V. CONCLUSION

In this note, we have considered the control of partially linear cascade nonlinear systems. We have given conditions under which switching of the gains of the linear controller can stabilize the cascade. Our condition, Assumption 1 for the existence of an invariance region is different than ISS and weaker than minimum phase assumptions. Of course, the tradeoff is that the switching policy depends on the state of both subsystems and that in general, only boundedness of trajectories is achieved. It would be of great interest to investigate the precise relationship between the conditions of this note and these other more familiar notions. For example, in the simulated system (27)–(29), the invariance condition (40) shows that the ratio k_2/k_1 grows with the radius R of the invariance region $\mathcal{G}$. The gains of a nonpeaking constant gain controller have this same property [1]. However, since the initial condition for z does not enter into that calculation we have more freedom in the design of the gains. It appears that the switching controller automatically adjusts the output gain α to find a nonpeaking controller.

Since the trajectories of the invariance controlled system are bounded by the prescribed invariance region one of the additional advantages of invariance control is that it may also be useful to enforce constraints on the states of the internal dynamics. There are many applications which require not only output regulation and internal stability but also that the internal states remain bounded below prescribed values. Further studies along these lines would also be valuable.

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On Stabilization of Bilinear Uncertain Time-Delay Stochastic Systems With Markovian Jumping Parameters

Zidong Wang, Hong Qiao, and K. J. Burnham

***Abstract*—In this note, we investigate the stochastic stabilization problem for a class of bilinear continuous time-delay uncertain systems with Markovian jumping parameters. Specifically, the stochastic bilinear jump system under study involves unknown state time-delay, parameter uncertainties, and unknown nonlinear deterministic disturbances. The jumping parameters considered here form a continuous-time discrete-state homogeneous Markov process. The whole system may be regarded as a stochastic bilinear hybrid system that includes both time-evolving and event-driven mechanisms. Our attention is focused on the design of a robust state-feedback controller such that, for all admissible uncertainties as well as nonlinear disturbances, the closed-loop system is stochastically exponentially stable in the mean square, independent of the time delay. Sufficient conditions are established to guarantee the existence of desired robust controllers, which are given in terms of the solutions to a set of either linear matrix inequalities (LMIs), or coupled quadratic matrix inequalities. The developed theory is illustrated by numerical simulation.**

***Index Terms*—Bilinear systems, linear matrix inequalities (LMIs), markovian jump, stochastic exponential stability, time-delay, uncertainty.**

I. INTRODUCTION

A lot of dynamical systems have variable structures subject to random abrupt changes, which may result from abrupt phenomena such as random failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, modification of the operating point of a linearized model of a nonlinear systems, etc. The hybrid systems, which involve both time-evolving and event-driven mechanisms, may be employed to model the above problems. A special class of hybrid systems is the so-called jump linear systems (JLSs). The jump linear system has many operation modes, and the system mode switching is governed by a Markov process. The parameter jumps among different modes may be seen as discrete events. The control of JLSs has been a research subject and attracted a lot of interest since the mid 1960s. The optimal regulator, controllability, observability, stability and stabilization problems have been extensively studied for JLSs, see [4], [11], [14], [16], [22], [26], [34], [36], [40], and the references therein.

It has been recognized that the time-delays and parameter uncertainties, which are inherent features of many physical processes, are very often the cause for instability and poor performance of systems [18]. In the past few years, considerable attention has been given to the robust and/or H_∞ controller design problems for linear uncertain state delayed systems. A great many of papers have appeared on this general

Manuscript received January 11, 2001; revised June 27, 2001 and November 6, 2001. Recommended by Associate Editor B. Chen. The work of Z. Wang was supported in part by the City University of Hong Kong, the University of Kaiserslautern of Germany, and the Alexander von Humboldt Foundation of Germany.

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Publisher Item Identifier S 0018-9286(02)03743-1.

topic, see, e.g., [25], [27], [35], [37], [38], and the references therein. As for the JLSs with parametric uncertainties, the issues of stability, stabilization, H_2 control, H_∞ control, H_2/H_∞ control, Kalman filtering have been well investigated, and recent results can be found in [1], [5], [6], [10], [13], [28]–[30], and [32]. Also, the control problem for time-delay uncertain JLSs has been tackled in [31] for the discrete-time case. In [20] and [21], the exponential stability analysis problem for a general class of linear/nonlinear stochastic jumping delay systems has been intensively studied, and a number of useful stability criteria have been established. In particular, for the linear case in [21], the exponential stability can be easily tested by checking the existence of the solution to a linear matrix inequality. Unfortunately, the parametric uncertainties and the nonlinear exogenous disturbance have not been considered in [20] and [21] for stabilization problem.

