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Spatially Periodic Solutions for Evolution Anisotropic Variable-Coefficient Navier–Stokes Equations: II. Serrin-Type Solutions

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ABSTRACT

We consider evolution (nonstationary) space-periodic solutions to the n-dimensional nonlinear Navier–Stokes equations of anisotropic fluids with the viscosity coefficient tensor variable in space and time and satisfying the relaxed ellipticity condition. Employing the Galerkin algorithm, we prove the existence of Serrin-type solutions, that is, the weak solutions with velocity in the periodic space $L_2\left(0, T; \dot{\mathbf{H}}^{n/2}_{\#\sigma}\right)$, $n \geq 2$. The solution uniqueness and regularity results are also discussed.

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1 | Introduction

Analysis of Stokes and Navier–Stokes equations is an established and active field of research in applied mathematical analysis; see, for example, [1–11] and many other publications. These works were mainly devoted to the flows of isotropic fluids with constant-viscosity coefficient, and some of the employed methods were heavily based on these properties.

On the other hand, in many cases, the fluid viscosity can vary in time and spatial coordinates, for example, due to variable ambient temperature. Moreover, from the point of view of rational mechanics of continuum, fluids can be anisotropic, and this feature is indeed observed in liquid crystals, electromagnetic fluids, and so forth; see, for example, [12] and references therein. In [13–18], the classical Navier–Stokes equations analysis has been extended to the transmission and boundary-value problems for *stationary* Stokes and Navier–Stokes equations of anisotropic fluids, particularly with relaxed ellipticity condition on the viscosity tensor.

In Part I, [19], we considered *evolution (nonstationary)* spatially periodic solutions in $\mathbb{R}^n$, $n \geq 2$, to the Navier–Stokes equations of an anisotropic fluid with the viscosity coefficient tensor variable in spatial coordinates and time and satisfying the relaxed ellipticity condition. We implemented the Galerkin algorithm but unlike the traditional approach, for example, in [10, 11], where the Galerkin basis consisted of the eigenfunctions of the corresponding isotropic constant-coefficient Stokes operator, we employed the basis constituted by the eigenfunctions of the periodic Bessel-potential operator having an advantage that it is universal, that is, independent of the analyzed anisotropic variable-coefficient Navier–Stokes operator. To analyze the solution in higher dimensions, the definition of the weak solution was generalized to some extent. Then, the periodic weak solution existence was considered in the spaces of Banach-valued functions mapping a finite-time interval to periodic Sobolev (Bessel-potential) spaces on n-dimensional flat torus, $L_\infty\left(0,T;\dot{\mathbf{H}}^0_{\#\sigma}\right)\cap L_2\left(0,T;\dot{\mathbf{H}}^1_{\#\sigma}\right)$. The periodic setting is interesting on its own, modeling fluid flow in periodic composite structures, and is also a common element of homogenization theories for inhomogeneous fluids and in the large eddy simulation.

In this paper, Part II, we prove the existence, uniqueness, and regularity of the weak solutions that belongs to the space $L_2\left(0,T;\dot{\mathbf{H}}^{n/2}_{\#\sigma}\right)$ (we call them Serrin-type solutions). It is well known that the regularity results available at the moment for evolution Navier–Stokes equations are rather different for dimensions $n=2$ and $n=3$, even for isotropic constant-viscosity fluids. The weak solution global regularity under arbitrarily large smooth input data for $n=2$ is proved and can be found, for example, in [1–4, 6–11]. However, for $n=3$, it is still an open question and constitutes one of the Clay Institute famous Millennium problems. Our motivation for considering arbitrary $n\geq 2$ is particularly to place the cases $n=2$ and $n=3$ in a more general set and to see which of them is an exception.

The paper material is presented as follows. In Section 1.1, we provide essentials on anisotropic Stokes and Navier–Stokes equations. Section 1.2 gives an introduction to the periodic Sobolev (Bessel-potential) functions spaces in spatial coordinates on n-dimensional flat torus and to the corresponding Banach-valued functions mapping a finite-time interval to these periodic Sobolev spaces. In Section 2, we describe the existence results for evolution spatially periodic anisotropic Navier–Stokes problem available from Part I, [19]. Sections 3–5 contain the main results of the paper. In Section 3, we define the Serrin-type solutions and prove the energy equality for them and also their uniqueness, for the n-dimensional periodic setting, $n\geq 2$. We also remark on their relations with the strong solutions and show that for $n=2$, any weak solution is a Serrin-type solution. In Section 4, we analyze the Serrin-type solution existence and regularity for constant anisotropic viscosity coefficients, while in Section 5, we generalize these results to variable anisotropic viscosity coefficients. In Section 6, we collect some technical results used in the main text of the paper, several of which might be new.

1.1 | Anisotropic Stokes and Navier–Stokes PDE Systems

Let $n\geq 2$ be an integer, $\mathbf{x}\in\mathbb{R}^n$ denote the space coordinate vector, and $t\in\mathbb{R}$ be time. Let $\mathfrak{L}$ denote the second-order differential operator represented in the component-wise divergence form as

$$(\mathfrak{L}\mathbf{u})_k := \partial_\alpha\left(a^{\alpha\beta}_{kj}E_{j\beta}(\mathbf{u})\right),\quad k=1,\dots,n, \tag{1.1}$$

where $\mathbf{u}=(u_1,\dots,u_n)^\top$, $E_{j\beta}(\mathbf{u}) := \frac{1}{2}(\partial_j u_\beta+\partial_\beta u_j)$ are the entries of the symmetric part, $\mathbb{E}(\mathbf{u})$, of the gradient, $\nabla\mathbf{u}$, in space coordinates, and $a^{\alpha\beta}_{kj}(\mathbf{x},t)$ are variable components of the tensor viscosity coefficient, $\mathbb{A}(\mathbf{x},t)=\left\{a^{\alpha\beta}_{kj}(\mathbf{x},t)\right\}_{1\leq i,j,\alpha,\beta\leq n}$, depending on the space coordinate vector $\mathbf{x}$ and on time t, cf. [12]. We also denoted $\partial_j=\frac{\partial}{\partial x_j}$, $\partial_t=\frac{\partial}{\partial t}$. Here and further on, the Einstein convention on summation in repeated indices from 1 to n is used unless stated otherwise.

The following symmetry conditions are assumed (see [20, (3.1) and (3.3)]), $a^{\alpha\beta}_{kj}(\mathbf{x},t)=a^{k\beta}_{\alpha j}(\mathbf{x},t)=a^{\alpha j}_{k\beta}(\mathbf{x},t)$. In addition, we require that the tensor $\mathbb{A}$ satisfies the relaxed ellipticity condition in terms of all *symmetric* matrices in $\mathbb{R}^{n\times n}$ with *zero matrix trace*; see [14, 15]. Thus, we assume that there exists a constant $C_{\mathbb{A}}>0$ such that

$$C_{\mathbb{A}}a^{\alpha\beta}_{kj}(\mathbf{x},t)\zeta_{k\alpha}\zeta_{j\beta}\geq|\zeta|^2\,,\quad \text{for a.e. } \mathbf{x},t,$$
$$\forall\,\zeta=\{\zeta_{k\alpha}\}_{k,\alpha=1,\dots,n}\in\mathbb{R}^{n\times n}\text{ such that }\zeta=\zeta^\top\text{ and }\sum_{k=1}^n\zeta_{kk}=0, \tag{1.2}$$

where $|\zeta|=|\zeta|_F:=(\zeta_{k\alpha}\zeta_{k\alpha})^{1/2}$ is the Frobenius matrix norm and the superscript $^\top$ denotes the transpose of a matrix. Note that in the more common, strong ellipticity condition (called S-ellipticity condition in [21, Definition 4.1]), inequality (1.2) should be satisfied for all matrices (not only symmetric with zero trace), which makes it much more restrictive (cf. also E-class in [20, Section 3.1], where condition (1.2) is assumed for all symmetric matrices).

We assume that $a^{\alpha\beta}_{ij}\in L_\infty(\mathbb{R}^n\times[0,T])$, where $[0,T]$ is some finite time interval, and the tensor $\mathbb{A}$ is endowed with the norm

$$\|\mathbb{A}\| := \|\mathbb{A}\|_{L_\infty(\mathbb{R}^n\times[0,T]),F} := \left|\left\{\|a^{\alpha\beta}_{ij}\|_{L_\infty(\mathbb{T}\times[0,T])}\right\}^n_{\alpha,\beta,i,j=1}\right|_F<\infty, \tag{1.3}$$

where $\left|\left\{b^{\alpha\beta}_{ij}\right\}^n_{\alpha,\beta,i,j=1}\right|_F := \left(b^{\alpha\beta}_{ij}b^{\alpha\beta}_{ij}\right)^{1/2}$ is the Frobenius norm of a fourth-order tensor.

Let $\mathbf{u}(\mathbf{x},t)$ be an unknown vector velocity field, $p(\mathbf{x},t)$ be an unknown (scalar) pressure field, and $\mathbf{f}(\mathbf{x},t)$ be a given vector field $\mathbb{R}^n$, where $t\in\mathbb{R}$ is the time variable. The nonlinear system

$$\partial_t\mathbf{u}-\mathfrak{L}\mathbf{u}+\nabla p+(\mathbf{u}\cdot\nabla)\mathbf{u}=\mathbf{f}\,,\quad \operatorname{div}\mathbf{u}=0$$

is the *evolution anisotropic incompressible Navier–Stokes system*, the main object of the analysis in this paper. Here, we use the notation $(\mathbf{u}\cdot\nabla):=u_j\partial_j$.

1.2 | Periodic Function Spaces

Let us introduce some function spaces on torus, that is, periodic function spaces (see, e.g., [22, p. 26], [23, 24], [25, Chapter 3], [7, Section 1.7.1], [10, Chapter 2] for more details).

Let $n\geq 1$ be an integer and $\mathbb{T}$ be the n-dimensional flat torus that can be parametrized as the semiopen cube $\mathbb{T}=\mathbb{T}^n=[0,1)^n\subset\mathbb{R}^n$; compare [26, p. 312]. In what follows, $\mathcal{D}(\mathbb{T})=C^\infty(\mathbb{T})$ denotes the (test) space of infinitely smooth real or complex functions on the torus. As usual, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ the set of natural numbers augmented by 0, and $\mathbb{Z}$ the set of integers.

Let $\xi\in\mathbb{Z}^n$ denote the n-dimensional vector with integer components. We will further need also the set $\dot{\mathbb{Z}}^n:=\mathbb{Z}^n\setminus\{\mathbf{0}\}$. Extending the torus parametrization to $\mathbb{R}^n$, it is often useful to identify $\mathbb{T}$ with the quotient space $\mathbb{R}^n\setminus\mathbb{Z}^n$. Then, the space of functions

$C^\infty(\mathbb{T})$ on the torus can be identified with the space of $\mathbb{T}$-periodic (1-periodic) functions $C^\infty_\# = C^\infty_\#(\mathbb{R}^n)$ that consists of functions $\phi \in C^\infty(\mathbb{R}^n)$ such that

$$\phi(\mathbf{x}+\xi) = \phi(\mathbf{x}) \quad \forall\, \xi \in \mathbb{Z}^n. \tag{1.4}$$

Similarly, the Lebesgue space on the torus $L_p(\mathbb{T})$, $1 \le p \le \infty$, can be identified with the periodic Lebesgue space $L_{p\#} = L_{p\#}(\mathbb{R}^n)$ that consists of functions $\phi \in L_{p,\mathrm{loc}}(\mathbb{R}^n)$, which satisfy the periodicity condition (1.4) for a.e. $\mathbf{x}$.

The space dual to $\mathcal{D}(\mathbb{T})$, that is, the space of linear bounded functionals on $\mathcal{D}(\mathbb{T})$, called the space of torus distributions, is denoted by $\mathcal{D}'(\mathbb{T})$ and can be identified with the space of periodic distributions $\mathcal{D}'_\#$ acting on $C^\infty_\#$.

The toroidal/periodic Fourier transform mapping a function $g \in C^\infty_\#$ to a set of its Fourier coefficients $\hat{g}$ is defined as (see, e.g., [25, Definition 3.1.8])

$$\hat{g}(\xi) = [\mathcal{F}_\mathbb{T} g](\xi) := \int_\mathbb{T} e^{-2\pi i \mathbf{x}\cdot\xi} g(\mathbf{x}) d\mathbf{x}, \quad \xi \in \mathbb{Z}^n,$$

and can be generalized to the Fourier transform acting on a distribution $g \in \mathcal{D}'_\#$.

For any $\xi \in \mathbb{Z}^n$, let $|\xi| := \left(\sum_{j=1}^n \xi_j^2\right)^{1/2}$ be the Euclidean norm in $\mathbb{Z}^n$ and let us denote $\varrho(\xi) := 2\pi(1+|\xi|^2)^{1/2}$. Evidently, $\frac{1}{2}\varrho(\xi)^2 \le |2\pi\xi|^2 \le \varrho(\xi)^2 \quad \forall\, \xi \in \dot{\mathbb{Z}}^n$.

Similar to [25, Definition 3.2.2], for $s \in \mathbb{R}$ we define the *periodic/toroidal Sobolev (Bessel-potential) spaces* $H^s_\# := H^s_\#(\mathbb{R}^n) := H^s(\mathbb{T})$ that consist of the torus distributions $g \in \mathcal{D}'(\mathbb{T})$, for which the norm

$$\|g\|_{H^s_\#} := \|\varrho^s \hat{g}\|_{\ell_2(\mathbb{Z}^n)} := \left(\sum_{\xi\in\mathbb{Z}^n} \varrho(\xi)^{2s} |\hat{g}(\xi)|^2\right)^{1/2} \tag{1.5}$$

is finite, that is, the series in (1.5) converges. Here, $\|\cdot\|_{\ell_2(\mathbb{Z}^n)}$ is the standard norm in the space of square summable sequences with indices in $\mathbb{Z}^n$. Evidently, $H^0_\# = L_{2\#}$.

For $g \in H^s_\#$, $s \in \mathbb{R}$, we can write $g(\mathbf{x}) = \sum_{\xi\in\mathbb{Z}^n} \hat{g}(\xi) e^{2\pi i \mathbf{x}\cdot\xi}$, where the Fourier series converges in the sense of norm (1.5). Moreover, because g is an arbitrary distribution from $H^s_\#$, this also implies that the space $C^\infty_\#$ is dense in $H^s_\#$ for any $s \in \mathbb{R}$ (cf. [25, Exercise 3.2.9]).

There holds the compact embedding $H^t_\# \hookrightarrow H^s_\#$ if $t > s$, embeddings $H^s_\# \subset C^m_\#$ if $m \in \mathbb{N}_0$, $s > m + \frac{n}{2}$, and moreover, $\bigcap_{s\in\mathbb{R}} H^s_\# = C^\infty_\#$ (cf. [25, Exercises 3.2.10 and 3.2.10, Corollary 3.2.11]). Note that the periodic norms on $H^s_\#$ are equivalent to the corresponding standard (nonperiodic) Bessel-potential norms on $\mathbb{T}$ as an n-cubic domain; see, for example, [23, Section 13.8.1].

Let

$$(\Lambda^r_\# \, g)(\mathbf{x}) := \sum_{\xi\in\mathbb{Z}^n} \varrho(\xi)^r \hat{g}(\xi) e^{2\pi i \mathbf{x}\cdot\xi} \quad \forall\, g \in H^s_\# \tag{1.6}$$

denote the periodic Bessel-potential operator of the order $r \in \mathbb{R}$. For any $s \in \mathbb{R}$, the operator

$$\Lambda^r_\# : H^s_\# \to H^{s-r}_\# \tag{1.7}$$

is continuous; see, for example, [23, Section 13.8.1].

By (1.5), $\|g\|^2_{H^s_\#} = |\hat{g}(\mathbf{0})|^2 + |g|^2_{H^s_\#}$, where

$$|g|_{H^s_\#} := \|\varrho^s \hat{g}\|_{\ell_2(\dot{\mathbb{Z}}^n)} := \left(\sum\nolimits_{\xi\in\dot{\mathbb{Z}}^n} \varrho(\xi)^{2s} |\hat{g}(\xi)|^2\right)^{1/2}$$

is the seminorm in $H^s_\#$. For any $s \in \mathbb{R}$, let us also introduce the space $\dot{H}^s_\# := \{g \in H^s_\# : \langle g, 1\rangle_\mathbb{T} = 0\}$. The definition implies that if $g \in \dot{H}^s_\#$, then $\hat{g}(\mathbf{0}) = 0$ and $\|g\|_{\dot{H}^s_\#} = \|g\|_{H^s_\#} = |g|_{H^s_\#} = \|\varrho^s \hat{g}\|_{\ell_2(\dot{\mathbb{Z}}^n)}$. The dual product between $g_1 \in \dot{H}^s_\#$ and $f_2 \in (\dot{H}^s_\#)^*$, $s \in \mathbb{R}$, is represented as $\langle g_1, f_2\rangle_\mathbb{T} := \sum_{\xi\in\dot{\mathbb{Z}}^n} \hat{g}_1(\xi) \hat{f}_2(-\xi)$. If $\hat{g}(\mathbf{0}) = 0$, then (1.6) implies that $\widehat{\Lambda^r_\# \, g}(\mathbf{0}) = 0$, and thus, the operator

$$\Lambda^r_\# : \dot{H}^s_\# \to \dot{H}^{s-r}_\# \tag{1.8}$$

is continuous as well. Due to the Riesz representation theorem, $(\dot{H}^s_\#)^* = \dot{H}^{-s}_\#$, as shown in [19, Section 2].

Denoting $\dot{C}^\infty_\# := \{g \in C^\infty_\# : \langle g, 1\rangle_\mathbb{T} = 0\}$, then $\bigcap_{s\in\mathbb{R}} \dot{H}^s_\# = \dot{C}^\infty_\#$. The corresponding spaces of n-component vector functions/distributions are denoted as $\mathbf{L}_{q\#} := (L_{q\#})^n$, $\mathbf{H}^s_\# := (H^s_\#)^n$, and so forth.

Note that the norm $\|\nabla(\cdot)\|_{\mathbf{H}^{s-1}_\#}$ is an equivalent norm in $\dot{H}^s_\#$. Indeed,

$$\nabla g(\mathbf{x}) = 2\pi i \sum_{\xi\in\dot{\mathbb{Z}}^n} \xi e^{2\pi i \mathbf{x}\cdot\xi} \hat{g}(\xi), \quad \widehat{\nabla g}(\xi) = 2\pi i \xi \hat{g}(\xi) \quad \forall\, g \in \dot{H}^s_\#,$$

and then,

$$\begin{aligned}
&\frac{1}{2}|g|^2_{H^s_\#} \le \|\nabla g\|^2_{\mathbf{H}^{s-1}_\#} \le |g|^2_{H^s_\#} \quad \forall\, g \in H^s_\#,\\
&\frac{1}{2}\|g\|^2_{H^s_\#} = \frac{1}{2}\|g\|^2_{\dot{H}^s_\#} = \frac{1}{2}|g|^2_{H^s_\#} \le \|\nabla g\|^2_{\mathbf{H}^{s-1}_\#} \le |g|^2_{H^s_\#}\\
&\qquad = \|g\|^2_{\dot{H}^s_\#} = \|g\|^2_{H^s_\#} \quad \forall\, g \in \dot{H}^s_\#.
\end{aligned} \tag{1.9}$$

The vector counterpart of (1.9) takes form

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{v}\|^2_{\mathbf{H}^s_\#} = \frac{1}{2}\|\mathbf{v}\|^2_{\dot{\mathbf{H}}^s_\#} \le \|\nabla\mathbf{v}\|^2_{(H^{s-1}_\#)^{n\times n}} \le \|\mathbf{v}\|^2_{\dot{\mathbf{H}}^s_\#}\\
&\qquad = \|\mathbf{v}\|^2_{\mathbf{H}^s_\#} \quad \forall\, \mathbf{v} \in \dot{\mathbf{H}}^s_\#.
\end{aligned} \tag{1.10}$$

Note that the second inequalities in (1.9) and (1.10) are valid also in wider spaces, that is, for $g \in H^s_\#$ and $\mathbf{v} \in \mathbf{H}^s_\#$, respectively.

Let us also define the Sobolev spaces of divergence-free functions and distributions,

$$\dot{\mathbf{H}}^s_{\#\sigma} := \left\{\mathbf{w} \in \dot{\mathbf{H}}^s_\# : \operatorname{div} \mathbf{w} = 0\right\}, \quad s \in \mathbb{R},$$

endowed with the same norm (1.5). Similarly, $\mathbf{C}^\infty_{\#\sigma}$ and $\mathbf{L}_{q\#\sigma}$ denote the subspaces of divergence-free vector functions from $\mathbf{C}^\infty_\#$ and $\mathbf{L}_{q\#}$, respectively.

In addition, see [19, Section 2], $(\dot{\mathbf{H}}^s_{\#\sigma})^* = \dot{\mathbf{H}}^{-s}_{\#\sigma}$. Note that for any $r, s \in \mathbb{R}$, the operator

$$\Lambda^r_\# : \dot{\mathbf{H}}^s_{\#\sigma} \to \dot{\mathbf{H}}^{s-r}_{\#\sigma} \tag{1.11}$$

defined as in (1.6) is continuous. Let us also introduce the space $\dot{\mathbf{H}}^s_{\#g} := \left\{\mathbf{w} = \nabla q,\ q \in \dot{H}^{s+1}_\#\right\}$, $s \in \mathbb{R}$, endowed with the norm (1.5).

The following assertion is produced in [19, Theorem 1].

Theorem 1.1. *Let $s \in \mathbb{R}$ and $n \geq 2$.*

a. *The space $\dot{\mathbf{H}}^s_\#$ has the Helmholtz–Weyl decomposition, $\dot{\mathbf{H}}^s_\# = \dot{\mathbf{H}}^s_{\#g} \oplus \dot{\mathbf{H}}^s_{\#\sigma}$, that is, any $\mathbf{F} \in \dot{\mathbf{H}}^s_\#$ can be uniquely represented as $\mathbf{F} = \mathbf{F}_g + \mathbf{F}_\sigma$, where $\mathbf{F}_g \in \dot{\mathbf{H}}^s_{\#g}$ and $\mathbf{F}_\sigma \in \dot{\mathbf{H}}^s_{\#\sigma}$.*

b. *The spaces $\dot{\mathbf{H}}^s_{\#g}$ and $\dot{\mathbf{H}}^s_{\#\sigma}$ are orthogonal subspaces of $\dot{\mathbf{H}}^s_\#$ in the sense of inner product, that is, $(\mathbf{w}, \mathbf{v})_{H^s_\#} = 0$ for any $\mathbf{w} \in \dot{\mathbf{H}}^s_{\#g}$ and $\mathbf{v} \in \dot{\mathbf{H}}^{-s}_{\#\sigma}$.*

c. *The spaces $\dot{\mathbf{H}}^s_{\#g}$ and $\dot{\mathbf{H}}^{-s}_{\#\sigma}$ are orthogonal in the sense of dual product, that is, $\langle \mathbf{w}, \mathbf{v} \rangle = 0$ for any $\mathbf{w} \in \dot{\mathbf{H}}^s_{\#g}$ and $\mathbf{v} \in \dot{\mathbf{H}}^{-s}_{\#\sigma}$.*

d. *There exist the bounded orthogonal projector operators $\mathbb{P}_g : \dot{\mathbf{H}}^s_\# \to \dot{\mathbf{H}}^s_{\#g}$ and $\mathbb{P}_\sigma : \dot{\mathbf{H}}^s_\# \to \dot{\mathbf{H}}^s_{\#\sigma}$ (the Leray projector), while $\mathbf{F} = \mathbb{P}_g \mathbf{F} + \mathbb{P}_\sigma \mathbf{F}$ for any $\mathbf{F} \in \dot{\mathbf{H}}^s_\#$.*

For the evolution problems, we will systematically use the spaces $L_q(0, T; H^s_\#)$, $s \in \mathbb{R}$, $1 \leq q \leq \infty$, $0 < T < \infty$, which consist of functions that map $t \in (0, T)$ to a function or distributions from $H^s_\#$. For $1 \leq q < \infty$, the space $L_q(0, T; H^s_\#)$ is endowed with the norm

$$\|h\|_{L_q(0,T;H^s_\#)} = \left(\int_0^T \|h(\cdot, t)\|^q_{H^s_\#} dt \right)^{1/q} = \left(\int_0^T \left[\sum_{\xi \in \mathbb{Z}^n} \varrho(\xi)^{2s} |\hat{h}(\xi, t)|^2 \right]^{q/2} dt \right)^{1/q} < \infty,$$

and for $q = \infty$ with the norm

$$\|h\|_{L_\infty(0,T;H^s_\#)} = \text{ess sup}_{t \in (0,T)} \|h(\cdot, t)\|_{H^s_\#} = \text{ess sup}_{t \in (0,T)} \left[\sum_{\xi \in \mathbb{Z}^n} \varrho(\xi)^{2s} |\hat{h}(\xi, t)|^2 \right]^{1/2} < \infty.$$

For a function (or distribution) $h(\mathbf{x}, t)$, we will use the notation

$$h'(\mathbf{x}, t) := \partial_t h(\mathbf{x}, t) := \frac{\partial}{\partial t} h(\mathbf{x}, t),$$
$$h^{(j)}(\mathbf{x}, t) := \partial^j_t h(\mathbf{x}, t) := \frac{\partial^j}{\partial t^j} h(\mathbf{x}, t),$$

for the partial derivatives in the time variable t.

Let X and Y be some Hilbert spaces. We will further need the space

$$W^1(0, T; X, Y) := \{u \in L_2(0, T; X) : u' \in L_2(0, T; Y)\}$$

endowed with the norm $\|u\|_{W^1(0,T;X,Y)} = (\|u\|^2_{L_2(0,T;X)} + \|u'\|^2_{L_2(0,T;Y)})^{1/2}$. Spaces of such type are considered in [27, Chapter 1, Section 2.2]. We will particularly need the spaces $W^1(0, T; H^s_\#, H^{s'}_\#)$ and their vector counterparts.

We will also employ the following spaces for $k \in \mathbb{N}$; compare, for example, [27, Chapter 1, Section 1.3, Remark 1.5],

$$W^k(0, T; X) := \{u \in L_2(0, T; X) : \partial^j_t u \in X,\ j = 1, \dots, k\},$$

endowed with the norm $\|u\|_{W^k(0,T;X)} = \left(\sum_{j=0}^k \|\partial^j_t u\|^2_{L_2(0,T;X)} \right)^{1/2}$.

Unless stated otherwise, we will assume in this paper that all vector and scalar variables are real valued (however, with complex-valued Fourier coefficients).

2 | Existence Results Available for Evolution Spatially Periodic Anisotropic Navier–Stokes Problem

Let us consider the following Navier–Stokes problem for the real-valued unknowns $(\mathbf{u}, p)$,

$$\mathbf{u}' - \mathfrak{L}\mathbf{u} + \nabla p + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{f} \quad \text{in } \mathbb{T} \times (0, T), \tag{2.1}$$

$$\text{div } \mathbf{u} = 0 \quad \text{in } \mathbb{T} \times (0, T), \tag{2.2}$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}^0 \text{ in } \mathbb{T}, \tag{2.3}$$

with given data $\mathbf{f} \in L_2(0, T; \dot{\mathbf{H}}^{-1}_\#)$, $\mathbf{u}^0 \in \dot{\mathbf{H}}^0_{\#\sigma}$. Note that the time-trace $\mathbf{u}(\cdot, 0)$ for $\mathbf{u}$ solving the weak form of (2.1–2.2) is well defined; see Definition 2.1 and Remark 2.3.

Let us introduce the bilinear form

$$a_\mathbb{T}(\mathbf{u}, \mathbf{v}) = a_\mathbb{T}(t; \mathbf{u}, \mathbf{v}) := \left\langle a^{\alpha\beta}_{ij}(\cdot, t) E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{v}) \right\rangle_\mathbb{T} \quad \forall\, \mathbf{u}, \mathbf{v} \in \dot{\mathbf{H}}^1_\#. \tag{2.4}$$

By the boundedness condition (1.3) and inequality (1.10), we have

$$\begin{aligned} |a_\mathbb{T}(t; \mathbf{u}, \mathbf{v})| &\leq \|\mathbb{A}\| \|\mathbb{E}(\mathbf{u})\|_{L^{n\times n}_{2\#}} \|\mathbb{E}(\mathbf{v})\|_{L^{n\times n}_{2\#}} \\ &\leq \|\mathbb{A}\| \|\nabla \mathbf{u}\|_{L^{n\times n}_{2\#}} \|\nabla \mathbf{v}\|_{L^{n\times n}_{2\#}} \\ &\leq \|\mathbb{A}\| \|\mathbf{u}\|_{\dot{\mathbf{H}}^1_\#} \|\mathbf{v}\|_{\dot{\mathbf{H}}^1_\#} \quad \forall\, \mathbf{u}, \mathbf{v} \in \dot{\mathbf{H}}^1_\#. \end{aligned} \tag{2.5}$$

If the relaxed ellipticity condition (1.2) holds, taking into account the relation $\sum_{i=1}^n E_{ii}(\mathbf{w}) = \text{div } \mathbf{w} = 0$ for $\mathbf{w} \in \dot{\mathbf{H}}^1_{\#\sigma}$, equivalence of the norm $\|\nabla(\cdot)\|_{L^{n\times n}_{2\#}}$ to the norm $\|\cdot\|_{\dot{\mathbf{H}}^1_{\#\sigma}}$ in $\dot{\mathbf{H}}^1_{\#\sigma}$, see (1.10), and the first Korn inequality (6.19), we obtain

$$\begin{aligned} a_\mathbb{T}(t; \mathbf{w}, \mathbf{w}) &= \left\langle a^{\alpha\beta}_{ij}(\cdot, t) E_{j\beta}(\mathbf{w}), E_{i\alpha}(\mathbf{w}) \right\rangle_\mathbb{T} \\ &\geq C^{-1}_\mathbb{A} \|\mathbb{E}(\mathbf{w})\|^2_{L^{n\times n}_{2\#}} \\ &\geq \frac{1}{2} C^{-1}_\mathbb{A} \|\nabla \mathbf{w}\|^2_{L^{n\times n}_{2\#}} \geq \frac{1}{4} C^{-1}_\mathbb{A} \|\mathbf{w}\|^2_{\dot{\mathbf{H}}^1_{\#\sigma}} \quad \forall\, \mathbf{w} \in \dot{\mathbf{H}}^1_{\#\sigma}. \end{aligned} \tag{2.6}$$

Then, (2.5) and (2.6) give

$$\frac{1}{4} C^{-1}_\mathbb{A} \|\mathbf{w}\|^2_{\dot{\mathbf{H}}^1_{\#\sigma}} \leq a_\mathbb{T}(t; \mathbf{w}, \mathbf{w}) \leq \|\mathbb{A}\| \|\mathbf{w}\|^2_{\dot{\mathbf{H}}^1_{\#\sigma}} \quad \forall\, \mathbf{w} \in \dot{\mathbf{H}}^1_{\#\sigma}. \tag{2.7}$$

Let us denote

$$\mathbf{F} := \mathbf{f} + \mathfrak{L}\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u}. \tag{2.8}$$

Let $\mathbf{u} \in \dot{\mathbf{H}}^1_{\#\sigma}$. Acting on (2.1) by the Leray projector $\mathbb{P}_\sigma$ and taking into account that $\mathbb{P}_\sigma \mathbf{u}' = \mathbf{u}'$ and $\mathbb{P}_\sigma \nabla p = \mathbf{0}$, we obtain

$$\mathbf{u}' = \mathbb{P}_\sigma \mathbf{F} = \mathbb{P}_\sigma[\mathbf{f} + \mathfrak{L}\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u}] \quad \text{in } \mathbb{T} \times (0, T). \tag{2.9}$$

On the other hand, acting on (2.1) by the projector $\mathbb{P}_g$ and taking into account that $\mathbb{P}_g\mathbf{u}' = 0$ and $\mathbb{P}_g\nabla p = \nabla p$, we obtain

$$\nabla p = \mathbb{P}_g\mathbf{F} = \mathbb{P}_g[\mathbf{f} + \mathfrak{L}\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u}] \quad \text{in } \mathbb{T}\times(0,T). \tag{2.10}$$

We use the following definition of weak solution given in [19, Definition 1].

