in \mathcal{V}_i(v_k)\} + N(v_k; \Omega) \tag{4.11}$$

$$+ \mu_0 \frac{\epsilon_k}{\lambda_k} I\!B_n,\ \ \mu_i \max_{u_i \in \mathcal{V}_i} h_i(v_k, u_i) = 0,\ i = 1, \ldots, m. \tag{4.12}$$

Since the problem (U) satisfies the $(\text{RQ})_\infty$, it holds that $\mu_0 > 0$. Denoting $\widetilde{\mu}_i := \frac{\mu_i}{\mu_0} \geq 0,\ i = 1, \ldots, m$, we derive from (4.11) and (4.12) that

$$\begin{aligned}
0 \in\ &\text{co}\{\partial_x f(v_k, \tau) \mid \tau \in \mathcal{T}(v_k)\} \\
&+ \sum_{i=1}^{m} \widetilde{\mu}_i \text{co}\{\partial_x h_i(v_k, u_i) \mid u_i \in \mathcal{V}_i(v_k)\} + N(v_k; \Omega) + \frac{\epsilon_k}{\lambda_k} I\!B_n, \qquad (4.13)\\
&\widetilde{\mu}_i \max_{u_i \in \mathcal{V}_i} h_i(v_k, u_i) = 0,\ i = 1, \ldots, m. \qquad (4.14)
\end{aligned}$$

We assert by (4.13) that there exists

$$x^* \in \text{co}\{\partial_x f(v_k, \tau) \mid \tau \in \mathcal{T}(v_k)\} + \sum_{i=1}^{m} \widetilde{\mu}_i \text{co}\{\partial_x h_i(v_k, u_i) \mid u_i \in \mathcal{V}_i(v_k)\} + N(v_k; \Omega)$$

such that $\|x^*\| \leq \frac{\epsilon_k}{\lambda_k}$ and thus,

$$\mathcal{R}(v_k) \leq \frac{\epsilon_k}{\lambda_k} \leq \frac{9}{k\|v_k\|},$$

which shows that $\|v_k\|\mathcal{R}(v_k) \to 0$ as $k \to \infty$.

Overall, we find a sequence $\{v_k\}_{k\in\mathbb{N}} \subset S$, $\|v_k\| \to \infty$, $F(v_k) \le F(\overline{x})$, $F(v_k) \to \omega$ and $\|v_k\|\mathcal{R}(v_k) \to 0$ as $k \to \infty$. This means that $\omega \in K^{\infty}_{F(\overline{x})}$(P), which results in a contradiction to $K^{\infty}_{F(\overline{x})}$(P) $= \emptyset$. So the implication (iii) $\Rightarrow$ (i) has been justified.

(iv) $\Rightarrow$ (i): Let the problem (U) be robust M-tame at $F(\overline{x})$, where $\overline{x}$ is a feasible point of problem (P), i.e., $\overline{x} \in S$. Suppose on the contrary that (U) is not robust proper at the sublevel $F(\overline{x})$. Then, as shown in the proof of (iii) $\Rightarrow$ (i), there exists a sequence $\{v_k\}_{k\in\mathbb{N}} \subset S$ such that $F(v_k) \le F(\overline{x})$ for all $k \in \mathbb{N}$, $\|v_k\| \to \infty$ and $F(v_k) \to \omega$ as $k \to \infty$, where $\omega := \liminf\limits_{x\in X,\, \|x\|\to\infty} F(x)$ is given as in (4.7).

Let $k \in \mathbb{N}$ and define the set $\widetilde{S}_k := \{x \in X \mid \|x\|^2 = \|v_k\|^2\}$, where X is given as in (4.6). Note that $\widetilde{S}_k$ is a nonempty set because of $v_k \in \widetilde{S}_k$. Moreover, since F is continuous and $\widetilde{S}_k$ is compact, we find $x_k^* \in \widetilde{S}_k$ such that

$$F(x_k^*) \le F(x) \text{ for all } x \in \widetilde{S}_k. \tag{4.15}$$

This entails that x_k^* is an optimal solution of the following problem:

$$\min\{F(x) \mid x \in \Omega,\ \|x\|^2 - \|v_k\|^2 = 0,\ H_i(x) \le 0,\ i = 1, \dots, m\}.$$