On the other hand, bilinear systems have been of great interest in the past three decades, since many real-world systems can be adequately approximated by a bilinear model. The application areas include nuclear, thermal, chemical processes, biology, socioeconomics, immunology, etc.; see [8], [23], and [24] for more details. In particular, the stochastic bilinear systems, also called state-dependent noise systems or multiplicative noise systems, have been dealt with by many authors. Among them, we quote DeKoning [12], Bernstein and Haddad [3], Yasuda *et al.* [41], Skelton *et al.* [33], Yaz [42], and Johnston and Krishnamurthy [17]. However, a literature search reveals that the issue of stabilization of jump bilinear systems with or without uncertainty and time-delay has not been fully investigated and remains important and challenging. This situation motivates the present study on the robust stabilization of bilinear continuous time-delay jump systems.

It is now worth pointing out that the essential differences between the JLS which has been extensively studied as mentioned above and the jump bilinear stochastic system (JBSS) that is to be considered in this note. For JLS, every mode corresponds to a *deterministic dynamics*, that is, when the mode is fixed, the system state evolves according to the corresponding deterministic dynamics. However, the JBSS can be regarded as the result of several *stochastic systems* (systems with multiplicative noises) switching from one to the others according to the movement of a Markov chain. For JBSS, every mode corresponds to a *stochastic dynamics*. Obviously, the JLS is a special case of the JBSS.

This note is concerned with the stochastic stabilization problem for a class of bilinear continuous time-delay uncertain systems with Markovian jumping parameters. We aim at designing a robust state-feedback controller such that, for all admissible uncertainties as well as nonlinear disturbances, the closed-loop system is stochastically exponentially stable in the mean square, independent of the time delay. We show that the analysis problem can be tackled in terms of the solutions to a set of linear matrix inequalities (LMIs) (see [15]), and the associated synthesis problem can be dealt with by solving a set of coupled quadratic matrix inequalities. We demonstrate the usefulness and applicability of the developed theory by means of a numerical simulation example.

Notation: The notations in this note are quite standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive–semidefinite (respectively, positive–definite). I is the identity matrix with compatible dimension. We let $h > 0$ and $C([-h, 0]; \mathbb{R}^n)$ denote the family of continuous functions φ from $[-h, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-h\leq\theta\leq 0} |\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. If A is a matrix, denote by $\|A\|$ its operator norm, i.e., $\|A\| = \sup\{|Ax|: |x| = 1\} = \sqrt{\lambda_{\max}(A^TA)}$ where $\lambda_{\max}(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) means the largest (respectively, smallest) eigenvalue of A. $l_2[0,\infty]$ is the space of square integrable vector. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). Denote by $L^p_{\mathcal{F}_0}([-h, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_0$-measurable $C([-h, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta): -h \leq \theta \leq 0\}$ such that $\sup_{-h\leq\theta\leq 0} \mathbb{E}|\xi(\theta)|^p < \infty$ where $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

II. Problem Formulation and Assumptions

Let $\{r(t), \quad t \geq 0\}$ be a right-continuous Markov process on the probability space which takes values in the finite space $\mathcal{S} = \{1, 2, \ldots, N\}$ with generator $\Pi = (\gamma_{ij})$ $(i, j \in \mathcal{S})$ given by

$$P\{r(t+\Delta) = j|r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j \end{cases}$$

where $\Delta > 0$ and $\lim_{\Delta\to 0} o(\Delta)/\Delta = 0$, $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ and $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$.

In this note, we consider the following class of bilinear uncertain continuous-time state delayed stochastic systems of the Itô type:

$$\begin{aligned} dx(t) =& [A(r(t)) + \Delta A(t, r(t))]x(t)dt \\ &+ \sum_{k=1}^{m} J_k(r(t))x(t)dw_k(t) \\ &+ [A_d(r(t))x(t-h) + B(r(t))u(t) \\ &+ D(r(t))f(r(t), x(t))]dt, \end{aligned} \tag{1}$$

$$x(t) = \varphi(t), \quad r(t) = r(0), \quad t \in [-h, 0] \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $f(\cdot): \mathbb{R}^n \to \mathbb{R}^{n_f}$ is an unknown nonlinear exogenous disturbance input, h denotes the *unknown* state delay, $\varphi(t)$ is a continuous vector valued initial function. Here, $w(t) := [w_1(t)\ w_2(t)\ \cdots\ w_m(t)]^T \in \mathbb{R}^m$ is an m-dimensional Brownian motion, and it is assumed that the Markov process $r(\cdot)$ is independent of $w_k(\cdot)$ $(k = 1, 2, \ldots, m)$. For fixed system mode, $A(r(t))$, $J_k(r(t))$ $(k = 1, 2, \ldots, n)$, $A_d(r(t))$, $B(r(t))$, $D(r(t))$ are known constant matrices with appropriate dimensions. $\Delta A(t, r(t))$ is a real-valued matrix function representing norm-bounded parameter uncertainty.