Definition 2.1. Let $n \geq 2$, $T > 0$, $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}_{\#}^{-1})$, and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^{0}$. A function $\mathbf{u} \in L_\infty\left(0,T;\dot{\mathbf{H}}_{\#\sigma}^{0}\right) \cap L_2\left(0,T;\dot{\mathbf{H}}_{\#\sigma}^{1}\right)$ is called a weak solution of the evolution space-periodic anisotropic Navier–Stokes initial value problem (2.1–2.3) if it solves the initial-variational problem

$$\begin{aligned}&\left\langle \mathbf{u}'(\cdot,t) + \mathbb{P}_\sigma[(\mathbf{u}(\cdot,t)\cdot\nabla)\mathbf{u}(\cdot,t)], \mathbf{w}\right\rangle_{\mathbb{T}} + a_{\mathbb{T}}(\mathbf{u}(\cdot,t),\mathbf{w})\\ &= \langle \mathbf{f}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}, \text{ for a.e. } t\in(0,T),\ \forall\ \mathbf{w}\in\dot{\mathbf{H}}_{\#\sigma}^{1},\end{aligned} \tag{2.11}$$

$$\langle \mathbf{u}(\cdot,0),\mathbf{w}\rangle_{\mathbb{T}} = \langle \mathbf{u}^0,\mathbf{w}\rangle_{\mathbb{T}},\ \forall\ \mathbf{w}\in\dot{\mathbf{H}}_{\#\sigma}^{0}. \tag{2.12}$$

The associated pressure p is a distribution on $\mathbb{T}\times(0,T)$ satisfying (2.1) in the sense of distributions, that is,

$$\begin{aligned}&\left\langle \mathbf{u}'(\cdot,t) + (\mathbf{u}(\cdot,t)\cdot\nabla)\mathbf{u}(\cdot,t),\mathbf{w}\right\rangle_{\mathbb{T}} + a_{\mathbb{T}}(\mathbf{u}(\cdot,t),\mathbf{w}) + \langle\nabla p(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}\\ &= \langle \mathbf{f}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}, \text{ for a.e. } t\in(0,T),\ \forall\ \mathbf{w}\in\mathbf{C}_{\#}^{\infty}.\end{aligned}$$

The following assertion is proved in [19, Lemma 1].

Lemma 2.2. *Let* $n \geq 2$, $T > 0$, $a_{ij}^{\alpha\beta} \in L_\infty(0,T;L_{\infty\#})$, $\mathbf{f}\in L_2(0,T;\dot{\mathbf{H}}_{\#}^{-1})$ *and* $\mathbf{u}^0\in\dot{\mathbf{H}}_{\#\sigma}^{0}$. *Let* $\mathbf{u}\in L_\infty\left(0,T;\dot{\mathbf{H}}_{\#\sigma}^{0}\right)\cap L_2\left(0,T;\dot{\mathbf{H}}_{\#\sigma}^{1}\right)$ *solve Equation (2.11).*

i. *Then,*

$$\begin{aligned}&\mathbf{Du} := \mathbf{u}' + \mathbb{P}_\sigma[(\mathbf{u}\cdot\nabla)\mathbf{u}] \in L_2(0,T;\dot{\mathbf{H}}_{\#\sigma}^{-1}) \quad \textit{and}\\ &\mathbf{Du}(\cdot,t)\in\dot{\mathbf{H}}_{\#\sigma}^{-1} \quad \textit{for a.e. } t\in[0,T],\end{aligned} \tag{2.13}$$

while

$$\begin{aligned}&(\mathbf{u}\cdot\nabla)\mathbf{u}\in L_2(0,T;\dot{\mathbf{H}}_{\#}^{-n/2}) \quad \textit{and}\\ &(\mathbf{u}\cdot\nabla)\mathbf{u}(\cdot,t)\in\dot{\mathbf{H}}_{\#}^{-n/2} \quad \textit{for a.e. } t\in[0,T],\\ &\mathbf{u}'\in L_2(0,T;\dot{\mathbf{H}}_{\#\sigma}^{-n/2}) \quad \textit{and} \quad \mathbf{u}'(\cdot,t)\in\dot{\mathbf{H}}_{\#\sigma}^{-n/2}\\ &\textit{for a.e. } t\in[0,T],\end{aligned}$$

and hence, $\mathbf{u}\in W^1(\dot{\mathbf{H}}_{\#\sigma}^{1},\dot{\mathbf{H}}_{\#\sigma}^{-n/2})$.
In addition, $\partial_t\|\mathbf{u}\|^2_{\dot{\mathbf{H}}_{\#\sigma}^{-(n-2)/4}} = 2\langle\Lambda_{\#}^{-n/2}\mathbf{u}',\Lambda_{\#}\mathbf{u}\rangle_{\mathbb{T}} = 2\langle\mathbf{u}',\Lambda_{\#}^{1-n/2}\mathbf{u}\rangle_{\mathbb{T}} = 2\langle\Lambda_{\#}^{1-n/2}\mathbf{u}',\mathbf{u}\rangle_{\mathbb{T}}$ *for a.e.* $t\in(0,T)$ *and also in the distribution sense on* $(0,T)$.

ii. *Moreover,* $\mathbf{u}$ *is almost everywhere on* $[0,T]$ *equal to a function* $\tilde{\mathbf{u}}\in C^0([0,T];\dot{\mathbf{H}}_{\#\sigma}^{-(n-2)/4})$, *and* $\tilde{\mathbf{u}}$ *is also* $\dot{\mathbf{H}}_{\#\sigma}^{0}$*-weakly continuous in time on* $[0,T]$, *that is,* $\lim_{t\to t_0}\langle\tilde{\mathbf{u}}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}} = \langle\tilde{\mathbf{u}}(\cdot,t_0),\mathbf{w}\rangle_{\mathbb{T}}\quad\forall\ \mathbf{w}\in\mathbf{H}_{\#}^{0},\ \forall\ t_0\in[0,T]$.

iii. *There exists the associated pressure* $p\in L_2(0,T;\dot{H}_{\#}^{-n/2+1})$ *that for the given* $\mathbf{u}$ *is the unique solution of Equation (2.1) in this space.*

Remark 2.3. The initial condition (2.12) should be understood for the function $\mathbf{u}$ redefined as the function $\tilde{\mathbf{u}}$ that was introduced in Lemma 2.2(ii) and is $\mathbf{H}_{\#}^{0}$-weakly continuous in time.

The following existence theorem was proved in [19, Theorem 2].

Theorem 2.4. (Existence). *Let* $n \geq 2$ *and* $T > 0$. *Let* $a_{ij}^{\alpha\beta}\in L_\infty(0,T;L_{\infty\#})$ *and the relaxed ellipticity condition (1.2) hold. Let* $\mathbf{f}\in L_2(0,T;\dot{\mathbf{H}}_{\#}^{-1})$, $\mathbf{u}^0\in\dot{\mathbf{H}}_{\#\sigma}^{0}$.

i. *Then, there exists a weak solution* $\mathbf{u}\in L_\infty\left(0,T;\dot{\mathbf{H}}_{\#\sigma}^{0}\right)\cap L_2\left(0,T;\dot{\mathbf{H}}_{\#\sigma}^{1}\right)$ *of the anisotropic Navier–Stokes initial value problem (2.1–2.3) in the sense of Definition 2.1. Particularly,* $\lim_{t\to0}\langle\mathbf{u}(\cdot,t),\mathbf{v}\rangle_{\mathbb{T}} = \langle\mathbf{u}^0,\mathbf{v}\rangle_{\mathbb{T}}\quad\forall\ \mathbf{v}\in\dot{\mathbf{H}}_{\#\sigma}^{0}$. *There exists also the unique pressure* $p\in L_2(0,T;\dot{H}_{\#}^{-n/2+1})$ *associated with the obtained* $\mathbf{u}$, *that is the solution of Equation (2.1) in* $L_2(0,T;\dot{H}_{\#}^{-n/2+1})$.

ii. *Moreover,* $\mathbf{u}$ *satisfies the following (strong) energy inequality,*

$$\begin{aligned}&\frac{1}{2}\|\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}_{2\#}} + \int_{t_0}^{t} a_{\mathbb{T}}(\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))d\tau\\ &\leq \frac{1}{2}\|\mathbf{u}(\cdot,t_0)\|^2_{\mathbf{L}_{2\#}} + \int_{t_0}^{t}\langle\mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}d\tau,\end{aligned}$$

for any $[t_0,t]\subset[0,T]$. *It particularly implies the standard energy inequality,*

$$\begin{aligned}&\frac{1}{2}\|\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}_{2\#}} + \int_{0}^{t} a_{\mathbb{T}}(\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))d\tau\\ &\leq \frac{1}{2}\|\mathbf{u}^0\|^2_{\mathbf{L}_{2\#}} + \int_{0}^{t}\langle\mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}\ d\tau \quad \forall\ t\in[0,T].\end{aligned} \tag{2.14}$$

3 | Serrin-Type Solutions and Their Properties

For the isotropic constant-coefficient homogeneous Navier–Stokes equations, it is well known that the weak solution satisfying the famous Ladyzhenskaya–Prodi–Serrin condition (accommodated here for the periodic setting)

$$\mathbf{u}\in L_{\tilde{q}}(0,T;\dot{\mathbf{L}}_{\#q}) \tag{3.1}$$

for some $\tilde{q}$ and q such that

$$\frac{2}{\tilde{q}} + \frac{n}{q} = 1, \quad n < q < \infty, \tag{3.2}$$

is unique in the class of weak solutions satisfying the energy inequality, for $n \leq 4$; the energy equality and the regularity results are also proved under the Prodi–Serrin conditions; see, for example, [28], [29], [6, Chapter 1, Theorem 6.9, and Remark 6.8], [4, Section 14], [7, Section 8.5], [8, Theorem 7.17], [9, Section 1.5].

In this paper, we limit ourself to the L_2-based Sobolev spaces with respect to the spatial variables and hence introduce a corresponding particular counterpart of the class of solutions satisfying the conditions close to (3.1) and (3.2) and leading to the Serrin-type results.

3.1 | Serrin-Type Solutions and Their Properties for $n \geq 2$

Definition 3.1. Let $n \geq 2$, $T > 0$, $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}_\#^{-1})$, and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^0$. If a solution $\mathbf{u}$ of the initial-variational problem (2.11)–(2.12) belongs to $L_2(0,T;\dot{\mathbf{H}}_{\#\sigma}^{n/2})$, we will call it a *Serrin-type solution*.

The inclusion $\mathbf{u} \in L_2(0,T;\dot{\mathbf{H}}_{\#\sigma}^{n/2})$ can be considered as counterpart of the Prodi–Serrin condition (3.1) in L_2-based Sobolev spaces. Indeed, by the Sobolev embedding theorem, Theorem 6.6, we obtain that the following continuous embeddings hold, $\dot{\mathbf{H}}_{\#\sigma}^{n/2} \subset \mathbf{H}_\#^{n/2} \subset \mathbf{L}_{\#q}$ for any $q \in (2,\infty)$. Hence, if $\mathbf{u} \in L_2(0,T;\dot{\mathbf{H}}_{\#\sigma}^{n/2})$, then for any $\epsilon > 0$, there exists $q_\epsilon \in (2,\infty)$ such that $\mathbf{u} \in L_{\widetilde{q}}(0,T;\mathbf{L}_{\#q_\epsilon})$ for $\widetilde{q} = 2$ and

$$\frac{2}{\widetilde{q}} + \frac{n}{q_\epsilon} = 1 + \epsilon. \tag{3.3}$$

Condition (3.3) is weaker than condition (3.2) by the arbitrarily small $\epsilon > 0$, but in spite of this, we will be able to show that the inclusion $\mathbf{u} \in L_2(0,T;\dot{\mathbf{H}}_{\#\sigma}^{n/2})$ for the weak solution is sufficient to prove for it the Serrin-type results about the energy equality, uniqueness, and regularity, which justifies the chosen Serrin-type solution name. We will also prove the existence of such solutions, under appropriate conditions.

Definition 3.2. Let $n \geq 2, T > 0, \mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}_\#^{-1})$, and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^0$. If a solution $\mathbf{u}$ of the initial-variational problem (2.11) and (2.12) belongs to $L_2(0,T;\dot{\mathbf{H}}_{\#\sigma}^2)$, we will call it a *a strong solution*.

The above definition of the strong solution is a bit weaker than, for example, in [7, Definition 6.1] or [6, Chapter 1, Section 6.7], because it does not explicitly require the additional inclusion $\mathbf{u} \in L_\infty\left(0,T;\dot{\mathbf{H}}_{\#\sigma}^1\right)$ or $\mathbf{u}' \in L_\infty\left(0,T;\dot{\mathbf{H}}_{\#\sigma}^0\right)$.

Remark 3.3. Definitions 3.1 and 3.2 imply that the strong solutions are also Serrin-type solutions if $n \in \{2,3,4\}$.

The Serrin-type solutions are also strong solutions if $n \geq 4$. Some sufficient conditions for the Serrin-type solution existence are provided in Section 5.2 further on in the paper.

If $n \in \{2,3\}$, then for a Serrin-type solution to be also a strong solution, the Serrin-type solution should have an additional regularity and the sufficient conditions for this are provided by the regularity theorems and corollaries in Sections 4 and 5, with the parameter $r \geq 1$ there.

Lemma 3.4. *Let $n \geq 2$, $T > 0$, $a_{ij}^{\alpha\beta} \in L_\infty(0,T;L_{\infty\#})$, $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}_\#^{-1})$, and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^0$. Let $\mathbf{u}$ be a Serrin-type solution. Then, $(\mathbf{u}\cdot\nabla)\mathbf{u} \in L_2(0,T;\mathbf{H}_\#^{-1})$, $\mathbf{u}' \in L_2(0,T;\mathbf{H}_{\#\sigma}^{-1})$, and hence, $\mathbf{u} \in W^1(\dot{\mathbf{H}}_{\#\sigma}^{n/2},\dot{\mathbf{H}}_{\#\sigma}^{-1})$ and $\mathbf{u} \in C^0([0,T];\dot{\mathbf{H}}_{\#\sigma}^{n/4-1/2}) \subset C^0([0,T];\dot{\mathbf{H}}_{\#\sigma}^0)$. Moreover,*

$$\begin{aligned}&\left\langle \mathbf{u}'(\cdot,t),\mathbf{w}\right\rangle_{\mathbb{T}} + \langle(\mathbf{u}(\cdot,t)\cdot\nabla)\mathbf{u}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}} + a_{\mathbb{T}}(t;\mathbf{u},\mathbf{w})\\ &= \langle\mathbf{f}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}}, \ \textit{for a.e. } t \in (0,T), \ \forall\, \mathbf{w} \in \dot{\mathbf{H}}_{\#\sigma}^1.\end{aligned} \tag{3.4}$$

The unique pressure p associated with the obtained $\mathbf{u}$ belongs to $L_2(0,T;\dot{H}_\#^0)$.

Proof. By relation (1.10), multiplication Theorem 6.1(b), and the Sobolev interpolation inequality (6.16),

$$\begin{aligned}\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{\mathbf{H}_\#^{-1}} &= \|\nabla\cdot(\mathbf{u}\otimes\mathbf{u})\|_{\mathbf{H}_\#^{-1}} \leq \|\mathbf{u}\otimes\mathbf{u}\|_{(H_\#^0)^{n\times n}}\\ &\leq C_{*n}\|\mathbf{u}\|^2_{\mathbf{H}_\#^{n/4}} \leq C_{*n}\|\mathbf{u}\|_{\mathbf{H}_\#^0}\|\mathbf{u}\|_{\mathbf{H}_\#^{n/2}},\end{aligned} \tag{3.5}$$

where $C_{*n} = C_*(n/4,n/4,n)$. Hence,

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L_2(0,T;\mathbf{H}_\#^{-1})} \leq C_{*n}\|\mathbf{u}\|_{L_\infty(0,T;\mathbf{H}_\#^0)}\|\mathbf{u}\|_{L_2(0,T;\mathbf{H}_\#^{n/2})}, \tag{3.6}$$

that is, $(\mathbf{u}\cdot\nabla)\mathbf{u} \in L_2(0,T;\mathbf{H}_\#^{-1})$ and (2.13) implies that $\mathbf{u}' \in L_2(0,T;\mathbf{H}_{\#\sigma}^{-1})$. Hence, $\mathbf{u} \in W^1(\dot{\mathbf{H}}_{\#\sigma}^{n/2},\dot{\mathbf{H}}_{\#\sigma}^{-1})$, and Theorem 6.8 implies that $\mathbf{u} \in C^0([0,T];\dot{\mathbf{H}}_{\#\sigma}^{n/4-1/2}) \subset C^0([0,T];\dot{\mathbf{H}}_{\#\sigma}^0)$. Because of this, Equation (2.11) for function $\mathbf{u}$ now reduces to (3.4).

To prove the lemma claim about the associated pressure p, we remark that it satisfies (2.10), where $\mathbf{F} \in L_2(0,T;\dot{\mathbf{H}}_\#^{-1})$ due to the lemma conditions and the inclusion $(\mathbf{u}\cdot\nabla)\mathbf{u} \in L_2(0,T;\mathbf{H}_\#^{-1})$. By Lemma 6.5 for gradient, with $s = 0$, Equation (2.10) has a unique solution p in $L_2(0,T;\dot{H}_\#^0)$. □

Let us prove, in the variable-coefficient anisotropic setting, the energy equality and solution uniqueness for the Serrin-type solutions.

Theorem 3.5. (Energy equality for Serrin-type solutions). *Let $n \geq 2, T > 0$, $a_{ij}^{\alpha\beta} \in L_\infty(0,T;L_{\infty\#})$, $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}_\#^{-1})$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^0$. If $\mathbf{u}$ is a Serrin-type solution of the initial-variational problem (2.11) and (2.12), then the following energy equality holds for any $[t_0,t] \subset [0,T]$,*

$$\begin{aligned}&\frac{1}{2}\|\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}_{2\#}} + \int_{t_0}^t a_{\mathbb{T}}(\tau;\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))d\tau =\\ &\frac{1}{2}\|\mathbf{u}(\cdot,t_0)\|^2_{\mathbf{L}_{2\#}} + \int_{t_0}^t \langle\mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}\,d\tau.\end{aligned} \tag{3.7}$$

It particularly implies the standard energy equality,

$$\begin{aligned}&\frac{1}{2}\|\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}_{2\#}} + \int_0^t a_{\mathbb{T}}(\tau;\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))d\tau =\\ &\frac{1}{2}\|\mathbf{u}^0\|^2_{\mathbf{L}_{2\#}} + \int_0^t \langle\mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}\ d\tau \quad \forall\, t \in [0,T].\end{aligned} \tag{3.8}$$

Proof. By Lemma 3.4, the function $\mathbf{u}$ satisfies Equation (3.4), where for a.e. $t \in (0,T)$, we can employ $\mathbf{u}$ as $\mathbf{w}$ to obtain

$$\begin{aligned}&\left\langle \mathbf{u}'(\cdot,t),\mathbf{u}(\cdot,t)\right\rangle_{\mathbb{T}} + \langle(\mathbf{u}(\cdot,t)\cdot\nabla)\mathbf{u}(\cdot,t),\mathbf{u}(\cdot,t)\rangle_{\mathbb{T}} + a_{\mathbb{T}}(t;\mathbf{u},\mathbf{u}(\cdot,t))\\ &= \langle\mathbf{f}(\cdot,t),\mathbf{u}(\cdot,t)\rangle_{\mathbb{T}}, \ \text{for a.e. } t \in (0,T).\end{aligned} \tag{3.9}$$

Taking into account Lemma 6.9 with $s = 1$, $s' = -1$ for the first dual product and relation (6.4) for the second dual product, we get

$$\begin{aligned}&\frac{1}{2}\partial_t\|\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}_{2\#}} + a_{\mathbb{T}}(t;\mathbf{u}(\cdot,t),\mathbf{u}(\cdot,t))\\ &= \langle\mathbf{f}(\cdot,t),\mathbf{u}(\cdot,t)\rangle_{\mathbb{T}}, \ \text{for a.e. } t \in (0,T).\end{aligned} \tag{3.10}$$

By the inclusions obtained in Lemma 3.4, each dual product and the bilinear form $a_{\mathbb{T}}$ in (3.9) and hence in (3.10) are integrable in t. After integrating (3.10), we obtain (3.7) for a.e. t_0.

By Lemma 3.4, $\mathbf{u} \in C^0([0,T];\dot{\mathbf{H}}^0_{\#\sigma})$, while the integrals in (3.7) are continuous in t_0 as well. Then, we conclude that the energy equality (3.7) holds for any $t_0 \in [0,T)$, implying also (3.8). □

Theorem 3.6. (Uniqueness of Serrin-type solutions). *Let $n \geq 2$, $T > 0$, $a_{ij}^{\alpha\beta} \in L_\infty(0,T;L_{\infty\#})$, $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#})$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}^0_{\#\sigma}$. Let $\mathbf{u}$ be a Serrin-type solution of the initial-variational problem (2.11)–(2.12) on the interval $[0,T]$ and $\mathbf{v}$ be any solution of the initial-variational problem (2.11)–(2.12) satisfying the energy inequality (2.14) on the interval $[0,T]$. Then, $\mathbf{u}=\mathbf{v}$ on $[0,T]$.*

Proof. We will here generalize the proof of Theorem 6.10 in [7].

By Lemma 3.4, the function $\mathbf{u}$ satisfies Equation (3.4), where for a.e. $t \in (0,T)$, we can employ $\mathbf{v}$ as $\mathbf{w}$ to obtain

$$\left\langle \mathbf{u}'(\cdot,t),\mathbf{v}(\cdot,t)\right\rangle_{\mathbb{T}} + \langle(\mathbf{u}(\cdot,t)\cdot\nabla)\mathbf{u}(\cdot,t),\mathbf{v}(\cdot,t)\rangle_{\mathbb{T}} + a_{\mathbb{T}}(t;\mathbf{u}(\cdot,t),\mathbf{v}(\cdot,t)) = \langle \mathbf{f}(\cdot,t),\mathbf{v}(\cdot,t)\rangle_{\mathbb{T}}. \tag{3.11}$$

On the other hand, Equation (2.11) for $\mathbf{v}$ with $\mathbf{u}$ employed for $\mathbf{w}$ can be written for a.e. t as

$$\left\langle \mathbf{v}'(\cdot,t),\mathbf{u}(\cdot,t)\right\rangle_{\mathbb{T}} + \langle(\mathbf{v}(\cdot,t)\cdot\nabla)\mathbf{v}(\cdot,t),\mathbf{u}(\cdot,t)\rangle_{\mathbb{T}} + a_{\mathbb{T}}(t;\mathbf{v}(\cdot,t),\mathbf{u}(\cdot,t)) = \langle \mathbf{f}(\cdot,t),\mathbf{u}(\cdot,t)\rangle_{\mathbb{T}}, \tag{3.12}$$

where we took into account that $\mathbf{u}(t) \in \dot{\mathbf{H}}^{n/2}_{\#\sigma} \subset \dot{\mathbf{H}}^1_{\#\sigma}$ for a.e. t. Adding Equations (3.11) and (3.12) and integrating in time, we obtain

$$\begin{aligned}
&\int_0^t \left[\langle \mathbf{u}'(\cdot,\tau),\mathbf{v}(\cdot,\tau)\rangle_{\mathbb{T}} + \langle \mathbf{v}'(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}\right] d\tau \\
&+ 2\int_0^t a_{\mathbb{T}}(\tau;\mathbf{u}(\cdot,\tau),\mathbf{v}(\cdot,\tau))dt \\
&+ \int_0^t \left[\langle(\mathbf{u}(\cdot,\tau)\cdot\nabla)\mathbf{u}(\cdot,\tau),\mathbf{v}(\cdot,\tau)\rangle_{\mathbb{T}} + \langle(\mathbf{v}(\cdot,\tau)\cdot\nabla)\mathbf{v}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}\right] d\tau \\
&= \int_0^t \langle \mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}} d\tau + \int_0^t \langle \mathbf{f}(\cdot,\tau),\mathbf{v}(\cdot,\tau)\rangle_{\mathbb{T}} d\tau.
\end{aligned} \tag{3.13}$$

By Lemma 2.2(i), $\mathbf{v} \in W^1(0,T;\dot{\mathbf{H}}^1_{\#\sigma},\dot{\mathbf{H}}^{-n/2}_{\#\sigma})$, and hence, by Lemma 6.8, the traces $\mathbf{v}(\cdot,0),\mathbf{v}(\cdot,t) \in \dot{\mathbf{H}}^{1/2-n/4}_{\#\sigma}$ are well defined. On the other hand, by Lemma 3.4, $\mathbf{u} \in W^1(0,T;\dot{\mathbf{H}}^{n/2}_{\#\sigma},\dot{\mathbf{H}}^{-1}_{\#\sigma})$, and hence, by Lemma 6.8, the traces $\mathbf{u}(\cdot,0),\mathbf{u}(\cdot,t) \in \dot{\mathbf{H}}^{n/4-1/2}_{\#\sigma}$ are well defined. Then, due to Lemma 6.9(ii) with $s=n/2$ and $s'=-1$,

$$\begin{aligned}
&\int_0^t \left[\langle \mathbf{u}'(\cdot,\tau),\mathbf{v}(\cdot,\tau)\rangle_{\mathbb{T}} + \langle \mathbf{v}'(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}\right] d\tau \\
&= \langle \mathbf{u}(\cdot,t),\mathbf{v}(\cdot,t)\rangle_{\mathbb{T}} - \langle \mathbf{u}(\cdot,0),\mathbf{v}(\cdot,0)\rangle_{\mathbb{T}}.
\end{aligned} \tag{3.14}$$

Let us denote $\widetilde{\mathbf{w}} := \mathbf{u}-\mathbf{v}$. Because $\mathbf{u}(\cdot,0)=\mathbf{v}(\cdot,0)=\mathbf{u}^0 \in \dot{\mathbf{H}}^0_{\#\sigma}$ and $\mathbf{u},\mathbf{v},\widetilde{\mathbf{w}} \in L_\infty\left(0,T;\dot{\mathbf{H}}^0_{\#\sigma}\right)$, we obtain

$$\langle \mathbf{u}(\cdot,0),\mathbf{v}(\cdot,0)\rangle_{\mathbb{T}} = \|\mathbf{u}^0\|^2_{\dot{\mathbf{H}}^0_{\#\sigma}} = \|\mathbf{v}^0\|^2_{\dot{\mathbf{H}}^0_{\#\sigma}}, \tag{3.15}$$

$$\begin{aligned}
\langle \mathbf{u}(\cdot,t),\mathbf{v}(\cdot,t)\rangle_{\mathbb{T}} &= \frac{1}{2}\langle \mathbf{u}(\cdot,t),\mathbf{u}(\cdot,t)\rangle_{\mathbb{T}} + \frac{1}{2}\langle \mathbf{v}(\cdot,t),\mathbf{v}(\cdot,t)\rangle_{\mathbb{T}} \\
&\quad - \frac{1}{2}\left\langle \widetilde{\mathbf{w}}(\cdot,t),\widetilde{\mathbf{w}}(\cdot,t)\right\rangle_{\mathbb{T}} \\
&= \frac{1}{2}\|\mathbf{u}(\cdot,t)\|^2_{\dot{\mathbf{H}}^0_{\#\sigma}} + \frac{1}{2}\|\mathbf{v}(\cdot,t)\|^2_{\dot{\mathbf{H}}^0_{\#\sigma}} \\
&\quad - \frac{1}{2}\left\|\widetilde{\mathbf{w}}(\cdot,t)\right\|^2_{\dot{\mathbf{H}}^0_{\#\sigma}} \ \text{for a.e. } t \in (0,T).
\end{aligned} \tag{3.16}$$

Due to (2.4),

$$\begin{aligned}
2a_{\mathbb{T}}(\tau;\mathbf{u}(\cdot,\tau),\mathbf{v}(\cdot,\tau)) &= a_{\mathbb{T}}(\tau;\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau)) \\
&\quad + a_{\mathbb{T}}(\tau;\mathbf{v}(\cdot,\tau),\mathbf{v}(\cdot,\tau)) - a_{\mathbb{T}}(\tau;\widetilde{\mathbf{w}}(\cdot,\tau),\widetilde{\mathbf{w}}(\cdot,\tau)).
\end{aligned}$$

By relations (6.3) and (6.4), we obtain

$$\begin{aligned}
&\langle(\mathbf{u}(\cdot,\tau)\cdot\nabla)\mathbf{u}(\cdot,\tau),\mathbf{v}(\cdot,\tau)\rangle_{\mathbb{T}} + \langle(\mathbf{v}(\cdot,\tau)\cdot\nabla)\mathbf{v}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}} \\
&= \langle(\mathbf{v}(\cdot,\tau)\cdot\nabla)\mathbf{v}(\cdot,\tau) - (\mathbf{u}(\cdot,\tau)\cdot\nabla)\mathbf{v}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}} \\
&= \langle(\widetilde{\mathbf{w}}(\cdot,\tau)\cdot\nabla)\widetilde{\mathbf{w}}(\cdot,\tau) - (\widetilde{\mathbf{w}}(\cdot,\tau)\cdot\nabla)\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}} \\
&= \langle(\widetilde{\mathbf{w}}(\cdot,\tau)\cdot\nabla)\widetilde{\mathbf{w}}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}.
\end{aligned} \tag{3.17}$$

Substituting (3.14–3.17) into (3.13), we get

$$\begin{aligned}
&\frac{1}{2}\left\|\widetilde{\mathbf{w}}(\cdot,t)\right\|^2_{\dot{\mathbf{H}}^0_{\#\sigma}} + \int_0^t a_{\mathbb{T}}(\tau;\widetilde{\mathbf{w}}(\cdot,\tau),\widetilde{\mathbf{w}}(\cdot,\tau))dt \\
&\quad - \int_0^t \langle(\widetilde{\mathbf{w}}(\cdot,\tau)\cdot\nabla)\widetilde{\mathbf{w}}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}} d\tau \\
&= A(\mathbf{u};t) + A(\mathbf{v};t) \text{for a.e. } t \in (0,T).
\end{aligned} \tag{3.18}$$

Here,

$$\begin{aligned}
A(\mathbf{u}) := &\frac{1}{2}\|\mathbf{u}(\cdot,t)\|^2_{\dot{\mathbf{H}}^0_{\#\sigma}} + \int_0^t a_{\mathbb{T}}(\tau;\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))dt \\
&- \frac{1}{2}\|\mathbf{u}^0\|^2_{\dot{\mathbf{H}}^0_{\#\sigma}} - \int_0^t \langle \mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}} d\tau = 0
\end{aligned} \tag{3.19}$$

by the energy equality condition (3.8) for $\mathbf{u}$, while

$$\begin{aligned}
A(\mathbf{v}) := &\frac{1}{2}\|\mathbf{v}(\cdot,t)\|^2_{\dot{\mathbf{H}}^0_{\#\sigma}} + \int_0^t a_{\mathbb{T}}(\tau;\mathbf{v}(\cdot,\tau),\mathbf{v}(\cdot,\tau))dt \\
&- \frac{1}{2}\|\mathbf{v}^0\|^2_{\dot{\mathbf{H}}^0_{\#\sigma}} - \int_0^t \langle \mathbf{f}(\cdot,\tau),\mathbf{v}(\cdot,\tau)\rangle_{\mathbb{T}} d\tau \leq 0
\end{aligned} \tag{3.20}$$

by the energy inequality condition (2.14) for $\mathbf{v}$.