Using Lemma 2.3, we find $(\mu_0, ..., \mu_{m+1}) \in \mathbb{R}^{m+2}\backslash\{0\}$ with $\mu_i \ge 0$, $i = 0, ..., m$ such that

$$\begin{aligned}
&0 \in \mu_0 \partial F(x_k^*) + \sum_{i=1}^{m} \mu_i \partial H_i(x_k^*) + 2\mu_{m+1} x_k^* + N(x_k^*; \Omega),\\
&\mu_i H_i(x_k^*) = 0,\ i = 1, ..., m.
\end{aligned}$$

As above, under the hypotheses (A) and (B), it holds that

$$\partial F(x_k^*) \subset \text{co}\{\partial_x f(x_k^*, \tau) \mid \tau \in \mathcal{T}(x_k^*)\},\ \partial H_i(x_k^*) \subset \text{co}\{\partial_x h_i(x_k^*, u_i) \mid u_i \in \mathcal{V}_i(x_k^*)\}.$$

Consequently, we arrive at

$$\begin{aligned}
&0 \in \mu_0 \text{co}\{\partial_x f(x_k^*, \tau) \mid \tau \in \mathcal{T}(x_k^*)\}\\
&\quad + \sum_{i=1}^{m} \mu_i \text{co}\{\partial h_i(x_k^*, u_i) \mid u_i \in \mathcal{V}_i(x_k^*)\} + 2\mu_{m+1} x_k^* + N(x_k^*; \Omega),\\
&\mu_i \max_{u_i \in \mathcal{V}_i} h_i(x_k^*, u_i) = 0,\ i = 1, ..., m,
\end{aligned}$$

which shows that $x_k^* \in \Lambda$(P).

Consequently, we find a sequence $\{x_k^*\}_{k\in\mathbb{N}} \subset \Lambda$(P) $\subset X$ such that

$$F(x_k^*) \le F(\overline{x}) \text{ for all } k \in \mathbb{N}$$

$\|x_k^*\| = \|v_k\| \to \infty$ as $k \to \infty$. On the one side, by taking (4.7) into account, we see that $\omega \le \liminf\limits_{k\to\infty} F(x_k^*)$. Besides this, it follows by (4.15) that $F(x_k^*) \le F(v_k)$ for all $k \in \mathbb{N}$, which yields $\limsup\limits_{k\to\infty} F(x_k^*) \le \omega$ inasmuch as $F(v_k) \to \omega$ as $k \to \infty$. Therefore, we assert that $F(x_k^*) \to \omega$ as $k \to \infty$.

In conclusion, $\omega \in T^{\infty}_{F(\overline{x})}$(P), which results in a contradiction due to $T^{\infty}_{F(\overline{x})}$(P) $= \emptyset$. So the implication (iv) $\Rightarrow$ (i) has been justified, which completes the proof of the theorem. $\square$

5 Solution existence for robust optimization problems

In this section, based on the asymptotic robust conditions, we establish necessary and sufficient conditions for the solution existence of the robust optimization problem (P).

Assume that $\overline{x}$ is a feasible point of the robust optimization problem (P) and put

$$C_{F(\overline{x})}(\mathrm{P}) := \{F(x) \in \mathbb{R} \mid x \in S,\ F(x) \leq F(\overline{x}),\ \mathcal{R}(x) = 0\},$$

where $F(x) := \max_{\tau \in \mathcal{T}} f(x, \tau)$, S is the feasible set of (P) defined by (1.1) and $\mathcal{R}(x)$ is the extended Rabier function given by (4.1).

Let us start by providing sufficient conditions in terms of asymptotic robust properties that guarantee the solution existence for the robust optimization problem (P).

Theorem 5.1 *Let the assumptions (A) and (B) hold for the problem* (P) *with* $\alpha^* := \inf$ (P) $> -\infty$ *and let* $\overline{x}$ *be a feasible point of* (P). *Assume that the problem* (U) *satisfies the* $(RQ)_\infty$. *Then, the problem* (P) *admits an optimal solution if one of the following conditions holds:*

(i) The problem (U) *is robust coercive.*
(ii) The problem (U) *is robust proper at a sublevel* $F(\overline{x})$.
(iii) The problem (U) *holds for the robust Palais-Smale condition at a sublevel* $F(\overline{x})$.
(iv) The problem (U) *holds for the weak robust Palais-Smale condition at a sublevel* $F(\overline{x})$.
(v) The problem (U) *is the robust M-tame at a sublevel* $F(\overline{x})$.