Assumption 1: The uncertain matrix $\Delta A(t, r(t))$ satisfies

$$\Delta A(t, r(t)) = M(r(t))F(t, r(t))N(r(t)) \tag{3}$$

where for fixed system mode, $M(r(t)) \in \mathbb{R}^{n\times i}$ and $N(r(t)) \in \mathbb{R}^{j\times n}$ are known real constant matrices which characterize how the deterministic uncertain parameter in $F(t, r(t))$ enters the nominal matrix $A(r(t))$; and $F(t, r(t)) \in \mathbb{R}^{i\times j}$ is an unknown time-varying matrix function meeting

$$F^T(t, r(t))F(t, r(t)) \leq I, \quad \forall\, t \geq 0;\ \ r(t) = i \in \mathcal{S}. \tag{4}$$

Assumption 2: For fixed system mode, there exists a known real constant matrix $H(r(t)) \in \mathbb{R}^{n\times n}$ such that the unknown nonlinear vector function $f(\cdot,\cdot)$ satisfies the following boundedness condition:

$$|f(r(t), x(t))| \leq |H(r(t))x(t)|, \quad \forall\,(r(t), x(t)) \in \mathcal{S} \times \mathbb{R}^n. \tag{5}$$

Assumption 3: For all $\delta \in [-h, 0]$, there exists a scalar $\sigma > 0$ such that $|x(t+\delta)| \leq \sigma|x(t)|$.

Remark 1: It is noted that, in the system model (1)–(2), there are two kinds of uncertainties acting on the nominal matrix $A(r(t))$, that is, the *deterministic* uncertainty $\Delta A(t, r(t))$ which can be regarded as the energy-bounded noise, and the *stochastic* perturbation $\sum_{k=1}^{m} J_k(r(t))dw_k(t)$ which is the multiplicative noise with known statistics. Both kinds of uncertainties have been extensively studied in the literature. If the multiplicative noise disappears and there are no time-delay and nonlinear exogenous disturbance, the system model (1)–(2) will be reduced to the usual jump linear system that has received considerable attention. Note that when the mode is fixed, the system (1)–(2) corresponds to a bilinear stochastic time-delay uncertain system.

Remark 2: The parameter uncertainty structure as in (3)–(4) has been widely used in the problems of robust control and robust filtering of uncertain systems (see, e.g., [31], [42] and references therein). We point out that the exogenous nonlinear time-varying disturbance term $f(r(t), x(t))$ in the system model (1)–(2) has not been taken into account in the research literature concerning jump systems. Such kind of disturbance may result from the linearization process of an originally highly nonlinear plant or may be an actual external nonlinear input. Also, as mentioned in [9], Assumption 3 is not restrictive as the scalar $\sigma > 0$ can be chosen arbitrarily.

Observe the system (1)–(2) and let $x(t;\xi)$ denote the state trajectory from the initial data $x(\theta) = \xi(\theta)$ on $-h \le \theta \le 0$ in $L^2_{\mathcal{F}_0}([-h, 0]; \mathbb{R}^n)$. Clearly, the system (1)–(2) admits a trivial solution $x(t; 0) \equiv 0$ corresponding to the initial data $\xi = 0$.

We now introduce the following stability concepts.

Definition 1: For the uncertain time-delay bilinear jump system (1)–(2) with $u(t) \equiv 0$ and every $\xi \in L^2_{\mathcal{F}_0}([-h, 0]; \mathbb{R}^n)$, the trivial solution is asymptotically stable in the mean square if

$$\lim_{t\to\infty} \mathbb{E}|x(t;\xi)|^2 = 0$$

and is exponentially stable in the mean square if there exist scalars $\alpha > 0$ and $\beta > 0$ such that

$$\mathbb{E}|x(t;\xi)|^2 \le \alpha e^{-\beta t} \sup_{-h\le\theta\le 0} \mathbb{E}|\xi(\theta)|^2. \quad (6)$$

Definition 2: We say that the system (1)–(2) is exponentially stabilizable in the mean square (respectively, asymptotically stabilizable in the mean square) if, for every $\xi \in L^2_{\mathcal{F}_0}([-h, 0]; \mathbb{R}^n)$, there exists a linear state feedback control law $u(t) = G(r(t))x(t)$ (the feedback gain $G(r(t))$ is constant for each fixed mode) such that the closed-loop system is exponentially stable in the mean square (respectively, asymptotically stable in the mean square).

In this note, we assume that all jump states $r(t) = i \in \mathcal{S}$ $(t \ge 0)$ and the system states $x(t)$ $(t \ge 0)$ are accessible, i.e., they are measurable for feedback.

The purpose of this note is to design a delay-independent memoryless state feedback controller of the form

$$\mathcal{G}(r(t)):\ \ u(t) = G(r(t))x(t) \quad (7)$$

based on the state $x(t)$ and the system mode $r(t)$, such that the following closed-loop system of (1)–(2) with $\mathcal{G}(r(t))$:

$$\begin{aligned} dx(t) =& [A(r(t)) + B(r(t))G(r(t)) + \Delta A(t, r(t))]x(t)dt \\ &+ \sum_{k=1}^{m} J_k(r(t))x(t)dw_k(t) \\ &+ [A_d(r(t))x(t-h) + D(r(t))f(r(t), x(t))]dt \end{aligned} \quad (8)$$

is exponentially stable in the mean square.