Taking into account inequality (2.6) for the quadratic form a, (3.18–3.20) imply

$$\begin{aligned}
&\frac{1}{2}\left\|\widetilde{\mathbf{w}}(\cdot,t)\right\|^2_{\dot{\mathbf{H}}^0_{\#\sigma}} + \frac{1}{4}C_{\mathbb{A}}^{-1}\int_0^t \|\widetilde{\mathbf{w}}(\cdot,\tau)\|^2_{\dot{\mathbf{H}}^1_{\#\sigma}} dt \\
&\leq \int_0^t \left|\langle(\widetilde{\mathbf{w}}(\cdot,\tau)\cdot\nabla)\widetilde{\mathbf{w}}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}\right| d\tau.
\end{aligned} \tag{3.21}$$

By multiplication Theorem 6.1(b), the Sobolev interpolation inequality (6.16), and Young's inequality, we obtain

$$
\begin{aligned}
&\left|\langle(\widetilde{\mathbf{w}}(\cdot,\tau)\cdot\nabla)\widetilde{\mathbf{w}}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}\right| \\
&\le \|(\widetilde{\mathbf{w}}(\cdot,\tau)\cdot\nabla)\widetilde{\mathbf{w}}(\cdot,\tau)\|_{\dot{\mathbf{H}}^{-n/2}_{\#\sigma}}\|\mathbf{u}(\cdot,\tau)\|_{\dot{\mathbf{H}}^{n/2}_{\#\sigma}} \\
&\le \|\nabla\cdot[\widetilde{\mathbf{w}}(\cdot,\tau)\otimes\widetilde{\mathbf{w}}(\cdot,\tau)]\|_{\dot{\mathbf{H}}^{-n/2}_{\#\sigma}}\|\mathbf{u}(\cdot,\tau)\|_{\dot{\mathbf{H}}^{n/2}_{\#\sigma}} \\
&\le \|\widetilde{\mathbf{w}}(\cdot,\tau)\otimes\widetilde{\mathbf{w}}(\cdot,\tau)]\|_{(H^{-n/2+1}_{\#})^{n\times n}}\|\mathbf{u}(\cdot,\tau)\|_{\dot{\mathbf{H}}^{n/2}_{\#\sigma}} \\
&\le C_*\|\widetilde{\mathbf{w}}(\cdot,\tau)\|^2_{\dot{\mathbf{H}}^{1/2}_{\#\sigma}}\|\mathbf{u}(\cdot,\tau)\|_{\dot{\mathbf{H}}^{n/2}_{\#\sigma}} \\
&\le C_*\|\widetilde{\mathbf{w}}(\cdot,\tau)\|_{\dot{\mathbf{H}}^{0}_{\#\sigma}}\|\widetilde{\mathbf{w}}(\cdot,\tau)\|_{\dot{\mathbf{H}}^{1}_{\#\sigma}}\|\mathbf{u}(\cdot,\tau)\|_{\dot{\mathbf{H}}^{n/2}_{\#\sigma}} \\
&\le \frac{1}{4}C_{\mathbb{A}}^{-1}\|\widetilde{\mathbf{w}}(\cdot,\tau)\|^2_{\dot{\mathbf{H}}^{1}_{\#\sigma}}+C_{\mathbb{A}}C_*^2\|\widetilde{\mathbf{w}}(\cdot,\tau)\|^2_{\dot{\mathbf{H}}^{0}_{\#\sigma}}\|\mathbf{u}(\cdot,\tau)\|^2_{\dot{\mathbf{H}}^{n/2}_{\#\sigma}},
\end{aligned}
\tag{3.22}
$$

where $C_* := C_*(1/2, 1/2, n)$ from Theorem 6.1(b). Implementing (3.22) in (3.21), we obtain

$$
\frac{1}{2}\|\widetilde{\mathbf{w}}(\cdot,t)\|^2_{\dot{\mathbf{H}}^{0}_{\#\sigma}} \le C_{\mathbb{A}}C_*^2\int_0^t\|\widetilde{\mathbf{w}}(\cdot,\tau)\|^2_{\dot{\mathbf{H}}^{0}_{\#\sigma}}\|\mathbf{u}(\cdot,\tau)\|^2_{\dot{\mathbf{H}}^{n/2}_{\#\sigma}}d\tau. \tag{3.23}
$$

Because

$$
\begin{aligned}
&\int_0^T\|\widetilde{\mathbf{w}}(\cdot,\tau)\|^2_{\dot{\mathbf{H}}^{0}_{\#\sigma}}\|\mathbf{u}(\cdot,\tau)\|^2_{\dot{\mathbf{H}}^{n/2}_{\#\sigma}}d\tau \\
&\le \|\widetilde{\mathbf{w}}\|^2_{L_\infty\left(0,T;\dot{\mathbf{H}}^{0}_{\#\sigma}\right)}\|\mathbf{u}\|^2_{L_2(0,T;\dot{\mathbf{H}}^{n/2}_{\#\sigma})}<\infty,
\end{aligned}
$$

we can employ to (3.23) the integral Gronwall's inequality from Lemma 6.14 to conclude that $\|\widetilde{\mathbf{w}}(\cdot,\tau)\|_{\dot{\mathbf{H}}^{0}_{\#\sigma}} = 0$. □

3.2 | Serrin-Type Property of the Two-Dimensional Weak Solution

By Definitions 2.1 and 3.1, any weak solution of the evolution space-periodic anisotropic Navier–Stokes initial value problem (2.1–2.3) is a Serrin-type solution for $n = 2$. Then, Lemmas 2.2 and 3.4 along with Theorems 3.5 and 3.6 lead to the following results for any $T > 0$ and arbitrarily large data (unlike the higher dimensions discussed further on).

Theorem 3.7. *Let $n = 2$, $T > 0$, $a_{ij}^{\alpha\beta} \in L_\infty(0,T; L_{\infty\#})$, and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}^{-1}_{\#})$, $\mathbf{u}^0 \in \dot{\mathbf{H}}^{0}_{\#\sigma}$.*

Then, the solution $\mathbf{u} \in L_\infty(0,T_*;\dot{\mathbf{H}}^{0}_{\#\sigma}) \cap L_2(0,T_*;\dot{\mathbf{H}}^{1}_{\#\sigma})$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) obtained in Theorem 2.4 is of Serrin type, and hence, $(\mathbf{u}\cdot\nabla)\mathbf{u} \in L_2(0,T;\mathbf{H}^{-1}_{\#})$, $\mathbf{u} \in W^1(\dot{\mathbf{H}}^{1}_{\#\sigma},\dot{\mathbf{H}}^{-1}_{\#\sigma})$, $\mathbf{u}$ is almost everywhere on $[0,T]$ equal to a function belonging to $C^0([0,T];\dot{\mathbf{H}}^{0}_{\#\sigma})$ and

$$
\lim_{t\to 0}\|\mathbf{u}(\cdot,t)-\mathbf{u}^0\|_{\dot{\mathbf{H}}^{0}_{\#\sigma}} = 0.
$$

In addition,

$$
\begin{aligned}
&\left\langle \mathbf{u}'(\cdot,t),\mathbf{w}\right\rangle_{\mathbb{T}} + \langle(\mathbf{u}(\cdot,t)\cdot\nabla)\mathbf{u}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}} + a_{\mathbb{T}}(t;\mathbf{u},\mathbf{w}) \\
&= \langle\mathbf{f}(\cdot,t),\mathbf{w}\rangle_{\mathbb{T}},\ \textit{for a.e. } t\in(0,T),\ \forall\ \mathbf{w}\in\dot{\mathbf{H}}^{1}_{\#\sigma},
\end{aligned}
$$

and the following energy equality holds,

$$
\begin{aligned}
&\frac{1}{2}\|\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}_{2\#}} + \int_0^t a_{\mathbb{T}}(\tau;\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))d\tau \\
&= \frac{1}{2}\|\mathbf{u}^0\|^2_{\mathbf{L}_{2\#}} + \int_0^t\langle\mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}\,d\tau \quad \forall\ t\in[0,T].
\end{aligned}
$$

Moreover, the solution $\mathbf{u}$ is unique in the class of solutions from $L_\infty\left(0,T;\dot{\mathbf{H}}^{0}_{\#\sigma}\right) \cap L_2\left(0,T;\dot{\mathbf{H}}^{1}_{\#\sigma}\right)$ satisfying the energy inequality (2.14). The unique pressure p associated with the obtained $\mathbf{u}$ belongs to $L_2(0,T;\dot{H}^0_{\#})$.

4 | Serrin-Type Solution Existence and Regularity for Constant Anisotropic Viscosity Coefficients

In this section, we analyze the existence and regularity of Serrin-type solutions for any $n \ge 2$ in the anisotropic constant-coefficient case. This gives a generalization of Theorem 10.1 in [7], where similar results were obtained for $n = 3$, for the smoothness index $r = 1/2$, and for the isotropic constant-viscosity coefficients.

4.1 | Vector Heat Equation

Let us first consider the spatially periodic Cauchy problem for the (vector) heat equation,

$$
\partial_t\mathbf{v} - \Delta\mathbf{v} = \mathbf{0} \quad \text{in } \mathbb{T}\times(0,\infty), \tag{4.1}
$$

$$
\mathbf{v}(\cdot,0) = \mathbf{u}^0 \quad \text{in } \mathbb{T}. \tag{4.2}
$$

Calculating the Fourier coefficients of the both sides of Equations (4.1) and (4.2) and solving the obtained ODE problem, the periodic solution of the Cauchy problem (4.1)–(4.2) can be written as

$$
\mathbf{v}(\mathbf{x},t) = (K\mathbf{u}^0)(\mathbf{x},t) := \sum_{\xi\in\mathbb{Z}^n}\hat{\mathbf{u}}^0(\xi)e^{-(2\pi|\xi|)^2t+2\pi i\mathbf{x}\cdot\xi}. \tag{4.3}
$$

If $\operatorname{div}\mathbf{u}^0(\mathbf{x}) = 0$, then $\operatorname{div}\mathbf{v}(\mathbf{x},t) = 0$. If $\hat{\mathbf{u}}^0(\mathbf{0}) = \mathbf{0}$, then $\hat{\mathbf{v}}(\mathbf{0},t) = \mathbf{0}$. Particularly, let us assume that $\mathbf{u}^0 \in \dot{\mathbf{H}}^r_{\#\sigma}$ for some $r\in\mathbb{R}$. Then, taking dual product of the both sides of Equation (4.1) with $\Lambda^{2r}_{\#}\mathbf{v}$ gives

$$
\langle\partial_t\Lambda^r_{\#}\mathbf{v},\Lambda^r_{\#}\mathbf{v}\rangle_{\mathbb{T}} + \langle\nabla\Lambda^r_{\#}\mathbf{v},\nabla\Lambda^r_{\#}\mathbf{v}\rangle_{\mathbb{T}} = \mathbf{0},
$$

implying

$$
\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2_{\mathbf{H}^r_{\#}} + \|\nabla\mathbf{v}\|^2_{(H^r_{\#})^{n\times n}} = 0.
$$

After integration, this gives the energy-type equality

$$
\frac{1}{2}\|\mathbf{v}(\cdot,t)\|^2_{\mathbf{H}^r_{\#}} + \int_0^t\|\nabla\mathbf{v}(\cdot,\tau)\|^2_{(H^r_{\#})^{n\times n}}d\tau = \frac{1}{2}\|\mathbf{u}^0\|^2_{\mathbf{H}^r_{\#}}, \quad t\ge 0. \tag{4.4}
$$

The solution representation (4.3) and the norm definition (1.5) imply that

$$
\|\mathbf{v}(\cdot,t)\|^2_{\mathbf{H}^r_{\#}} \le \|\mathbf{u}^0\|^2_{\mathbf{H}^r_{\#}} \quad t\ge 0. \tag{4.5}
$$

On the other hand, (1.10) and (4.4) lead to

$$\int_0^t \|K\mathbf{u}^0\|^2_{\dot{\mathbf{H}}^{r+1}_\#} d\tau = \int_0^t \|\mathbf{v}\|^2_{\dot{\mathbf{H}}^{r+1}_\#} d\tau \le 2\int_0^t \|\nabla \mathbf{v}\|^2_{(H^r_\#)^{n\times n}} d\tau \\ \le \|\mathbf{u}^0\|^2_{\dot{\mathbf{H}}^r_\#}, \quad t \ge 0. \tag{4.6}$$

Estimates (4.5) and (4.6) mean that $\mathbf{v} \in L_\infty(0,T;\dot{\mathbf{H}}^r_{\#\sigma}) \cup L_2(0,T;\dot{\mathbf{H}}^{r+1}_{\#\sigma})$ for any $T > 0$ and

$$\|\mathbf{v}\|^2_{L_\infty(0,T;\mathbf{H}^r_\#)} = \|K\mathbf{u}^0\|^2_{L_\infty(0,T;\mathbf{H}^r_\#)} \le \|\mathbf{u}^0\|^2_{\mathbf{H}^r_\#}, \tag{4.7}$$

$$\|\mathbf{v}\|^2_{L_2(0,T;\mathbf{H}^{r+1}_\#)} = \|K\mathbf{u}^0\|^2_{L_2(0,T;\mathbf{H}^{r+1}_\#)} \le \|\mathbf{u}^0\|^2_{\mathbf{H}^r_\#}, \quad \forall\, T > 0. \tag{4.8}$$

This implies that the operator $K : \dot{\mathbf{H}}^r_{\#\sigma} \to L_\infty(0,\infty;\dot{\mathbf{H}}^r_{\#\sigma}) \cup L_2(0,\infty;\dot{\mathbf{H}}^{r+1}_{\#\sigma})$ is continuous.

4.2 | Preliminary Results for Constant Anisotropic Viscosity Coefficients

For some $n \ge 2$, $r \ge n/2 - 1$, and $T > 0$, let the coefficients $a^{\alpha\beta}_{ij}$ be constant and the relaxed ellipticity condition (1.2) hold. Let also $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}^{r-1}_\#)$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}^r_{\#\sigma}$.

Let us employ, as usual, the Galerkin approximation, with the sequence $\{\mathbf{w}_\ell\} \subset \dot{\mathbf{C}}^\infty_{\#\sigma}$ of eigenfunctions of the Bessel-potential operator $\Lambda_\#$ in $\dot{\mathbf{H}}^0_{\#\sigma}$, corresponding to eigenvalues λ_ℓ and constituting an orthonormal basis in $\dot{\mathbf{H}}^0_{\#\sigma}$; see Section 6.3. They also constitute an orthogonal basis in $\dot{\mathbf{H}}^r_{\#\sigma}$ and $\dot{\mathbf{H}}^{r+1}_{\#\sigma}$; see Theorem 6.4. Let us construct the m-term approximation to $\mathbf{u}^0$,

$$\mathbf{u^0_m} := P_m\mathbf{u^0_m} = \sum_{\ell=1}^{m} \langle \mathbf{u}^0, \mathbf{w}_\ell\rangle_{\mathbb{T}} \mathbf{w}_\ell,$$

where P_m is the orthogonal projector from $\mathbf{H}^r_{\#\sigma}$ to $\mathrm{Span}\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$, compare (6.13), that converges in $\dot{\mathbf{H}}^0_{\#\sigma}$ and $\dot{\mathbf{H}}^r_{\#\sigma}$, as $m \to \infty$. Due to the basis orthogonality, we have the inequalities

$$\|\mathbf{u^0_m}\|_{\dot{\mathbf{H}}^0_{\#\sigma}} \le \|\mathbf{u}^0\|_{\dot{\mathbf{H}}^0_{\#\sigma}}, \quad \|\mathbf{u^0_m}\|_{\dot{\mathbf{H}}^r_{\#\sigma}} \le \|\mathbf{u}^0\|_{\dot{\mathbf{H}}^r_{\#\sigma}}.$$

Let $\{\mathbf{u}_m\}$ be the sequence employed to prove Theorem 2 in [19], given here as Theorem 2.4. The sequence $\{\mathbf{u}_m\}$ converges to a solution $\mathbf{u} \in L_\infty\left(0,T;\dot{\mathbf{H}}^0_{\#\sigma}\right) \cap L_2\left(0,T;\dot{\mathbf{H}}^1_{\#\sigma}\right)$ of the initial-variational problem (2.11) and (2.12) weakly in $L_2\left(0,T;\dot{\mathbf{H}}^1_{\#\sigma}\right)$, weakly star in $L_\infty\left(0,T;\dot{\mathbf{H}}^0_{\#\sigma}\right)$, and strongly in $L_2\left(0,T;\dot{\mathbf{H}}^0_{\#\sigma}\right)$. Particularly, $\mathbf{u}_m(\mathbf{x},t) = \sum_{\ell=1}^m \eta_{\ell,m}(t)\mathbf{w}_\ell$ and solves the following nonlinear ODE problem from Theorem 2 in [19]:

$$\langle \partial_t \mathbf{u}_m, \mathbf{w}_k\rangle_{\mathbb{T}} + a_{\mathbb{T}}(t;\mathbf{u}_m,\mathbf{w}_k) + \langle (\mathbf{u}_m\cdot\nabla)\mathbf{u}_m, \mathbf{w}_k\rangle_{\mathbb{T}} \\ = \langle \mathbf{f}, \mathbf{w}_k\rangle_{\mathbb{T}}, \text{ a.e. } t \in (0,T), \ \forall\, k \in \{1, \dots, m\}, \tag{4.9}$$

$$\langle \mathbf{u}_m, \mathbf{w}_k\rangle_{\mathbb{T}}(\cdot,0) = \langle \mathbf{u}^0, \mathbf{w}_k\rangle_{\mathbb{T}}, \quad \forall\, k \in \{1, \dots, m\}. \tag{4.10}$$

Similarly to the proof of Theorem 10.1 in [7], let us define $\mathbf{v}_m(\mathbf{x},t) := P_m\mathbf{v} = \sum_{k=1}^m \langle \mathbf{v}, \mathbf{w}_k\rangle_{\mathbb{T}}\mathbf{w}_k$ for $\mathbf{v}$ given by (4.3). Acting by the projector P_m on (4.1) and (4.2) and then taking the dual product with $\mathbf{w}_k$, we obtain that for any $m > 1$, $\mathbf{v}_m$ solves the initial value ODE problem

$$\langle \partial_t \mathbf{v}_m, \mathbf{w}_k\rangle_{\mathbb{T}} + \langle \nabla\mathbf{v}_m, \nabla\mathbf{w}_k\rangle_{\mathbb{T}} = \mathbf{0}, \ \forall\, t \in (0,T), \ \forall\, k \in \{1, \dots, m\}, \tag{4.11}$$

$$\langle \mathbf{v}_m, \mathbf{w}_k\rangle_{\mathbb{T}}(\cdot,0) = \langle \mathbf{u}^0, \mathbf{w}_k\rangle_{\mathbb{T}}, \quad \forall\, k \in \{1, \dots, m\}, \tag{4.12}$$

and by (4.7) satisfies the estimates

$$\|\mathbf{v}_m\|^2_{L_\infty(0,T;\mathbf{H}^r_\#)} \le \|\mathbf{v}\|^2_{L_\infty(0,T;\mathbf{H}^r_\#)} \le \|\mathbf{u}^0\|^2_{\mathbf{H}^r_\#}, \quad \forall\, T > 0, \tag{4.13}$$

$$\|\mathbf{v}_m\|^2_{L_2(0,T;\mathbf{H}^{r+1}_\#)} \le \|\mathbf{v}\|^2_{L_2(0,T;\mathbf{H}^{r+1}_\#)} \le \|\mathbf{u}^0\|^2_{\mathbf{H}^r_\#}, \quad \forall\, T > 0. \tag{4.14}$$

To reduce the problem (4.9)–(4.10) to the one with zero initial conditions, let us represent $\mathbf{u}_m = \mathbf{v}_m + \widetilde{\mathbf{u}}_m$. Then, due to (4.9) and (4.10), the auxiliary function $\widetilde{\mathbf{u}}_m(\mathbf{x},t) = \sum_{\ell=1}^m \tilde\eta_{\ell,m}(t)\mathbf{w}_\ell$ satisfies the ODE problem

$$\langle \partial_t \widetilde{\mathbf{u}}_m, \mathbf{w}_k\rangle_{\mathbb{T}} + a_{\mathbb{T}}(t;\widetilde{\mathbf{u}}_m,\mathbf{w}_k) + \langle (\mathbf{u}_m\cdot\nabla)\mathbf{u}_m, \mathbf{w}_k\rangle_{\mathbb{T}} \\ = \langle \mathbf{f}, \mathbf{w}_k\rangle_{\mathbb{T}} + \langle \nabla\mathbf{v}_m, \nabla\mathbf{w}_k\rangle_{\mathbb{T}} - a_{\mathbb{T}}(t;\mathbf{v}_m,\mathbf{w}_k), \ \forall\, k \in \{1, \dots, m\}, \tag{4.15}$$

$$\langle \widetilde{\mathbf{u}}_m, \mathbf{w}_k\rangle_{\mathbb{T}}(\cdot,0) = \mathbf{0}, \quad \forall\, k \in \{1, \dots, m\}. \tag{4.16}$$

After multiplying by λ_k^{2r} and taking into account the property $\Lambda^{2r}_\#\mathbf{w}_k = \lambda_k^{2r}\mathbf{w}_k$, relation (2.4), and that the operator $\Lambda^r_\#$ commutate with operators ∇ and $E_{j\beta}$, Equation (4.15) leads to

$$\begin{aligned} &\langle \partial_t\widetilde{\mathbf{u}}_m, \Lambda^{2r}_\#\mathbf{w}_k\rangle_{\mathbb{T}} + \langle E_{j\beta}(\widetilde{\mathbf{u}}_m), a^{\alpha\beta}_{ij}\Lambda^r_\# E_{i\alpha}(\Lambda^r_\#\mathbf{w}_k)\rangle_{\mathbb{T}} \\ &+ \langle (\mathbf{u}_m\cdot\nabla)\mathbf{u}_m, \Lambda^{2r}_\#\mathbf{w}_k\rangle_{\mathbb{T}} \\ &= \langle \mathbf{f}, \Lambda^{2r}_\#\mathbf{w}_k\rangle_{\mathbb{T}} + \langle \nabla\mathbf{v}_m, \Lambda^r_\#\nabla\Lambda^r_\#\mathbf{w}_k\rangle_{\mathbb{T}} \\ &- \left\langle E_{j\beta}(\mathbf{v}_m), a^{\alpha\beta}_{ij}\Lambda^r_\# E_{i\alpha}(\Lambda^r_\#\mathbf{w}_k)\right\rangle_{\mathbb{T}}, \quad \forall\, k \in \{1, \dots, m\}. \end{aligned} \tag{4.17}$$

These equations can be rewritten as

$$\begin{aligned} &\langle \partial_t\Lambda^r_\#\widetilde{\mathbf{u}}_m, \Lambda^r_\#\mathbf{w}_k\rangle_{\mathbb{T}} + \left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\Lambda^r_\#\widetilde{\mathbf{u}}_m), E_{i\alpha}(\Lambda^r_\#\mathbf{w}_k)\right\rangle_{\mathbb{T}} \\ &+ \langle \Lambda^{r-1}_\#[(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m], \Lambda^{r+1}_\#\mathbf{w}_k\rangle_{\mathbb{T}} \\ &= \langle \Lambda^{r-1}_\#\mathbf{f}, \Lambda^{r+1}_\#\mathbf{w}_k\rangle_{\mathbb{T}} + \langle \nabla\Lambda^r_\#\mathbf{v}_m, \nabla\Lambda^r_\#\mathbf{w}_k\rangle_{\mathbb{T}} \\ &- \left\langle E_{j\beta}(\mathbf{v}_m), a^{\alpha\beta}_{ij}\Lambda^r_\# E_{i\alpha}(\Lambda^r_\#\mathbf{w}_k)\right\rangle_{\mathbb{T}} \quad \forall\, k \in \{1, \dots, m\}. \end{aligned} \tag{4.18}$$

Multiplying equations in (4.18) by $\tilde\eta_{k,m}(t)$ and summing them up over $k \in \{1, \dots, m\}$, we obtain

$$\begin{aligned} &\frac{1}{2}\partial_t\|\Lambda^r_\#\widetilde{\mathbf{u}}_m\|^2_{\mathbf{H}^0_\#} + a_{\mathbb{T}}(\Lambda^r_\#\widetilde{\mathbf{u}}_m, \Lambda^r_\#\widetilde{\mathbf{u}}_m) = \langle \Lambda^{r-1}_\#\mathbf{f}, \Lambda^{r+1}_\#\widetilde{\mathbf{u}}_m\rangle_{\mathbb{T}} \\ &+ \langle \nabla\Lambda^r_\#\mathbf{v}_m, \nabla\Lambda^r_\#\widetilde{\mathbf{u}}_m\rangle_{\mathbb{T}} \\ &- \left\langle E_{j\beta}(\mathbf{v}_m), a^{\alpha\beta}_{ij}\Lambda^r_\# E_{i\alpha}(\Lambda^r_\#\widetilde{\mathbf{u}}_m)\right\rangle_{\mathbb{T}} \\ &- \langle \Lambda^{r-1}_\#[(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m], \Lambda^{r+1}_\#\widetilde{\mathbf{u}}_m\rangle_{\mathbb{T}}. \end{aligned} \tag{4.19}$$

From (2.7), we have

$$a_{\mathbb{T}}(\Lambda^r_\#\widetilde{\mathbf{u}}_m, \Lambda^r_\#\widetilde{\mathbf{u}}_m) \ge \frac{1}{4}C^{-1}_{\mathbb{A}}\|\Lambda^r_\#\widetilde{\mathbf{u}}_m\|^2_{\dot{\mathbf{H}}^1_{\#\sigma}} = \frac{1}{4}C^{-1}_{\mathbb{A}}\|\widetilde{\mathbf{u}}_m\|^2_{\dot{\mathbf{H}}^{r+1}_{\#\sigma}}. \tag{4.20}$$

Let us now estimate the terms in the right-hand side of (4.19). First,

$$\langle \Lambda_{\#}^{r-1}\mathbf{f}, \Lambda_{\#}^{r+1}\widetilde{\mathbf{u}}_m\rangle_{\mathbb{T}} \le \|\Lambda_{\#}^{r-1}\mathbf{f}\|_{\dot{\mathbf{H}}_{\#}^{0}}\|\Lambda_{\#}^{r+1}\widetilde{\mathbf{u}}_m\|_{\dot{\mathbf{H}}_{\#}^{0}} \le \|\mathbf{f}\|_{\dot{\mathbf{H}}_{\#}^{r-1}}\|\widetilde{\mathbf{u}}_m\|_{\dot{\mathbf{H}}_{\#}^{r+1}}. \tag{4.21}$$

Next, inequality (1.10) implies

$$\begin{aligned}\langle \nabla\Lambda_{\#}^{r}\mathbf{v}_m, \nabla\Lambda_{\#}^{r}\widetilde{\mathbf{u}}_m\rangle_{\mathbb{T}} &\le \|\nabla\Lambda_{\#}^{r}\mathbf{v}_m\|_{L_{2\#}^{n\times n}}\|\nabla\Lambda_{\#}^{r}\widetilde{\mathbf{u}}_m\|_{L_{2\#}^{n\times n}} \\ &\le \|\Lambda_{\#}^{r}\mathbf{v}_m\|_{\dot{\mathbf{H}}_{\#}^{1}}\|\Lambda_{\#}^{r}\widetilde{\mathbf{u}}_m\|_{\dot{\mathbf{H}}_{\#}^{1}} \le \|\mathbf{v}_m\|_{\dot{\mathbf{H}}_{\#}^{r+1}}\|\widetilde{\mathbf{u}}_m\|_{\dot{\mathbf{H}}_{\#}^{r+1}}.\end{aligned} \tag{4.22}$$

Further, we obtain by inequality (1.10),

$$\begin{aligned}&\left|\left\langle E_{j\beta}(\mathbf{v}_m), a_{ij}^{\alpha\beta}\Lambda_{\#}^{r}E_{i\alpha}(\Lambda_{\#}^{r}\widetilde{\mathbf{u}}_m)\right\rangle_{\mathbb{T}}\right| \\ &\le \|E_{j\beta}(\mathbf{v}_m)\|_{(H_{\#}^{r})^{n\times n}}\|a_{ij}^{\alpha\beta}\Lambda_{\#}^{r}E_{i\alpha}(\Lambda_{\#}^{r}\widetilde{\mathbf{u}}_m)\|_{(H_{\#}^{-r})^{n\times n}} \\ &\le |\mathbb{A}|\|\mathbf{v}_m\|_{\dot{\mathbf{H}}_{\#}^{r+1}}\|\widetilde{\mathbf{u}}_m\|_{\dot{\mathbf{H}}_{\#}^{r+1}},\end{aligned} \tag{4.23}$$

where $|\mathbb{A}| := \left|\left\{a_{ij}^{\alpha\beta}\right\}_{\alpha,\beta,i,j=1}^{n}\right|_{F}$. Finally,

$$\begin{aligned}&|\langle \Lambda_{\#}^{r-1}[(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m], \Lambda_{\#}^{r+1}\widetilde{\mathbf{u}}_m\rangle_{\mathbb{T}}| \\ &\le \|\Lambda_{\#}^{r-1}[(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m]\|_{\dot{\mathbf{H}}_{\#}^{0}}\|\Lambda_{\#}^{r+1}\widetilde{\mathbf{u}}_m\|_{\dot{\mathbf{H}}_{\#}^{0}} \\ &\le \|(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m\|_{\dot{\mathbf{H}}_{\#}^{r-1}}\|\widetilde{\mathbf{u}}_m\|_{\dot{\mathbf{H}}_{\#}^{r+1}}.\end{aligned} \tag{4.24}$$

Implementing (4.20–4.24) in (4.19) and using Young's inequality, we obtain

$$\begin{aligned}&\frac{d}{dt}\|\widetilde{\mathbf{u}}_m\|_{\mathbf{H}_{\#}^{r}}^2 + \frac{1}{2}C_{\mathbb{A}}^{-1}\|\widetilde{\mathbf{u}}_m\|_{\mathbf{H}_{\#}^{r+1}}^2 \\ &\le 2\Big(\|\mathbf{f}\|_{\mathbf{H}_{\#}^{r-1}} + [|\mathbb{A}|+1]\|\mathbf{v}_m\|_{\dot{\mathbf{H}}_{\#}^{r+1}} + \left\|(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m\right\|_{\mathbf{H}_{\#}^{r-1}}\Big)\|\widetilde{\mathbf{u}}_m\|_{\mathbf{H}_{\#}^{r+1}} \\ &\le 4C_{\mathbb{A}}\Big(\|\mathbf{f}\|_{\mathbf{H}_{\#}^{r-1}} + [|\mathbb{A}|+1]\|\mathbf{v}_m\|_{\dot{\mathbf{H}}_{\#}^{r+1}} + \left\|(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m\right\|_{\mathbf{H}_{\#}^{r-1}}\Big)^2 \\ &\quad + \frac{1}{4}C_{\mathbb{A}}^{-1}\|\widetilde{\mathbf{u}}_m\|_{\mathbf{H}_{\#}^{r+1}}^2.\end{aligned}$$

Hence, by the inequality $(\sum_{i=1}^{k}a_i)^2 \le k\sum_{i=1}^{k}a_i^2$ (following from the Cauchy–Schwarz inequality),

$$\begin{aligned}&\frac{d}{dt}\|\widetilde{\mathbf{u}}_m\|_{\mathbf{H}_{\#}^{r}}^2 + \frac{1}{4}C_{\mathbb{A}}^{-1}\|\widetilde{\mathbf{u}}_m\|_{\mathbf{H}_{\#}^{r+1}}^2 \le \\ &16C_{\mathbb{A}}\left(\|\mathbf{f}\|_{\mathbf{H}_{\#}^{r-1}}^2 + \left[|\mathbb{A}|^2+1\right]\|\mathbf{v}_m\|_{\dot{\mathbf{H}}_{\#}^{r+1}}^2 + \left\|(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m\right\|_{\mathbf{H}_{\#}^{r-1}}^2\right).\end{aligned} \tag{4.25}$$

Note that by the similar reasoning, but without employing in (4.9) and (4.10) the function $\mathbf{v}$, we obtain that $\mathbf{u}_m$ satisfies the differential inequality

$$\begin{aligned}&\frac{d}{dt}\|\mathbf{u}_m\|_{\mathbf{H}_{\#}^{r}}^2 + \frac{1}{4}C_{\mathbb{A}}^{-1}\|\mathbf{u}_m\|_{\mathbf{H}_{\#}^{r+1}}^2 \\ &\le 8C_{\mathbb{A}}\left(\|\mathbf{f}\|_{\mathbf{H}_{\#}^{r-1}}^2 + \left\|(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m\right\|_{\mathbf{H}_{\#}^{r-1}}^2\right).\end{aligned} \tag{4.26}$$

Let us also estimate the last term in (4.25) and (4.26) for the case $n/2-1 \le r < n/2$. By relation (1.10), multiplication Theorem 6.1(b), and the Sobolev interpolation inequality (6.16), we obtain

$$\begin{aligned}&\left\|(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m\right\|_{\mathbf{H}_{\#}^{r-1}}^2 = \left\|\nabla\cdot(\mathbf{u}_m\otimes\mathbf{u}_m)\right\|_{\mathbf{H}_{\#}^{r-1}}^2 \\ &\le \left\|\mathbf{u}_m\otimes\mathbf{u}_m)\right\|_{(H_{\#}^{r})^{n\times n}}^2 \\ &\le C_{*rn}^2\|\mathbf{u}_m\|_{\mathbf{H}_{\#}^{r/2+n/4}}^4 \le C_{*rn}^2\|\mathbf{u}_m\|_{\mathbf{H}_{\#}^{r}}^2\|\mathbf{u}_m\|_{\mathbf{H}_{\#}^{n/2}}^2,\end{aligned} \tag{4.27}$$

where $C_{*rn} = C_*(r/2+n/4, r/2+n/4, n)$.

4.3 | Serrin-Type Solution Existence for Constant Anisotropic Viscosity Coefficients

Employing the results from Section 4.2 for $r = n/2-1$, we are now in the position to prove the existence of Serrin-type solutions.

