and

$$\begin{aligned} v(iX, jT) &= R_1^{-1} M_1^T x_d(iX, jT) + u_d(iX, jT) \\ &= [\,1.008 \times 10^{-3} \quad 0.026 \times 10^{-3}\,] x_d(iX, jT) \\ &\quad + u_d(iX, jT). \end{aligned} \tag{36}$$

The state responses of the designed sampled-data system and the optimal control input $u_d(iX, jT)$ solved from (36) are shown in Figs. 3 and 4, respectively. We observe that the values of $x_d^h(iX, jT)$ and $x_d^v(iX, jT)$ approach zero rapidly. Finally, the minimal cost function can be obtained from (32)–(34) as $J^*_{X=0.1,\,T=0.1} = 0.7348$.

Whenever, the relative difference between two consecutive cost functions is smaller than some acceptable tolerance error for different sampling intervals, one can regard the digital controller as the acceptable continuous-time controller. To show the viewpoint of Remark 1, some sampling intervals versus their minimal cost functions are given in Table I.

V. Conclusions

This brief presents a novel approach to design an optimal digital regulator for continuous-time two-dimensional (2-D) systems described by linear partial differential equations (PDEs). The basic idea is to convert a system of PDEs into the linear 2-D state-space form with both horizontal and vertical states. By gridding the finite space-time domain of interest and assuming piecewise-constant control input over a each gridded rectangular zone, the equivalent discrete version of this linear continuous-time 2-D state-space model results in a Roesser model. To solve the optimal digital regulator for the discrete-time equivalent system described by Roesser model, the paper transforms the 2-D model into an equivalent 1-D model, which is in the descriptor form. With this 1-D descriptor state space model, we are able to apply Bellman's principle of optimality from the concept dynamic programming to derive the optimal control law for the 2-D system. Also, whenever the sampling time intervals are sufficiently small enough, it almost preserves the identical responses between the discretized quadratic optimal controlled system and the well-designed continuous-time system. The proposed approach in this paper is able to achieve the goal of preserving the original system performance in the optimally controlled hybrid 2-D systems.

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Robust Filtering for Uncertain Linear Systems With Delayed States and Outputs

Zidong Wang and Fuwen Yang

***Abstract*—This brief deals with the robust filtering problem for uncertain linear systems with delayed states and outputs. Both time-invariant and time-varying cases are considered. For the time-invariant case, an algebraic Riccati matrix inequality approach is proposed to design a robust H_∞ filter such that the filtering process remains asymptotically stable for all admissible uncertainties, and the transfer function from the disturbance inputs to error state outputs satisfies the prespecified H_∞ norm upper bound constraint. We establish the conditions under which the desired robust H_∞ filters exist, and derive the explicit expression of these filters. For the time-varying case, we develop a differential Riccati inequality method to design the robust filters. A numerical example is provided to demonstrate the validity of the proposed design approach.**

***Index Terms*—Differential Riccati inequality, H_∞ filtering, parameter uncertainty, quadratic matrix inequality, robust filtering, time-delay systems.**

I. Introduction

One of the problems with optimal Kalman filters, which has now been well recognized, is that they can be sensitive to the system data and the spectral densities of noise processes, or in other words, they may lack robustness [1]. Therefore, in the past decade, a number of papers have attempted to develop robust filters that are capable of guaranteeing satisfactory estimation in the presence of modeling errors and unknown signal statistics.

Concerning the energy bounded deterministic noise inputs, the H_∞ filtering theory has been developed which provides a bound for the worst-case estimation error without the need for knowledge of noise statistics [7], [14]. It has been demonstrated by means of examples that H_∞ filtering has the advantages of being less sensitive than Kalman filtering to uncertainties of the underlying systems, see e.g., [15]. Furthermore, the robust H_∞ filtering problem has recently received considerable attention. The aim of this problem is to pursue the enforcement of the upper bound constraint on the H_∞ norm where the system

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is affected by parameter uncertainties. A lot of papers have appeared on this topic, see e.g., [2], [6].

In the case when there exist plant parameter uncertainties and the disturbance inputs are assumed as zero mean white noises, the study of the so-called cost guaranteed filters has recently gained growing interest. The main idea is to minimize an easy-to-compute upper bound on the worst performance. A lot of results have been obtained on such a robust H_2 or H_2/H_∞ filtering problem, and the corresponding applications in signal processing have also been reported, see [3], [4], [13], [15]–[17], [20].