III. Main Results and Proofs

Let us first give the following lemmas which will be frequently used in the proofs of our main results in this note.

Lemma 1: (Schur Complement): Given constant matrices Ω_1, Ω_2, Ω_3 where $\Omega_1 = \Omega_1^T$ and $0 < \Omega_2 = \Omega_2^T$, then $\Omega_1 + \Omega_3^T\Omega_2^{-1}\Omega_3 < 0$ if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0$$

or, equivalently

$$\begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^T & \Omega_1 \end{bmatrix} < 0.$$

Lemma 2: (See, e.g., [39]): Let M, N and F be real matrices of appropriate dimensions with $F^TF \le I$ where F may be time varying. Then, for any scalar $\mu > 0$, we have

$$MFN + N^TF^TM^T \le \mu^2 MM^T + \mu^{-2}N^TN.$$

Recall that the Markov process $\{r(t),\ t \ge 0\}$ takes values in the finite space $\mathcal{S} = \{1, 2, \ldots, N\}$. For the sake of simplicity, we write

$$\begin{aligned} &A(i) := A_i \quad A_d(i) := A_{di} \quad B(i) := B_i \quad J_k(i) := J_{ki} \\ &D(i) := D_i \quad M(i) := M_i \quad N(i) := N_i \quad H(i) := H_i \\ &G(i) := G_i \quad f(i, x(t)) := f_i(x(t)) \quad \forall\, i \in \mathcal{S}. \end{aligned}$$

and

$$A_{ci} := A(i) + B(i)G(i) = A_i + B_iG_i \quad (9)$$

and then for the mode $r(t) = i$, the closed-loop system (8) becomes

$$\begin{aligned} dx(t) = [A_{ci} + M_iF(t, i)N_i]x(t)dt + \sum_{k=1}^{m} J_{ki}x(t)dw_k(t) \\ + [A_{di}x(t-h) + D_if_i(x(t))]dt. \end{aligned} \quad (10)$$

In the following theorem, we establish the analysis results, i.e., for a given controller, we derive the sufficient conditions under which the closed-loop system (10) is exponentially stable in the mean square.

Theorem 1: Let the controller gain $G(r(t))$ be given. If there exists a positive scalar $\mu > 0$ such that the following N matrix inequalities:

$$\begin{aligned} A_{ci}^TP_i + P_iA_{ci} + \sum_{k=1}^{m} J_{ki}^TP_iJ_{ki} + \sum_{j=1}^{N} \gamma_{ij}P_j + P_i\Big(A_{di}A_{di}^T \\ + D_iD_i^T + \mu^2M_iM_i^T\Big)P_i + \mu^{-2}N_i^TN_i + H_i^TH_i + I < 0 \end{aligned} \quad (11)$$

have positive–definite solutions $P_i > 0$ $(i \in \mathcal{S})$, then the system (10) is exponentially stable in the mean square.

Proof: First, we let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}; \mathbb{R}_+)$ denote the family of all nonnegative functions $Y(x, t, i)$ on $\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}$ which are continuously twice differentiable in x and once differentiable in t.

Fix $\xi \in L^2_{\mathcal{F}_0}([-h, 0]; \mathbb{R}^n)$ arbitrarily and write $x(t;\xi) = x(t)$. Define a Lyapunov function candidate $Y(x, t, i) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}; \mathbb{R}_+)$ by

$$Y(x(t), t, i) = x^T(t)P_ix(t) + \int_{t-h}^{t} x^T(s)x(s)ds. \quad (12)$$

It can be derived by Itô's formula (see, e.g., [19]) that

$$\mathbb{E}Y(x(s), s, i) := \mathbb{E}Y(x(0), 0, r(0)) + \mathbb{E}\int_0^s \mathcal{L}Y(x(t), t, i)dt \quad (13)$$

where

$$\begin{aligned}\mathcal{L}Y(x(t),t,i) =& x^T(t)\Bigg[(A_{ci}+\Delta A(t,i))^T P_i \\ &+ P_i(A_{ci}+\Delta A(t,i)) \\ &+ \sum_{k=1}^{m} J_{ki}^T P_i J_{ki} + \sum_{j=1}^{N}\gamma_{ij}P_j + I\Bigg]x(t) \\ &+ x^T(t-h)A_{di}^T P_i x(t) \\ &+ x^T(t)P_i A_{di}x(t-h) + x^T(t)P_i D_i f_i(x(t)) \\ &+ f_i^T(x(t))D_i^T P_i x(t) - x^T(t-h)x(t-h).\end{aligned} \tag{14}$$

Note that $\Delta A(t,i) = M_i F(t,i) N_i$ and $F^T(t,i)F(t,i) \le I$. It follows from Lemma 2 that, for any scalar $\mu > 0$:

$$\begin{aligned}P_i(\Delta A(t,i)) + (\Delta A(t,i))^T P_i =& (P_i M_i)F(t,i)N_i \\ &+ N_i^T F^T(t,i)(P_i M_i)^T \\ \le& \mu^2 P_i M_i M_i^T P_i \\ &+ \mu^{-2}N_i^T N_i.\end{aligned} \tag{15}$$