Proof Since (i) implies (ii) and under the current assumptions, we assert by Theorem 4.1 that the assertions in (ii)-(v) are equivalent. Therefore, it suffices to prove that the problem (P) has an optimal solution whenever (ii) holds. To do so, we assume that the problem (U) is robust proper at the sublevel $F(\overline{x})$. Let us consider a set X given by

$$X := \{x \in \Omega \mid F(x) - F(\overline{x}) \leq 0,\ H_i(x) \leq 0,\ i = 1, \ldots, m\},$$

where $F(x) := \max_{\tau \in \mathcal{T}} f(x, \tau)$ and $H_i(x) := \max_{u_i \in \mathcal{V}_i} h_i(x, u_i)$, $i = 1, \ldots, m$ for $x \in \mathbb{R}^n$. It should be noted that $X \subset S$ and X is a nonempty set because of $\overline{x} \in X$. If X is unbounded, then there exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset S$ such that

$$F(x_k) \leq F(\overline{x}),\ \forall k \in \mathbb{N},\ \|x_k\| \to \infty \text{ as } k \to \infty.$$

However, we have $\alpha^* \leq F(x_k) \leq F(\overline{x})$ for every $k \in \mathbb{N}$, which means that $\{|F(x_k)|\}_{k\in\mathbb{N}}$ is bounded. This leads to a contradiction to the assumption that the problem (U) is robust proper at the sublevel $F(\overline{x})$. Consequently, X is bounded and hence compact as it is closed. Moreover, F is continuous. Thus, there is $x^* \in X$ such $F(x^*) \leq F(x)$ for all $x \in X$. This entails that $x^* \in S$ and $F(x^*) \leq F(x)$ for all $x \in S$, i.e., the problem (P) has an optimal solution x^*. □

The following example illustrates that the asymptotic robust conditions in Theorem 5.1 can be used to justify the solution existence for a robust optimization problem.

Example 5.1 Let $f : \mathbb{R}^2 \times \mathcal{T} \to \mathbb{R}$ and $h_i : \mathbb{R}^2 \times \mathcal{V}_i \to \mathbb{R},\ i = 1, 2$ be defined respectively by

$$f(x, \tau) := 2\tau|x_1| + |x_2|,\ h_1(x, u_1) := |x_1| - |x_2| + u_1,\ \tau \in \mathcal{T},$$
$$h_2(x, u_2) := x_1x_2 + u_2,\ x := (x_1, x_2) \in \mathbb{R}^2,\ u_i \in \mathcal{V}_i,\ i = 1, 2,$$

where $\mathcal{T} := [0, 1]$, $\mathcal{V}_1 := [-1, 0]$ and $\mathcal{V}_2 := [-3, 0]$. Let $\Omega := \mathbb{R}^2$ and consider an uncertain optimization problem in the form of (U) as follows:

$$\min_{x\in\mathbb{R}^2}\{f(x,\tau) \mid x\in\Omega,\ h_i(x,u_i)\le 0,\ i=1,2\}, \tag{EU6}$$

where $\tau \in \mathcal{T}$ and $u_i \in \mathcal{V}_i$, $i = 1, 2$ are uncertain. The robust counterpart of (EU6) can be captured by

$$\min_{x\in\mathbb{R}^2}\left\{\max_{\tau\in\mathcal{T}} f(x,\tau) \mid x\in\Omega,\ h_i(x,u_i)\le 0,\ \forall u_i\in\mathcal{V}_i,\ i=1,2\right\}. \tag{EP6}$$

We can check that the assumptions (A) and (B) hold for the problem (EP6) and the feasible set S of (EP6) is given by

$$S := \{x\in\Omega \mid h_i(x,u_i)\le 0,\ \forall u_i\in\mathcal{V}_i,\ i=1,2\} = \{x\in\mathbb{R}^2 \mid |x_1|\le|x_2|,\ x_1x_2\le 0\},$$

which is depicted in Fig. 2. Moreover, we can verify that $\alpha^* := \inf$ (EP6) ≥ 0 and the problem (EU6) satisfies the (RQ) at every point $x \in S\backslash\{(0, 0)\}$ and so it satisfies the $(\mathrm{RQ})_\infty$.