On the other hand, in addition to the system uncertainties, it is well known that the time delay is also often the main cause of instability and poor performance of systems [9]. In the past few decades increased attention has been devoted to the problem of robust stability and stabilization of linear systems with delayed state and parameter uncertainty, see [10] for a survey. However, the "dual" filter/observer design problems of uncertain time-delay systems have received *much less* attention although they are important in control design and signal processing applications. In [18], the robust H_∞ observer design problem has been studied for *discrete* time-delay systems. Very recently, Pila et al. [11] have considered the problem of H_∞ filtering for linear time-varying system with time-delay measurements, but the system uncertainty has not been taken into account. So far, the robust H_∞ filtering problem for uncertain continuous-time systems with time-delays in *both* state and output equations has not been fully investigated and remains to be important and challenging.

In this brief, we are concerned with the robust filtering problem for uncertain linear system with delayed states and outputs. Both time-invariant and time-varying cases are considered. For the time-invariant case, an algebraic Riccati matrix inequality approach is proposed to design a robust H_∞ filter such that the filtering process remains asymptotically stable for all admissible uncertainties, and the transfer function from the disturbance inputs to error state outputs satisfies the prespecified H_∞ norm upper bound constraint. We establish the conditions under which the desired robust H_∞ filters exist, and derive the explicit expression of these filters. For the time-varying case, we develop a differential Riccati inequality method to design the robust filters. A numerical example is provided to demonstrate the validity of the proposed design approach.

II. Problem Formulation for the Time-Invariant Case

Consider a linear uncertain continuous time-delay system described by

$$\dot{x}(t) = (A+\Delta A)x(t) + (A_d+\Delta A_d)x(t-h) + D_1 w(t) \quad (1)$$
$$x(t) = \phi(t), \qquad t\in[-h,0] \quad (2)$$
$$y(t) = (C+\Delta C)x(t) + (C_d+\Delta C_d)x(t-h) + D_2 w(t) \quad (3)$$

where $x(t)\in R^n$ is the state, $w(t)\in R^r$ is a square integrable exogenous disturbance, $y(t)\in R^m$ is the measurement. A, A_d, C, C_d, D_1, D_2 are known constant matrices with appropriate dimensions, h denotes the unknown state delay, $\phi(t)$ is a continuous vector valued initial function. $\Delta A, \Delta A_d, \Delta C, \Delta C_d$ are real valued constant matrices representing norm-bounded parameter uncertainties and satisfy

$$\begin{bmatrix}\Delta A & \Delta A_d\\ \Delta C & \Delta C_d\end{bmatrix} = \begin{bmatrix} M_1\\ M_2\end{bmatrix} F[N_1 \quad N_2] \quad (4)$$

where $F\in R^{i\times j}$ is a real uncertain time-invariant matrix and meets $FF^T\le I$, and M_1, M_2, N_1, N_2 are known matrices with appropriate dimensions.

Assumption 1: The system matrix A is asymptotically stable.

Assumption 2: The matrix D_2 is of full row rank.

In this brief, the full order linear filter is of the form

$$\dot{\hat{x}}(t) = G\hat{x}(t) + Ky(t) \quad (5)$$

where $\hat{x}$ is the state estimate, and the constant matrices G and K are filter parameters to be designed.

Define the error estimate as $e(t)=x(t)-\hat{x}(t)$. It follows from (1)–(3), and (5) that

$$\begin{aligned}\dot{e}(t) &= Ge(t) + [(A+\Delta A) - K(C+\Delta C) - G]x(t)\\ &\quad + [(A_d+\Delta A_d) - K(C_d+\Delta C_d)]\\ &\quad \times x(t-h) + (D_1 - KD_2)w(t). \end{aligned} \quad (6)$$

Let $z(t)=Le(t)$ represent the output error state where L is a known constant matrix. We now give the following definitions:

$$\begin{aligned} x_f(t) &:= \begin{bmatrix} x(t)\\ e(t)\end{bmatrix}\\ A_{df} &:= \begin{bmatrix} A_d & 0\\ A_d - KC_d & 0\end{bmatrix}\\ A_f &:= \begin{bmatrix} A & 0\\ A-KC-G & G\end{bmatrix} \quad (7)\\ D_f &:= \begin{bmatrix} D_1\\ D_1 - KD_2\end{bmatrix}\\ M_f &:= \begin{bmatrix} M_1\\ M_1-KM_2\end{bmatrix} \quad (8)\\ N_f &:= [N_1 \quad 0]\\ \Delta A_f &:= M_f F N_f\\ M_{df} &:= M_f \quad (9)\\ N_{df} &:= [N_2 \quad 0]\\ \Delta A_{df} &:= M_{df} F N_{df}\\ C_f &:= [0 \quad L]. \quad (10)\end{aligned}$$