Moreover, it results from (5) and the following inequality:

$$\left(f_i^T(x(t)) - x^T(t)P_i D_i\right)\left(f_i^T(x(t)) - x^T(t)P_i D_i\right)^T \ge 0$$

that

$$\begin{aligned}&x^T(t)P_i D_i f_i(x(t)) + f_i^T(x(t))D_i^T P_i x(t) \\ &\quad\le f_i^T(x(t))f_i(x(t)) + x^T(t)P_i D_i D_i^T P_i x(t) \\ &\quad\le x^T(t)\left(H_i^T H_i + P_i D_i D_i^T P_i\right)x(t).\end{aligned} \tag{16}$$

Denote

$$\begin{aligned}\Theta_i :=& A_{ci}^T P_i + P_i A_{ci} + \mu^2 P_i M_i M_i^T P_i \\ &+ \mu^{-2}N_i^T N_i + \sum_{k=1}^{m} J_{ki}^T P_i J_{ki} + \sum_{j=1}^{N}\gamma_{ij}P_j \\ &+ H_i^T H_i + P_i D_i D_i^T P_i + I\end{aligned} \tag{17}$$

$$S_i := \begin{bmatrix}\Theta_i & P_i A_{di} \\ A_{di}^T P_i & -I\end{bmatrix}. \tag{18}$$

Then, substituting (15) and (16) into (14) results in

$$\begin{aligned}\mathcal{L}Y(x(t),t,i) \le& x^T(t)\Theta x(t) + x^T(t-h)A_{di}^T P_i x(t) \\ &+ x^T(t)P_i A_{di}x(t-h) - x^T(t-h)x(t-h) \\ =& [\, x^T(t) \quad x^T(t-h)\,]\, S_i \begin{bmatrix} x(t) \\ x(t-h)\end{bmatrix} \\ =& x_e^T(t)S_i x_e(t)\end{aligned} \tag{19}$$

where $x_e(t) := [x^T(t)\ x^T(t-h)]^T$.

From the Schur Complement Lemma (Lemma 1), we know that $S_i < 0$ if and only if

$$\Theta_i + P_i A_{di}A_{di}^T P_i < 0 \tag{20}$$

which is the same as the inequality (11). Therefore, we arrive at the conclusion that $\mathcal{L}Y(x(t),t,i) < 0$.

Note that $|x(t)| \le |x_e(t)|$, $S_i < 0$, and $P_i > 0$. It follows from Assumption 3 that

$$\begin{aligned}\frac{\mathcal{L}Y(x(t),t,i)}{Y(x(t),t,i)} \le& \frac{x_e^T(t)S_i x_e(t)}{x^T(t)P_i x(t) + \int_{t-h}^{t} x^T(s)x(s)ds} \\ \le& -\min_{i\in\mathcal{S}}\left\{\frac{\lambda_{\min}(-S_i)}{\lambda_{\max}(P_i)+h\sigma^2}\right\} := -\kappa\end{aligned}$$

and, therefore, $\kappa > 0$ and $\mathcal{L}Y(x(t),t,i) \le -\kappa Y(x(t),t,i)$. Then, similar to the proof of Theorem 1 in [9], by employing the Dynkin's formula and the Gronwall–Bellman lemma, we can easily show that, the uncertain time-delay bilinear jump system (10) is asymptotically stable in the mean square provided that the inequality (11) is met.

Based on the inequality (19), the exponential stability (in the mean square) of the system (10) can be proved as follows by using the techniques developed in [20] and [21].

Define

$$\begin{aligned}\lambda_P &= \max_{i\in\mathcal{S}}\lambda_{\max}(P_i) \quad \lambda_S = \min_{i\in\mathcal{S}}(-\lambda_{\max}(S_i)) \\ \lambda_p &= \min_{i\in\mathcal{S}}\lambda_{\min}(P_i)\end{aligned}$$

where $P_i > 0$ is the solution to (11) and S_i is defined in (18). Let δ be the unique root to

$$\delta\left(\lambda_P + he^{\delta h}\right) = \lambda_S + \min\left(1, \lambda_S e^{\delta h}\right).$$

To prove the mean square exponential stability, we modify the Lyapunov function candidate (12) as

$$Y_1(x(t),t,i) = e^{\delta t}\left(x^T(t)P_i x(t) + \int_{t-h}^{t}|x(s)|^2 ds\right). \tag{21}$$

Along the similar line for the proof [21, Th. 3.1], we can show that

$$e^{\delta t}\lambda_p \mathbb{E}|x(t)|^2 \le \left(\lambda_P + h(1+e^{\delta h})\right)\mathbb{E}\|\xi\|^2$$

or

$$\lim_{t\to\infty}\sup\frac{1}{t}\log(\mathbb{E}|x(t,\xi)|^2) \le -\delta.$$

This indicates that the trivial solution of the system (10) is exponentially stable in the mean square. This completes the proof of this theorem. ■

Remark 3: Theorem 1 provides the analysis results for the exponential stability of the system (10). It can be seen from (11) that we need to check whether there exist a positive scalar μ and N positive–definite matrices $P_i > 0$ $(i = 1, 2, \ldots, N)$ meeting the N coupled matrix inequalities. This may be done by converting the N coupled nonlinear (on P_i and μ) inequalities into the associated LMIs [7], and then we are able to determine exponential stability of the system (10) readily by checking the solvability of the LMIs [15].