Take any sequence $\{x_k\}_{k\in\mathbb{N}} \subset S$ such that $\|x_k\| \to \infty$ as $k \to \infty$. We see that $F(x_k) \to +\infty$ as $k \to \infty$, because $F(x) := \max_{\tau\in\mathcal{T}} f(x, \tau) = 2|x_1| + |x_2|$ for $x := (x_1, x_2) \in \mathbb{R}^2$. So the problem (EU6) is robust coercive. Invoking Theorem 5.1, we conclude that the problem (EP6) admits an optimal solution. In fact, we can check that $x^* := (0, 0)$ is an optimal solution of problem (EP6).

The next theorem provides characterizations of the solution existence for the robust optimization problem (P).

Theorem 5.2 *Let the assumptions (A) and (B) hold for the problem* (P) *and denote* $\alpha^* := \inf$ (P). *Assume that the problem* (U) *satisfies the (RQ). Then, the following statements are equivalent to each other:*

(i) The problem (P) *admits an optimal solution.*
(ii) There exists a feasible point $\overline{x}$ *of* (P) *such that* $\widetilde{K}^{\infty}_{F(\overline{x})}(\mathrm{P}) \subset C_{F(\overline{x})}(\mathrm{P})$ *and* $\alpha^* > -\infty$.
(iii) There exists a feasible point $\overline{x}$ *of* (P) *such that* $K^{\infty}_{F(\overline{x})}(\mathrm{P}) \subset C_{F(\overline{x})}(\mathrm{P})$ *and* $\alpha^* > -\infty$.
(iv) There exists a feasible point $\overline{x}$ *of* (P) *such that* $T^{\infty}_{F(\overline{x})}(\mathrm{P}) \subset C_{F(\overline{x})}(\mathrm{P})$ *and* $\alpha^* > -\infty$.

Proof We first show that (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iv). To do this, assume that (i) holds. This means that the problem (P) admits an optimal solution, say $\overline{x}$. This, in particular, ensures that $\alpha^* = F(\overline{x}) > -\infty$.

To justify (ii), we need to show that

$$\widetilde{K}^{\infty}_{F(\overline{x})}(\mathrm{P}) \subset C_{F(\overline{x})}(\mathrm{P}). \tag{5.1}$$

This inclusion is obvious if $\widetilde{K}^{\infty}_{F(\overline{x})}(\mathrm{P}) = \emptyset$. Now, take any $y \in \widetilde{K}^{\infty}_{F(\overline{x})}(\mathrm{P})$. Then, there exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset S$ such that $\|x_k\| \to \infty$, $F(x_k) \to y$, $\mathcal{R}(x_k) \to 0$ as $k \to \infty$ and

$$F(x_k) \le F(\overline{x}) \ \text{ for all } \ k \in \mathbb{N}.$$

This entails that $F(x_k) = F(\overline{x})$ for all $k \in \mathbb{N}$ because $\overline{x}$ is an optimal solution of (P). Since $F(x_k) \to y$ as $k \to \infty$, we conclude that $y = F(\overline{x})$. By Lemma 2.3, there exists $(\mu_0, ..., \mu_m) \in \mathbb{R}^{m+1}_+\backslash\{0\}$ such that

$$0 \in \mu_0\partial F(\overline{x}) + \sum_{i=1}^{m} \mu_i\partial H_i(\overline{x}) + N(\overline{x}; \Omega),$$

$$\mu_i H_i(\overline{x}) = 0,\ i = 1, ..., m,$$

where $F(x) := \max_{\tau \in \mathcal{T}} f(x, \tau)$ and $H_i(x) := \max_{u_i \in \mathcal{V}_i} h_i(x, u_i)$ for $x \in \mathbb{R}^n$. Under the hypotheses (A) and (B), we get by similar arguments as in the proof of Theorem 3.1 that

$$\partial F(\overline{x}) \subset \text{co}\{\partial_x f(\overline{x}, \tau) \mid \tau \in \mathcal{T}(\overline{x})\},\ \partial H_i(\overline{x}) \subset \text{co}\{\partial_x h_i(\overline{x}, u_i) \mid u_i \in \mathcal{V}_i(\overline{x})\},\ i = 1, ..., m.$$

Therefore, we arrive at

$$\begin{aligned}
&0 \in \mu_0 \text{co}\{\partial_x f(\overline{x}, \tau) \mid \tau \in \mathcal{T}(\overline{x})\} + \sum_{i=1}^{m} \mu_i \text{co}\{\partial_x h_i(\overline{x}, u_i) \mid u_i \in \mathcal{V}_i(\overline{x})\} + N(\overline{x}; \Omega), \\
&\mu_i \max_{u_i \in \mathcal{V}_i} h_i(\overline{x}, u_i) = 0,\ i = 1, ..., m.
\end{aligned} \tag{5.2}$$