Theorem 4.1. *Let $n \ge 2$ and $T > 0$. Let the coefficients $a_{ij}^{\alpha\beta}$ be constant and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}_{\#}^{n/2-2})$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^{n/2-1}$.*

i. *Then, there exist constants $A_1 \ge 0$, $A_2 \ge 0$, and $A_3 > 0$ that are independent of $\mathbf{f}$ and $\mathbf{u}^0$ but may depend on n, $|\mathbb{A}|$ and $C_{\mathbb{A}}$, such that if $\mathbf{f}$, $\mathbf{u}^0$ and $T_* \in (0,T]$ satisfy the inequality*

$$\begin{aligned}&\int_0^{T_*}\|\mathbf{f}(\cdot,t)\|_{\mathbf{H}_{\#}^{n/2-2}}^2dt + \\ &\left(A_1\|\mathbf{u}^0\|_{\mathbf{H}_{\#}^{n/2-1}}^2 + A_2\right)\int_0^{T_*}\|(K\mathbf{u}^0)(\cdot,t)\|_{\dot{\mathbf{H}}_{\#}^{n/2}}^2dt < A_3.\end{aligned} \tag{4.28}$$

where K is the operator defined in (4.3), then there exists a solution $\mathbf{u}$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) in $L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2-1}) \cap L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2})$, which is thus a Serrin-type solution.

ii. *In addition, $\mathbf{u}' \in L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2-2})$, $\mathbf{u} \in C^0([0,T_*];\dot{\mathbf{H}}_{\#\sigma}^{n/2-1})$, $\lim_{t\to 0}\|\mathbf{u}(\cdot,t)-\mathbf{u}^0\|_{\dot{\mathbf{H}}_{\#\sigma}^{n/2-1}} = 0$, and $p \in L_2(0,T_*;\dot{H}_{\#}^{n/2-1})$.*

iii. *Moreover, $\mathbf{u}$ satisfies the following energy equality for any $[t_0,t] \subset [0,T_*]$,*

$$\begin{aligned}&\frac{1}{2}\|\mathbf{u}(\cdot,t)\|_{\mathbf{L}_{2\#}}^2 + \int_{t_0}^{t}a_{\mathbb{T}}(\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))d\tau = \\ &\frac{1}{2}\|\mathbf{u}(\cdot,t_0)\|_{\mathbf{L}_{2\#}}^2 + \int_{t_0}^{t}\langle\mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}d\tau.\end{aligned} \tag{4.29}$$

It particularly implies the standard energy equality,

$$\begin{aligned}&\frac{1}{2}\|\mathbf{u}(\cdot,t)\|_{\mathbf{L}_{2\#}}^2 + \int_0^{t}a_{\mathbb{T}}(\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))d\tau = \\ &\frac{1}{2}\|\mathbf{u}^0\|_{\mathbf{L}_{2\#}}^2 + \int_0^{t}\langle\mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}\,d\tau \quad \forall\, t \in [0,T_*].\end{aligned} \tag{4.30}$$

iv. *The solution $\mathbf{u}$ is unique in the class of solutions from $L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{0}) \cap L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{1})$ satisfying the energy inequality (2.14) on the interval $[0,T_*]$.*

Proof.

i. Let $r = n/2-1$. Estimate (4.27) implies

$$\begin{aligned}&\left\|(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m\right\|_{\mathbf{H}_{\#}^{r-1}}^2 \\ &\le C_{*rn}^2\|\mathbf{u}_m\|_{\mathbf{H}_{\#}^{r/2+n/4}}^4 \le 8C_{*rn}^2(\|\widetilde{\mathbf{u}}_m\|_{\mathbf{H}_{\#}^{r/2+n/4}}^4 + \|\mathbf{v}_m\|_{\mathbf{H}_{\#}^{r/2+n/4}}^4) \\ &\le 8C_{*rn}^2\|\widetilde{\mathbf{u}}_m\|_{\mathbf{H}_{\#}^{r}}^2\|\widetilde{\mathbf{u}}_m\|_{\mathbf{H}_{\#}^{n/2}}^2 + 8C_{*rn}^2\|\mathbf{v}_m\|_{\mathbf{H}_{\#}^{r}}^2\|\mathbf{v}_m\|_{\mathbf{H}_{\#}^{n/2}}^2 \\ &= 8C_{*rn}^2\|\widetilde{\mathbf{u}}_m\|_{\mathbf{H}_{\#}^{n/2-1}}^2\|\widetilde{\mathbf{u}}_m\|_{\mathbf{H}_{\#}^{n/2}}^2 + 8C_{*rn}^2\|\mathbf{v}_m\|_{\mathbf{H}_{\#}^{n/2-1}}^2\|\mathbf{v}_m\|_{\mathbf{H}_{\#}^{n/2}}^2,\end{aligned} \tag{4.31}$$

where $C_{*rn}^2 := C_{*n/2-1,n} = C_*(n/2-1/2, n/2-1/2, n)$. Then, by (4.31), we obtain from (4.25),

$$\frac{d}{dt}\|\widetilde{\mathbf{u}}_m\|^2_{\mathbf{H}_\#^{n/2-1}} + \frac{1}{4}C_{\mathbb{A}}^{-1}\|\widetilde{\mathbf{u}}_m\|^2_{\mathbf{H}_\#^{n/2}} \le 128C_{*rn}^2 C_{\mathbb{A}}\|\widetilde{\mathbf{u}}_m\|^2_{\mathbf{H}_\#^{n/2}}\|\widetilde{\mathbf{u}}_m\|^2_{\dot{\mathbf{H}}_\#^{n/2-1}}$$
$$+ 16C_{\mathbb{A}}\left(\|\mathbf{f}\|^2_{\mathbf{H}_\#^{n/2-2}} + 8C_{*rn}^2\|\mathbf{v}_m\|^2_{\mathbf{H}_\#^{n/2-1}}\|\mathbf{v}_m\|^2_{\mathbf{H}_\#^{n/2}} + \left[|\mathbb{A}|^2+1\right]\|\mathbf{v}_m\|^2_{\dot{\mathbf{H}}_\#^{n/2}}\right). \tag{4.32}$$

Let us now apply to (4.32) Lemma 6.12 with

$$\eta = \|\widetilde{\mathbf{u}}_m\|^2_{\mathbf{H}_\#^{n/2-1}},\ \eta_0 = 0,$$
$$y = \|\widetilde{\mathbf{u}}_m\|^2_{\mathbf{H}_\#^{n/2}},\ b = \frac{1}{4}C_{\mathbb{A}}^{-1},\ c = 128C_{*rn}^2 C_{\mathbb{A}},$$
$$\psi = 16C_{\mathbb{A}}\left(\|\mathbf{f}\|^2_{\mathbf{H}_\#^{n/2-2}} + 8C_{*rn}^2\|\mathbf{v}_m\|^2_{\mathbf{H}_\#^{n/2-1}}\|\mathbf{v}_m\|^2_{\mathbf{H}_\#^{n/2}} + \left[|\mathbb{A}|^2+1\right]\|\mathbf{v}_m\|^2_{\dot{\mathbf{H}}_\#^{n/2}}\right),$$

to conclude that if T_* is such that

$$\int_0^{T_*}\left(\|\mathbf{f}(\cdot,t)\|^2_{\mathbf{H}_\#^{n/2-2}} + \left(8C_{*rn}^2\|\mathbf{v}_m(\cdot,t)\|^2_{\mathbf{H}_\#^{n/2-1}} + \left[|\mathbb{A}|^2+1\right]\right)\|\mathbf{v}_m(\cdot,t)\|^2_{\dot{\mathbf{H}}_\#^{n/2}}\right)dt < \left(512eC_{\mathbb{A}}^2C_{*rn}^2\right)^{-1}, \tag{4.33}$$

then

$$\|\widetilde{\mathbf{u}}_m\|^2_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2-1})} \le \left(512C_{\mathbb{A}}^2C_{*rn}^2\right)^{-1},$$
$$\|\widetilde{\mathbf{u}}_m\|^2_{L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2})} \le \left(128C_{\mathbb{A}}C_{*rn}^2\right)^{-1},$$

and hence,

$$\|\mathbf{u}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2-1})} \le \|\widetilde{\mathbf{u}}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2-1})} + \|\mathbf{v}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2-1})} \le \left(16\sqrt{2}C_{\mathbb{A}}C_{*rn}\right)^{-1} + \|\mathbf{u}^0\|_{\dot{\mathbf{H}}_{\#\sigma}^{n/2-1}}, \tag{4.34}$$

$$\|\mathbf{u}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2})} \le \|\widetilde{\mathbf{u}}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2})} + \|\mathbf{v}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2})} \le \left(8\sqrt{2C_{\mathbb{A}}}C_{*rn}\right)^{-1} + \|\mathbf{u}^0\|_{\dot{\mathbf{H}}_{\#\sigma}^{n/2-1}}. \tag{4.35}$$

Estimates (4.13) and (4.14) were taken into account in (4.34) and (4.35).
Taking into account inequalities (4.13) and (4.14) again, we obtain that condition (4.33) is satisfied if T_* is such that

$$\int_0^{T_*}\|\mathbf{f}(\cdot,t)\|^2_{\mathbf{H}_\#^{n/2-2}}dt + \left(8C_{*rn}^2\|\mathbf{u}^0\|^2_{\mathbf{H}_\#^{n/2-1}} + \left[|\mathbb{A}|^2+1\right]\right)\int_0^{T_*}\|\mathbf{v}(\cdot,t)\|^2_{\dot{\mathbf{H}}_\#^{n/2}}dt < \left(512eC_{\mathbb{A}}^2C_{*rn}^2\right)^{-1}. \tag{4.36}$$

Note that condition (4.36) gives condition (4.28) with

$$A_1 = 8C_{*rn}^2,\quad A_2 = |\mathbb{A}|^2+1,\quad A_3 = \left(512eC_{\mathbb{A}}^2C_{*rn}^2\right)^{-1}.$$

Inequalities (4.34) and (4.35) imply that there exists a subsequence of $\{\mathbf{u}_m\}$ converging weakly in $L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2})$ and weakly star in $L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2-1})$ to a function $\mathbf{u}^\dagger \in L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2}) \cup L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2-1})$. Then, the subsequence converges to $\mathbf{u}^\dagger$ also weakly in $L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^1)$ and weakly star in $L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^0)$. Because $\{\mathbf{u}_m\}$ is the subsequence of the sequence that converges weakly in $L_2\left(0,T;\dot{\mathbf{H}}_{\#\sigma}^1\right)$ and weakly star in $L_\infty\left(0,T;\dot{\mathbf{H}}_{\#\sigma}^0\right)$ to the weak solution, $\mathbf{u}$, of problem (2.1–2.3) on $\left[0,T_*\right]$, we conclude that $\mathbf{u} = \mathbf{u}^\dagger \in L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2-1}) \cup L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2})$.
This implies that $\mathbf{u}$ is a Serrin-type solution on the interval $[0,T_*]$, and we thus proved item (i) of the theorem.

ii. Repeating for $\mathbf{u}$ the reasoning related to inequality (4.27), we obtain

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|^2_{\mathbf{H}_\#^{n/2-2}} \le C_{*rn}^2\|\mathbf{u}\|^2_{\mathbf{H}_\#^{n/2-1}}\|\mathbf{u}\|^2_{\mathbf{H}_\#^{n/2}}.$$

Hence,

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L_2(0,T_*;\mathbf{H}_\#^{n/2-2})} \le C_{*rn}\|\mathbf{u}\|_{L_\infty(0,T_*;\mathbf{H}_\#^{n/2-1})}\|\mathbf{u}\|_{L_2(0,T_*;\mathbf{H}_\#^{n/2})}, \tag{4.37}$$

that is, $(\mathbf{u}\cdot\nabla)\mathbf{u} \in L_2(0,T_*;\dot{\mathbf{H}}_\#^{n/2-2})$. By (1.1) and (1.3), we have

$$\|\mathfrak{L}\mathbf{u}\|^2_{\mathbf{H}_\#^{n/2-2}} \le \|a_{ij}^{\alpha\beta}E_{i\alpha}(\mathbf{u})\|_{(H_\#^{n/2-1})^{n\times n}} \le |\mathbb{A}|^2\|\mathbf{u}\|^2_{\mathbf{H}_\#^{n/2}},$$

and thus,

$$\|\mathfrak{L}\mathbf{u}\|^2_{L_2(0,T_*;\dot{\mathbf{H}}_\#^{n/2-2})} \le |\mathbb{A}|^2\|\mathbf{u}\|^2_{L_2(0,T_*;\mathbf{H}_\#^{n/2})},$$

that is, $\mathfrak{L}\mathbf{u} \in L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2-2})$. We also have $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}_\#^{n/2-2})$.
Then, (2.9) implies that $\mathbf{u}' \in L_2(0,T_*;\mathbf{H}_{\#\sigma}^{n/2-2})$, and hence, by Theorem 6.8, we obtain that $\mathbf{u} \in C^0([0,T_*];\mathbf{H}_{\#\sigma}^{n/2-1})$, which also means that $\|\mathbf{u}(\cdot,t)-\mathbf{u}^0\|_{\dot{\mathbf{H}}_{\#\sigma}^{n/2-1}} \to 0$ as $t \to 0$. To prove the theorem claim about the associated pressure p, we remark that it satisfies (2.10), where $\mathbf{F} \in L_2(0,T;\dot{\mathbf{H}}_\#^{n/2-2})$ due to the theorem conditions and the inclusion $(\mathbf{u}\cdot\nabla)\mathbf{u} \in L_2(0,T_*;\dot{\mathbf{H}}_\#^{n/2-2})$. By Lemma 6.5 for gradient, with $s = n/2-1$, Equation (2.10) has a unique solution p in $L_2(0,T_*;\dot{H}_\#^{n/2-1})$.

iii. The energy equalities (4.29) and (4.30) immediately follow from Theorem 3.5.

iv. The solution uniqueness follows from Theorem 3.6. □

Remark 4.2. Because $\|\mathbf{f}(\cdot,t)\|^2_{\dot{\mathbf{H}}_\#^{n/2-2}}$ is integrable on $(0,T]$ by the theorem condition and $\|(K\mathbf{u}^0)(\cdot,t)\|^2_{\dot{\mathbf{H}}_\#^{n/2}}$ is integrable on $(0,\infty)$ by inequality (4.6), we conclude that due to the absolute continuity of the Lebesgue integrals, for arbitrarily large data $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}_\#^{n/2-2})$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^{n/2-1}$, there exists $T_* > 0$ such that condition (4.28) holds.

Estimating the integrand in the second integral in (4.28) according to (4.5), we arrive at the following assertion allowing an explicit estimate of T_* for arbitrarily large data if $\mathbf{f} \in L_\infty(0,T;\dot{\mathbf{H}}_\#^{n/2-2})$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^{n/2}$.

Corollary 4.3. (Serrin-type solution for arbitrarily large data but small time or vice versa). *Let $n \ge 2$ and $T > 0$. Let* *the coefficients $a_{ij}^{\alpha\beta}$ be constant and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in L_\infty(0,T;\dot{\mathbf{H}}_\#^{n/2-2})$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^{n/2}$.*

Then, there exist constants $A_1, A_2, A_3 > 0$ that are independent of $\mathbf{f}$ and $\mathbf{u}^0$ but may depend on T, n, $|\mathbb{A}|$ and $C_\mathbb{A}$, such that if $T_* \in (0,T]$ satisfies the inequality

$$T_*\left[\|\mathbf{f}\|^2_{L_\infty(0,T;\dot{\mathbf{H}}_\#^{n/2-2})} + \left(A_1\|\mathbf{u}^0\|^2_{\dot{\mathbf{H}}_\#^{n/2-1}} + A_2\right)\|\mathbf{u}^0\|^2_{\dot{\mathbf{H}}_\#^{n/2}}\right] < A_3, \tag{4.38}$$

then there exists a Serrin-type solution $\mathbf{u} \in L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2-1}) \cap L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2})$ of the anisotropic Navier–Stokes initial value problem. This solution satisfies items (ii)–(iv) in Theorem 4.1.

Estimating the second integral in (4.28) according to (4.8), we arrive at the following assertion.

Corollary 4.4. (Existence of Serrin-type solution for arbitrary time but small data). *Let $n \geq 2$ and $T > 0$. Let the coefficients $a_{ij}^{\alpha\beta}$ be constant and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}_\#^{n/2-2})$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^{n/2-1}$.*

Then, there exist constants $A_1, A_2, A_3 > 0$ that are independent of $\mathbf{f}$ and $\mathbf{u}^0$ but may depend on n, $|\mathbb{A}|$, and $C_\mathbb{A}$, such that if $\mathbf{f}$ and $\mathbf{u}^0$ satisfy the inequality

$$\|\mathbf{f}\|^2_{L_2(0,T;\dot{\mathbf{H}}_\#^{n/2-2})} + \left(A_1\|\mathbf{u}^0\|^2_{\dot{\mathbf{H}}_\#^{n/2-1}} + A_2\right)\|\mathbf{u}^0\|^2_{\dot{\mathbf{H}}_\#^{n/2-1}} < A_3, \tag{4.39}$$

then there exists a Serrin-type solution $\mathbf{u} \in L_\infty(0,T;\dot{\mathbf{H}}_{\#\sigma}^{n/2-1}) \cap L_2(0,T;\dot{\mathbf{H}}_{\#\sigma}^{n/2})$ of the anisotropic Navier–Stokes initial value problem. This solution satisfies items (ii)–(iv) in Theorem 4.1 with $T_* = T$.

4.4 | Spatial Regularity of Serrin-Type Solutions for Constant Anisotropic Viscosity Coefficients

Theorem 4.5. (Spatial regularity of Serrin-type solution for arbitrarily large data). *Let $n \geq 2$, $r > n/2-1$, and $T > 0$. Let the coefficients $a_{ij}^{\alpha\beta}$ be constant and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}_\#^{r-1})$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^{r}$, while $\mathbf{f}$, $\mathbf{u}^0$, and $T_* \in (0,T]$ satisfy inequality (4.28) from Theorem 4.1.*

Then, the Serrin-type solution $\mathbf{u}$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) belongs to $L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r}) \cap L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r+1})$. In addition, $\mathbf{u}' \in L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r-1})$, $\mathbf{u} \in C^0([0,T_*];\dot{\mathbf{H}}_{\#\sigma}^{r})$, $\lim_{t\to 0}\|\mathbf{u}(\cdot,t) - \mathbf{u}^0\|_{\dot{\mathbf{H}}_{\#\sigma}^{r}} = 0$, and $p \in L_2(0,T_*;\dot{H}_\#^{r})$.

Proof. The existence of the Serrin-type solution $\mathbf{u} \in L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2-1}) \cap L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2})$ is proved in Theorem 4.1(i), and we will prove that it has a higher smoothness. We will employ the same Galerkin approximation used in Section 4.2 and in the proof of Theorem 4.1(i).

Step a. Let us estimate the last term in (4.26) for the case $n/2-1 < r < n/2$. By (4.27), we obtain from (4.26),

$$\begin{aligned}&\frac{d}{dt}\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{r}} + \frac{1}{4}C_\mathbb{A}^{-1}\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{r+1}}\\ &\leq 8C_\mathbb{A}C_{*rn}^2\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{n/2}}\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{r}} + 8C_\mathbb{A}\|\mathbf{f}\|^2_{\dot{\mathbf{H}}_\#^{r-1}},\end{aligned} \tag{4.40}$$

implying

$$\frac{d}{dt}\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{r}} \leq 8C_\mathbb{A}C_{*rn}^2\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{n/2}}\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{r}} + 8C_\mathbb{A}\|\mathbf{f}\|^2_{\dot{\mathbf{H}}_\#^{r-1}}. \tag{4.41}$$

By Gronwall's inequality (6.22), we obtain from (4.41) that

$$\begin{aligned}\|\mathbf{u}_m\|^2_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r})} &\leq \exp\left(8C_\mathbb{A}C_{*rn}^2\|\mathbf{u}_m\|^2_{L_2(0,T_*;\dot{\mathbf{H}}_\#^{n/2})}\right)\\ &\times\left[\|\mathbf{u}_m(\cdot,0)\|^2_{\dot{\mathbf{H}}_\#^{r}} + 8C_\mathbb{A}\|\mathbf{f}\|^2_{L_2(0,T_*;\dot{\mathbf{H}}_\#^{r-1})}\right].\end{aligned} \tag{4.42}$$

We have $\|\mathbf{u}_m(\cdot,0)\|_{\dot{\mathbf{H}}_\#^{r}} \leq \|\mathbf{u}^0\|_{\dot{\mathbf{H}}_\#^{r}}$, and by (4.35), the sequence $\|\mathbf{u}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}_\#^{n/2})}$ is bounded. Then, (4.42) implies that the sequence $\|\mathbf{u}_m\|^2_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r})}$ is bounded as well. Integrating (4.40), we conclude that

$$\begin{aligned}\|\mathbf{u}_m\|^2_{L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r+1})} &\leq 32C_\mathbb{A}^2C_{*rn}^2\|\mathbf{u}_m\|^2_{L_2(0,T_*;\dot{\mathbf{H}}_\#^{n/2})}\|\mathbf{u}_m\|^2_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r})}\\ &+ 4C_\mathbb{A}\|\mathbf{u}_m(\cdot,0)\|^2_{\dot{\mathbf{H}}_\#^{r}} + 32C_\mathbb{A}^2\|\mathbf{f}\|^2_{L_2(0,T_*;\dot{\mathbf{H}}_\#^{r-1})}.\end{aligned} \tag{4.43}$$

Inequalities (4.42) and (4.43) mean that the sequences

$$\{\|\mathbf{u}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r})}\}_{m=1}^\infty \text{ and } \{\|\mathbf{u}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r+1})}\}_{m=1}^\infty$$
$$\text{are bounded for } n/2-1 < r < n/2. \tag{4.44}$$

Step b. Let now $r = n/2$. Then, by multiplication Theorem 6.1(a) and relation (1.10),

$$\begin{aligned}&\|(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{n/2-1}} = \|\nabla\cdot(\mathbf{u}_m\otimes\mathbf{u}_m)\|^2_{\dot{\mathbf{H}}_\#^{n/2-1}}\\ &\leq \|\mathbf{u}_m\otimes\mathbf{u}_m)\|^2_{(H_\#^{n/2})^{n\times n}}\\ &\leq C_{*rn}^2\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{n/2}}\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{n/2+1/2}},\end{aligned} \tag{4.45}$$

where $C_{*rn} = C_*(n/2, n/2+1/2, n)$.
Then, by (4.45), we obtain from (4.26),

$$\begin{aligned}&\frac{d}{dt}\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{n/2}} + \frac{1}{4}C_\mathbb{A}^{-1}\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{n/2+1}} \leq\\ &8C_\mathbb{A}C_{*rn}^2\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{n/2+1/2}}\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{n/2}} + 8C_\mathbb{A}\|\mathbf{f}\|^2_{\dot{\mathbf{H}}_\#^{n/2-1}}\end{aligned} \tag{4.46}$$

implying

$$\begin{aligned}\frac{d}{dt}\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{n/2}} &\leq 8C_\mathbb{A}C_{*rn}^2\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{n/2+1/2}}\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{n/2}}\\ &+ 8C_\mathbb{A}\|\mathbf{f}\|^2_{\dot{\mathbf{H}}_\#^{n/2-1}}.\end{aligned} \tag{4.47}$$

By Gronwall's inequality (6.22), we obtain from (4.47) that

$$\begin{aligned}\|\mathbf{u}_m\|^2_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2})} &\leq \exp\left(8C_\mathbb{A}C_{*rn}^2\|\mathbf{u}_m\|^2_{L_2(0,T_*;\dot{\mathbf{H}}_\#^{n/2+1/2})}\right)\\ &\times\left[\|\mathbf{u}_m(\cdot,0)\|^2_{\dot{\mathbf{H}}_\#^{n/2}} + 8C_\mathbb{A}\|\mathbf{f}\|^2_{L_2(0,T_*;\dot{\mathbf{H}}_\#^{n/2-1})}\right].\end{aligned} \tag{4.48}$$

We have $\|\mathbf{u}_m(\cdot,0)\|_{\mathbf{H}_\#^{n/2}} \le \|\mathbf{u}^0\|_{\mathbf{H}_\#^{n/2}}$, and by (4.44), the sequence $\|\mathbf{u}_m\|_{L_2(0,T_*;\mathbf{H}_\#^{n/2+1/2})}$ is bounded as well. Then, (4.48) implies that the sequence $\|\mathbf{u}_m\|^2_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2})}$ is also bounded. Integrating (4.46), we conclude that

$$\|\mathbf{u}_m\|^2_{L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2+1})} \le 32C_{\mathbb{A}}^2C_{*rn}^2\|\mathbf{u}_m\|^2_{L_2(0,T_*;\mathbf{H}_\#^{n/2+1/2})}\|\mathbf{u}_m\|^2_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2})} + 4C_{\mathbb{A}}\|\mathbf{u}_m(\cdot,0)\|^2_{\mathbf{H}_\#^{n/2}} + 32C_{\mathbb{A}}^2\|\mathbf{f}\|^2_{L_2(0,T_*;\mathbf{H}_\#^{n/2-1})} \le C < \infty. \tag{4.49}$$

Inequalities (4.48) and (4.49) mean that the sequences

$$\{\|\mathbf{u}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^r)}\}_{m=1}^\infty \text{ and } \{\|\mathbf{u}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r+1})}\}_{m=1}^\infty \text{ are bounded for } r=n/2. \tag{4.50}$$

Step c. Let now $kn/2 < r \le (k+1)n/2$, $k=1,2,3,\dots$ By multiplication Theorem 6.1(a) and relation (1.10),

$$\|(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m\|^2_{\mathbf{H}_\#^{r-1}} \le C_{*rn}^2\|\mathbf{u}_m\|^2_{\mathbf{H}_\#^r}\|\nabla\mathbf{u}_m\|^2_{(H_\#^{r-1})^{n\times n}} \le C_{*rn}^2\|\mathbf{u}_m\|^2_{\mathbf{H}_\#^r}\|\mathbf{u}_m\|^2_{\mathbf{H}_\#^r}. \tag{4.51}$$

where $C_{*rn} = C_*(r-1,r,n)$.
Then, by (4.51), we obtain from (4.26),

$$\frac{d}{dt}\|\mathbf{u}_m\|^2_{\mathbf{H}_\#^r} + \frac{1}{4}C_{\mathbb{A}}^{-1}\|\mathbf{u}_m\|^2_{\mathbf{H}_\#^{r+1}} \le 8C_{\mathbb{A}}C_{*rn}^2\|\mathbf{u}_m\|^2_{\mathbf{H}_\#^r}\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^r} + 8C_{\mathbb{A}}\|\mathbf{f}\|^2_{\mathbf{H}_\#^{r-1}}, \tag{4.52}$$

implying

$$\frac{d}{dt}\|\mathbf{u}_m\|^2_{\mathbf{H}_\#^r} \le 8C_{\mathbb{A}}C_{*rn}^2\|\mathbf{u}_m\|^2_{\mathbf{H}_\#^r}\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^r} + 8C_{\mathbb{A}}\|\mathbf{f}\|^2_{\mathbf{H}_\#^{r-1}}. \tag{4.53}$$

By Gronwall's inequality (6.22), we obtain from (4.53) that

$$\|\mathbf{u}_m\|^2_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^r)} \le \exp\left(8C_{\mathbb{A}}C_{*rn}^2\|\mathbf{u}_m\|^2_{L_2(0,T_*;\mathbf{H}_\#^r)}\right) \times \left[\|\mathbf{u}_m(\cdot,0)\|^2_{\mathbf{H}_\#^r} + 8C_{\mathbb{A}}\|\mathbf{f}\|^2_{L_2(0,T_*;\mathbf{H}_\#^{r-1})}\right], \tag{4.54}$$

where $\|\mathbf{u}_m(\cdot,0)\|_{\mathbf{H}_\#^r} \le \|\mathbf{u}^0\|_{\mathbf{H}_\#^r}$.
If $k=1$, then the sequence $\|\mathbf{u}_m\|_{L_2(0,T_*;\mathbf{H}_\#^r)}$ in (4.54) is bounded due to (4.44) and (4.50). Then, (4.54) implies that the sequence $\|\mathbf{u}_m\|^2_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^r)}$ is bounded as well. Integrating (4.52), we also conclude that for $k=1$,

$$\|\mathbf{u}_m\|^2_{L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r+1})} \le 32C_{\mathbb{A}}^2C_{*rn}^2\|\mathbf{u}_m\|^2_{L_2(0,T_*;\mathbf{H}_\#^r)}\|\mathbf{u}_m\|^2_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^r)} + 4C_{\mathbb{A}}\|\mathbf{u}_m(\cdot,0)\|^2_{\mathbf{H}_\#^r} + 32C_{\mathbb{A}}^2\|\mathbf{f}\|^2_{L_2(0,T_*;\mathbf{H}_\#^{r-1})}, \quad kn/2 < r \le (k+1)n/2. \tag{4.55}$$

Inequalities (4.54) and (4.55) mean that for $k=1$, the sequences

$$\{\|\mathbf{u}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^r)}\}_{m=1}^\infty \, and \, \{\|\mathbf{u}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r+1})}\}_{m=1}^\infty \text{ are bounded for } kn/2 < r \le (k+1)n/2. \tag{4.56}$$

If we assume that properties (4.56) hold for some integer $k \ge 1$, then by the similar argument, properties (4.56) hold with k replaced by $k+1$, and thus, by induction, they hold for any integer k. Hence, collecting properties (4.44), (4.50), and (4.56), we conclude that the sequences

$$\{\|\mathbf{u}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^r)}\}_{m=1}^\infty \text{ and } \{\|\mathbf{u}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r+1})}\}_{m=1}^\infty \text{ are bounded for } n/2-1 < r. \tag{4.57}$$

Properties (4.57) imply that there exists a subsequence of $\{\mathbf{u}_m\}$ converging weakly in $L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r+1})$ and weakly star in $L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^r)$ to a function $\mathbf{u}^\dagger \in L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r+1}) \cup L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^r)$. Then, the subsequence converges to $\mathbf{u}^\dagger$ also weakly in $L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^1)$ and weakly star in $L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^0)$. Because $\{\mathbf{u}_m\}$ is the subsequence of the sequence that converges weakly in $L_2\left(0,T;\dot{\mathbf{H}}_{\#\sigma}^1\right)$ and weakly star in $L_\infty\left(0,T;\dot{\mathbf{H}}_{\#\sigma}^0\right)$ to the weak solution, $\mathbf{u}$, of problem (2.1–2.3) on $[0,T_*]$, we conclude that $\mathbf{u}=\mathbf{u}^\dagger \in L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r+1}) \cup L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^r)$, for any $r > n/2-1$, and we thus finished proving that

$$\mathbf{u} \in L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^r) \cap L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r+1}). \tag{4.58}$$

Step d. Repeating for $\mathbf{u}$ the reasoning related to inequalities (4.27), (4.45), and (4.51), corresponding to the considered r, we obtain

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|^2_{\mathbf{H}_\#^{r-1}} \le C_{*rn}^2\|\mathbf{u}\|^2_{\mathbf{H}_\#^r}\|\mathbf{u}\|^2_{\mathbf{H}_\#^{r+1}}.$$

Hence,

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L_2(0,T;\mathbf{H}_\#^{r-1})} \le C_{*rn}\|\mathbf{u}\|_{L_\infty(0,T_*;\mathbf{H}_\#^r)}\|\mathbf{u}\|_{L_2(0,T_*;\mathbf{H}_\#^{r+1})}. \tag{4.59}$$

Due to (4.58), then $(\mathbf{u}\cdot\nabla)\mathbf{u} \in L_2(0,T_*;\mathbf{H}_\#^{r-1})$. By (1.1) and (1.3), we have

$$\|\mathfrak{L}\mathbf{u}\|^2_{\mathbf{H}_\#^{r-1}} \le \|a_{ij}^{\alpha\beta}E_{i\alpha}(\mathbf{u})\|^2_{(H_\#^r)^{n\times n}} \le |\mathbb{A}|^2\|\mathbf{u}\|^2_{\mathbf{H}_\#^{r+1}}$$

for a.e. $t \in (0,T)$,

and thus,

$$\|\mathfrak{L}\mathbf{u}\|^2_{L_2(0,T_*;\dot{\mathbf{H}}_\#^{r-1})} \le |\mathbb{A}|^2\|\mathbf{u}\|^2_{L_2(0,T_*;\mathbf{H}_\#^{r+1})},$$

that is, $\mathfrak{L}\mathbf{u} \in L_2(0,T_*;\dot{\mathbf{H}}_\#^{r-1})$. We also have $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}_\#^{r-1})$. Thus, $\mathbf{F}$ defined by (2.8) belongs to $L_2(0,T;\dot{\mathbf{H}}_\#^{r-1})$. Then, (2.9) implies that $\mathbf{u}' \in L_2(0,T_*;\mathbf{H}_{\#\sigma}^{r-1})$, and because $\mathbf{u} \in L_2(0,T_*;\mathbf{H}_{\#\sigma}^{r+1})$, we obtain by Theorem 6.8 that $\mathbf{u} \in C^0([0,T_*];\dot{\mathbf{H}}_{\#\sigma}^r)$, which also means that $\|\mathbf{u}(\cdot,t)-\mathbf{u}^0\|_{\dot{\mathbf{H}}_{\#\sigma}^r} \to 0$ as $t\to 0$.
To prove the theorem claim about the associated pressure p, we remark that p satisfies (2.10). By Lemma 6.5 for gradient, with $s=r$, Equation (2.10) has a unique solution p in $L_2(0,T_*;\dot{H}_\#^r)$.

□

As in Corollaries 4.3 and 4.4, condition (4.28) in Theorem 4.5 can be replaced by simpler conditions for particular cases, which leads to the following two assertions.