The following theorem offers a LMI representation of Theorem 1.

Theorem 2: Let the controller gain $G(r(t))$ be given. If there exist a positive scalar $\varepsilon > 0$ and N positive–definite matrices $P_i > 0$ $(i \in \mathcal{S})$ satisfying the following LMIs:

$$\begin{bmatrix} \Lambda_i & P_i A_{di} & P_i D_i & \varepsilon N_i^T & P_i M_i \\ A_{di}^T P_i & -I & 0 & 0 & 0 \\ D_i^T P_i & 0 & -I & 0 & 0 \\ \varepsilon N_i & 0 & 0 & -\varepsilon I & 0 \\ M_i^T P_i & 0 & 0 & 0 & -\varepsilon I \end{bmatrix} < 0 \tag{22}$$

where Λ_i is defined by

$$\Lambda_i := A_{ci}^T P_i + P_i A_{ci} + \sum_{k=1}^{m} J_{ki}^T P_i J_{ki} + \sum_{j=1}^{N} \gamma_{ij} P_j + H_i^T H_i + I \tag{23}$$

then the system (10) is exponentially stable in the mean square.

Proof: To begin with, we rewrite (11) as

$$\Lambda_i + [\, P_i A_{di} \quad P_i D_i \quad \mu^{-1} N_i^T \quad \mu P_i M_i \,] \times \begin{bmatrix} A_{di}^T P_i \\ D_i^T P_i \\ \mu^{-1} N_i \\ \mu M_i^T P_i \end{bmatrix} < 0. \tag{24}$$

If follows from the Schur Complement Lemma (Lemma 1) that the previous inequality holds if and only if

$$\begin{bmatrix} \Lambda_i & P_i A_{di} & P_i D_i & \mu^{-1} N_i^T & \mu P_i M_i \\ A_{di}^T P_i & -I & 0 & 0 & 0 \\ D_i^T P_i & 0 & -I & 0 & 0 \\ \mu^{-1} N_i & 0 & 0 & -I & 0 \\ \mu M_i^T P_i & 0 & 0 & 0 & -I \end{bmatrix} < 0. \tag{25}$$

Note that (25) is not linear in μ. Let $\varepsilon := \mu^{-2}$. Pre- and postmultiplying the inequality (25) by $\mathrm{diag}\{I, I, I, \mu^{-1} I, \mu^{-1} I\}$ yield (22). The proof follows from Theorem 1 immediately. ∎

Remark 4: It is observed that the inequality (22) is linear in ε and $P_i > 0$ $(i = 1, 2, \ldots, N)$, and thus the standard LMI techniques can be exploited to check the exponential stability of the closed-loop system (10) when the controller is given. The analysis result given in Theorem 3 is also useful to determine the exponential stability of the free system (1)–(2) (i.e., $u(t) \equiv 0$).

Finally, the following result solves the addressed stochastic stabilization problem of bilinear continuous time-delay jump uncertain systems in terms of quadratic matrix inequalities.

Theorem 3: Consider the system (1)–(2) satisfying the Assumption 1 and Assumption 2. Let $\rho > 0$ be a positive scalar. If there exist a scalar $\varepsilon > 0$ and N positive–definite matrices $P_i > 0$ $(i \in \mathcal{S})$ such that the following quadratic matrix inequalities:

$$\begin{aligned} A_i^T P_i + P_i A_i &+ \sum_{k=1}^{m} J_{ki}^T P_i J_{ki} + \sum_{j=1}^{N} \gamma_{ij} P_j \\ &+ P_i \left(-2\rho B_i B_i^T + A_{di} A_{di}^T + D_i D_i^T + \mu^2 M_i M_i^T \right) P_i \\ &+ \mu^{-2} N_i^T N_i + H_i^T H_i + I < 0 \end{aligned} \tag{26}$$

hold, then the uncertain time-delay jump system (1)–(2) with nonlinear disturbances can be exponentially stabilized (in the mean square) by the memoryless state feedback controller of the form (7) with the gain matrix

$$G_i = -\rho B_i^T P_i \tag{27}$$

for all admissible parameter uncertainty.