Since the problem (U) satisfies the (RQ), it guarantees that $\mu_0 > 0$ and so we may assume without loss of generality that $\mu_0 = 1$. By the definition of the extended Rabier function in (4.1), we get by (5.2) that $\mathcal{R}(\overline{x}) = 0$. Therefore, $y = F(\overline{x}) \in C_{F(\overline{x})}(\text{P})$, which shows that the inclusion (5.1) is valid and so (ii) holds. Similarly, we can verify that $T^{\infty}_{F(\overline{x})}(\text{P}) \subset C_{F(\overline{x})}(\text{P})$ and so (iv) holds as well.

By definition, $K^{\infty}_{F(\overline{x})}(\text{P}) \subset \widetilde{K}^{\infty}_{F(\overline{x})}(\text{P})$, we see that (ii) $\Rightarrow$ (iii) holds.

(iii) $\Rightarrow$ (i): Assume that (iii) holds, i.e., there exists a feasible point $\overline{x}$ of (P) such that $K^{\infty}_{F(\overline{x})}(\text{P}) \subset C_{F(\overline{x})}(\text{P})$ and $\alpha^* > -\infty$. To show that (i) holds, one considers the set X given by

$$X := \{x \in \Omega \mid F(x) - F(\overline{x}) \leq 0,\ H_i(x) \leq 0,\ i = 1, \ldots, m\}. \tag{5.3}$$

It should be noted that $X \subset S$ and it is a closed nonempty set because of $\overline{x} \in X$. If X is bounded and hence compact, then there is $x^* \in X$ such $F(x^*) \leq F(x)$ for all $x \in X$. This entails that $x^* \in S$ and $F(x^*) \leq F(x)$ for all $x \in S$, i.e., the problem (P) has an optimal solution x^*.

Otherwise, X is unbounded. Then, there exists a sequence $\{x_k\}_{k \in \mathbb{N}} \subset S$ such that

$$F(x_k) \leq F(\overline{x}),\ \forall k \in \mathbb{N},\ \|x_k\| \to \infty \text{ as } k \to \infty. \tag{5.4}$$

Since $\alpha^* > -\infty$, it follows that $\liminf_{x \in X,\ \|x\| \to \infty} F(x)$ exists and is finite, denoted by ω as in (4.7). Note that the problem (U) satisfies the (RQ) and thus it satisfies the $(\text{RQ})_\infty$. Following similar arguments as in the proof of (iii) $\Rightarrow$ (i) in Theorem 4.1, we obtain $\omega \in K^{\infty}_{F(\overline{x})}(\text{P})$, which shows that $\omega \in C_{F(\overline{x})}(\text{P})$. Thus, there exists $x^0 \in S$ such that $F(x^0) \leq F(\overline{x})$ and $\omega = F(x^0)$. As $\alpha^* \leq \omega$, we have the following possibilities:

Case 1: $\alpha^* = \omega$. This shows that $\alpha^* = F(x^0)$ with $x^0 \in S$ and so the problem (P) has an optimal solution x^0.

Case 2: $\alpha^* < \omega$. Take a number $q \in \mathbb{R}$ such that $\alpha^* < q < \omega$ and considering the set

$$X_q := \{x \in \Omega \mid F(x) - q \leq 0,\ H_i(x) \leq 0,\ i = 1, \ldots, m\}.$$

By the definition of $\alpha^* := \inf_{x \in S} F(x)$, for $\epsilon := q - \alpha^* > 0$, there exists $x_q \in S$ such that $F(x_q) < \alpha^* + \epsilon = q$ and so $x_q \in X_q$. Moreover, we can verify that the nonempty set X_q is compact. Hence, there exists $x^1 \in X_q$ such that $F(x^1) \leq F(x)$ for all $x \in X_q$. This in turn implies that $F(x^1) \leq F(x)$ for all $x \in S$ as $X_q \subset S$, i.e., the problem (P) has an optimal solution x^1. Consequently, in all cases, (i) holds.