Corollary 4.6. (Serrin-type solution for arbitrarily large data but small time or vice versa). *Let $n \geq 2$ and $T > 0$. Let the coefficients $a_{ij}^{\alpha\beta}$ be constant and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in L_2(0, T; \dot{\mathbf{H}}_{\#}^{r-1}) \cap L_\infty(0, T; \dot{\mathbf{H}}_{\#}^{n/2-2})$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^{r} \cap \dot{\mathbf{H}}_{\#\sigma}^{n/2}$, $r > n/2 - 1$. Let $T_* \in (0, T)$ satisfies inequality (4.38) in Corollary 4.3.*

Then, the Serrin-type solution $\mathbf{u}$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) belongs to $L_\infty(0, T_*; \dot{\mathbf{H}}_{\#\sigma}^{r}) \cap L_2(0, T_*; \dot{\mathbf{H}}_{\#\sigma}^{r+1})$. In addition, $\mathbf{u}' \in L_2(0, T_*; \dot{\mathbf{H}}_{\#\sigma}^{r-1})$, $\mathbf{u} \in C^0([0, T_*]; \dot{\mathbf{H}}_{\#\sigma}^{r})$, $\lim_{t\to 0} \|\mathbf{u}(\cdot, t) - \mathbf{u}^0\|_{\dot{\mathbf{H}}_{\#\sigma}^{r}} = 0$, and $p \in L_2(0, T_*; \dot{H}_{\#}^{r})$.

Corollary 4.7. (Serrin-type solution for arbitrary time but small data). *Let $n \geq 2$, $r \geq n/2 - 1$, and $T > 0$. Let the coefficients $a_{ij}^{\alpha\beta}$ be constant and the relaxed ellipticity condition (1.2) hold. Let the data $\mathbf{f} \in L_2(0, T; \dot{\mathbf{H}}_{\#}^{r-1})$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^{r}$ satisfy inequality (4.39) in Corollary 4.4.*

Then, the Serrin-type solution $\mathbf{u}$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) belongs to $L_\infty(0, T; \dot{\mathbf{H}}_{\#\sigma}^{r}) \cap L_2(0, T; \dot{\mathbf{H}}_{\#\sigma}^{r+1})$. In addition, $\mathbf{u}' \in L_2(0, T; \dot{\mathbf{H}}_{\#\sigma}^{r-1})$, $\mathbf{u} \in C^0([0, T]; \dot{\mathbf{H}}_{\#\sigma}^{r})$, $\lim_{t\to 0} \|\mathbf{u}(\cdot, t) - \mathbf{u}^0\|_{\dot{\mathbf{H}}_{\#\sigma}^{r}} = 0$, and $p \in L_2(0, T; \dot{H}_{\#}^{r})$.

Theorem 4.5 leads also to the following infinite regularity assertion.

Corollary 4.8. *Let $T > 0$ and $n \geq 2$. Let the coefficients $a_{ij}^{\alpha\beta}$ be constant and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in L_2(0, T; \dot{\mathbf{C}}_{\#}^{\infty})$ and $\mathbf{u}^0 \in \dot{\mathbf{C}}_{\#\sigma}^{\infty}$, while $\mathbf{f}$, $\mathbf{u}^0$, and $T_* \in (0, T]$ satisfy inequality (4.28) from Theorem 4.1.*

Then, the Serrin-type solution $\mathbf{u}$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) is such that $\mathbf{u} \in C^0([0, T_*]; \dot{\mathbf{C}}_{\#\sigma}^{\infty})$, $\mathbf{u}' \in L_2(0, T_*; \dot{\mathbf{C}}_{\#\sigma}^{\infty})$, and $p \in L_2(0, T_*; \dot{C}_{\#}^{\infty})$.

Proof. Taking into account that $\dot{\mathbf{C}}_{\#}^{\infty} = \bigcap_{r\in\mathbb{R}} \dot{\mathbf{H}}_{\#\sigma}^{r}$, Theorem 4.5 implies that $\mathbf{u} \in L_\infty(0, T_*; \dot{\mathbf{H}}_{\#\sigma}^{r}) \cap L_2(0, T_*; \dot{\mathbf{H}}_{\#}^{r+1})$, $\mathbf{u}' \in L_2(0, T_*; \dot{\mathbf{H}}_{\#\sigma}^{r-1})$, $p \in L_2(0, T; \dot{H}_{\#}^{r})$, $\forall\, r \in \mathbb{R}$. Hence, $\mathbf{u} \in C^0([0, T_*]; \dot{\mathbf{C}}_{\#\sigma}^{\infty})$, $\mathbf{u}' \in L_2(0, T_*; \dot{\mathbf{C}}_{\#\sigma}^{\infty})$, and $p \in L_2(0, T_*; \dot{C}_{\#}^{\infty})$. □

4.5 | Spatial-Temporal Regularity of Serrin-Type Solutions for Constant Anisotropic Viscosity Coefficients

Theorem 4.9. *Let $T > 0$ and $n \geq 2$. Let $r \geq n/2 - 1$ if $n \geq 3$, while $r > n/2 - 1$ if $n = 2$. Let the coefficients $a_{ij}^{\alpha\beta}$ be constant and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in L_\infty(0, T; \dot{\mathbf{H}}_{\#}^{r-2}) \cap L_2(0, T; \dot{\mathbf{H}}_{\#}^{r-1})$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^{r}$, while $\mathbf{f}$, $\mathbf{u}^0$, and $T_* \in (0, T]$ satisfy inequality (4.28) from Theorem 4.1.*

Then, the Serrin-type solution $\mathbf{u}$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) is such that $\mathbf{u}' \in L_\infty(0, T_*; \dot{\mathbf{H}}_{\#\sigma}^{r-2}) \cup L_2(0, T_*; \dot{\mathbf{H}}_{\#\sigma}^{r-1})$, while $p \in L_\infty(0, T_*; \dot{H}_{\#}^{r-1}) \cap L_2(0, T_*; \dot{H}_{\#}^{r})$.

Proof. By Theorems 4.1 and 4.5, we have the inclusions $\mathbf{u} \in L_\infty(0, T_*; \dot{\mathbf{H}}_{\#\sigma}^{r})$, $\mathbf{u}' \in L_2(0, T_*; \dot{\mathbf{H}}_{\#\sigma}^{r-1})$, and $p \in L_2(0, T_*; \dot{H}_{\#}^{r})$. Then, we only need to prove the inclusions $\mathbf{u}' \in L_\infty(0, T_*; \dot{\mathbf{H}}_{\#\sigma}^{r-2})$ and $p \in L_\infty(0, T_*; \dot{H}_{\#}^{r-1})$.

Let, first, $n/2 - 1 \leq r < n/2 + 1$ (and also $0 < r$ if $n = 2$). By relation (1.10) and multiplication Theorem 6.1(b), we have

$$\|(\mathbf{u} \cdot \nabla)\mathbf{u}\|_{\mathbf{H}_{\#}^{r-2}} = \|\nabla \cdot (\mathbf{u} \otimes \mathbf{u})\|_{\mathbf{H}_{\#}^{r-2}} \leq \|\mathbf{u} \otimes \mathbf{u}\|_{(H_{\#}^{r-1})^{n\times n}} \leq C'_{*rn}\|\mathbf{u}\|^2_{\mathbf{H}_{\#}^{r/2+n/4-1/2}} \leq C'_{*rn}\|\mathbf{u}\|^2_{\mathbf{H}_{\#}^{r}},$$

where $C'_{*rn} = C_*(r/2 + n/4, r/2 + n/4, n)$.

Let now $r \geq n/2 + 1$. Again, by relation (1.10) and by multiplication Theorem 6.1(a), we have

$$\|(\mathbf{u} \cdot \nabla)\mathbf{u}\|_{\mathbf{H}_{\#}^{r-2}} = \|\nabla \cdot (\mathbf{u} \otimes \mathbf{u})\|_{\mathbf{H}_{\#}^{r-2}} \leq \|\mathbf{u} \otimes \mathbf{u}\|_{(H_{\#}^{r-1})^{n\times n}} \leq C''_{*rn}\|\mathbf{u}\|_{\mathbf{H}_{\#}^{r-1}}\|\mathbf{u}\|_{\mathbf{H}_{\#}^{r}} \leq C''_{*rn}\|\mathbf{u}\|^2_{\mathbf{H}_{\#}^{r}},$$

where $C''_{*rn} = C_*(r - 1, r, n)$. Hence, in both cases,

$$\|(\mathbf{u} \cdot \nabla)\mathbf{u}\|_{L_\infty(0,T_*;\mathbf{H}_{\#}^{r-2})} \leq C_{*rn}\|\mathbf{u}\|^2_{L_\infty(0,T_*;\mathbf{H}_{\#}^{r})}, \tag{4.60}$$

where C_{*rn} is C'_{*rn} or C''_{*rn}, respectively.

By (1.1) and (1.3), we have

$$\|\mathfrak{L}\mathbf{u}\|_{\mathbf{H}_{\#}^{r-2}} \leq \|a_{ij}^{\alpha\beta} E_{i\alpha}(\mathbf{u})\|_{(H_{\#}^{r-1})^{n\times n}} \leq |\mathbb{A}|\|\mathbf{u}\|_{\mathbf{H}_{\#}^{r}}.$$

Thus,

$$\|\mathfrak{L}\mathbf{u}\|_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#}^{r-2})} \leq |\mathbb{A}|\|\mathbf{u}\|_{L_\infty(0,T_*;\mathbf{H}_{\#}^{r})},$$

that is, $\mathfrak{L}\mathbf{u} \in L_\infty(0, T_*; \dot{\mathbf{H}}_{\#}^{r-2})$. We also have $\mathbf{f} \in L_\infty(0, T; \dot{\mathbf{H}}_{\#}^{r-2})$.

Then, (2.9) implies that $\mathbf{u}' \in L_\infty(0, T_*; \mathbf{H}_{\#\sigma}^{r-2})$, while (2.10) and Lemma 6.5 for gradient, with $s = r - 1$, imply that $p \in L_\infty(0, T_*; \dot{H}_{\#}^{r-1})$. □

Theorem 4.10. *Let $T > 0$ and $n \geq 2$. Let $r > n/2 - 1$. Let the coefficients $a_{ij}^{\alpha\beta}$ be constant and the relaxed ellipticity condition (1.2) hold. Let $k \in [1, r + 1)$ be an integer. Let $\mathbf{f}^{(l)} \in L_\infty(0, T; \dot{\mathbf{H}}_{\#}^{r-2-2l}) \cap L_2(0, T; \dot{\mathbf{H}}_{\#}^{r-1-2l})$, $l = 0, 1, \dots, k - 1$, and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^{r}$, while $\mathbf{f}$, $\mathbf{u}^0$, and $T_* \in (0, T]$ satisfy inequality (4.28) from Theorem 4.1.*

Then, the Serrin-type solution $\mathbf{u}$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) is such that $\mathbf{u}^{(l)} \in L_\infty(0, T_*; \dot{\mathbf{H}}_{\#\sigma}^{r-2l}) \cap L_2(0, T_*; \dot{\mathbf{H}}_{\#\sigma}^{r+1-2l})$, $l = 0, \dots, k$, while $p^{(l)} \in L_\infty(0, T_*; \dot{H}_{\#}^{r-1-2l}) \cap L_2(0, T_*; \dot{H}_{\#\sigma}^{r-2l})$, $l = 0, \dots, k - 1$.

Proof. Some parts of the following proof are inspired by [30, Theorem 3.1] and [11, Chapter 3, Section 3.6], see also [7, Section 7.2].

We will employ the mathematical induction argument in the proof. We first remark that by Theorems 4.1, 4.5, and 4.9, if $\mathbf{f} \in L_\infty(0, T; \dot{\mathbf{H}}_{\#}^{r-2}) \cap L_2(0, T; \dot{\mathbf{H}}_{\#}^{r-1})$, then $\mathbf{u} \in L_\infty(0, T_*; \dot{\mathbf{H}}_{\#\sigma}^{r}) \cap L_2(0, T_*; \dot{\mathbf{H}}_{\#\sigma}^{r+1})$, $\mathbf{u}' \in L_\infty(0, T_*; \dot{\mathbf{H}}_{\#\sigma}^{r-2}) \cap L_2(0, T_*; \dot{\mathbf{H}}_{\#\sigma}^{r-1})$, while $p \in$

$L_\infty(0,T_*;\dot{H}_\#^{r-1})\cap L_2(0,T_*;\dot{H}_\#^{r})$. This means that the theorem holds true for $k=1$.

Let us assume that the theorem holds true for some $k'=k-1\in[1,r)$, that is,

$$\mathbf{u}^{(l)}\in L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r-2l})\cap L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r+1-2l}),\quad l=0,\,\dots\,,k-1, \tag{4.61}$$

and prove that it holds also for $l=k$. To this end, let us differentiate Equation (2.9) $k-1$ times in t (in the distribution sense) to obtain

$$\mathbf{u}^{(k)}=\mathbb{P}_\sigma\mathbf{F}^{(k-1)}\quad\text{in}\quad\mathbb{T}\times(0,T), \tag{4.62}$$

where

$$\mathbf{F}^{(l)}:=\mathbf{f}^{(l)}+\partial_t^l\mathfrak{L}\mathbf{u}-\partial_t^l[(\mathbf{u}\cdot\nabla)\mathbf{u}]\quad\forall\,l\in\mathbb{N}. \tag{4.63}$$

Let us denote $s_{l1}:=r-2\max\{(k-1-l),l\}$, $s_{l2}:=r-2\min\{(k-1-l),l\}$. Then,

$$s_{l1}\le r-k+1\le s_{l2},$$
$$s_{l1}+s_{l2}=r-2(k-1-l)+r-2l=2(r-k+1),\quad\forall\,l=0,\,\dots\,,k-1.$$

The theorem condition $r-k+1>0$ implies $s_{l1}+s_{l2}>0$.

Step 1. Convection term.

$$\partial_t^{k-1}[(\mathbf{u}\cdot\nabla)\mathbf{u}]=\sum_{l=0}^{k-1}C_{k-1}^l(\mathbf{u}^{(k-1-l)}\cdot\nabla)\mathbf{u}^{(l)}, \tag{4.64}$$

where C_{k-1}^l are the binomial coefficients.

Case A. Let $0\le l\le(k-1)/2$.
Then, $s_{l1}=r-2(k-1-l)$, $s_{l2}=r-2l$.

Subcase A1. Let $n/2-1<r\le n/2+2l$. Then, $s_{l2}=r-2l\le n/2$. By the theorem conditions, $r+1-n/2>0$ and $r-k+1>0$, and hence, there exists $\epsilon\in(0,\min\{r+1-n/2,2(r-k+1)\})$, and thus,

$$(r+1-n/2)-\epsilon>0,\quad s_{l1}-\epsilon/2\le s_{l2}-\epsilon/2<n/2,$$
$$s_{l1}+s_{l2}-\epsilon=2(r-k+1)-\epsilon>0.$$

By relation (1.10) and multiplication Theorem 6.1(b), we have

$$\begin{aligned}
&\left\|(\mathbf{u}^{(k-1-l)}\cdot\nabla)\mathbf{u}^{(l)}\right\|_{\mathbf{H}_\#^{r-2k}}\\
&=\left\|\nabla\cdot(\mathbf{u}^{(k-1-l)}\otimes\mathbf{u}^{(l)})\right\|_{\mathbf{H}_\#^{r-2k}}\\
&\le\left\|\mathbf{u}^{(k-1-l)}\otimes\mathbf{u}^{(l)})\right\|_{(H_\#^{r+1-2k})^{n\times n}}\\
&\le\left\|\mathbf{u}^{(k-1-l)}\otimes\mathbf{u}^{(l)})\right\|_{(H_\#^{r+1-2k+(r+1-n/2)-\epsilon})^{n\times n}}\\
&=\left\|\mathbf{u}^{(k-1-l)}\otimes\mathbf{u}^{(l)})\right\|_{(H_\#^{s_{l1}+s_{l2}-\epsilon-n/2})^{n\times n}}\\
&\le C'_{*rn}\left\|\mathbf{u}^{(k-1-l)}\right\|_{\mathbf{H}_\#^{s_{l1}-\epsilon/2}}\left\|\mathbf{u}^{(l)})\right\|_{\mathbf{H}_\#^{s_{l2}-\epsilon/2}}\\
&\le C'_{*rn}\left\|\mathbf{u}^{(k-1-l)}\right\|_{\mathbf{H}_\#^{r-2(k-1-l)}}\left\|\mathbf{u}^{(l)}\right\|_{\mathbf{H}_\#^{r-2l}},
\end{aligned}$$

$$\begin{aligned}
&\left\|(\mathbf{u}^{(k-1-l)}\cdot\nabla)\mathbf{u}^{(l)}\right\|_{\mathbf{H}_\#^{r+1-2k}}\\
&\le\left\|(\mathbf{u}^{(k-1-l)}\cdot\nabla)\mathbf{u}^{(l)}\right\|_{\mathbf{H}_\#^{r+1-2k+(r+1-n/2)-\epsilon}}\\
&=\left\|(\mathbf{u}^{(k-1-l)}\cdot\nabla)\mathbf{u}^{(l)}\right\|_{\mathbf{H}_\#^{s_{l1}+s_{l2}-\epsilon-n/2}}\\
&\le C'_{*rn}\left\|\mathbf{u}^{(k-1-l)}\right\|_{\mathbf{H}_\#^{s_{l1}-\epsilon/2}}\left\|\nabla\mathbf{u}^{(l)})\right\|_{(H_\#^{s_{l2}-\epsilon/2})^{n\times n}}\\
&\le C'_{*rn}\left\|\mathbf{u}^{(k-1-l)}\right\|_{\mathbf{H}_\#^{r-2(k-1-l)}}\left\|\mathbf{u}^{(l)}\right\|_{\mathbf{H}_\#^{r+1-2l}},
\end{aligned}$$

where $C'_{*rn}=C_*(s_{l1}-\epsilon/2,s_{l2}-\epsilon/2,n)$.

Subcase A2. Let $r>n/2+2l$.
Then, $s_{l2}=r-2l>n/2$. Hence, by relation (1.10) and multiplication Theorem 6.1(a), we have

$$\begin{aligned}
&\left\|(\mathbf{u}^{(k-1-l)}\cdot\nabla)\mathbf{u}^{(l)}\right\|_{\mathbf{H}_\#^{r-2k}}\\
&=\left\|\nabla\cdot(\mathbf{u}^{(k-1-l)}\otimes\mathbf{u}^{(l)})\right\|_{\mathbf{H}_\#^{r-2k}}\\
&\le\left\|\mathbf{u}^{(k-1-l)}\otimes\mathbf{u}^{(l)})\right\|_{(H_\#^{r+1-2k})^{n\times n}}\\
&\le\left\|\mathbf{u}^{(k-1-l)}\otimes\mathbf{u}^{(l)})\right\|_{(H_\#^{r-2(k-1-l)})^{n\times n}}\\
&\le C''_{*rn}\left\|\mathbf{u}^{(k-1-l)}\right\|_{\mathbf{H}_\#^{r-2(k-1-l)}}\left\|\mathbf{u}^{(l)})\right\|_{\mathbf{H}_\#^{r-2l}},
\end{aligned}$$

$$\begin{aligned}
&\left\|(\mathbf{u}^{(k-1-l)}\cdot\nabla)\mathbf{u}^{(l)}\right\|_{\mathbf{H}_\#^{r+1-2k}}\\
&\le\left\|(\mathbf{u}^{(k-1-l)}\cdot\nabla)\mathbf{u}^{(l)}\right\|_{\mathbf{H}_\#^{r-2(k-1-l)}}\\
&\le C''_{*rn}\left\|\mathbf{u}^{(k-1-l)}\right\|_{\mathbf{H}_\#^{r-2(k-1-l)}}\left\|\nabla\mathbf{u}^{(l)})\right\|_{\mathbf{H}_\#^{r-2l}}\\
&\le C''_{*rn}\left\|\mathbf{u}^{(k-1-l)}\right\|_{\mathbf{H}_\#^{r-2(k-1-l)}}\left\|\mathbf{u}^{(l)})\right\|_{\mathbf{H}_\#^{r+1-2l}},
\end{aligned}$$

where $C''_{*rn}=C_*(s_{l1},s_{l2},n)$.
Thus, combining Cases (A1) and (A2), we obtain that for any $r>n/2-1$ and for $0\le l\le(k-1)/2$,

$$\left\|(\mathbf{u}^{(k-1-l)}\cdot\nabla)\mathbf{u}^{(l)}\right\|_{\mathbf{H}_\#^{r-2k}}\le C_{*rn}\left\|\mathbf{u}^{(k-1-l)}\right\|_{\mathbf{H}_\#^{r-2(k-1-l)}}\left\|\mathbf{u}^{(l)}\right\|_{\mathbf{H}_\#^{r-2l}}, \tag{4.65}$$

$$\left\|(\mathbf{u}^{(k-1-l)}\cdot\nabla)\mathbf{u}^{(l)}\right\|_{\mathbf{H}_\#^{r+1-2k}}\le C_{*rn}\left\|\mathbf{u}^{(k-1-l)}\right\|_{\mathbf{H}_\#^{r-2(k-1-l)}}\left\|\mathbf{u}^{(l)})\right\|_{\mathbf{H}_\#^{r+1-2l}}, \tag{4.66}$$

where C_{*rn} is C'_{*rn} or C''_{*rn}, respectively.

Case B. Let $(k-1)/2\le l\le k-1$. Then, taking into account that

$$\begin{aligned}
\left\|(\mathbf{u}^{(k-1-l)}\cdot\nabla)\mathbf{u}^{(l)}\right\|_{\mathbf{H}_\#^{r-2k}}&=\left\|\nabla\cdot(\mathbf{u}^{(k-1-l)}\otimes\mathbf{u}^{(l)})\right\|_{\mathbf{H}_\#^{r-2k}}\\
&=\left\|(\mathbf{u}^{(l)}\cdot\nabla)\mathbf{u}^{(k-1-l)}\right\|_{\mathbf{H}_\#^{r-2k}},
\end{aligned}$$

we arrive at Case (A) for $l'=k-1-l$ and finally to the same estimates (4.65) and (4.66).

Thus, for any $r > n/2 - 1$ and any integer $l \in [0, k-1]$, we obtain the estimates

$$\left\|(\mathbf{u}^{(k-1-l)} \cdot \nabla)\mathbf{u}^{(l)}\right\|_{L_\infty(0,T_*;\mathbf{H}_\#^{r-2k})} \le C_{*rn}\left\|\mathbf{u}^{(k-1-l)}\right\|_{L_\infty(0,T_*;\mathbf{H}_\#^{r-2(k-1-l)})}\left\|\mathbf{u}^{(l)}\right\|_{L_\infty(0,T_*;\mathbf{H}_\#^{r-2l})},$$

$$\left\|(\mathbf{u}^{(k-1-l)} \cdot \nabla)\mathbf{u}^{(l)}\right\|_{L_2(0,T_*;\mathbf{H}_\#^{r+1-2k})} \le C_{*rn}\left\|\mathbf{u}^{(k-1-l)}\right\|_{L_\infty(0,T_*;\mathbf{H}_\#^{r-2(k-1-l)})}\left\|\mathbf{u}^{(l)}\right\|_{L_2(0,T_*;\mathbf{H}_\#^{r+1-2l})}.$$

Hence, by (4.64) and (4.61),

$$\partial_t^{k-1}[(\mathbf{u}\cdot\nabla)\mathbf{u}] \in L_\infty(0,T_*;\mathbf{H}_\#^{r-2k}) \cap L_2(0,T_*;\mathbf{H}_\#^{r+1-2k}). \tag{4.67}$$

Step 2. Linear terms and right-hand side.
Due to (4.61),

$$\partial_t^{k-1}\mathfrak{L}\mathbf{u} = \mathfrak{L}\mathbf{u}^{(k-1)} \in L_\infty(0,T_*;\mathbf{H}_\#^{r-2k}) \cap L_2(0,T_*;\mathbf{H}_\#^{r+1-2k}). \tag{4.68}$$

We also have $\mathbf{f}^{(k-1)} \in L_\infty(0,T;\dot{\mathbf{H}}_\#^{r-2k}) \cap L_2(0,T;\dot{\mathbf{H}}_\#^{r+1-2k})$. Then, (4.63), (4.67), and (4.68) imply that

$$\mathbf{F}^{(k-1)} \in L_\infty(0,T;\dot{\mathbf{H}}_\#^{r-2k}) \cap L_2(0,T;\dot{\mathbf{H}}_\#^{r+1-2k}). \tag{4.69}$$

Thus, by (4.62), $\mathbf{u}^{(k)} \in L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r-2k}) \cap L_2(0,T_*;\dot{\mathbf{H}}_\#^{r+1-2k})$.

Step 3. Pressure.
The associated pressure p satisfies (2.10). Differentiating it in time, we obtain

$$\nabla p^{(l)} = \mathbb{P}_g\mathbf{F}^{(l)} \qquad \text{in } \mathbb{T}\times(0,T), \quad l = 0,1,\dots,k-1. \tag{4.70}$$

By the same reasoning as in the proof of (4.69), the similar inclusions for junior derivatives also hold:

$$\mathbf{F}^{(l)} \in L_\infty(0,T;\dot{\mathbf{H}}_\#^{r-2-2l}) \cap L_2(0,T;\dot{\mathbf{H}}_\#^{r-1-2l}), \quad l = 0,1,\dots,k-1.$$

By Lemma 6.5 for gradient, with $s = r-1-2l$ and $s = r-2l$, respectively, Equation (4.70) implies that $p^{(l)} \in L_\infty(0,T_*;\dot{H}_\#^{r-1-2l}) \cap L_2(0,T_*;\dot{H}_\#^{r-2l})$. □

Corollary 4.11. *Let $T > 0$ and $n \ge 2$. Let the coefficients $a_{ij}^{\alpha\beta}$ be constant and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in C^\infty(0,T;\dot{\mathbf{C}}_\#^\infty)$ and $\mathbf{u}^0 \in \dot{\mathbf{C}}_{\#\sigma}^\infty$, while $\mathbf{f}$, $\mathbf{u}^0$, and $T_* \in (0,T]$ satisfy inequality (4.28) from Theorem 4.1.*

Then, the Serrin-type solution $\mathbf{u}$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) is such that $\mathbf{u} \in C^\infty(0,T_*;\dot{\mathbf{C}}_{\#\sigma}^\infty)$, $p \in C^\infty(0,T_*;\dot{C}_\#^\infty)$.

Proof. Taking into account that $\dot{\mathbf{C}}_\#^\infty = \bigcap_{r\in\mathbb{R}}\dot{\mathbf{H}}_{\#\sigma}^r$, Theorem 4.10 implies that for any integer $k \ge 0$, $\mathbf{u}^{(k)} \in L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r-2k}) \cap L_2(0,T_*;\dot{\mathbf{H}}_\#^{r+1-2k})$, $p^{(k)} \in L_\infty(0,T_*;\dot{H}_\#^{r-1-2k}) \cap L_2(0,T_*;\dot{H}_{\#\sigma}^{r-2k})$, for any $r \in \mathbb{R}$. Hence, $\mathbf{u} \in C^\infty(0,T_*;\dot{\mathbf{C}}_{\#\sigma}^\infty)$, $p \in C^\infty(0,T_*;\dot{C}_\#^\infty)$. □

4.6 | Regularity of Two-Dimensional Weak Solution for Constant-Viscosity Coefficients

The regularity results of Sections 4.4 and 4.5 hold for $n = 2$, but as for the isotropic constant-coefficient case (cf., e.g., [11, Chapter 3, Sections 3.3 and 3.5.1], [7, Section 6.5]), these results can be essentially improved for $n = 2$ also in the anisotropic setting with constant coefficients.

Let us give a counterpart of Theorem 4.5 that for $n = 2$, it is valid on any time interval $[0,T]$ (and not only on its special subinterval $[0,T_*]$).

Theorem 4.12. (Spatial regularity of solution for arbitrarily large data). *Let $n = 2$, $r > 0$, and $T > 0$. Let the coefficients $a_{ij}^{\alpha\beta}$ be constant and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}_\#^{r-1})$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^r$.*

Then, the solution $\mathbf{u}$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) obtained in Theorem 2.4 is of Serrin type and belongs to $L_\infty(0,T;\dot{\mathbf{H}}_{\#\sigma}^r) \cap L_2(0,T;\dot{\mathbf{H}}_{\#\sigma}^{r+1})$. In addition, $\mathbf{u}' \in L_2(0,T;\dot{\mathbf{H}}_{\#\sigma}^{r-1})$, $\mathbf{u} \in C^0([0,T];\dot{\mathbf{H}}_{\#\sigma}^r)$, $\lim_{t\to 0}\|\mathbf{u}(\cdot,t)-\mathbf{u}^0\|_{\dot{\mathbf{H}}_{\#\sigma}^r} = 0$, and $p \in L_2(0,T_*;\dot{H}_\#^r)$.

Proof. The proof coincides word for word with the proof of Theorem 4.5 if we take there $n = 2$ while replacing T_* by T and the reference to (4.35) for the boundedness of the sequence $\|\mathbf{u}_m\|_{L_2(0,T_*;\mathbf{H}_\#^{n/2})}$ for $n = 2$ by the reference to the corresponding inequality

$$\|\mathbf{u}_m\|^2_{L_2(0,T;\dot{\mathbf{H}}_\#^1)} \le 4C_{\mathbb{A}}\left(\|\mathbf{u}^0\|^2_{\mathbf{L}_{2\#}} + 4C_{\mathbb{A}}\|\mathbf{f}\|^2_{L_2(0,T;\dot{\mathbf{H}}_\#^{-1})}\right).$$

obtained as inequality (59) in our paper [19]. □

The following assertion can be proved similarly to Corollary 4.8.

Corollary 4.13. *Let $T > 0$ and $n = 2$. Let the coefficients $a_{ij}^{\alpha\beta}$ be constant and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in L_2(0,T;\dot{\mathbf{C}}_\#^\infty)$ and $\mathbf{u}^0 \in \dot{\mathbf{C}}_{\#\sigma}^\infty$.*

Then, the solution $\mathbf{u}$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) obtained in Theorem 2.4 is of Serrin type and is such that $\mathbf{u} \in C^0([0,T];\dot{\mathbf{C}}_{\#\sigma}^\infty)$, $\mathbf{u}' \in L_2(0,T;\dot{\mathbf{C}}_{\#\sigma}^\infty)$, and $p \in L_2(0,T;\dot{C}_\#^\infty)$.

The next three assertions on spatial-temporal regularity for $n = 2$ are the corresponding counterparts of Theorems 4.9 and 4.10 and Corollary 4.11 and are proved in a similar way after replacing there T_* by T.

Theorem 4.14. *Let $T > 0$, $n = 2$, and $r > 0$. Let the coefficients $a_{ij}^{\alpha\beta}$ be constant and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in L_\infty(0,T;\dot{\mathbf{H}}_\#^{r-2}) \cap L_2(0,T;\dot{\mathbf{H}}_\#^{r-1})$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^r$.*

Then, the solution $\mathbf{u}$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) obtained in Theorem 2.4 is of Serrin type and is such that $\mathbf{u}' \in L_\infty(0,T;\dot{\mathbf{H}}^{r-2}_{\#\sigma}) \cup L_2(0,T;\dot{\mathbf{H}}^{r-1}_{\#\sigma})$, $p \in L_\infty(0,T;\dot{H}^{r-1}_{\#}) \cap L_2(0,T;\dot{H}^{r}_{\#})$.

Theorem 4.15. *Let $T>0$, $n=2$, and $r>0$. Let the coefficients $a^{\alpha\beta}_{ij}$ be constant and the relaxed ellipticity condition (1.2) hold. Let $k\in[1,r+1)$ be an integer. Let $\mathbf{f}^{(l)} \in L_\infty(0,T;\dot{\mathbf{H}}^{r-2-2l}_{\#}) \cap L_2(0,T;\dot{\mathbf{H}}^{r-1-2l}_{\#})$, $l=0,1,\dots,k-1$, and $\mathbf{u}^0 \in \dot{\mathbf{H}}^{r}_{\#\sigma}$.*

Then, the solution $\mathbf{u}$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) obtained in Theorem 2.4 is of Serrin type and is such that $\mathbf{u}^{(l)} \in L_\infty(0,T;\dot{\mathbf{H}}^{r-2l}_{\#\sigma}) \cap L_2(0,T;\dot{\mathbf{H}}^{r+1-2l}_{\#\sigma})$, $l=0,\dots,k$; $p^{(l)} \in L_\infty(0,T;\dot{H}^{r-1-2l}_{\#}) \cap L_2(0,T;\dot{H}^{r-2l}_{\#\sigma})$, $l=0,\dots,k-1$.

Corollary 4.16. *Let $T>0$ and $n=2$. Let the coefficients $a^{\alpha\beta}_{ij}$ be constant and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in C^\infty(0,T;\dot{\mathbf{C}}^{\infty}_{\#})$ and $\mathbf{u}^0 \in \dot{\mathbf{C}}^{\infty}_{\#\sigma}$.*

Then, the solution $\mathbf{u}$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) obtained in Theorem 2.4 is of Serrin type and is such that $\mathbf{u} \in C^\infty(0,T;\dot{\mathbf{C}}^{\infty}_{\#\sigma})$, $p \in C^\infty(0,T;\dot{C}^{\infty}_{\#})$.

5 | Serrin-Type Solution Existence and Regularity for Variable Anisotropic Viscosity Coefficients

In this section, we generalize to the anisotropic variable viscosity coefficients the analysis of the existence and regularity of Serrin-type solutions for any $n\ge 2$ given in Section 4 for the anisotropic constant-viscosity coefficients.