Proof: The proof follows from Theorem 1 immediately by substituting (27) into (11). ∎

Remark 5: It is shown in Theorem 3 that the robust stochastic exponentially stabilization of system (1)–(2) with (7) is guaranteed if the inequalities (26) are valid. Note that when the system (1)–(2) is linear (i.e., $J_k = 0$ for $k = 1, \ldots, m$), the time-delay term disappears, and the parameter uncertainty $\Delta A(r(t))$ equals zero, Theorem 3 will reduce to the results in [16] and [22]. Furthermore, if the modes $r(t)$ are all equal to 1, Theorem 3 will recover the results of those, for example, [25]. Also, if the matrices $A_d(r(t))$ and $B(r(t))$ in system (1)–(2) contain parameter uncertainties, say $\Delta A_d(r(t))$ and $\Delta B(r(t))$, similar results to Theorem 3 can be obtained.

Remark 6: It is worth mentioning that, bilinear systems in real world are often bilinear of the states and controlled inputs of the systems. The research on such kind of bilinear systems should be interesting, which gives us one of future important research topics.

IV. Numerical Simulation

In this section, for the purpose of illustrating the usefulness and flexibility of the theory developed in this note, we present a simulation example. Attention is focused on the design of a robust stabilizing controller for an uncertain time-delay jump bilinear system that is assumed to have two modes. The Markov process that governs the mode switching has generator $\Pi = (\gamma_{ij})$ $(i, j = 1, 2)$.

The system data of (1)–(2) are as follows:

$$\begin{aligned} A_1 &= \begin{bmatrix} -2.1 & 0.1 \\ 0.1 & 1.1 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1.9 & -0.1 \\ -0.1 & 0.9 \end{bmatrix} \\ B_1 &= I_2 \quad B_2 = \begin{bmatrix} 0.9 & 0 \\ 0 & 1.1 \end{bmatrix} \\ A_{d1} &= 0.1 I_2 \quad A_{d2} = -0.1 I_2 \\ J_{11} &= 0.2 I_2, \quad J_{12} = 0.2 I_2, \\ D_1 &= \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 \\ -0.2 \end{bmatrix} \\ H_1 &= 0.1 I_2 \quad H_2 = 0.1 I_2 \\ M_1 &= 0.3 I_2 \quad M_2 = -0.3 I_2, \\ N_1 &= 0.4 I_2 \quad N_2 = -0.4 I_2 \\ \Pi &= \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \quad f_1(x(t)) = 0.1 \sin x_1(t) \\ f_2(x(t)) &= 0.1 \sin x_2(t) \quad F(t,1) = F(t,2) = \sin t I_2 \\ h &= 0.1 \quad \varphi(t) = 0.1. \end{aligned}$$

We choose $\mu = 0.5$ and $\rho = 1$. Solving (26) gives

$$P_1 = \begin{bmatrix} 1.4208 & 0.0448 \\ 0.0448 & 2.8533 \end{bmatrix} \quad P_2 = \begin{bmatrix} 1.5751 & -0.0425 \\ -0.0425 & 2.4390 \end{bmatrix}$$

and then a set of gain matrices can be obtained as

$$G_1 = \begin{bmatrix} -1.4208 & -0.0448 \\ -0.0448 & -2.8533 \end{bmatrix}, \quad G_2 = \begin{bmatrix} -1.4176 & 0.0382 \\ 0.0467 & -2.6829 \end{bmatrix}.$$

The responses of closed-loop system dynamics to initial conditions are shown in Figs. 1 and 2. The simulation results imply that the desired goal is well achieved.