Combining (1)–(3), (4) and (6), we obtain the following augmented system:

$$\begin{aligned}\dot{x}_f &= (A_f+\Delta A_f)x_f(t) + (A_{df}\\ &\quad + \Delta A_{df})x_f(t-h) + D_f w(t) \quad (11)\\ z(t) &= C_f x_f(t). \quad (12)\end{aligned}$$

The transfer function from the disturbance $w(t)$ to the error state output $C_f x_f(t)$ is given by

$$H_{zw}(s) = C_f(sI - (A_f+\Delta A_f) - (A_{df}+\Delta A_{df})e^{-sh})^{-1}D_f. \quad (13)$$

Our goal is to design the filter parameters, G and K, such that for all admissible parameter uncertainties $\Delta A, \Delta A_d, \Delta C, \Delta C_d$, the augmented system (11)–(12) is asymptotically stable and the following specified H_∞-norm upper bound constraint $\|H_{zw}(s)\|_\infty \le \gamma$ is simultaneously guaranteed, independent of the unknown time delay h, where $\|H_{zw}(s)\|_\infty := \sup_{\omega\in R}\sigma_{\max}[H_{zw}(j\omega)]$ and $\sigma_{\max}[\cdot]$ denotes the largest singular value of $[\cdot]$; and $\gamma<1$ is a given positive constant.

III. Main Results for Time-Invariant Case

The following lemmas play a crucial role in designing a desired robust H_∞ filter for the uncertain linear time-delay system (1)–(3).

Lemma 1 [19]: For an arbitrary positive scalar $\varepsilon_1>0$ and a positive definite matrix $P>0$, we have

$$(\Delta A_f)^T P + P(\Delta A_f) \le \varepsilon_1 P M_f M_f^T P + \varepsilon_1^{-1} N_f^T N_f.$$

Lemma 2 [19]: Let a positive scalar $\varepsilon_2 > 0$ and a positive definite matrix $Q > 0$ be such that $N_{df} Q^{-1} N_{df}^T < \varepsilon_2 I$. Then

$$(A_{df} + \Delta A_{df}) Q^{-1} (A_{df} + \Delta A_{df})^T \leq A_{df} \left(Q - \varepsilon_2^{-1} N_{df}^T N_{df}\right)^{-1} A_{df}^T + \varepsilon_2 M_{df} M_{df}^T.$$

The following lemma is easily accessible.

Lemma 3: For a given negative definite matrix $\Gamma < 0$ $(\Gamma \in R^{n \times n})$, there always exists a matrix $S \in R^{n \times p} (p \leq n)$ such that $\Gamma + SS^T < 0$.

The next lemma can be readily proved along the same line of the proof for Theorem 1 in [8].

Lemma 4: For a given positive constant γ and a positive definite matrix Q, if there exists a positive definite matrix P satisfying the inequality

$$\begin{aligned} &(A_f + \Delta A_f)^T P + P(A_f + \Delta A_f) + P(A_{df} \\ &\quad + \Delta A_{df}) Q^{-1} (A_{df} + \Delta A_{df})^T P \\ &\quad + Q + C_f^T C_f + \gamma^{-2} P D_f D_f^T P < 0 \end{aligned} \tag{14}$$

for all admissible parameter uncertainties ΔA_f and ΔA_{df}, then the system (11)–(12) is robustly asymptotically stable and meet $\|H_{zw}(s)\|_\infty \leq \gamma$.

For presentation convenience, we make the following definitions:

$$\Phi\colon = A_d \left(Q_1 - \varepsilon_2^{-1} N_2^T N_2\right)^{-1} A_d^T + \varepsilon_2 M_1 M_1^T \tag{15}$$

$$\hat{A}\colon = A + \varepsilon_1 M_1 M_1^T P_1 + \Phi P_1 + \gamma^{-2} D_1 D_1^T P_1 \tag{16}$$

$$\begin{aligned} \hat{C}\colon = &C + (\varepsilon_1 + \varepsilon_2) M_2 M_1^T P_1 + C_d (Q_1 - \varepsilon_2^{-1} N_2^T N_2)^{-1} \\ &\times A_d^T P_1 + \gamma^{-2} D_2 D_1^T P_1 \end{aligned} \tag{17}$$

$$\begin{aligned} R: = &(\varepsilon_1 + \varepsilon_2) M_2 M_2^T + C_d (Q_1 - \varepsilon_2^{-1} N_2^T N_2)^{-1} C_d^T \\ &+ \gamma^{-2} D_2 D_2^T \end{aligned} \tag{18}$$

$$\begin{aligned} \Theta\colon = &\hat{C} + (\varepsilon_1 + \varepsilon_2) M_2 M_1^T P_2 + \gamma^{-2} D_2 D_1^T P_2 + C_d (Q_1 \\ &- \varepsilon_2^{-1} N_2^T N_2)^{-1} A_d^T P_2 \end{aligned} \tag{19}$$

$$\begin{aligned} \Omega\colon = &(\varepsilon_1 + \varepsilon_2) M_2 M_1^T + \gamma^{-2} D_2 D_1^T \\ &+ C_d (Q_1 - \varepsilon_2^{-1} N_2^T N_2)^{-1} A_d^T. \end{aligned} \tag{20}$$

We are now ready to give our main results.