5.1 | Preliminary Results for Variable Anisotropic Viscosity Coefficients

For some $n\ge 2$, $r\ge n/2-1$, and $T>0$, let $a^{\alpha\beta}_{ij} \in L_\infty([0,T];H^{\tilde\sigma+1}_{\#})$, $\tilde\sigma > \frac{n}{2}+\max\{|r-1|,|n/2-2|\}$, and the relaxed ellipticity condition (1.2) hold. Let also $\mathbf{f}\in L_2(0,T;\dot{\mathbf{H}}^{r-1}_{\#})$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}^{r}_{\#\sigma}$.

We employ the Galerkin approximation as in Section 4.2 and repeating the same arguments arrive at the same Equation (4.17) but now with the variable coefficients $a^{\alpha\beta}_{ij}(\mathbf{x},t)$. These equations can be now rewritten as

$$\begin{aligned}
&\langle \partial_t \Lambda^r_{\#}\tilde{\mathbf{u}}_m, \Lambda^r_{\#}\mathbf{w}_k\rangle_{\mathbb{T}} + \left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\Lambda^r_{\#}\tilde{\mathbf{u}}_m), E_{i\alpha}(\Lambda^r_{\#}\mathbf{w}_k)\right\rangle_{\mathbb{T}} \\
&\quad + \langle \Lambda^{r-1}_{\#}[(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m], \Lambda^{r+1}_{\#}\mathbf{w}_k\rangle_{\mathbb{T}} \\
&= \langle \Lambda^{r-1}_{\#}\mathbf{f}, \Lambda^{r+1}_{\#}\mathbf{w}_k\rangle_{\mathbb{T}} + \langle \nabla\Lambda^r_{\#}\mathbf{v}_m, \nabla\Lambda^r_{\#}\mathbf{w}_k\rangle_{\mathbb{T}} \\
&\quad - \left\langle E_{j\beta}(\mathbf{v}_m), a^{\alpha\beta}_{ij}\Lambda^r_{\#}E_{i\alpha}(\Lambda^r_{\#}\mathbf{w}_k)\right\rangle_{\mathbb{T}} \\
&\quad - \left\langle E_{j\beta}(\tilde{\mathbf{u}}_m), a^{\alpha\beta}_{ij}\Lambda^r_{\#}E_{i\alpha}(\Lambda^r_{\#}\mathbf{w}_k) - \Lambda^r_{\#}[a^{\alpha\beta}_{ij}E_{i\alpha}(\Lambda^r_{\#}\mathbf{w}_k)]\right\rangle_{\mathbb{T}} \\
&\quad \forall\, k\in\{1,\dots,m\}.
\end{aligned} \tag{5.1}$$

Multiplying equations in (5.1) by $\tilde\eta_{k,m}(t)$ and summing them up over $k\in\{1,\dots,m\}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t \|\Lambda^r_{\#}\tilde{\mathbf{u}}_m\|^2_{\mathbf{H}^0_{\#}} + a_{\mathbb{T}}(t;\Lambda^r_{\#}\tilde{\mathbf{u}}_m,\Lambda^r_{\#}\tilde{\mathbf{u}}_m) \\
&= \langle \Lambda^{r-1}_{\#}\mathbf{f}, \Lambda^{r+1}_{\#}\tilde{\mathbf{u}}_m\rangle_{\mathbb{T}} + \langle \nabla\Lambda^r_{\#}\mathbf{v}_m, \nabla\Lambda^r_{\#}\tilde{\mathbf{u}}_m\rangle_{\mathbb{T}} \\
&\quad - \left\langle E_{j\beta}(\mathbf{v}_m), a^{\alpha\beta}_{ij}\Lambda^r_{\#}E_{i\alpha}(\Lambda^r_{\#}\tilde{\mathbf{u}}_m)\right\rangle_{\mathbb{T}} \\
&\quad - \left\langle E_{j\beta}(\tilde{\mathbf{u}}_m), a^{\alpha\beta}_{ij}\Lambda^r_{\#}E_{i\alpha}(\Lambda^r_{\#}\tilde{\mathbf{u}}_m) - \Lambda^r_{\#}[a^{\alpha\beta}_{ij}E_{i\alpha}(\Lambda^r_{\#}\tilde{\mathbf{u}}_m)]\right\rangle_{\mathbb{T}} \\
&\quad - \langle \Lambda^{r-1}_{\#}[(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m], \Lambda^{r+1}_{\#}\tilde{\mathbf{u}}_m\rangle_{\mathbb{T}}.
\end{aligned} \tag{5.2}$$

From (2.7), we have

$$a_{\mathbb{T}}(t;\Lambda^r_{\#}\tilde{\mathbf{u}}_m,\Lambda^r_{\#}\tilde{\mathbf{u}}_m) \ge \frac{1}{4}C^{-1}_{\mathbb{A}}\|\Lambda^r_{\#}\tilde{\mathbf{u}}_m\|^2_{\dot{\mathbf{H}}^1_{\#\sigma}} = \frac{1}{4}C^{-1}_{\mathbb{A}}\|\tilde{\mathbf{u}}_m\|^2_{\dot{\mathbf{H}}^{r+1}_{\#\sigma}}. \tag{5.3}$$

Let us now estimate the terms in the right-hand side of (5.2). For the first two terms and for the last one, estimates (4.21), (4.22), and (4.24) still hold.

Further, because $\tilde\sigma+1>n/2$, we obtain by Theorem 6.1(a) and inequality (1.10),

$$\begin{aligned}
&\left|\left\langle E_{j\beta}(\mathbf{v}_m), a^{\alpha\beta}_{ij}\Lambda^r_{\#}E_{i\alpha}(\Lambda^r_{\#}\tilde{\mathbf{u}}_m)\right\rangle_{\mathbb{T}}\right| \\
&\le \|E_{j\beta}(\mathbf{v}_m)\|_{(H^r_{\#})^{n\times n}} \|a^{\alpha\beta}_{ij}\Lambda^r_{\#}E_{i\alpha}(\Lambda^r_{\#}\tilde{\mathbf{u}}_m)\|_{(H^{-r}_{\#})^{n\times n}} \\
&\le \|E_{j\beta}(\mathbf{v}_m)\|_{(H^r_{\#})^{n\times n}} C_{*\tilde\sigma rn}\|a^{\alpha\beta}_{ij}\|_{H^{\tilde\sigma+1}_{\#}}\|\Lambda^r_{\#}E_{i\alpha}(\Lambda^r_{\#}\tilde{\mathbf{u}}_m)\|_{(H^{-r}_{\#})^{n\times n}} \\
&\le C_{*\tilde\sigma rn}\|\mathbb{A}\|_{H^{\tilde\sigma+1}_{\#},F}\ \|\mathbf{v}_m\|_{\dot{\mathbf{H}}^{r+1}_{\#}}\|\tilde{\mathbf{u}}_m\|_{\dot{\mathbf{H}}^{r+1}_{\#}},
\end{aligned} \tag{5.4}$$

where $C_{*\tilde\sigma rn} := C_*(-r,\tilde\sigma+1,n)$,

$$\|\mathbb{A}(\cdot,t)\|_{H^{\tilde\sigma+1}_{\#},F} := \left|\left\{\|a^{\alpha\beta}_{ij}(\cdot,t)\|_{H^{\tilde\sigma+1}_{\#}}\right\}^n_{\alpha,\beta,i,j=1}\right|_F.$$

By Theorem 6.3 with $\theta=r$, $s=0$,

$$\begin{aligned}
&\left|\left\langle E_{j\beta}(\tilde{\mathbf{u}}_m), a^{\alpha\beta}_{ij}\Lambda^r_{\#}E_{i\alpha}(\Lambda^r_{\#}\tilde{\mathbf{u}}_m) - \Lambda^r_{\#}[a^{\alpha\beta}_{ij}E_{i\alpha}(\Lambda^r_{\#}\tilde{\mathbf{u}}_m)]\right\rangle_{\mathbb{T}}\right| \\
&\le \|E_{j\beta}(\tilde{\mathbf{u}}_m)\|_{(H^{r-1}_{\#})^{n\times n}}\|a^{\alpha\beta}_{ij}\Lambda^r_{\#}E_{i\alpha}(\Lambda^r_{\#}\tilde{\mathbf{u}}_m) \\
&\quad - \Lambda^r_{\#}[a^{\alpha\beta}_{ij}E_{i\alpha}(\Lambda^r_{\#}\tilde{\mathbf{u}}_m)]\|_{(H^{-r+1}_{\#})^{n\times n}} \\
&\le \overline{C}_{0,r,\tilde\sigma}\|E_{j\beta}(\tilde{\mathbf{u}}_m)\|_{(H^{r-1}_{\#})^{n\times n}}|a^{\alpha\beta}_{ij}|_{H^{\tilde\sigma+1}_{\#}}\|E_{i\alpha}(\Lambda^r_{\#}\tilde{\mathbf{u}}_m)\|_{(H^0_{\#}))^{n\times n}} \\
&\le \overline{C}_{0,r,\tilde\sigma}|\mathbb{A}|_{H^{\tilde\sigma+1}_{\#},F}\|\tilde{\mathbf{u}}_m\|_{\dot{\mathbf{H}}^r_{\#}}\|\tilde{\mathbf{u}}_m\|_{\dot{\mathbf{H}}^{r+1}_{\#}},
\end{aligned} \tag{5.5}$$

where

$$|\mathbb{A}(\cdot,t)|_{H^{\tilde\sigma+1}_{\#},F} := \left|\left\{|a^{\alpha\beta}_{ij}(\cdot,t)|_{H^{\tilde\sigma+1}_{\#}}\right\}^n_{\alpha,\beta,i,j=1}\right|_F \le \|\mathbb{A}(\cdot,t)\|_{H^{\tilde\sigma+1}_{\#},F}.$$

Implementing estimates (4.21), (4.22), (4.24), (5.3), (5.4), and (5.5) in (5.2) and using Young's inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\tilde{\mathbf{u}}_m\|^2_{\mathbf{H}^r_{\#}} + \frac{1}{2}C^{-1}_{\mathbb{A}}\|\tilde{\mathbf{u}}_m\|^2_{\mathbf{H}^{r+1}_{\#}} \\
&\le 2(\|\mathbf{f}\|_{\mathbf{H}^{r-1}_{\#}} + \left[C_{*\tilde\sigma rn}\|\mathbb{A}\|_{H^{\tilde\sigma+1}_{\#},F}+1\right]\|\mathbf{v}_m\|_{\dot{\mathbf{H}}^{r+1}_{\#}} \\
&\quad + \overline{C}_{0,r,\tilde\sigma}|\mathbb{A}|_{H^{\tilde\sigma+1}_{\#},F}\|\tilde{\mathbf{u}}_m\|_{\dot{\mathbf{H}}^r_{\#}} + \|(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m\|_{\mathbf{H}^{r-1}_{\#}})\|\tilde{\mathbf{u}}_m\|_{\mathbf{H}^{r+1}_{\#}} \\
&\le 4C_{\mathbb{A}}(\|\mathbf{f}\|_{\mathbf{H}^{r-1}_{\#}} + \left[C_{*\tilde\sigma rn}\|\mathbb{A}\|_{H^{\tilde\sigma+1}_{\#},F}+1\right]\|\mathbf{v}_m\|_{\dot{\mathbf{H}}^{r+1}_{\#}} \\
&\quad + \overline{C}_{0,r,\tilde\sigma}|\mathbb{A}|_{H^{\tilde\sigma+1}_{\#},F}\|\tilde{\mathbf{u}}_m\|_{\dot{\mathbf{H}}^r_{\#}} + \|(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m\|_{\mathbf{H}^{r-1}_{\#}})^2 \\
&\quad + \frac{1}{4}C^{-1}_{\mathbb{A}}\|\tilde{\mathbf{u}}_m\|^2_{\mathbf{H}^{r+1}_{\#}}.
\end{aligned}$$

Hence, by the inequality $\left(\sum_{i=1}^k a_i\right)^2 \le k\sum_{i=1}^k a_i^2$ (following from the Cauchy–Schwarz inequality),

$$\begin{aligned}
&\frac{d}{dt}\|\tilde{\mathbf{u}}_m\|^2_{\mathbf{H}^r_\#} + \frac{1}{4}C_{\mathbb{A}}^{-1}\|\tilde{\mathbf{u}}_m\|^2_{\mathbf{H}^{r+1}_\#}\\
&\le 20C_{\mathbb{A}}\Bigg(\|\mathbf{f}\|^2_{\mathbf{H}^{r-1}_\#} + \left[C^2_{*\tilde{\sigma}rn}\|\mathbb{A}\|^2_{H^{\tilde{\sigma}+1}_\#,F}+1\right]\|\mathbf{v}_m\|^2_{\dot{\mathbf{H}}^{r+1}_\#} \\
&\quad + \overline{C}^2_{0,r,\tilde{\sigma}}|\mathbb{A}|^2_{H^{\tilde{\sigma}+1}_\#,F}\|\tilde{\mathbf{u}}_m\|^2_{\dot{\mathbf{H}}^r_\#} + \|(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m\|^2_{\mathbf{H}^{r-1}_\#}\Bigg).
\end{aligned} \tag{5.6}$$

Note that by the similar reasoning, but without employing in (4.9) and (4.10) the function $\mathbf{v}$, we obtain that $\mathbf{u}_m$ satisfies the differential inequality

$$\begin{aligned}
&\frac{d}{dt}\|\mathbf{u}_m\|^2_{\mathbf{H}^r_\#} + \frac{1}{4}C_{\mathbb{A}}^{-1}\|\mathbf{u}_m\|^2_{\mathbf{H}^{r+1}_\#}\\
&\le 12C_{\mathbb{A}}\left(\|\mathbf{f}\|^2_{\mathbf{H}^{r-1}_\#} + \overline{C}^2_{0,r,\tilde{\sigma}}|\mathbb{A}|^2_{H^{\tilde{\sigma}+1}_\#,F}\|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}^r_\#} + \|(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m\|^2_{\mathbf{H}^{r-1}_\#}\right).
\end{aligned} \tag{5.7}$$

5.2 | Serrin-Type Solution Existence for Variable Anisotropic Viscosity Coefficients

Employing the results from Section 5.1 for $r = n/2-1$, we are now in the position to prove the existence of Serrin-type solutions.

Theorem 5.1. *Let $n \ge 2$ and $T > 0$. Let $a^{\alpha\beta}_{ij} \in L_\infty([0,T]; H^{\tilde{\sigma}+1}_\#)$, $\tilde{\sigma} > \frac{n}{2} + |n/2-2|$, and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}^{n/2-2}_\#)$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}^{n/2-1}_{\#\sigma}$.*

i. *Then, there exist constants $A_1 \ge 0$, $A_2 \ge 0$, and $A_3 > 0$ that are independent of $\mathbf{f}$ and $\mathbf{u}^0$ but may depend on T, n, $\|\mathbb{A}\|$, and $C_{\mathbb{A}}$, such that if $\mathbf{f}$, $\mathbf{u}^0$, and $T_* \in (0,T]$ satisfy the inequality*

$$\begin{aligned}
&\int_0^{T_*}\|\mathbf{f}(\cdot,t)\|^2_{\dot{\mathbf{H}}^{n/2-2}_\#}dt + \left(A_1\|\mathbf{u}^0\|^2_{\dot{\mathbf{H}}^{n/2-1}_\#} + A_2\right)\\
&\int_0^{T_*}\|(K\mathbf{u}^0)(\cdot,t)\|^2_{\dot{\mathbf{H}}^{n/2}_\#}dt < A_3,
\end{aligned} \tag{5.8}$$

where K is the operator defined in (4.3), then there exists a solution $\mathbf{u}$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) in $L_\infty(0,T_*;\dot{\mathbf{H}}^{n/2-1}_{\#\sigma}) \cap L_2(0,T_*;\dot{\mathbf{H}}^{n/2}_{\#\sigma})$, which is thus a Serrin-type solution.

ii. *In addition, $\mathbf{u}' \in L_2(0,T_*;\dot{\mathbf{H}}^{n/2-2}_{\#\sigma})$, $\mathbf{u} \in C^0([0,T_*];\dot{\mathbf{H}}^{n/2-1}_{\#\sigma})$, $\lim_{t\to 0}\|\mathbf{u}(\cdot,t)-\mathbf{u}^0\|_{\dot{\mathbf{H}}^{n/2-1}_{\#\sigma}} = 0$, and $p \in L_2(0,T_*;\dot{H}^{n/2-1}_\#)$.*

iii. *Moreover, $\mathbf{u}$ satisfies the following energy equality for any $[t_0,t] \subset [0,T_*]$,*

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}_{2\#}} + \int_{t_0}^t a_{\mathbb{T}}(\tau;\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))d\tau\\
&= \frac{1}{2}\|\mathbf{u}(\cdot,t_0)\|^2_{\mathbf{L}_{2\#}} + \int_{t_0}^t \langle\mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}d\tau,
\end{aligned} \tag{5.9}$$

It particularly implies the standard energy equality,

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{u}(\cdot,t)\|^2_{\mathbf{L}_{2\#}} + \int_0^t a_{\mathbb{T}}(\tau;\mathbf{u}(\cdot,\tau),\mathbf{u}(\cdot,\tau))d\tau\\
&= \frac{1}{2}\|\mathbf{u}^0\|^2_{\mathbf{L}_{2\#}} + \int_0^t \langle\mathbf{f}(\cdot,\tau),\mathbf{u}(\cdot,\tau)\rangle_{\mathbb{T}}\,d\tau \quad \forall\, t \in [0,T_*].
\end{aligned} \tag{5.10}$$

iv. *The solution $\mathbf{u}$ is unique in the class of solutions from $L_\infty(0,T_*;\dot{\mathbf{H}}^0_{\#\sigma}) \cap L_2(0,T_*;\dot{\mathbf{H}}^1_{\#\sigma})$ satisfying the energy inequality (2.14) on the interval $[0,T_*]$.*

Proof.

i. Let $r = n/2-1$. The estimate (4.27) still holds. Let us fix any small $\tilde{\sigma}_n$ such that $\tilde{\sigma} \ge \tilde{\sigma}_n > n/2 + |n/2-2| = \max\{2, n-2\}$. Then, by (4.31), we obtain from (5.6),

$$\begin{aligned}
&\frac{d}{dt}\|\tilde{\mathbf{u}}_m\|^2_{\mathbf{H}^{n/2-1}_\#} + \frac{1}{4}C_{\mathbb{A}}^{-1}\|\tilde{\mathbf{u}}_m\|^2_{\mathbf{H}^{n/2}_\#}\\
&\le \left(160C^2_{*rn}C_{\mathbb{A}}\|\tilde{\mathbf{u}}_m\|^2_{\mathbf{H}^{n/2}_\#} + 20C_{\mathbb{A}}\overline{C^2_n}|\mathbb{A}|^2_{H^{\tilde{\sigma}_n+1}_\#,F}\right)\|\tilde{\mathbf{u}}_m\|^2_{\dot{\mathbf{H}}^{n/2-1}_\#}\\
&\quad + 20C_{\mathbb{A}}\Bigg(\|\mathbf{f}\|^2_{\mathbf{H}^{n/2-2}_\#} + 8C^2_{*rn}\|\mathbf{v}_m\|^2_{\mathbf{H}^{n/2-1}_\#}\|\mathbf{v}_m\|^2_{\mathbf{H}^{n/2}_\#}\\
&\quad + \left[\tilde{C}^2_{*n}\|\mathbb{A}\|^2_{H^{\tilde{\sigma}_n+1}_\#,F}+1\right]\|\mathbf{v}_m\|^2_{\dot{\mathbf{H}}^{n/2}_\#}\Bigg),
\end{aligned} \tag{5.11}$$

where $\overline{C}_n := \overline{C}_{0,n/2-1,\tilde{\sigma}_n}$, $\tilde{C}_{*n} := C_{*\tilde{\sigma}_n,n/2-1,n} = C_*(-n/2+1,\tilde{\sigma}_n+1,n)$.

Let us now apply to (5.11) Lemma 6.13 with

$$\begin{aligned}
&\eta = \|\tilde{\mathbf{u}}_m\|^2_{\mathbf{H}^{n/2-1}_\#},\ \eta_0 = 0,\ y = \|\tilde{\mathbf{u}}_m\|^2_{\mathbf{H}^{n/2}_\#},\ b = \frac{1}{4}C_{\mathbb{A}}^{-1},\\
&c = 160C^2_{*rn}C_{\mathbb{A}},\ \phi = 20C_{\mathbb{A}}\overline{C^2_n}|\mathbb{A}|^2_{H^{\tilde{\sigma}_n+1}_\#,F},\\
&\psi = 20C_{\mathbb{A}}\Bigg(\|\mathbf{f}\|^2_{\mathbf{H}^{n/2-2}_\#} + 8C^2_{*rn}\|\mathbf{v}_m\|^2_{\mathbf{H}^{n/2-1}_\#}\|\mathbf{v}_m\|^2_{\mathbf{H}^{n/2}_\#}\\
&\qquad + \left[\tilde{C}^2_{*n}\|\mathbb{A}\|^2_{H^{\tilde{\sigma}_n+1}_\#,F}+1\right]\|\mathbf{v}_m\|^2_{\dot{\mathbf{H}}^{n/2}_\#}\Bigg),
\end{aligned}$$

to conclude that if T_* is such that

$$\begin{aligned}
&\int_0^{T_*} e^{\Phi(T_*)-\Phi(t)}\Bigg(\|\mathbf{f}(\cdot,t)\|^2_{\mathbf{H}^{n/2-2}_\#} + \Bigg(8C^2_{*rn}\|\mathbf{v}_m(\cdot,t)\|^2_{\mathbf{H}^{n/2-1}_\#}\\
&\quad + \left[\tilde{C}^2_{*n}\|\mathbb{A}\|^2_{H^{\tilde{\sigma}_n+1}_\#,F}+1\right]\Bigg)\|\mathbf{v}_m(\cdot,t)\|^2_{\dot{\mathbf{H}}^{n/2}_\#}\Bigg)dt\\
&< \left(640eC^2_{\mathbb{A}}C^2_{*rn}\right)^{-1},
\end{aligned} \tag{5.12}$$

where

$$\Phi(s) := \int_0^s \phi(\tau)d\tau = 20C_{\mathbb{A}}\overline{C^2_n}\int_0^s|\mathbb{A}(\cdot,\tau)|^2_{H^{\tilde{\sigma}_n+1}_\#,F}d\tau,$$

then

$$\begin{aligned}
&\|\mathbf{u}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}^{n/2-1}_{\#\sigma})} \le \|\tilde{\mathbf{u}}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}^{n/2-1}_{\#\sigma})} + \|\mathbf{v}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}^{n/2-1}_{\#\sigma})}\\
&\le \left(8\sqrt{10}C_{\mathbb{A}}C_{*rn}\right)^{-1} + \|\mathbf{u}^0\|_{\dot{\mathbf{H}}^{n/2-1}_{\#\sigma}},
\end{aligned} \tag{5.13}$$

$$\|\mathbf{u}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}^{n/2}_{\#\sigma})} \le \|\widetilde{\mathbf{u}}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}^{n/2}_{\#\sigma})} + \|\mathbf{v}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}^{n/2}_{\#\sigma})}$$
$$\le \left(4\sqrt{10C_{\mathbb{A}}}C_{*rn}\right)^{-1} + \|\mathbf{u}^0\|_{\dot{\mathbf{H}}^{n/2-1}_{\#\sigma}}. \quad (5.14)$$

Estimates (4.13) and (4.14) were taken into account in (5.13) and (5.14).

Taking into account inequality (4.13), we obtain that condition (5.12) is satisfied if T_* is such that

$$\int_0^{T_*}\|\mathbf{f}(\cdot,t)\|^2_{\mathbf{H}^{n/2-2}_\#}dt + \left(8C^2_{*rn}\|\mathbf{u}^0\|^2_{\mathbf{H}^{n/2-1}_\#} + \left[\tilde{C}^2_{*n}\|\mathbb{A}\|^2_{L_\infty(0,T;H^{\tilde{\sigma}_n+1}_\#),F}+1\right]\right)\int_0^{T_*}\|\mathbf{v}(\cdot,t)\|^2_{\dot{\mathbf{H}}^{n/2}_\#}dt$$
$$< \left(640C^2_{\mathbb{A}}C^2_{*rn}\right)^{-1}\exp\left(-1-20C_{\mathbb{A}}\overline{C^2_n}\|\mathbb{A}\|^2_{L_\infty(0,T;H^{\tilde{\sigma}_n+1}_\#),F}\,T\right). \quad (5.15)$$

Note that condition (5.15) gives condition (5.8) with

$$A_1 = 8C^2_{*rn}, \quad A_2 = \tilde{C}^2_{*n}\|\mathbb{A}\|^2_{L_\infty(0,T;H^{\tilde{\sigma}_n+1}_\#),F}+1,$$
$$A_3 = \left(640C^2_{\mathbb{A}}C^2_{*rn}\right)^{-1}\exp\left(-1-20C_{\mathbb{A}}\overline{C^2_n}\|\mathbb{A}\|^2_{L_\infty(0,T;H^{\tilde{\sigma}+1}_\#),F}\,T\right).$$

Inequalities (5.13) and (5.14) imply that there exists a subsequence of $\{\mathbf{u}_m\}$ converging weakly in $L_2(0,T_*;\dot{\mathbf{H}}^{n/2}_{\#\sigma})$ and weakly star in $L_\infty(0,T_*;\dot{\mathbf{H}}^{n/2-1}_{\#\sigma})$ to a function $\mathbf{u}^\dagger \in L_2(0,T_*;\dot{\mathbf{H}}^{n/2}_{\#\sigma}) \cup L_\infty(0,T_*;\dot{\mathbf{H}}^{n/2-1}_{\#\sigma})$. Then, the subsequence converges to $\mathbf{u}^\dagger$ also weakly in $L_2(0,T_*;\dot{\mathbf{H}}^1_{\#\sigma})$ and weakly star in $L_\infty(0,T_*;\dot{\mathbf{H}}^0_{\#\sigma})$. Because$\{\mathbf{u}_m\}$ is the subsequence of the sequence that converges weakly in $L_2\left(0,T;\dot{\mathbf{H}}^1_{\#\sigma}\right)$ and weakly star in $L_\infty\left(0,T;\dot{\mathbf{H}}^0_{\#\sigma}\right)$ to the weak solution, $\mathbf{u}$, of problem (2.1–2.3) on $[0,T_*]$, we conclude that $\mathbf{u}=\mathbf{u}^\dagger \in L_\infty(0,T_*;\dot{\mathbf{H}}^{n/2-1}_{\#\sigma}) \cup L_2(0,T_*;\dot{\mathbf{H}}^{n/2}_{\#\sigma})$.

This implies that $\mathbf{u}$ is a Serrin-type solution on the interval $[0,T_*]$, and we thus proved item (i) of the theorem.

ii. As in step (ii) of the proof of Theorem 5.1, estimate (4.37) implies that $(\mathbf{u}\cdot\nabla)\mathbf{u} \in L_2(0,T_*;\dot{\mathbf{H}}^{n/2-2}_\#)$. By (1.1) and (1.3), we have

$$\|\mathfrak{L}\mathbf{u}\|^2_{\mathbf{H}^{n/2-2}_\#} \le \|a^{\alpha\beta}_{ij}E_{i\alpha}(\mathbf{u})\|_{(H^{n/2-1}_\#)^{n\times n}} \le \|\mathbb{A}\|^2_{H^{\tilde{\sigma}+1}_\#,F}\|\mathbf{u}\|^2_{\mathbf{H}^{n/2}_\#},$$

and thus,

$$\|\mathfrak{L}\mathbf{u}\|^2_{L_2(0,T_*;\mathbf{H}^{n/2-2}_\#)} \le \|\mathbb{A}\|^2_{L_\infty(0,T_*;H^{\tilde{\sigma}+1}_\#),F}\|\mathbf{u}\|^2_{L_2(0,T_*;\mathbf{H}^{n/2}_\#)},$$

that is, $\mathfrak{L}\mathbf{u} \in L_2(0,T_*;\dot{\mathbf{H}}^{n/2-2}_{\#\sigma})$. We also have $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}^{n/2-2}_\#)$.

Then, (2.9) implies that $\mathbf{u}' \in L_2(0,T_*;\dot{\mathbf{H}}^{n/2-2}_{\#\sigma})$, and hence, by Theorem 6.8, we obtain that $\mathbf{u} \in C^0([0,T_*];\dot{\mathbf{H}}^{n/2-1}_{\#\sigma})$, which also means that $\|\mathbf{u}(\cdot,t)-\mathbf{u}^0\|_{\dot{\mathbf{H}}^{n/2-1}_{\#\sigma}} \to 0$ as $t\to 0$.

To prove the theorem claim about the associated pressure p, we remark that it satisfies (2.10), where $\mathbf{F} \in L_2(0,T;\dot{\mathbf{H}}^{n/2-2}_\#)$ due to the theorem conditions and the inclusion $(\mathbf{u}\cdot\nabla)\mathbf{u} \in L_2(0,T_*;\dot{\mathbf{H}}^{n/2-2}_\#)$. By Lemma 6.5 for gradient, with $s=n/2-1$, Equation (2.10) has a unique solution p in $L_2(0,T_*;\dot{H}^{n/2-1}_\#)$.

iii. The energy equalities (5.9) and (5.10) immediately follow from Theorem 3.5.

iv. The solution uniqueness follows from Theorem 3.6. □

Remark 5.2. Note that by the Sobolev embedding theorem, the condition $a^{\alpha\beta}_{ij} \in L_\infty([0,T];H^{\tilde{\sigma}+1}_\#)$, $\tilde{\sigma} > |n/2-2| + \frac{n}{2}$, in Theorem 5.1 and further on implies $a^{\alpha\beta}_{ij} \in L_\infty([0,T];C^0_\#) \subset L_\infty([0,T];L_{\infty\#})$.

Remark 5.3. Because $\|\mathbf{f}(\cdot,t)\|^2_{\dot{\mathbf{H}}^{n/2-2}_\#}$ is integrable on $(0,T]$ by the theorem condition and $\|(K\mathbf{u}^0)(\cdot,t)\|^2_{\dot{\mathbf{H}}^{n/2}_\#}$ is integrable on $(0,\infty)$ by the inequality (4.5), we conclude that due to the absolute continuity of the Lebesgue integrals, for arbitrarily large data $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}^{n/2-2}_\#)$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}^{n/2-1}_{\#\sigma}$, there exists $T_* > 0$ such that condition (5.8) holds.

Estimating the integrand in the second integral in (5.8) according to (4.5), we arrive at the following assertion allowing an explicit estimate of T_* for arbitrarily large data if $\mathbf{f} \in L_\infty(0,T;\dot{\mathbf{H}}^{n/2-2}_\#)$.

Corollary 5.4. (Serrin-type solution for arbitrarily large data but small time or vice versa). *Let $n \ge 2$ and $T > 0$. Let $a^{\alpha\beta}_{ij} \in L_\infty([0,T];H^{\tilde{\sigma}+1}_\#)$, $\tilde{\sigma} > |n/2-2| + \frac{n}{2}$, and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in L_\infty(0,T;\dot{\mathbf{H}}^{n/2-2}_\#)$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}^{n/2}_{\#\sigma}$.*

Then, there exist constants $A_1, A_2, A_3 > 0$ that are independent of $\mathbf{f}$ and $\mathbf{u}^0$ but may depend on T, n, $\|\mathbb{A}\|$, and $C_{\mathbb{A}}$, such that if $T_* \in (0,T]$ satisfies the inequality

$$T_*\left[\|\mathbf{f}\|^2_{L_\infty(0,T;\dot{\mathbf{H}}^{n/2-2}_\#)} + \left(A_1\|\mathbf{u}^0\|^2_{\mathbf{H}^{n/2-1}_\#} + A_2\right)\|\mathbf{u}^0\|^2_{\mathbf{H}^{n/2}_\#}\right] < A_3, \quad (5.16)$$

then there exists a Serrin-type solution $\mathbf{u} \in L_\infty(0,T_*;\dot{\mathbf{H}}^{n/2-1}_{\#\sigma}) \cap L_2(0,T_*;\dot{\mathbf{H}}^{n/2}_{\#\sigma})$ of the anisotropic Navier–Stokes initial value problem. This solution satisfies items (ii)–(iv) in Theorem 5.1.

Estimating the second integral in (5.8) according to (4.8), we arrive at the following assertion.