To finish the proof of the theorem, we need to justify that (iv) $\Rightarrow$ (i) holds. Assume that there exists a feasible point $\overline{x}$ of (P) such that $T^{\infty}_{F(\overline{x})}(\text{P}) \subset C_{F(\overline{x})}(\text{P})$ and $\alpha^* > -\infty$. Consider the nonempty set X given as in (5.3). If X is bounded, then the problem (P) has an optimal solution x^* as shown in the proof of (iii) $\Rightarrow$ (i). In the case, where X is unbounded, we have the existence of ω in (4.7) and we can find a sequence $\{x_k\}_{k\in\mathbb{N}} \subset S$ such that (5.4) holds.

With the help of the (RQ), we use similar arguments as in the proof of (iv) $\Rightarrow$ (i) of Theorem 4.1 to come to an assertion that $\omega \in T^{\infty}_{F(\overline{x})}(\mathrm{P})$ and so $\omega \in C_{F(\overline{x})}(\mathrm{P})$. Now, the existence of an optimal solution for the problem (P) is guaranteed by considering Cases 1 and 2 as in the proof of (iii) $\Rightarrow$ (i). So the proof of the theorem is complete. $\square$

The following example shows how we can use our characterizations in Theorem 5.2 to determine if a robust optimization problem has an optimal solution or not.

Example 5.2 Let $f : \mathbb{R}^2 \times \mathcal{T} \to \mathbb{R}$ and $h_i : \mathbb{R}^2 \times \mathcal{V}_i \to \mathbb{R},\ i = 1, 2, 3$ be defined respectively by

$$\begin{aligned}
&f(x,\tau) := (1 - x_1x_2)^2 + |x_2| + \tau,\\
&h_1(x,u_1) := -x_1 - |x_2| + u_1,\ h_2(x,u_2) := -x_1x_2^2 + x_2 - u_2,\\
&h_3(x,u_3) := -x_1 - x_2 - 2u_3,\ x := (x_1,x_2) \in \mathbb{R}^2,\ \tau \in \mathcal{T},\ u_i \in \mathcal{V}_i,\ i = 1,2,3,
\end{aligned}$$

where $\mathcal{T} := [-5, 0],\ \mathcal{V}_1 := [-2, 0],\ \mathcal{V}_2 := [0, 5]$ and $\mathcal{V}_3 := [0, 5]$. Consider an uncertain optimization problem of the form (U) as

$$\min_{x\in\mathbb{R}^2}\{f(x,\tau) \mid x \in \Omega,\ h_i(x,u_i) \le 0,\ i = 1,2,3\}, \tag{EU7}$$

where $\Omega := \mathbb{R}^2$ and $\tau \in \mathcal{T}$ and $u_i \in \mathcal{V}_i,\ i = 1, 2, 3$ are uncertain parameters, and its robust counterpart as

$$\min_{x\in\mathbb{R}^2}\left\{\max_{\tau\in\mathcal{T}} f(x,\tau) \mid x \in \Omega,\ h_i(x,u_i) \le 0,\ \forall u_i \in \mathcal{V}_i,\ i = 1,2,3\right\}. \tag{EP7}$$

In this setting, we can verify that the assumptions (A) and (B) are satisfied, and the feasible set of (EP7) is calculated by

$$\begin{aligned}
S &:= \{x \in \Omega \mid h_i(x,u_i) \le 0,\ \forall u_i \in \mathcal{V}_i,\ i = 1,2,3\}\\
&= \{x \in \mathbb{R}^2 \mid x_2 > 0,\ 1 - x_1x_2 \le 0\} \cup \{x \in \mathbb{R}^2 \mid x_1 + x_2 \ge 0,\ x_2 \le 0\}.
\end{aligned}$$

For each $x := (x_1, x_2) \in S$, by direct calculation, we obtain

$$\begin{aligned}
&\partial_x f(x,\tau) = (2x_1x_2^2 - 2x_2, 2x_1^2x_2 - 2x_1 + 1) \text{ if } x_2 > 0,\ \tau \in \mathcal{T}(x),\\
&\partial_x h_1(x,u_1) = \{(-1,1),(-1,-1)\} \text{ if } x_2 = 0,\\
&\partial_x h_1(x,u_1) = (-1,-1) \text{ if } x_2 > 0,\ \partial_x h_1(x,u_1) = (-1,1) \text{ if } x_2 < 0,\\
&\partial_x h_2(x,u_2) = (-x_2^2, 1 - 2x_1x_2),\ \partial_x h_3(x,u_3) = (-1,-1),\\
&u_1 \in \mathcal{V}_1(x),\ u_2 \in \mathcal{V}_2(x),\ u_3 \in \mathcal{V}_3(x),\ N(x;\Omega) = \{(0,0)\},
\end{aligned}$$

where $\mathcal{T}(x) = \mathcal{V}_1(x) = \mathcal{V}_2(x) = \mathcal{V}_3(x) = \{0\}$. Then, we can verify that the problem (EU7) satisfies the (RQ).