Theorem 1: Let σ be a sufficiently small positive constant and Q_1 be a positive definite matrix. Assume that there exist positive scalars $\varepsilon_1, \varepsilon_2$ such that $N_2 Q_1^{-1} N_2^T < \varepsilon_2 I$ and the following two Riccati matrix inequalities:

$$\begin{aligned} &A^T P_1 + P_1 A + P_1 (\varepsilon_1 M_1 M_1^T + \gamma^{-2} D_1 D_1^T + \Phi) P_1 \\ &\quad + \varepsilon_1^{-1} N_1^T N_1 + Q_1 < 0 \end{aligned} \tag{21}$$

$$\begin{aligned} \Gamma := &(\hat{A} - \Omega^T R^{-1} \hat{C})^T P_2 + P_2 (\hat{A} - \Omega^T R^{-1} \hat{C}) \\ &+ P_2 \left(\varepsilon_1 M_1 M_1^T + \gamma^{-2} D_1 D_1^T + \Phi \right. \\ &\left. - \Omega^T R^{-1} \Omega\right) P_2 + L^T L - \hat{C}^T R^{-1} \hat{C} + \sigma I < 0 \end{aligned} \tag{22}$$

have positive definite solutions $P_1 > 0$ and $P_2 > 0$, respectively, where the matrices $\Phi, \hat{A}, \hat{C}, R, \Theta, \Omega$ are defined respectively in (15)–(20). Furthermore, let $U \in R^{p \times p}$ be an arbitrary orthogonal matrix (i.e., $UU^T = I$) and $S \in R^{n \times p}$ be an arbitrary matrix meeting $\Gamma + SS^T < 0$ (see Lemma 3). Then, the filter (5) with parameters

$$K = P_2^{-1} \left(\Theta R^{-1} + SUR^{-1/2}\right) \tag{23}$$

$$G = \hat{A} - K\hat{C} \tag{24}$$

will be such that, independent of the time delay h, 1) the augmented system (11)–(12) is asymptotically stable, and 2) $\|H_{zw}(s)\|_\infty \leq \gamma$.

Proof: By the Assumption 2, we know R^{-1} exists. From Lemma 1 and Lemma 2, we have

$$\begin{aligned} &(A_f + \Delta A_f)^T P + P(A_f + \Delta A_f) \\ &\quad + P(A_{df} + \Delta A_{df}) Q^{-1} (A_{df} + \Delta A_{df})^T P \\ &\quad \leq A_f^T P + P A_f + \varepsilon_1 P M_f M_f^T P + \varepsilon_1^{-1} N_f^T N_f \\ &\quad + P \left[A_{df} \left(Q - \varepsilon_2^{-1} N_{df}^T N_{df}\right)^{-1} A_{df}^T + \varepsilon_2 M_{df} M_{df}^T\right] P. \end{aligned} \tag{25}$$

Put

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} > 0 \quad Q = \begin{bmatrix} Q_1 & 0 \\ 0 & \sigma I \end{bmatrix}. \tag{26}$$

Using the definitions (7)–(10) and (15)–(20), we get

$$\begin{aligned} \Sigma\colon = &A_f^T P + P A_f + \varepsilon_1 P M_f M_f^T P + \varepsilon_1^{-1} N_f^T N_f \\ &+ P \left[A_{df} \left(Q - \varepsilon_2^{-1} N_{df}^T N_{df}\right)^{-1} A_{df}^T + \varepsilon_2 M_{df} M_{df}^T\right] P \\ &+ Q + C_f^T C_f + \gamma^{-2} P D_f D_f^T P := \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix} \end{aligned} \tag{27}$$

where

$$\begin{aligned} \Sigma_{11} = &A^T P_1 + P_1 A + \varepsilon_1 P_1 M_1 M_1^T P_1 + \varepsilon_1^{-1} N_1^T N_1 \\ &+ P_1 \Phi P_1 + Q_1 + \gamma^{-2} P_1 D_1 D_1^T P_1 \end{aligned} \tag{28}$$