Corollary 5.5. (Existence of Serrin-type solution for arbitrary time but small data). *Let $n \ge 2$ and $T > 0$. Let $a^{\alpha\beta}_{ij} \in L_\infty([0,T];H^{\tilde{\sigma}+1}_\#)$, $\tilde{\sigma} > |n/2-2| + \frac{n}{2}$, and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}^{n/2-2}_\#)$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}^{n/2-1}_{\#\sigma}$.*

Then, there exist constants $A_1, A_2, A_3 > 0$ that are independent of $\mathbf{f}$ and $\mathbf{u}^0$ but may depend on T, n, $\|\mathbb{A}\|$, and $C_{\mathbb{A}}$, such that if $\mathbf{f}$ and $\mathbf{u}^0$ satisfy the inequality

$$\|\mathbf{f}\|^2_{L_2(0,T;\dot{\mathbf{H}}^{n/2-2}_\#)} + \left(A_1\|\mathbf{u}^0\|^2_{\mathbf{H}^{n/2-1}_\#} + A_2\right)\|\mathbf{u}^0\|^2_{\mathbf{H}^{n/2-1}_\#} < A_3, \quad (5.17)$$

then there exists a Serrin-type solution $\mathbf{u} \in L_\infty(0,T;\dot{\mathbf{H}}^{n/2-1}_{\#\sigma}) \cap L_2(0,T;\dot{\mathbf{H}}^{n/2}_{\#\sigma})$ of the anisotropic Navier–Stokes initial value problem. This solution satisfies items (ii)–(iv) in Theorem 5.1 with $T_* = T$.

5.3 | Spatial Regularity of Serrin-Type Solutions for Variable Anisotropic Viscosity Coefficients

Theorem 5.6. (Spatial regularity of Serrin-type solution for arbitrarily large data). *Let $n \geq 2$, $r > n/2-1$, and $T > 0$. Let $a_{ij}^{\alpha\beta} \in L_\infty([0,T]; H_\#^{\tilde\sigma+1})$, $\tilde\sigma > \frac{n}{2} + \max\{|r-1|, |n/2-2|\}$, and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in L_2(0,T; \dot{\mathbf{H}}_\#^{r-1})$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}_{\#\sigma}^r$, while $\mathbf{f}$, $\mathbf{u}^0$, and $T_* \in (0,T]$ satisfy inequality (5.8) from Theorem 5.1.*

Then, the Serrin-type solution $\mathbf{u}$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) belongs to $L_\infty(0,T_*; \dot{\mathbf{H}}_{\#\sigma}^r) \cap L_2(0,T_*; \dot{\mathbf{H}}_{\#\sigma}^{r+1})$. In addition, $\mathbf{u}' \in L_2(0,T_*; \dot{\mathbf{H}}_{\#\sigma}^{r-1})$, $\mathbf{u} \in C^0([0,T_*]; \dot{\mathbf{H}}_{\#\sigma}^r)$, $\lim_{t\to 0} \|\mathbf{u}(\cdot,t) - \mathbf{u}^0\|_{\dot{\mathbf{H}}_{\#\sigma}^r} = 0$, and $p \in L_2(0,T_*; \dot{H}_\#^r)$.

Proof. The existence of the Serrin-type solution $\mathbf{u} \in L_\infty(0,T_*; \dot{\mathbf{H}}_{\#\sigma}^{n/2-1}) \cap L_2(0,T_*; \dot{\mathbf{H}}_{\#\sigma}^{n/2})$ is proved in Theorem 5.1(i), and we will prove that it has a higher smoothness. We will employ the same Galerkin approximation used in Section 5.1 and in the proof of Theorem 5.1(i).

Step a. Let us estimate the last term in (5.7) for the case $n/2-1 < r < n/2$. By (4.27), we obtain from (5.7),

$$\frac{d}{dt}\|\mathbf{u}_m\|_{\mathbf{H}_\#^r}^2 + \frac{1}{4}C_{\mathbb{A}}^{-1}\|\mathbf{u}_m\|_{\mathbf{H}_\#^{r+1}}^2 \leq 12C_{\mathbb{A}}\left(\overline{C}_{0,r,\tilde\sigma}^2|\mathbb{A}|_{H_\#^{\tilde\sigma+1},F}^2 + C_{*rn}^2\|\mathbf{u}_m\|_{\mathbf{H}_\#^{n/2}}^2\right)\|\mathbf{u}_m\|_{\dot{\mathbf{H}}_\#^r}^2 + 12C_{\mathbb{A}}\|\mathbf{f}\|_{\mathbf{H}_\#^{r-1}}^2, \tag{5.18}$$

implying

$$\frac{d}{dt}\|\mathbf{u}_m\|_{\mathbf{H}_\#^r}^2 \leq 12C_{\mathbb{A}}\left(\overline{C}_{0,r,\tilde\sigma}^2|\mathbb{A}|_{H_\#^{\tilde\sigma+1},F}^2 + C_{*rn}^2\|\mathbf{u}_m\|_{\mathbf{H}_\#^{n/2}}^2\right)\|\mathbf{u}_m\|_{\dot{\mathbf{H}}_\#^r}^2 + 12C_{\mathbb{A}}\|\mathbf{f}\|_{\mathbf{H}_\#^{r-1}}^2. \tag{5.19}$$

By Gronwall's inequality (6.22), we obtain from (5.19) that

$$\|\mathbf{u}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^r)}^2 \leq \exp\left[12C_{\mathbb{A}}\left(\overline{C}_{0,r,\tilde\sigma}^2 T_*\||\mathbb{A}|_{H_\#^{\tilde\sigma+1},F}\|_{L_\infty(0,T_*)}^2 + C_{*rn}^2\|\mathbf{u}_m\|_{L_2(0,T_*;\mathbf{H}_\#^{n/2})}^2\right)\right] \times\left[\|\mathbf{u}_m(\cdot,0)\|_{\mathbf{H}_\#^r}^2 + 12C_{\mathbb{A}}\|\mathbf{f}\|_{L_2(0,T_*;\mathbf{H}_\#^{r-1})}^2\right]. \tag{5.20}$$

We have $\|\mathbf{u}_m(\cdot,0)\|_{\mathbf{H}_\#^r} \leq \|\mathbf{u}^0\|_{\mathbf{H}_\#^r}$ and by (5.14), the sequence $\|\mathbf{u}_m\|_{L_2(0,T_*;\mathbf{H}_\#^{n/2})}$ is bounded. Then, (5.20) implies that the sequence $\|\mathbf{u}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^r)}^2$ is bounded as well. Integrating (5.18), we conclude that

$$\|\mathbf{u}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r+1})}^2 \leq 48C_{\mathbb{A}}^2\left(\overline{C}_{0,r,\tilde\sigma}^2 T_*\||\mathbb{A}|_{H_\#^{\tilde\sigma+1},F}\|_{L_\infty(0,T_*)}^2 + C_{*rn}^2\|\mathbf{u}_m\|_{L_2(0,T_*;\mathbf{H}_\#^{n/2})}^2\right)\|\mathbf{u}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^r)}^2 + 4C_{\mathbb{A}}\|\mathbf{u}_m(\cdot,0)\|_{\mathbf{H}_\#^r}^2 + 48C_{\mathbb{A}}^2\|\mathbf{f}\|_{L_2(0,T_*;\mathbf{H}_\#^{r-1})}^2. \tag{5.21}$$

Inequalities (5.20) and (5.21) mean that the sequences

$$\{\|\mathbf{u}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^r)}\}_{m=1}^\infty \text{ and } \{\|\mathbf{u}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r+1})}\}_{m=1}^\infty \tag{5.22}$$

are bounded for $n/2-1 < r < n/2$.

Step b. Let now $r = n/2$. By (4.45), we obtain from (5.7),

$$\frac{d}{dt}\|\mathbf{u}_m\|_{\mathbf{H}_\#^{n/2}}^2 + \frac{1}{4}C_{\mathbb{A}}^{-1}\|\mathbf{u}_m\|_{\mathbf{H}_\#^{n/2+1}}^2 \leq 12C_{\mathbb{A}}\left(\overline{C}_{0,r,\tilde\sigma}^2|\mathbb{A}|_{H_\#^{\tilde\sigma+1},F}^2 + C_{*rn}^2\|\mathbf{u}_m\|_{\mathbf{H}_\#^{n/2+1/2}}^2\right)\|\mathbf{u}_m\|_{\dot{\mathbf{H}}_\#^{n/2}}^2 + 12C_{\mathbb{A}}\|\mathbf{f}\|_{\mathbf{H}_\#^{n/2-1}}^2, \tag{5.23}$$

implying

$$\frac{d}{dt}\|\mathbf{u}_m\|_{\mathbf{H}_\#^{n/2}}^2 \leq 12C_{\mathbb{A}}\left(\overline{C}_{0,r,\tilde\sigma}^2|\mathbb{A}|_{H_\#^{\tilde\sigma+1},F}^2 + C_{*rn}^2\|\mathbf{u}_m\|_{\mathbf{H}_\#^{n/2+1/2}}^2\right) \times\|\mathbf{u}_m\|_{\dot{\mathbf{H}}_\#^{n/2}}^2 + 12C_{\mathbb{A}}\|\mathbf{f}\|_{\mathbf{H}_\#^{n/2-1}}^2. \tag{5.24}$$

By Gronwall's inequality (6.22), we obtain from (5.24) that

$$\|\mathbf{u}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2})}^2 \leq \exp\left[12C_{\mathbb{A}}\left(\overline{C}_{0,r,\tilde\sigma}^2 T_*\||\mathbb{A}|_{H_\#^{\tilde\sigma+1},F}\|_{L_\infty(0,T_*)}^2 + C_{*rn}^2\|\mathbf{u}_m\|_{L_2(0,T_*;\mathbf{H}_\#^{n/2+1/2})}^2\right)\right] \times\left[\|\mathbf{u}_m(\cdot,0)\|_{\mathbf{H}_\#^{n/2}}^2 + 12C_{\mathbb{A}}\|\mathbf{f}\|_{L_2(0,T_*;\mathbf{H}_\#^{n/2-1})}^2\right]. \tag{5.25}$$

We have $\|\mathbf{u}_m(\cdot,0)\|_{\mathbf{H}_\#^{n/2}} \leq \|\mathbf{u}^0\|_{\mathbf{H}_\#^{n/2}}$, and by (5.22), the sequence $\|\mathbf{u}_m\|_{L_2(0,T_*;\mathbf{H}_\#^{n/2+1/2})}$ is bounded as well. Then, (5.25) implies that the sequence $\|\mathbf{u}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2})}^2$ is also bounded. Integrating (5.23), we conclude that

$$\|\mathbf{u}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2+1})}^2 \leq 48C_{\mathbb{A}}^2\left(\overline{C}_{0,r,\tilde\sigma}^2 T_*\||\mathbb{A}|_{H_\#^{\tilde\sigma+1},F}\|_{L_\infty(0,T_*)}^2 + C_{*rn}^2\|\mathbf{u}_m\|_{L_2(0,T_*;\mathbf{H}_\#^{n/2+1/2})}^2\right)\|\mathbf{u}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{n/2})}^2 + 4C_{\mathbb{A}}\|\mathbf{u}_m(\cdot,0)\|_{\mathbf{H}_\#^{n/2}}^2 + 48C_{\mathbb{A}}^2\|\mathbf{f}\|_{L_2(0,T_*;\mathbf{H}_\#^{n/2-1})}^2 \leq C < \infty. \tag{5.26}$$

Inequalities (5.25) and (5.26) mean that the sequences

$$\{\|\mathbf{u}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^r)}\}_{m=1}^\infty \text{ and } \{\|\mathbf{u}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}_{\#\sigma}^{r+1})}\}_{m=1}^\infty \tag{5.27}$$

are bounded for $r = n/2$.

Step c. Let now $kn/2 < r \leq (k+1)n/2$, $k = 1, 2, 3, \dots$ By (4.51), we obtain from (5.7),

$$\frac{d}{dt}\|\mathbf{u}_m\|_{\mathbf{H}_\#^r}^2 + \frac{1}{4}C_{\mathbb{A}}^{-1}\|\mathbf{u}_m\|_{\mathbf{H}_\#^{r+1}}^2 \leq 12C_{\mathbb{A}}\left(\overline{C}_{0,r,\tilde\sigma}^2|\mathbb{A}|_{H_\#^{\tilde\sigma+1},F}^2 + C_{*rn}^2\|\mathbf{u}_m\|_{\mathbf{H}_\#^r}^2\right)\|\mathbf{u}_m\|_{\dot{\mathbf{H}}_\#^r}^2 + 12C_{\mathbb{A}}\|\mathbf{f}\|_{\mathbf{H}_\#^{r-1}}^2, \tag{5.28}$$

implying

$$\frac{d}{dt}\|\mathbf{u}_m\|^2_{\mathbf{H}^r_\#} \le 12C_{\mathbb{A}}\left(\overline{C}^2_{0,r,\tilde{\sigma}}|\mathbb{A}|^2_{H^{\tilde{\sigma}+1}_\#,F} + C^2_{*rn}\|\mathbf{u}_m\|^2_{\mathbf{H}^r_\#}\right)\|\mathbf{u}_m\|^2_{\mathbf{H}^r_\#} + 12C_{\mathbb{A}}\|\mathbf{f}\|^2_{\mathbf{H}^{r-1}_\#}. \tag{5.29}$$

By Gronwall's inequality (6.22), we obtain from (5.29) that

$$\|\mathbf{u}_m\|^2_{L_\infty(0,T_*;\dot{\mathbf{H}}^r_{\#\sigma})} \le \exp\left[12C_{\mathbb{A}}\left(\overline{C}^2_{0,r,\tilde{\sigma}}T_*\||\mathbb{A}|_{H^{\tilde{\sigma}+1}_\#,F}\|^2_{L_\infty(0,T_*)} + C^2_{*rn}\|\mathbf{u}_m\|^2_{L_2(0,T_*;\mathbf{H}^r_\#)}\right)\right] \times\left[\|\mathbf{u}_m(\cdot,0)\|^2_{\mathbf{H}^r_\#} + 12C_{\mathbb{A}}\|\mathbf{f}\|^2_{L_2(0,T_*;\mathbf{H}^{r-1}_\#)}\right], \tag{5.30}$$

where $\|\mathbf{u}_m(\cdot,0)\|_{\mathbf{H}^r_\#} \le \|\mathbf{u}^0\|_{\mathbf{H}^r_\#}$.
If $k=1$, then the sequence $\|\mathbf{u}_m\|_{L_2(0,T_*;\mathbf{H}^r_\#)}$ in (5.30) is bounded due to (5.22) and (5.27). Then, (5.30) implies that the sequence $\|\mathbf{u}_m\|^2_{L_\infty(0,T_*;\dot{\mathbf{H}}^r_{\#\sigma})}$ is bounded as well. Integrating (5.28), we also conclude that for $k=1$,

$$\|\mathbf{u}_m\|^2_{L_2(0,T_*;\dot{\mathbf{H}}^{r+1}_{\#\sigma})} \le 48C^2_{\mathbb{A}}\left(\overline{C}^2_{0,r,\tilde{\sigma}}T_*\||\mathbb{A}|_{H^{\tilde{\sigma}+1}_\#,F}\|^2_{L_\infty(0,T_*)} + C^2_{*rn}\|\mathbf{u}_m\|^2_{L_2(0,T_*;\mathbf{H}^r_\#)}\right)\|\mathbf{u}_m\|^2_{L_\infty(0,T_*;\dot{\mathbf{H}}^r_{\#\sigma})} + 4C_{\mathbb{A}}\|\mathbf{u}_m(\cdot,0)\|^2_{\mathbf{H}^r_\#} + 48C^2_{\mathbb{A}}\|\mathbf{f}\|^2_{L_2(0,T_*;\mathbf{H}^{r-1}_\#)}, \quad kn/2 < r \le (k+1)n/2. \tag{5.31}$$

Inequalities (5.30) and (5.31) mean that for $k=1$, the sequences

$$\{\|\mathbf{u}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}^r_{\#\sigma})}\}_{m=1}^\infty \text{ and } \{\|\mathbf{u}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}^{r+1}_{\#\sigma})}\}_{m=1}^\infty \text{ are bounded for } kn/2 < r \le (k+1)n/2. \tag{5.32}$$

If we assume that properties (5.32) hold for some integer $k \ge 1$, then by the similar argument, properties (5.32) hold with k replaced by $k+1$, and thus, by induction, they hold for any integer k. Hence, collecting properties (5.22), (5.27), and (5.32), we conclude that the sequences

$$\{\|\mathbf{u}_m\|_{L_\infty(0,T_*;\dot{\mathbf{H}}^r_{\#\sigma})}\}_{m=1}^\infty \text{ and } \{\|\mathbf{u}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}^{r+1}_{\#\sigma})}\}_{m=1}^\infty \text{ are bounded for } n/2-1 < r. \tag{5.33}$$

Properties (5.33) imply that there exists a subsequence of $\{\mathbf{u}_m\}$ converging weakly in $L_2(0,T_*;\dot{\mathbf{H}}^{r+1}_{\#\sigma})$ and weakly star in $L_\infty(0,T_*;\dot{\mathbf{H}}^r_{\#\sigma})$ to a function $\mathbf{u}^\dagger \in L_2(0,T_*;\dot{\mathbf{H}}^{r+1}_{\#\sigma}) \cup L_\infty(0,T_*;\dot{\mathbf{H}}^r_{\#\sigma})$. Then, the subsequence converges to $\mathbf{u}^\dagger$ also weakly in $L_2(0,T_*;\dot{\mathbf{H}}^1_{\#\sigma})$ and weakly star in $L_\infty(0,T_*;\dot{\mathbf{H}}^0_{\#\sigma})$. Because $\{\mathbf{u}_m\}$ is the subsequence of the sequence that converges weakly in $L_2\left(0,T;\dot{\mathbf{H}}^1_{\#\sigma}\right)$ and weakly star in $L_\infty\left(0,T;\dot{\mathbf{H}}^0_{\#\sigma}\right)$ to the weak solution, $\mathbf{u}$, of problem (2.1–2.3) on $\left[0,T_*\right]$, we conclude that $\mathbf{u} = \mathbf{u}^\dagger \in L_2(0,T_*;\dot{\mathbf{H}}^{r+1}_{\#\sigma}) \cup L_\infty(0,T_*;\dot{\mathbf{H}}^r_{\#\sigma})$, for any $r > n/2-1$, and we thus finished proving that

$$\mathbf{u} \in L_\infty(0,T_*;\dot{\mathbf{H}}^r_{\#\sigma}) \cap L_2(0,T_*;\dot{\mathbf{H}}^{r+1}_{\#\sigma}). \tag{5.34}$$

Step d. Estimate (4.59) and inclusion (5.34) imply that $(\mathbf{u}\cdot\nabla)\mathbf{u} \in L_2(0,T_*;\mathbf{H}^{r-1}_\#)$. By (1.1) and (1.3), we have

$$\|\mathfrak{L}\mathbf{u}\|^2_{\mathbf{H}^{r-1}_\#} \le \|a^{\alpha\beta}_{ij}E_{i\alpha}(\mathbf{u})\|^2_{(H^r_\#)^{n\times n}} \le \|\mathbb{A}\|^2_{H^{\tilde{\sigma}+1}_\#,F}\|\mathbf{u}\|^2_{\mathbf{H}^{r+1}_\#} \quad \text{for a.e. } t\in(0,T),$$

and thus,

$$\|\mathfrak{L}\mathbf{u}\|^2_{L_2(0,T_*;\dot{\mathbf{H}}^{r-1}_\#)} \le \|\mathbb{A}\|^2_{L_\infty(0,T_*;H^{\tilde{\sigma}+1}_\#),F}\|\mathbf{u}\|^2_{L_2(0,T_*;\mathbf{H}^{r+1}_\#)},$$

that is, $\mathfrak{L}\mathbf{u} \in L_2(0,T_*;\dot{\mathbf{H}}^{r-1}_\#)$. We also have $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}^{r-1}_\#)$. Thus, $\mathbf{F}$ defined by (2.8) belongs to $L_2(0,T;\dot{\mathbf{H}}^{r-1}_\#)$. Then, (2.9) implies that $\mathbf{u}' \in L_2(0,T_*;\mathbf{H}^{r-1}_{\#\sigma})$, and because $\mathbf{u} \in L_2(0,T_*;\mathbf{H}^{r+1}_{\#\sigma})$, we obtain by Theorem 6.8 that $\mathbf{u} \in C^0([0,T_*];\dot{\mathbf{H}}^r_{\#\sigma})$, which also means that $\|\mathbf{u}(\cdot,t)-\mathbf{u}^0\|_{\dot{\mathbf{H}}^r_{\#\sigma}} \to 0$ as $t\to 0$.
To prove the theorem claim about the associated pressure p, we remark that p satisfies (2.10). By Lemma 6.5 for gradient, with $s=r$, Equation (2.10) has a unique solution p in $L_2(0,T_*;\dot{H}^r_\#)$. □

As in Corollaries 5.4 and 5.5, condition (5.8) in Theorem 5.6 can be replaced by simpler conditions for particular cases, which leads to the following two assertions.

Corollary 5.7. (Serrin-type solution for arbitrarily large data but small time or vice versa). *Let $n \ge 2$ and $T>0$. Let $a^{\alpha\beta}_{ij} \in L_\infty([0,T];H^{\tilde{\sigma}+1}_\#)$, $\tilde{\sigma} > |r-1| + \frac{n}{2}$, and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}^{r-1}_\#) \cap L_\infty(0,T;\dot{\mathbf{H}}^{n/2-2}_\#)$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}^r_{\#\sigma} \cap \dot{\mathbf{H}}^{n/2}_{\#\sigma}$, $r > n/2-1$. Let $T_* \in (0,T)$ satisfy inequality (5.16) in Corollary 5.4.*

Then, the Serrin-type solution $\mathbf{u}$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) belongs to $L_\infty(0,T_*;\dot{\mathbf{H}}^r_{\#\sigma}) \cap L_2(0,T_*;\dot{\mathbf{H}}^{r+1}_{\#\sigma})$. In addition, $\mathbf{u}' \in L_2(0,T_*;\dot{\mathbf{H}}^{r-1}_{\#\sigma})$, $\mathbf{u} \in C^0([0,T_*];\dot{\mathbf{H}}^r_{\#\sigma})$, $\lim_{t\to 0}\|\mathbf{u}(\cdot,t)-\mathbf{u}^0\|_{\dot{\mathbf{H}}^r_{\#\sigma}} = 0$, and $p \in L_2(0,T_*;\dot{H}^r_\#)$.

Corollary 5.8. (Serrin-type solution for arbitrary time but small data). *Let $n\ge 2$, $r \ge n/2-1$, and $T>0$. Let $a^{\alpha\beta}_{ij} \in L_\infty([0,T];H^{\tilde{\sigma}+1}_\#)$, $\tilde{\sigma} > |r-1| + \frac{n}{2}$, and the relaxed ellipticity condition (1.2) hold. Let the data $\mathbf{f} \in L_2(0,T;\dot{\mathbf{H}}^{r-1}_\#)$ and $\mathbf{u}^0 \in \dot{\mathbf{H}}^r_{\#\sigma}$ satisfy inequality (5.17) in Corollary 5.5.*

Then, the Serrin-type solution $\mathbf{u}$ of the anisotropic Navier–Stokes initial value problem (2.1–2.3) belongs to $L_\infty(0,T;\dot{\mathbf{H}}^r_{\#\sigma}) \cap L_2(0,T;\dot{\mathbf{H}}^{r+1}_{\#\sigma})$. In addition, $\mathbf{u}' \in L_2(0,T;\dot{\mathbf{H}}^{r-1}_{\#\sigma})$, $\mathbf{u} \in C^0([0,T];\dot{\mathbf{H}}^r_{\#\sigma})$, $\lim_{t\to 0}\|\mathbf{u}(\cdot,t)-\mathbf{u}^0\|_{\dot{\mathbf{H}}^r_{\#\sigma}} = 0$, and $p \in L_2(0,T;\dot{H}^r_\#)$.

Theorem 5.6 leads also to the following infinite regularity assertion.

Corollary 5.9. *Let $T>0$ and $n\ge 2$. Let $a^{\alpha\beta}_{ij} \in C^\infty([0,T];C^\infty_\#)$ and the relaxed ellipticity condition (1.2) hold. Let $\mathbf{f} \in L_2(0,T;\dot{\mathbf{C}}^\infty_\#)$ and $\mathbf{u}^0 \in \dot{\mathbf{C}}^\infty_{\#\sigma}$, while $\mathbf{f}$, $\mathbf{u}^0$, and $T_* \in (0,T]$ satisfy inequality (5.8) from Theorem 5.1.*

Then, the Serrin-type solution $\mathbf{u}$ *of the anisotropic Navier–Stokes initial value problem (2.1–2.3) is such that* $\mathbf{u} \in C^0([0,T_*]; \dot{\mathbf{C}}^\infty_{\#\sigma})$, $\mathbf{u}' \in L_2(0,T_*; \dot{\mathbf{C}}^\infty_{\#\sigma})$*, and* $p \in L_2(0,T_*; \dot{C}^\infty_{\#})$.

Proof. Taking into account that $\dot{\mathbf{C}}^\infty_{\#} = \bigcap_{r\in\mathbb{R}} \dot{\mathbf{H}}^r_{\#\sigma}$, Theorem 5.6 implies that $\mathbf{u} \in L_\infty(0,T_*; \dot{\mathbf{H}}^r_{\#\sigma}) \cap L_2(0,T_*; \dot{\mathbf{H}}^{r+1}_{\#})$, $\mathbf{u}' \in L_2(0,T_*; \dot{\mathbf{H}}^{r-1}_{\#\sigma})$, $p \in L_2(0,T; \dot{H}^r_{\#})$, $\forall\, r \in \mathbb{R}$. Hence, $\mathbf{u} \in C^0([0,T_*]; \dot{\mathbf{C}}^\infty_{\#\sigma})$, $\mathbf{u}' \in L_2(0,T_*; \dot{\mathbf{C}}^\infty_{\#\sigma})$ and $p \in L_2(0,T_*; \dot{C}^\infty_{\#})$. □

5.4 | Spatial-Temporal Regularity of Serrin-Type Solutions for Variable Anisotropic Viscosity Coefficients

Theorem 5.10. *Let* $T > 0$ *and* $n \geq 2$. *Let* $r \geq n/2 - 1$ *if* $n \geq 3$, *while* $r > n/2 - 1$ *if* $n = 2$. *Let* $a^{\alpha\beta}_{ij} \in L_\infty([0,T]; H^{\tilde\sigma+1}_{\#})$, $\tilde\sigma > |r - 1| + \frac{n}{2}$, *and the relaxed ellipticity condition (1.2) hold. Let* $\mathbf{f} \in L_\infty(0,T; \dot{\mathbf{H}}^{r-2}_{\#}) \cap L_2(0,T; \dot{\mathbf{H}}^{r-1}_{\#})$ *and* $\mathbf{u}^0 \in \dot{\mathbf{H}}^r_{\#\sigma}$, *while* $\mathbf{f}$, $\mathbf{u}^0$, *and* $T_* \in (0,T]$ *satisfy inequality (5.8) from Theorem 5.1.*

Then, the Serrin-type solution $\mathbf{u}$ *of the anisotropic Navier–Stokes initial value problem (2.1–2.3) is such that* $\mathbf{u}' \in L_\infty(0,T_*; \dot{\mathbf{H}}^{r-2}_{\#\sigma}) \cup L_2(0,T_*; \dot{\mathbf{H}}^{r-1}_{\#\sigma})$, *while* $p \in L_\infty(0,T_*; \dot{H}^{r-1}_{\#}) \cap L_2(0,T_*; \dot{H}^r_{\#})$.

Proof. By Theorems 5.1 and 5.6, we have the inclusions $\mathbf{u} \in L_\infty(0,T_*; \dot{\mathbf{H}}^r_{\#\sigma})$, $\mathbf{u}' \in L_2(0,T_*; \dot{\mathbf{H}}^{r-1}_{\#\sigma})$, and $p \in L_2(0,T_*; \dot{H}^r_{\#})$. Then, we only need to prove the inclusions $\mathbf{u}' \in L_\infty(0,T_*; \dot{\mathbf{H}}^{r-2}_{\#\sigma})$ and $p \in L_\infty(0,T_*; \dot{H}^{r-1}_{\#})$.

As in the proof of Theorem 5.10, we arrive at estimate (4.60).

By (1.1), (1.3), and multiplication Theorem 6.1(a), we have

$$\|\mathfrak{L}\mathbf{u}\|_{\mathbf{H}^{r-2}_{\#}} \leq \|a^{\alpha\beta}_{ij} E_{i\alpha}(\mathbf{u})\|_{(H^{r-1}_{\#})^{n\times n}} \leq C_*(r-1, \tilde\sigma+1, n)\|\mathbb{A}\|_{H^{\tilde\sigma+1}_{\#},F}\|\mathbf{u}\|_{\mathbf{H}^r_{\#}},$$

where we took into account that $\tilde\sigma + 1 > n/2$ and $\tilde\sigma + 1 > r$. Thus,

$$\|\mathfrak{L}\mathbf{u}\|_{L_\infty(0,T_*;\dot{\mathbf{H}}^{r-2}_{\#})} \leq C_*(r-1, \tilde\sigma+1, n)\|\mathbb{A}\|_{L_\infty(0,T_*;H^{\tilde\sigma+1}_{\#}),F}\|\mathbf{u}\|_{L_\infty(0,T_*;\mathbf{H}^r_{\#})},$$

that is, $\mathfrak{L}\mathbf{u} \in L_\infty(0,T_*; \dot{\mathbf{H}}^{r-2}_{\#})$. We also have $\mathbf{f} \in L_\infty(0,T; \dot{\mathbf{H}}^{r-2}_{\#})$.

Then, (2.9) implies that $\mathbf{u}' \in L_\infty(0,T_*; \mathbf{H}^{r-2}_{\#\sigma})$, while (2.10) and Lemma 6.5 for gradient, with $s = r-1$, imply that $p \in L_\infty(0,T_*; \dot{H}^{r-1}_{\#})$. □

To simplify the following two assertions we assume there that the viscosity coefficients are infinitely smooth it time and in the space coordinates. This smoothness condition can be relaxed if we instead assume that all the norms of these coefficients encountered in the proof are bounded.

Theorem 5.11. *Let* $T > 0$ *and* $n \geq 2$. *Let* $r > n/2 - 1$. *Let* $a^{\alpha\beta}_{ij} \in C^\infty([0,T]; C^\infty_{\#})$ *and the relaxed ellipticity condition (1.2) hold. Let* $k \in [1, r+1)$ *be an integer. Let* $\mathbf{f}^{(l)} \in L_\infty(0,T; \dot{\mathbf{H}}^{r-2-2l}_{\#}) \cap L_2(0,T; \dot{\mathbf{H}}^{r-1-2l}_{\#})$, $l = 0, 1, \dots, k-1$, *and* $\mathbf{u}^0 \in \dot{\mathbf{H}}^r_{\#\sigma}$, *while* $\mathbf{f}$, $\mathbf{u}^0$, *and* $T_* \in (0,T]$ *satisfy inequality (5.8) from Theorem 5.1.*

Then, the Serrin-type solution $\mathbf{u}$ *of the anisotropic Navier–Stokes initial value problem (2.1–2.3) is such that* $\mathbf{u}^{(l)} \in L_\infty(0,T_*; \dot{\mathbf{H}}^{r-2l}_{\#\sigma}) \cap L_2(0,T_*; \dot{\mathbf{H}}^{r+1-2l}_{\#\sigma})$, $l = 0, \dots, k$, *while* $p^{(l)} \in L_\infty(0,T_*; \dot{H}^{r-1-2l}_{\#}) \cap L_2(0,T_*; \dot{H}^{r-2l}_{\#\sigma})$, $l = 0, \dots, k-1$.

Proof. The proof coincide with the proof of the corresponding constant-coefficient Theorem (4.10), except the proof of inclusion (4.68) in Step 2, that for the variable coefficients is replaced by the following argument.

$$\partial_t^{k-1}\mathfrak{L}\mathbf{u} = \sum_{l=0}^{k-1} C^l_{k-1}\nabla\cdot[\mathbb{A}^{(k-1-l)}\mathbb{E}(\mathbf{u}^{(l)})]. \tag{5.35}$$

By (1.1), (1.3), and Theorem 6.1(a), we have

$$\begin{aligned}
&\|\nabla\cdot[\mathbb{A}^{(k-1-l)}\mathbb{E}(\mathbf{u}^{(l)})]\|_{\dot{\mathbf{H}}^{r-2k}_{\#\sigma}} \\
&\leq \|\mathbb{A}^{(k-1-l)}\mathbb{E}(\mathbf{u}^{(l)})\|_{(H^{r+1-2k}_{\#})^{n\times n}} \\
&\leq \|\mathbb{A}^{(k-1-l)}\mathbb{E}(\mathbf{u}^{(l)})\|_{(H^{r-1-2l}_{\#})^{n\times n}} \\
&\leq C_{*l1}\|\mathbb{A}^{(k-1-l)}\|_{H^{\tilde\sigma_{l1}}_{\#},F}\|\mathbb{E}(\mathbf{u}^{(l)})\|_{(H^{r-1-2l}_{\#})^{n\times n}} \\
&\leq C_{*l1}\|\mathbb{A}^{(k-1-l)}\|_{H^{\tilde\sigma_{l1}}_{\#},F}\|\mathbf{u}^{(l)}\|_{\mathbf{H}^{r-2l}_{\#}},
\end{aligned}$$

where $\tilde\sigma_{l1} > \max\{n/2, 2l+1-r\}$, $C_{*l1} = C_*(r-1-2l, \tilde\sigma_{l1}, n)$. Thus,

$$\begin{aligned}
&\|\nabla\cdot[\mathbb{A}^{(k-1-l)}\mathbb{E}(\mathbf{u}^{(l)})]\|_{L_\infty(0,T_*;\dot{\mathbf{H}}^{r-2k}_{\#\sigma})} \\
&\leq C_{*l1}\|\mathbb{A}^{(k-1-l)}\|_{L_\infty(0,T_*;H^{\tilde\sigma_{l1}}_{\#}),F}\|\mathbf{u}^{(l)}\|_{L_\infty(0,T_*;\mathbf{H}^{r-2l}_{\#})},
\end{aligned}$$

that is, due to (4.61), $\nabla\cdot[\mathbb{A}^{(k-1-l)}\mathbb{E}(\mathbf{u}^{(l)})] \in L_\infty(0,T_*; \dot{\mathbf{H}}^{r-2k}_{\#\sigma})$, $l = 0, \dots, k-1$.