Take any feasible point $\overline{x}$ of problem (EP7), i.e., $\overline{x} \in S$. We justify that

$$\widetilde{K}^{\infty}_{F(\overline{x})}(\mathrm{EP7}) \not\subset C_{F(\overline{x})}(\mathrm{EP7}). \tag{5.5}$$

To see this, observe first that $F(x) := \max_{\tau\in\mathcal{T}} f(x,\tau) = (1 - x_1x_2)^2 + |x_2| > 0$ for all $x \in S$. This entails that $0 \notin C_{F(\overline{x})}(\mathrm{EP7})$. Now, take a sequence $\{x_k\}_{k\in\mathbb{N}}$ with $x_k := \left(\dfrac{k}{F(\overline{x})}, \dfrac{F(\overline{x})}{k}\right)$.

Then, it holds that $\{x_k\}_{k\in\mathbb{N}} \subset S$, $F(x_k) = \dfrac{F(\overline{x})}{k} \leq F(\overline{x})$ for all $k \in \mathbb{N}$ and $\|x_k\| \to \infty$ and $F(x_k) \to 0$ as $k \to \infty$. For each $k \in \mathbb{N}$, we have

$$
\begin{aligned}
\mathcal{R}(x_k) :=\ & \inf\Big\{\|x^*\| \ \mid\ x^* \in\ \mathrm{co}\{\partial_x f(x_k,\tau)\,|\,\tau \in \mathcal{T}(x_k)\} \\
&+\sum_{i=1}^{3}\mu_i \mathrm{co}\{\partial_x h_i(x_k,u_i)\mid u_i \in V_i(x_k)\} \\
&\ + N(x_k;S),\ \mu_i \geq 0,\ \mu_i \max_{u_i\in\mathcal{V}_i} h_i(x_k,u_i) = 0,\ i=1,2,3\Big\} \\
&= \inf\Big\{\|x^*\| \ \mid\ x^* \in (0,1) + \mu_1(-1,-1) + \mu_2\Big(-\frac{F^2(\overline{x})}{k^2}, -1\Big) + \mu_3(-1,-1),\ \ \mu_i \geq 0, \\
&\mu_i \max_{u_i\in\mathcal{V}_i} h_i(x_k,u_i) = 0,\ i=1,2,3\Big\}.
\end{aligned}
$$

Note by $\mu_i \max_{u_i\in\mathcal{V}_i} h_i(x_k,u_i) = 0$, $i = 1,2,3$ that $\mu_1 = \mu_3 = 0$. Hence, we assert that $\mathcal{R}(x_k) \leq \dfrac{F^2(\overline{x})}{k^2}$, which ensures that $\mathcal{R}(x_k) \to 0$ as $k \to \infty$. Hence, $0 \in \widetilde{K}^{\infty}_{F(\overline{x})}$(EP7) and consequently, (5.5) holds.

By (ii) of Theorem 5.2, we conclude that the problem (EP7) does not have an optimal solution.

6 Conclusions

In this paper, we have examined the existence of global optimal solutions for nonconvex and nonsmooth robust optimization problems. Our approach is to first introduce a concept called extended tangency variety and then show how a robust optimization problem with a constraint set can be transformed into a minimizing problem of the extended tangency variety. We have employed the extended tangency variety together with a constraint qualification condition and the boundedness of the objective function to establish relationships among the notions of robust properness, robust M-tamesness and robust Palais-Smale condition related to the considered problem. The obtained results are then employed to derive necessary and sufficient conditions for the existence of global optimal solutions to the underlying robust optimization problem.

It would be interesting to see how we can develop numerical schemes to verify criteria that guarantee the solution existence or optimal solutions for the underlying robust optimization problem. Moreover, analyzing and developing the obtained results to investigate the existence of optimal solutions for a more general class of robust vector/set-valued optimization problems is worth further study.

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