$$\begin{aligned} \Sigma_{12} = &(A - G - KC)^T P_2 + \varepsilon_1 P_1 M_1 (M_1 - KM_2)^T P_2 \\ &+ P_1 \left[A_d \left(Q_1 - \varepsilon_2^{-1} N_2^T N_2\right)^{-1} (A_d - KC_d)^T \right. \\ &\left. + \varepsilon_2 M_1 (M_1 - KM_2)^T\right] P_2 \\ &+ \gamma^{-2} P_1 D_1 (D_1 - GD_2)^T P_2 \end{aligned} \tag{29}$$

$$\begin{aligned} \Sigma_{22} = &G^T P_2 + P_2 G + \varepsilon_1 P_2 (M_1 - KM_2) \\ &\times (M_1 - KM_2)^T P_2 + L^T L \\ &+ P_2 \left[(A_d - KC_d) \left(Q_1 - \varepsilon_2^{-1} N_2^T N_2\right)^{-1} \right. \\ &\left. \times (A_d - KC_d)^T + \varepsilon_2 (M_1 - KM_2)(M_1 - KM_2)^T\right] \\ &\times P_2 + \sigma I + \gamma^{-2} P_2 (D_1 - KD_2)(D_1 - KD_2)^T P_2. \end{aligned} \tag{30}$$

From (21) we immediately see that $\Sigma_{11} < 0$. Now we consider Σ_{22}. In the light of (24), replacing G by $\hat{A} - K\hat{C}$ in (30) gives

$$\begin{aligned} \Sigma_{22} = &\hat{A}^T P_2 + P_2 \hat{A} + P_2 \left(\varepsilon_1 M_1 M_1^T + \gamma^{-2} D_1 D_1^T\right) \\ &\times P_2 + LL^T + \sigma I + P_2 \left[\varepsilon_2 M_1 M_1^T \right. \\ &\left. + A_d \left(Q_1 - \varepsilon_2^{-1} N_2^T N_2\right)^{-1} A_d^T\right] P_2 - (P_2 K) \\ &\cdot \left[\hat{C} + \varepsilon_1 M_2 M_1^T P_2 + \varepsilon_2 M_2 M_1^T P_2 + \gamma^{-2} D_2 D_1^T P_2 \right. \\ &\left. + C_d \left(Q_1 - \varepsilon_2^{-1} N_2^T N_2\right)^{-1} A_d^T P_2\right] \\ &- \left[\hat{C} + \varepsilon_1 M_2 M_1^T P_2 + \varepsilon_2 M_2 M_1^T P_2 + \gamma^{-2} D_2 D_1^T P_2 \right. \\ &\left. + C_d \left(Q_1 - \varepsilon_2^{-1} N_2^T N_2\right)^{-1} A_d^T P_2\right]^T (P_2 K)^T \\ &+ (P_2 K) \left[\varepsilon_1 M_2 M_2^T + C_d \left(Q_1 - \varepsilon_2^{-1} N_2^T N_2\right)^{-1} C_d^T \right. \\ &\left. + \gamma^{-2} D_2 D_2^T + \varepsilon_2 M_2 M_2^T\right] (P_2 K)^T. \end{aligned} \tag{31}$$

Noticing the definitions of R, Θ, Γ, respectively, in (18), (19), and (22), we can rewrite (31) as

$$\Sigma_{22} = \Gamma + [(P_2K)R^{1/2} - \Theta^T R^{-1/2}] \times [(P_2K)R^{1/2} - \Theta^T R^{-1/2}]^T. \quad (32)$$

Using the expression of K in (23), we can see that

$$[(P_2K)R^{1/2} - \Theta^T R^{-1/2}] \times [(P_2K)R^{1/2} - \Theta^T R^{-1/2}]^T = SS^T. \quad (33)$$

Therefore, it follows from the definition of S in this theorem that $\Sigma_{22} < 0$. It is also not difficult to verify $\Sigma_{12} = 0$ by putting (24) into (29). We now arrive at the conclusion that $\Sigma < 0$. By Lemma 4, the system (11)–(12) is robustly asymptotically stable and $\|H_{zw}(s)\|_\infty \leq \gamma$. This completes the proof of this theorem. ■

Remark 1: Theorem 1 shows that the robust H_∞ stability constraint on the uncertain time system (1)–(3) can be guaranteed when two positive definite solutions P_1, P_2 respectively to the the quadratic matrix inequalities (QMIs) (21)–(22) are known to exist for some positive definite scalars $\varepsilon_1 > 0, \varepsilon_2 > 0$ and positive definite matrix Q. For general solving algorithm of QMIs, we refer the reader to [12] and references therein.