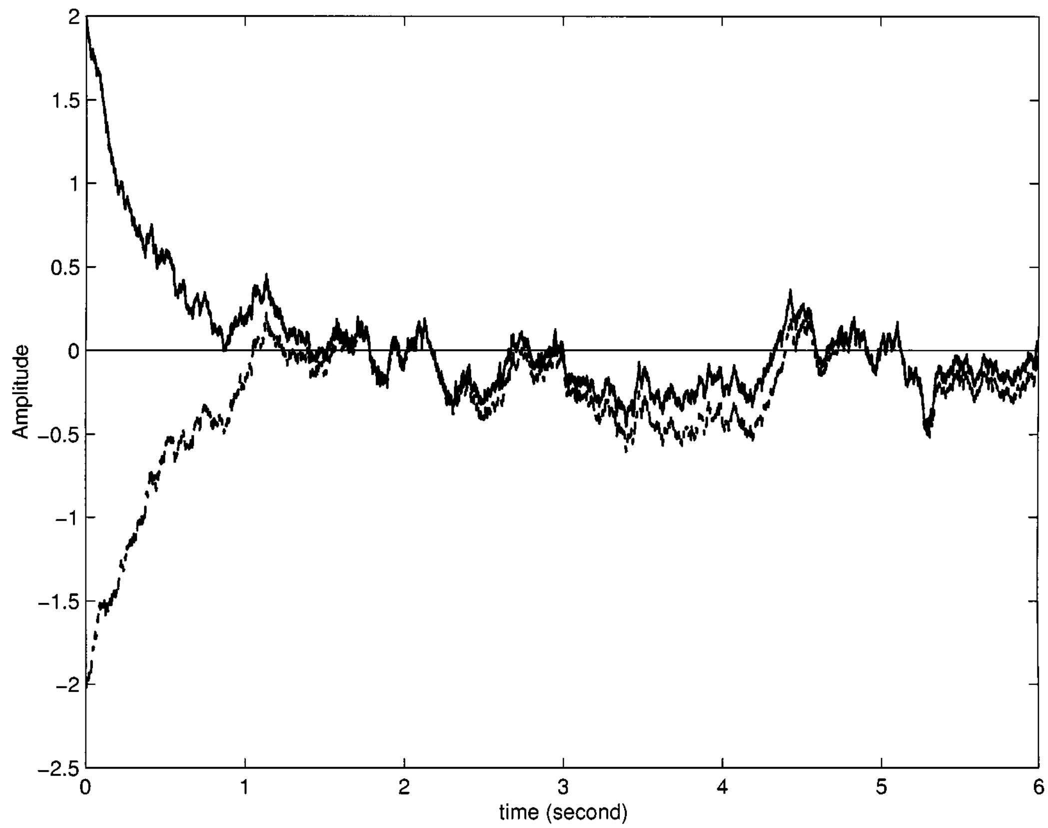

Fig. 1. x_1 (solid), x_2 (dashed). Mode 1: responses of system dynamics to initial conditions.

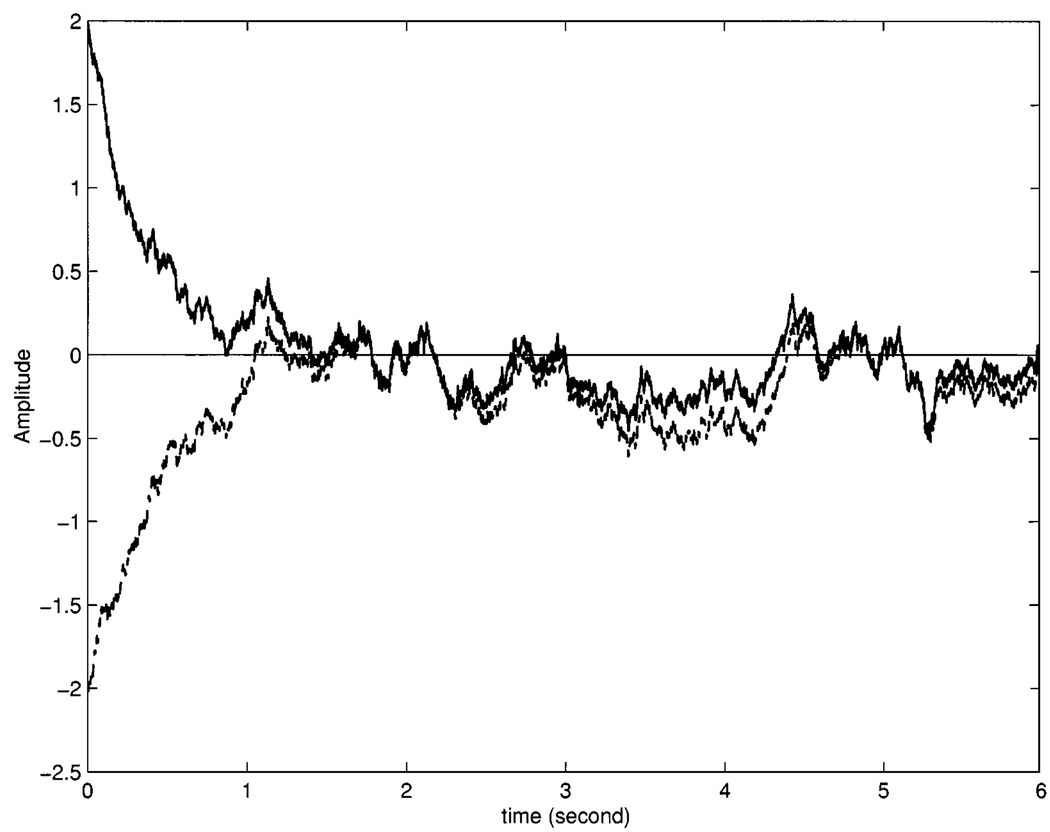

Fig. 2. x_1 (solid), x_2 (dashed). Mode 2: responses of system dynamics to initial conditions.

V. Conclusion

This note has introduced an algebraic matrix inequality approach to the robust stabilization for a class of bilinear continuous time-delay uncertain systems with Markovian jumping parameters. We have focused on the design of a robust state-feedback controller such that, for all admissible uncertainties as well as nonlinear disturbances, the closed-loop system is stochastically exponentially stable in the mean square, independent of the time delay. Sufficient conditions have been derived to ensure the existence of desired robust controllers, which are given in terms of the solutions to a set of either LMIs, or coupled quadratic matrix inequalities. A numerical example has demonstrated the applicability of the proposed approach.

Acknowledgment

The first author is grateful to J. Lam of The University of Hong Kong for helpful discussions, to H. Unbehauen of Ruhr-University Bochum of Germany for detailed comments, and to D. Prätzel-Wolters of the University of Kaiserslautern of Germany for useful suggestions.

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