Remark 2: It is worth mentioning that the result of Theorem 1 may be conservative due to the use of the inequalities in Lemma 1, Lemma 2 and Lemma 4. However, the conservatism can be significantly reduced by properly selecting the parameters ε_1 and ε_2 in a matrix norm sense. The relevant discussion and corresponding optimization algorithm can be found in [20] and references therein.

Remark 3: It should be pointed out that, in the present design procedure of robust H_∞ filters for time-delay systems, there exists much *explicit* freedom, such as the choices of the positive definite matrix $Q_1 > 0$, the free parameters S ($S \in R^{n\times p}$ satisfies $\Gamma + SS^T < 0$) and orthogonal matrix U, etc. The remaining freedom provides the possibility for considering more performance constraints (e.g., the transient requirement and reliability behavior on the filtering process) which requires further investigations.

IV. Robust Filtering for Uncertain Time-Varying System With Time-Delays

Consider the following linear continuous uncertain time-varying system with state and output delays

$$\dot{x}(t) = [A(t) + \Delta A(t)]x(t) + [A_d(t) + \Delta A_d(t)]x(t-h) + D_1(t)w(t) \quad (34)$$

$$z(t) = L(t)x(t) \quad (35)$$

$$y(t) = [C(t) + \Delta C(t)]x(t) + [C_d(t) + \Delta C_d(t)]x(t-h) + D_2(t)w(t) \quad (36)$$

where $x(t)$ and $y(t)$ have the same meanings as those in Section 2. $z(t) \in R^m$ is a linear combination of the state to be estimated and $w(t) \in R^q$ is a disturbance signal. $A(t), A_d(t), C(t), C_d(t), D_1(t), D_2(t), L(t)$ are known time-varying matrices that describe the nominal system. $\Delta A(t), \Delta C(t), \Delta A_d(t), \Delta C_d(t)$ are parameter uncertainties that are time varying and satisfy the following constraints

$$\begin{bmatrix} \Delta A(t) & \Delta A_d(t) \\ \Delta C(t) & \Delta C_d(t) \end{bmatrix} = \begin{bmatrix} M_1(t) \\ M_2(t) \end{bmatrix} F(t) [N_1(t) \quad N_2(t)] \quad (37)$$

where $M_1(t), M_2(t), N_1(t), N_2(t)$ are time-varying matrices with appropriate dimensions and $F(t) \in R^{i\times j}$ is a perturbation matrix with Lebesgue measurable elements and satisfies $F(t)F^T(t) \leq I$.

Consider a filter for the system (34)–(36) of the form

$$\dot{\hat{x}}(t) = G(t)\hat{x}(t) + K(t)y(t) \quad (38)$$

$$\hat{z}(t) = L(t)\hat{x}(t) \quad (39)$$

where $\hat{x} \in R^n$ is the state estimate, $\hat{z} \in R^m$ is an estimate for $z(t), G(t)$ and $K(t)$ are filter parameters to be determined.

We denote the state estimate error, the output estimate error, and an augmented state vector by $e(t), e_z(t)$ and $x_f(t)$, respectively, which are defined as follows:

$$e(t) = x(t) - \hat{x}(t)$$
$$e_z(t) = z(t) - \hat{z}(t)$$
$$x_f(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}.$$

From (34)–(36) and (38)–(39), we can obtain an augmented system

$$\dot{x}_f(t) = [A_f(t) + \Delta A_f(t)]x_f(t) + [A_{df}(t) + \Delta A_{df}(t)]x_f(t-h) + D_f(t)w(t) \quad (40)$$

$$e_z(t) = C_f(t)x_f(t) \quad (41)$$

where $A_f(t), D_f(t), A_{df}(t), M_f(t), N_f(t), N_{df}(t), \Delta A_f(t), \Delta A_{df}(t), M_{df}(t)$, and $C_f(t)$ have the same forms as in (7)–(10) except that all variables here should be time varying.

The robust filtering problem addressed here is to seek the filter parameters $G(t)$ and $K(t)$ such that for all admissible uncertainties $\Delta A(t), \Delta A_d(t), \Delta C(t), \Delta C_d(t)$, the system defined in (40) is asymptotically stable.

Theorem 2: Given a constant positive definite matrix $Q > 0$. If the following differential Riccati inequalities

$$\frac{d}{dt}P(t) + [A_f(t) + \Delta A_f(t)]^T P(t) + P(t)[A_f(t) + \Delta A_f(t)] + Q + P(t)[A_{df}(t) + \Delta A_{df}(t)]Q^{-1}[A_{df}(t) + \Delta A_{df}(t)]^T P(t) < 0 \quad (42)$$

has a positive definite solution $P(t)$ for all admissible uncertainties, then the system (40) is robustly asymptotically stable.

Proof: Define a Lyapunov function as

$$V(x_f(t), t) = x_f^T(t)P(t)x_f(t) + \int_{t-h}^{t} x_f^T(s)Qx_f(s)\,ds$$

with $w(t) = 0$. The time derivative of $V(x_f(t), t)$ along a given trajectory is obtained as

$$\frac{d}{dt}V(x_f(t), t) = \begin{bmatrix} x_f(t) \\ x_f(t-h) \end{bmatrix}^T \times \begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_2^T & \Delta_3 \end{bmatrix} \begin{bmatrix} x_f(t) \\ x_f(t-h) \end{bmatrix} \quad (43)$$

where

$$\Delta_1 = \frac{d}{dt}P(t) + \left[A_f(t) + \Delta A_f^T(t)\right] P(t) + P(t)[A_f(t) + \Delta A_f(t)] + Q$$
$$\Delta_2 = P(t)[A_{df}(t) + \Delta A_{df}(t)]$$
$$\Delta_3 = -Q.$$

It is easy to see from (42) that $\Delta_1 - \Delta_2\Delta_3^{-1}\Delta_1 < 0$. Noting $\Delta_3 < 0$, we conclude from [5] that the matrix in (43) is negative definite, and thus the system (40) is asymptotically stable according to the Lyapunov stability theory. ■

Following the same line of the proof of Theorem 1, we can obtain the following parallel results for the robust filtering problem in the time-varying case.

Theorem 3: Let σ be a sufficiently small positive constants and Q_1 be a positive definite matrix. Assume that there exist positive scalars $\varepsilon_1, \varepsilon_2$ such that $N_2(t)Q_1^{-1}N_2^T(t) < \varepsilon_2 I$ and the following two differential Riccati matrix inequalities:

$$\begin{aligned} &\frac{d}{dt}P_1(t) + A(t)^T P_1(t) + P_1(t)A(t) \\ &\quad + P_1(t)\left[\varepsilon_1 M_1(t)M_1^T(t) + \Phi(t)\right]P_1(t) \\ &\quad + \varepsilon_1^{-1}N_1^T(t)N_1(t) + Q_1 < 0 \end{aligned} \tag{44}$$

$$\begin{aligned} \tilde{\Gamma}(t) := &\frac{d}{dt}P_2(t) + [\hat{A}(t) - \Omega^T(t)R^{-1}(t)\hat{C}(t)]^T P_2(t) \\ &+ P_2(t)[\hat{A}(t) - \Omega^T(t)R^{-1}(t)\hat{C}(t)] \\ &+ P_2(t)\Big[\varepsilon_1 M_1(t)M_1^T(t) + \Phi(t) \\ &- \Omega^T(t)R^{-1}(t)\Omega(t)\Big]P_2(t) \\ &- \hat{C}^T(t)R^{-1}(t)\hat{C}(t) + \sigma I < 0 \end{aligned} \tag{45}$$

have positive definite solutions $P_1 > 0$ and $P_2 > 0$, respectively, where

$$\begin{aligned} \Phi(t) :&= A_d(t)\left(Q_1 - \varepsilon_2^{-1}N_2^T(t)N_2(t)\right)^{-1}A_d^T(t) \\ &\quad + \varepsilon_2 M_1(t)M_1^T(t) \\ \hat{A}(t) :&= A(t) + \varepsilon_1 M_1(t)M_1^T(t)P_1(t) + \Phi(t)P_1(t) \\ \hat{C}(t) :&= C(t) + (\varepsilon_1 + \varepsilon_2)M_2(t)M_1^T(t)P_1(t) \\ &\quad + C_d(t)\left(Q_1 - \varepsilon_2^{-1}N_2^T(t)N_2(t)\right)^{-1}A_d^T(t)P_1(t) \\ R(t) :&= (\varepsilon_1 + \varepsilon_2)M_2(t)M_2^T(t) + C_d(t) \\ &\quad \times \left(Q_1 - \varepsilon_2^{-1}N_2^T(t)N_2(t)\right)^{-1}C_d^T(t) \\ \Theta(t) :&= \hat{C}(t) + (\varepsilon_1 + \varepsilon_2)M_2(t)M_1^T(t)P_2(t) \\ &\quad + C_d(t)\left(Q_1 - \varepsilon_2^{-1}N_2^T(t)N_2(t)\right)^{-1}A_d^T(t)P_2(t) \\ \Omega(t) :&= (\varepsilon_1 + \varepsilon_2)M_2(t)M_1^T(t) \\ &\quad + C_d(t)\left(Q_1 - \varepsilon_2^{-1}N_2^T(t)N_2(t)\right)^{-1}A_d^T(t). \end{aligned}$$

Furthermore, let $U \in R^{p\times p}$ be an arbitrary orthogonal matrix (i.e., $UU^T = I$) and $S \in R^{n\times p}$ be an arbitrary matrix meeting $\tilde{\Gamma}(t) + SS^T < 0$. Then, the filter (38)–(39) with parameters

$$\begin{aligned} K(t) &= P_2^{-1}(t)[\Theta(t)R^{-1}(t) + SUR^{-1/2}(t)], \\ G(t) &= \hat{A}(t) - K(t)\hat{C}(t) \end{aligned} \tag{46}$$

will be such that, independent of the time delay h, the augmented system (40) is robustly asymptotically stable in the presence of all admissible uncertainties.

V. A Numerical Example

In this section, we shall give a numerical example to demonstrate the theoretical result obtained. Consider the system (1)–(3) with system data given as follows:

$$\begin{aligned} A &= \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix} \\ A_d &= \begin{bmatrix} -1 & 0 \\ -0.8 & -1 \end{bmatrix} \\ D_1 &= \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} \\ C &= [1 \quad 0] \\ D_2 &= [0.5 \quad 0.8] \\ M_1 &= \begin{bmatrix} 0.1 & 0.05 \\ -0.02 & 0.1 \end{bmatrix} \\ M_2 &= [-0.2 \quad 0.8] \\ C_d &= [0.5 \quad 1] \\ L &= [0.5 \quad 0.4] \\ N_1 &= \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix} \\ N_2 &= \begin{bmatrix} 0.02 & 0.01 \\ 0.2 & 0.5 \end{bmatrix} \\ S &= \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \end{aligned}$$

We focus on designing the robust H_∞ filter of structure (5) which depend on neither the uncertainties nor the time-delay, such that for all admissible parameter perturbations, the filtering process is asymptotically stable and the transfer function from exogenous disturbance to error state output meets the prespecified H_∞-norm upper bound constraints $\|H_{zw}(s)\|_\infty \le \gamma = 0.8$.

Considering the constraint $N_2Q_1^{-1}N_2^T < \varepsilon_2 I$, we choose $\varepsilon_1 = 0.1, \varepsilon_2 = 0.4, \sigma = 10, Q_1 = I_2$. Solving the QMI (21), we obtain the positive definite solution P_1, and subsequently $\hat{A}, \hat{C}$ and R, respectively, as follows:

$$\begin{aligned} P_1 &= \begin{bmatrix} 1.3101 & -0.1123 \\ -0.1123 & 0.7524 \end{bmatrix} \\ \hat{A} &= \begin{bmatrix} -0.3446 & 1.3501 \\ 2.9935 & 1.0506 \end{bmatrix} \\ \hat{C} &= [-0.3445 \quad -3.3918] \\ R &= 5.0868. \end{aligned}$$

Then, solve the QMI (22) to obtain

$$P_2 = \begin{bmatrix} 3.9720 & -0.0591 \\ -0.0591 & 1.3823 \end{bmatrix} \quad \Theta = [-5.7490 \quad -9.8341].$$

Note that the dimension $p = 1$, the only choices for U satisfying $UU^T = I$ are $U = 1$ (case 1) and $U = -1$ (case 2). In these two cases, we get the following two set of solutions for K and G:

$$\begin{aligned} &\text{Case 1: } K = \begin{bmatrix} -0.3402 \\ -1.4616 \end{bmatrix} \quad G = \begin{bmatrix} 1.1929 & 0.1963 \\ 2.4900 & -3.9068 \end{bmatrix} \\ &\text{Case 2: } K = \begin{bmatrix} -0.6487 \\ -3.1069 \end{bmatrix} \quad G = \begin{bmatrix} 1.0866 & -0.8500 \\ 1.9232 & -9.4872 \end{bmatrix}. \end{aligned}$$

It is not difficult to verify that the specified robust stability as well as H_∞ disturbance rejection constraints are achieved.

VI. Conclusion

The robust filtering problem of uncertain linear time-invariant (time-varying) system with delay states and outputs has been studied in this paper. For the time-invariant case a linear filter structure which does not depend on the uncertainties has been proposed, and a matrix Riccati inequality approach has been used to solve the problem. The effectiveness of the designed filter has been demonstrated by a numerical example. For the time-varying case, a differential Riccati inequality approach has been developed to design the robust filter. We point out that the results obtained can also be extended to the discrete-time system and sampled-data systems within the same framework.

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