On the other hand, by (1.1), (1.3), and Theorem 6.1(a), we have

$$\begin{aligned}
&\|\nabla\cdot[\mathbb{A}^{(k-1-l)}\mathbb{E}(\mathbf{u}^{(l)})]\|_{\dot{\mathbf{H}}^{r+1-2k}_{\#\sigma}} \\
&\leq \|\mathbb{A}^{(k-1-l)}\mathbb{E}(\mathbf{u}^{(l)})\|_{(H^{r+2-2k}_{\#})^{n\times n}} \\
&\leq \|\mathbb{A}^{(k-1-l)}\mathbb{E}(\mathbf{u}^{(l)})\|_{(H^{r-2l}_{\#})^{n\times n}} \\
&\leq C_{*l2}\|\mathbb{A}^{(k-1-l)}\|_{H^{\tilde\sigma_{l2}}_{\#},F}\|\mathbb{E}(\mathbf{u}^{(l)})\|_{(H^{r-2l}_{\#})^{n\times n}} \\
&\leq C_{*l2}\|\mathbb{A}^{(k-1-l)}\|_{H^{\tilde\sigma_{l2}}_{\#},F}\|\mathbf{u}^{(l)}\|_{\mathbf{H}^{r+1-2l}_{\#}},
\end{aligned}$$

where $\tilde\sigma_{l2} > \max\{n/2, 2l-r\}$, $C_{*l2} = C_*(r-2l, \tilde\sigma_{l2}, n)$. Thus,

$$\begin{aligned}
&\|\nabla\cdot[\mathbb{A}^{(k-1-l)}\mathbb{E}(\mathbf{u}^{(l)})]\|_{L_\infty(0,T_*;\dot{\mathbf{H}}^{r+1-2k}_{\#\sigma})} \\
&\leq C_{*l2}\|\mathbb{A}^{(k-1-l)}\|_{L_\infty(0,T_*;H^{\tilde\sigma_{l2}}_{\#}),F}\|\mathbf{u}^{(l)}\|_{L_2(0,T_*;\mathbf{H}^{r+1-2l}_{\#})},
\end{aligned}$$

that is, due to (4.61), $\nabla\cdot[\mathbb{A}^{(k-1-l)}\mathbb{E}(\mathbf{u}^{(l)})] \in L_2(0,T_*; \dot{\mathbf{H}}^{r+1-2k}_{\#\sigma})$, $l = 0, \dots, k-1$. Hence, by (5.35),

$$\partial_t^{k-1}\mathfrak{L}\mathbf{u} \in L_\infty(0,T_*; \mathbf{H}^{r-2k}_{\#}) \cap L_2(0,T_*; \mathbf{H}^{r+1-2k}_{\#}). \tag{5.36}$$

□

Corollary 5.12. *Let* $T > 0$ *and* $n \geq 2$. *Let* $a^{\alpha\beta}_{ij} \in C^\infty([0,T]; C^\infty_{\#})$ *and the relaxed ellipticity condition (1.2) hold. Let* $\mathbf{f} \in C^\infty(0,T; \dot{\mathbf{C}}^\infty_{\#})$, *and* $\mathbf{u}^0 \in \dot{\mathbf{C}}^\infty_{\#\sigma}$, *while* $\mathbf{f}$, $\mathbf{u}^0$, *and* $T_* \in (0,T]$ *satisfy inequality (5.8) from Theorem 5.1.*

Then, the Serrin-type solution $\mathbf{u}$ *of the anisotropic Navier–Stokes initial value problem (2.1–2.3) is such that* $\mathbf{u} \in C^\infty(0, T_*; \dot{\mathbf{C}}^\infty_{\#\sigma})$, $p \in C^\infty(0, T_*; \dot{C}^\infty_\#)$.

Proof. Taking into account that $\dot{\mathbf{C}}^\infty_\# = \bigcap_{r\in\mathbb{R}} \dot{\mathbf{H}}^r_{\#\sigma}$, Theorem 5.11 implies that for any integer $k \geq 0$, $\mathbf{u}^{(k)} \in L_\infty(0, T_*; \dot{\mathbf{H}}^{r-2k}_{\#\sigma}) \cap L_2(0, T_*; \dot{\mathbf{H}}^{r+1-2k}_\#)$, $p^{(k)} \in L_\infty(0, T_*; \dot{H}^{r-1-2k}_\#) \cap L_2(0, T_*; \dot{H}^{r-2k}_{\#\sigma})$, for any $r \in \mathbb{R}$. Hence, $\mathbf{u} \in C^\infty(0, T_*; \dot{\mathbf{C}}^\infty_{\#\sigma})$, $p \in C^\infty(0, T_*; \dot{C}^\infty_\#)$. □

5.5 | Regularity of Two-Dimensional Weak Solution for Variable Anisotropic Viscosity Coefficients

Here, we provide a counterpart of Section 4.6 generalized to variable viscosity coefficients.

Theorem 5.13. (Spatial regularity of solution for arbitrarily large data). *Let* $n = 2$, $r > 0$, *and* $T > 0$. *Let* $a^{\alpha\beta}_{ij} \in L_\infty([0, T]; H^{\tilde{\sigma}+1}_\#)$, $\tilde{\sigma} > 1 + \max\{|r-1|, 1\}$, *and the relaxed ellipticity condition (1.2) hold. Let* $\mathbf{f} \in L_2(0, T; \dot{\mathbf{H}}^{r-1}_\#)$ *and* $\mathbf{u}^0 \in \dot{\mathbf{H}}^r_{\#\sigma}$.

Then, the solution $\mathbf{u}$ *of the anisotropic Navier–Stokes initial value problem (2.1–2.3) obtained in Theorem 2.4 is of Serrin type and belongs to* $L_\infty(0, T; \dot{\mathbf{H}}^r_{\#\sigma}) \cap L_2(0, T; \dot{\mathbf{H}}^{r+1}_{\#\sigma})$. *In addition,* $\mathbf{u}' \in L_2(0, T; \dot{\mathbf{H}}^{r-1}_{\#\sigma})$, $\mathbf{u} \in C^0([0, T]; \dot{\mathbf{H}}^r_{\#\sigma})$, $\lim_{t\to 0} \|\mathbf{u}(\cdot, t) - \mathbf{u}^0\|_{\dot{\mathbf{H}}^r_{\#\sigma}} = 0$, *and* $p \in L_2(0, T_*; \dot{H}^r_\#)$.

Proof. The proof coincides word for word with the proof of Theorem 5.6 if we take there $n = 2$ while replacing T_* by T and the reference to (5.14) for the boundedness of the sequence $\|\mathbf{u}_m\|_{L_2(0,T_*;\dot{\mathbf{H}}^{n/2}_\#)}$ for $n = 2$ by the reference to the corresponding inequality

$$\|\mathbf{u}_m\|^2_{L_2(0,T;\dot{\mathbf{H}}^1_\#)} \leq 4C_\mathbb{A}\left(\|\mathbf{u}^0\|^2_{\mathbf{L}_{2\#}} + 4C_\mathbb{A}\|\mathbf{f}\|^2_{L_2(0,T;\dot{\mathbf{H}}^{-1}_\#)}\right).$$

obtained as inequality (59) in our paper [19]. □

The following assertion is proved similar to Corollary 5.9.

Corollary 5.14. *Let* $T > 0$ *and* $n = 2$. *Let* $a^{\alpha\beta}_{ij} \in C^\infty([0, T]; C^\infty_\#)$ *and the relaxed ellipticity condition (1.2) hold. Let* $\mathbf{f} \in L_2(0, T; \dot{\mathbf{C}}^\infty_\#)$ *and* $\mathbf{u}^0 \in \dot{\mathbf{C}}^\infty_{\#\sigma}$.

Then, the solution $\mathbf{u}$ *of the anisotropic Navier–Stokes initial value problem (2.1–2.3) obtained in Theorem 2.4 is of Serrin type and is such that* $\mathbf{u} \in C^0([0, T]; \dot{\mathbf{C}}^\infty_{\#\sigma})$, $\mathbf{u}' \in L_2(0, T; \dot{\mathbf{C}}^\infty_{\#\sigma})$, *and* $p \in L_2(0, T; \dot{C}^\infty_\#)$.

The next three assertions on spatial-temporal regularity for $n = 2$ are the corresponding counterparts of Theorems 5.10 and 5.11 and Corollary 5.12 and are proved in a similar way after replacing there T_* by T.

Theorem 5.15. *Let* $T > 0$, $n \geq 2$, *and* $r > 0$. *Let* $a^{\alpha\beta}_{ij} \in L_\infty([0, T]; H^{\tilde{\sigma}+1}_\#)$, $\tilde{\sigma} > |r-1| + 1$, *and the relaxed ellipticity condition (1.2) hold. Let* $\mathbf{f} \in L_\infty(0, T; \dot{\mathbf{H}}^{r-2}_\#) \cap L_2(0, T; \dot{\mathbf{H}}^{r-1}_\#)$ *and* $\mathbf{u}^0 \in \dot{\mathbf{H}}^r_{\#\sigma}$.

Then, the solution $\mathbf{u}$ *of the anisotropic Navier–Stokes initial value problem (2.1–2.3) obtained in Theorem 2.4 is of Serrin type and is such that* $\mathbf{u}' \in L_\infty(0, T; \dot{\mathbf{H}}^{r-2}_{\#\sigma}) \cup L_2(0, T; \dot{\mathbf{H}}^{r-1}_{\#\sigma})$, $p \in L_\infty(0, T; \dot{H}^{r-1}_\#) \cap L_2(0, T; \dot{H}^r_\#)$.

Theorem 5.16. *Let* $T > 0$, $n = 2$, *and* $r > 0$. *Let* $a^{\alpha\beta}_{ij} \in C^\infty([0, T]; C^\infty_\#)$ *and the relaxed ellipticity condition (1.2) hold. Let* $k \in [1, r+1)$ *be an integer. Let* $\mathbf{f}^{(l)} \in L_\infty(0, T; \dot{\mathbf{H}}^{r-2-2l}_\#) \cap L_2(0, T; \dot{\mathbf{H}}^{r-1-2l}_\#)$, $l = 0, 1, \ldots, k-1$, *and* $\mathbf{u}^0 \in \dot{\mathbf{H}}^r_{\#\sigma}$.

Then, the solution $\mathbf{u}$ *of the anisotropic Navier–Stokes initial value problem (2.1–2.3) obtained in Theorem 2.4 is of Serrin type and is such that* $\mathbf{u}^{(l)} \in L_\infty(0, T; \dot{\mathbf{H}}^{r-2l}_{\#\sigma}) \cap L_2(0, T; \dot{\mathbf{H}}^{r+1-2l}_{\#\sigma})$, $l = 0, \ldots, k$; $p^{(l)} \in L_\infty(0, T; \dot{H}^{r-1-2l}_\#) \cap L_2(0, T; \dot{H}^{r-2l}_{\#\sigma})$, $l = 0, \ldots, k-1$.

Corollary 5.17. *Let* $T > 0$ *and* $n = 2$. *Let* $a^{\alpha\beta}_{ij} \in C^\infty([0, T]; C^\infty_\#)$ *and the relaxed ellipticity condition (1.2) hold. Let* $\mathbf{f} \in C^\infty(0, T; \dot{\mathbf{C}}^\infty_\#)$ *and* $\mathbf{u}^0 \in \dot{\mathbf{C}}^\infty_{\#\sigma}$.

Then, the solution $\mathbf{u}$ *of the anisotropic Navier–Stokes initial value problem (2.1–2.3) obtained in Theorem 2.4 is of Serrin type and is such that* $\mathbf{u} \in C^\infty(0, T; \dot{\mathbf{C}}^\infty_{\#\sigma})$, $p \in C^\infty(0, T; \dot{C}^\infty_\#)$.

6 | Auxiliary Results

6.1 | Advection Term Properties

The divergence theorem and periodicity imply the following identity for any $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbf{C}^\infty_\#$.

$$\begin{aligned} \langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2, \mathbf{v}_3\rangle_\mathbb{T} &= \int_\mathbb{T} \nabla\cdot\big(\mathbf{v}_1(\mathbf{v}_2\cdot\mathbf{v}_3)\big)d\mathbf{x} \\ &\quad - \langle(\nabla\cdot\mathbf{v}_1)\mathbf{v}_3 + (\mathbf{v}_1\cdot\nabla)\mathbf{v}_3, \mathbf{v}_2\rangle_\mathbb{T} \\ &= -\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_3, \mathbf{v}_2\rangle_\mathbb{T} - \langle(\nabla\cdot\mathbf{v}_1)\mathbf{v}_3, \mathbf{v}_2\rangle_\mathbb{T}. \end{aligned} \tag{6.1}$$

Hence, for any $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{C}^\infty_\#$,

$$\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2, \mathbf{v}_2\rangle_\mathbb{T} = -\frac{1}{2}\langle(\nabla\cdot\mathbf{v}_1)\mathbf{v}_2, \mathbf{v}_2\rangle_\mathbb{T} = -\frac{1}{2}\big\langle \operatorname{div}\mathbf{v}_1, |\mathbf{v}_2|^2\big\rangle_\mathbb{T}. \tag{6.2}$$

In view of (6.1), we obtain the identity

$$\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2, \mathbf{v}_3\rangle_\mathbb{T} = -\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_3, \mathbf{v}_2\rangle_\mathbb{T} \quad \forall\ \mathbf{v}_1 \in \mathbf{C}^\infty_{\#\sigma},\ \mathbf{v}_2,\ \mathbf{v}_3 \in \mathbf{C}^\infty_\# \tag{6.3}$$

and hence, the following well-known formula for any $\mathbf{v}_1 \in \mathbf{C}^\infty_{\#\sigma}$, $\mathbf{v}_2 \in \mathbf{C}^\infty_\#$,

$$\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2, \mathbf{v}_2\rangle_\mathbb{T} = 0. \tag{6.4}$$

Equations (6.3) and (6.4) evidently hold also for $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ from the more general spaces, for which the dual products in (6.3) and (6.4) are bounded and in which $\mathbf{C}^\infty_{\#\sigma}$ and $\mathbf{C}^\infty_\#$, respectively, are densely embedded.

6.2 | Some Point-Wise Multiplication Results

Let us accommodate to the periodic function spaces in $\mathbb{R}^n$, $n \geq 1$, a particular case of a much more general Theorem 1 in Section 4.6.1 of [31] about point-wise products of functions/distributions.

Theorem 6.1. *Assume $n \geq 1$, $s_1 \leq s_2$, and $s_1 + s_2 > 0$. Then, there exists a constant $C_*(s_1, s_2, n) > 0$ such that for any $f_1 \in H_{\#}^{s_2}$ and $f_2 \in H_{\#}^{s_1}$,*

a. $f_1 \cdot f_2 \in H_{\#}^{s_1}$ *and*
$\|f_1 \cdot f_2\|_{H_{\#}^{s_1}} \leq C_*(s_1, s_2, n)\|f_1\|_{H_{\#}^{s_1}}\ \|f_2\|_{H_{\#}^{s_2}}$
if $s_2 > n/2$;

b. $f_1 \cdot f_2 \in H_{\#}^{s_1+s_2-n/2}$ *and*
$\|f_1 \cdot f_2\|_{H_{\#}^{s_1+s_2-n/2}} \leq C_*(s_1, s_2, n)\|f_1\|_{H_{\#}^{s_1}}\ \|f_2\|_{H_{\#}^{s_2}}$
if $s_2 < n/2$.

Proof. Items (a) and (b) follow, respectively, from items (i) and (iii) of [31, Theorem 1 in Section 4.6.1] when we take into account the norm equivalence in the standard and periodic Sobolev spaces. □

Let

$$(u \star v)(\xi) := \sum_{\eta \in \mathbb{Z}^n} u(\eta) v(\xi - \eta), \quad \xi \in \mathbb{Z}^n,$$

be the convolution in $\mathbb{Z}^n$. We will need the following Young's inequality for discrete convolution of sequences in $\mathbb{Z}^n$. For other choices of parameters, see, for example, [32, p. 316] and references therein.

Lemma 6.2. *Let $n \in \mathbb{N}$, $1 \leq q \leq \infty$. Then, the convolution of sequences $u \in \ell_1(\mathbb{Z}^n)$ and $v \in \ell_q(\mathbb{Z}^n)$ belongs to $\ell_q(\mathbb{Z}^n)$ and*

$$\|u \star v\|_{\ell_q(\mathbb{Z}^n)} \leq \|u\|_{\ell_1(\mathbb{Z}^n)} \|v\|_{\ell_q(\mathbb{Z}^n)}. \tag{6.5}$$

Proof. By the triangle inequality, we obtain

$$\begin{aligned}
\|u \star v\|_{\ell_q(\mathbb{Z}^n)} &= \left\| \sum_{\eta \in \mathbb{Z}^n} u(\eta) v(\cdot - \eta) \right\|_{\ell_q} \\
&\leq \sum_{\eta \in \mathbb{Z}^n} \|u(\eta) v(\cdot - \eta)\|_{\ell_q} \\
&= \sum_{\eta \in \mathbb{Z}^n} |u(\eta)| \|v\|_{\ell_q} \\
&= \|u\|_{\ell_1(\mathbb{Z}^n)} \|v\|_{\ell_q(\mathbb{Z}^n)}.
\end{aligned}$$

□

Theorem 6.3. *Assume $n \geq 1$. Let $s, \theta \in \mathbb{R}$, $w \in H_{\#}^{s}$, $g \in H_{\#}^{\tilde{\sigma}+1}$, $\tilde{\sigma} > \max\{|s|, |s - \theta + 1|\} + \frac{n}{2} = |s - \frac{\theta-1}{2}| + |\frac{\theta-1}{2}| + \frac{n}{2}$. Then, $\Lambda_{\#}^{\theta}(gw) - g\Lambda_{\#}^{\theta} w \in H_{\#}^{s-\theta+1}$ and*

$$\|\Lambda_{\#}^{\theta}(gw) - g\Lambda_{\#}^{\theta} w\|_{H_{\#}^{s-\theta+1}} \leq \overline{C}_{s,\theta,\tilde{\sigma}} |g|_{H_{\#}^{\tilde{\sigma}+1}} \|w\|_{H_{\#}^{s}},$$

where the constant $\overline{C}_{s,\theta,\tilde{\sigma}}$ does not depend on g or w but may depend on s, θ, and $\tilde{\sigma}$.

Proof. By (1.6), we have

$$\begin{aligned}
K(\xi) &:= \mathcal{F}_{\mathbb{T}}[\Lambda_{\#}^{\theta}(gw) - g\Lambda_{\#}^{\theta} w](\xi) \\
&= (2\pi)^{\theta}(1 + |\xi|^2)^{\theta/2}(\hat{g} \star \hat{w})(\xi) - (\hat{g} \star \mathcal{F}_{\mathbb{T}}[\Lambda_{\#}^{\theta} w])(\xi) \\
&= (2\pi)^{\theta} \sum_{\eta \in \mathbb{Z}^n} [(1 + |\xi|^2)^{\theta/2} - (1 + |\xi - \eta|^2)^{\theta/2}] \hat{g}(\eta) \hat{w}(\xi - \eta) \\
&= (2\pi)^2 \sum_{\eta \in \mathbb{Z}^n} [(\eta \cdot \xi + \eta \cdot (\xi - \eta)] f_{\theta}(\xi, \xi - \eta) \hat{g}(\eta) \hat{w}(\xi - \eta) \\
&= \frac{2\pi}{i} \sum_{\eta \in \mathbb{Z}^n} \widehat{\nabla g}(\eta) \cdot (\xi + \xi - \eta) f_{\theta}(\xi, \xi - \eta) \hat{w}(\xi - \eta).
\end{aligned}$$

Here,

$$\begin{aligned}
f_{\theta}(\xi, \xi - \eta) &:= (2\pi)^{\theta-2} \frac{(1 + |\xi|^2)^{\theta/2} - (1 + |\xi - \eta|^2)^{\theta/2}}{|\xi|^2 - |\xi - \eta|^2} \\
&= \frac{\varrho^{\theta}(\xi) - \varrho^{\theta}(\xi - \eta)}{\varrho^2(\xi) - \varrho^2(\xi - \eta)},
\end{aligned}$$

and we took into account that $|\xi|^2 - |\xi - \eta|^2 = \eta \cdot \xi + \eta \cdot (\xi - \eta)$. Because the inequality $|c_1^{\beta} - c_2^{\beta}| \leq |\beta||c_1 - c_2|(c_1^{\beta-1} + c_2^{\beta-1})$ holds for any $c_1, c_2 > 0$, $\beta \in \mathbb{R}$, we have

$$|f_{\theta}(\xi, \xi - \eta)| \leq \frac{|\varrho^{\theta-1}(\xi) + \varrho^{\theta-1}(\xi - \eta)|}{|\varrho(\xi) + \varrho(\xi - \eta)|}.$$

Hence,

$$\begin{aligned}
&2\pi|(\xi + \xi - \eta) f_{\theta}(\xi, \xi - \eta)| \\
&\leq 2\pi(|\xi| + |\xi - \eta|)|f_{\theta}(\xi, \xi - \eta)| \\
&\leq 2\pi|\theta| \left[\varrho^{\theta-1}(\xi) + \varrho^{\theta-1}(\xi - \eta)\right] \frac{|\xi| + |\xi - \eta|}{\varrho(\xi) + \varrho(\xi - \eta)} \\
&\leq |\theta| \left[\varrho^{\theta-1}(\xi) + \varrho^{\theta-1}(\xi - \eta)\right],
\end{aligned}$$

for any $\theta \in \mathbb{R}$. Then,

$$\begin{aligned}
|K(\xi)| &\leq |\theta| \sum_{\eta \in \mathbb{Z}^n} \left[\varrho^{\theta-1}(\xi) + \varrho^{\theta-1}(\xi - \eta)\right] |\widehat{\nabla g}(\eta) \hat{w}(\xi - \eta)| \\
&= |\theta| \sum_{\eta \in \mathbb{Z}^n} \left[\varrho^{\theta-1}(\xi) |\widehat{\nabla g}(\eta) \hat{w}(\xi - \eta)| + |\widehat{\nabla g}(\eta) \varrho^{\theta-1}(\xi - \eta) \hat{w}(\xi - \eta)|\right] \\
&= |\theta| \left[\varrho^{\theta-1}(\xi)\{|\widehat{\nabla g}| \star |\hat{w}|\}(\xi) + \{|\widehat{\nabla g}| \star |\varrho^{\theta-1} \hat{w}|\}(\xi)\right].
\end{aligned}$$

Taking into account Petree's inequality

$$\varrho^{s}(\xi) \leq \frac{2^{|s|/2}}{(2\pi)^{|s|}} \varrho^{|s|}(\eta) \varrho^{s}(\xi - \eta)\ \forall\ \xi, \eta \in \mathbb{Z}^n,\ \forall\ s \in \mathbb{R},$$

and the discrete Young's inequality (6.5) for convolutions with $q = 2$, we obtain

$$\begin{aligned}
&\|\Lambda_{\#}^{\theta}(gw) - g\Lambda_{\#}^{\theta} w\|_{H_{\#}^{s-\theta+1}} = \|\varrho^{s-\theta+1} K\|_{\ell_2} \\
&\leq |\theta| \left\| \varrho^{s}\{|\widehat{\nabla g}| \star |\hat{w}|\} + \varrho^{s-\theta+1}\{|\widehat{\nabla g}| \star |\varrho^{\theta-1} \hat{w}|\} \right\|_{\ell_2} \\
&\leq \frac{2^{|s|/2}|\theta|}{(2\pi)^{|s|}} \left\| |\varrho^{|s|} \widehat{\nabla g}| \star |\varrho^{s} \hat{w}| + |\varrho^{|s-\theta+1|} \widehat{\nabla g}| \star |\varrho^{s} \hat{w}| \right\|_{\ell_2} \\
&\leq \frac{2^{|s|/2}|\theta|}{2\pi} \left(\|\varrho^{|s|} \widehat{\nabla g}\|_{\ell_1} + \|\varrho^{|s-\theta+1|} \widehat{\nabla g}\|_{\ell_1} \right) \|\varrho^{s} \hat{w}\|_{\ell_2}.
\end{aligned}$$

By the Schwarz inequality for any $\tilde{\sigma} > |s| + n/2$, we have

$$\|\varrho^{|s|}\widehat{\nabla g}\|_{\ell_1} = \sum_{\xi\in\mathbb{Z}^n} \varrho^{|s|}(\xi)|\widehat{\nabla g}(\xi)| \leq \left[\sum_{\xi\in\mathbb{Z}^n} \varrho^{2\tilde{\sigma}}(\xi)|\widehat{\nabla g}(\xi)|^2\right]^{1/2}\left[\sum_{\xi\in\mathbb{Z}^n} \varrho^{2|s|-2\tilde{\sigma}}(\xi)\right]^{1/2}. \tag{6.6}$$

Similarly, for any $\tilde{\sigma} > |s-\theta+1| + n/2$, we have

$$\|\varrho^{|s-\theta+1|}\widehat{\nabla g}\|_{\ell_1} \leq \left[\sum_{\xi\in\mathbb{Z}^n} \varrho^{2\tilde{\sigma}}(\xi)|\widehat{\nabla g}(\xi)|^2\right]^{1/2}\left[\sum_{\xi\in\mathbb{Z}^n} \varrho^{2|s-\theta+1|-2\tilde{\sigma}}(\xi)\right]^{1/2}. \tag{6.7}$$

Then, for $\tilde{\sigma} > \tilde{\sigma}_0 := \max\{|s| + \frac{n}{2}, |s-\theta+1| + \frac{n}{2}\} = \left|s - \frac{\theta-1}{2}\right| + \left|\frac{\theta-1}{2}\right| + \frac{n}{2}$, we obtain

$$\|\Lambda_\#^\theta(gw) - g\Lambda_\#^\theta w\|_{H_\#^{s-\theta+1}} \leq \overline{C}_{s,\theta,\tilde{\sigma}}\|\nabla g\|_{H_\#^{\tilde{\sigma}}}\|\hat{w}\|_{H_\#^s} \leq \overline{C}_{s,\theta,\tilde{\sigma}}|g|_{H_\#^{\tilde{\sigma}+1}}\|\hat{w}\|_{H_\#^s},$$

where

$$\overline{C}_{s,\theta,\tilde{\sigma}} := \frac{2^{|s|/2}}{2\pi}|\theta|\left[\sum_{\xi\in\mathbb{Z}^n} \varrho^{2\tilde{\sigma}_0-n-2\tilde{\sigma}}(\xi)\right]^{1/2},$$

and (1.9) is taken into account. □

6.3 | Spectrum of the Periodic Bessel-Potential Operator

In this section, we assume that vector functions/distributions $\mathbf{u}$ are generally complex valued and the Sobolev spaces $\dot{\mathbf{H}}_{\#\sigma}^s$ are complex. Let us recall the definition

$$\left(\Lambda_\#^r \mathbf{u}\right)(\mathbf{x}) := \sum_{\xi\in\dot{\mathbb{Z}}^n} \varrho(\xi)^r \hat{\mathbf{u}}(\xi) e^{2\pi i \mathbf{x}\cdot\xi} \quad \forall\, \mathbf{u} \in \dot{\mathbf{H}}_{\#\sigma}^s,\ s, r \in \mathbb{R} \tag{6.8}$$

of the continuous periodic Bessel-potential operator $\Lambda_\#^r : \dot{\mathbf{H}}_{\#\sigma}^s \to \dot{\mathbf{H}}_{\#\sigma}^{s-r}$, $r \in \mathbb{R}$; see (1.6), (1.8), and (1.11).

The following assertion is given in [19, Theorem 4, Remark 2].

Theorem 6.4. *Let $r \in \mathbb{R}$, $r \neq 0$.*

i. *Then, the operator $\Lambda_\#^r$ in $\dot{\mathbf{H}}_{\#\sigma}^0$ possesses a (nonstrictly) monotone sequence of real eigenvalues $\lambda_j^{(r)} = \lambda_j^r$ and a real orthonormal sequence of associated eigenfunctions $\mathbf{w}_j$ such that*

$$\Lambda_\#^r \mathbf{w}_j = \lambda_j^r \mathbf{w}_j,\ j \geq 1,\ \lambda_j > 0, \tag{6.9}$$

$$\lambda_j \to +\infty,\ j \to +\infty, \tag{6.10}$$

$$\mathbf{w}_j \in \dot{\mathbf{C}}_{\#\sigma}^\infty, \quad (\mathbf{w}_j, \mathbf{w}_k)_{\dot{\mathbf{H}}_{\#\sigma}^0} = \delta_{jk} \quad \forall\, j, k > 0. \tag{6.11}$$

ii. *Moreover, the sequence $\{\mathbf{w}_j\}$ is an orthonormal basis in $\dot{\mathbf{H}}_{\#\sigma}^0$, that is,*

$$\mathbf{u} = \sum_{i=1}^\infty \langle \mathbf{u}, \mathbf{w}_j\rangle_{\mathbb{T}} \mathbf{w}_j, \tag{6.12}$$

where the series converges in $\dot{\mathbf{H}}_{\#\sigma}^0$ for any $\mathbf{u} \in \dot{\mathbf{H}}_{\#\sigma}^0$.

iii. *In addition, the sequence $\{\mathbf{w}_j\}$ is also an orthogonal basis in $\dot{\mathbf{H}}_{\#\sigma}^r$ with*

$$(\mathbf{w}_j, \mathbf{w}_k)_{\dot{\mathbf{H}}_{\#\sigma}^r} = \lambda_j^r \lambda_k^r \delta_{jk} \quad \forall\, j, k > 0,$$

and for any $\mathbf{u} \in \dot{\mathbf{H}}_{\#\sigma}^r$ series (6.12) converges also in $\dot{\mathbf{H}}_{\#\sigma}^r$, that is, the sequence of partial sums

$$P_m \mathbf{u} := \sum_{j=1}^m \langle \mathbf{u}, \mathbf{w}_j\rangle_{\mathbb{T}} \mathbf{w}_j \tag{6.13}$$

converges to $\mathbf{u}$ in $\dot{\mathbf{H}}_{\#\sigma}^r$ as $m \to \infty$. The operator P_m defined by (6.13) is for any $r \in \mathbb{R}$ the orthogonal projector operator from $\mathbf{H}_\#^r$ to $\mathrm{Span}\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$.

6.4 | Isomorphism of Divergence and Gradient Operators in Periodic Spaces

The following assertion proved in [19, Lemma 2] provides for arbitrary $s \in \mathbb{R}$ and dimension $n \geq 2$ the periodic version of Bogovskii/deRham-type results well known for nonperiodic domains and particular values of s; see, for example, [33, 34] and references therein.

Lemma 6.5. *Let $s \in \mathbb{R}$ and $n \geq 2$. The following operators are isomorphisms,*

$$\mathrm{div} \ : \dot{\mathbf{H}}_{\#g}^{s+1} \to \dot{H}_\#^s, \tag{6.14}$$

$$\mathrm{grad} \ : \dot{H}_\#^s \to \dot{\mathbf{H}}_{\#g}^{s-1}. \tag{6.15}$$

6.5 | Some Functional Analysis Results

Let us provide the Sobolev embedding theorem that can be considered, for example, as a particular case of [31, Section 2.2.4, Corollary 2] adapted to the periodic spaces.

Theorem 6.6. *Let $n \in \mathbb{N}$ be the dimension, $q_0 \leq q_1 < \infty$ and $q_1 \geq 1$. The periodic Bessel-potential space $H_{\#q_0}^s$ is continuously embedded in $L_{\#q_1}$ if and only if $\frac{s}{n} \geq \frac{1}{q_0} - \frac{1}{q_1}$.*

The following version of the Sobolev interpolation inequality without a multiplicative constant, generalized also to any real (including negative) smoothness indices, on periodic Bessel-potential spaces was given in [19, Theorem 5].

Theorem 6.7. *Let s, s_1, s_2, θ_1, θ_2 be real numbers such that $0 \leq \theta_1, \theta_2 \leq 1$, $\theta_1 + \theta_2 = 1$ and $s = \theta_1 s_1 + \theta_2 s_2$. Then, for any $g \in H_\#^{s_1} \cap H_\#^{s_2}$,*

$$\|g\|_{H_\#^s} \leq \|g\|_{H_\#^{s_1}}^{\theta_1}\|g\|_{H_\#^{s_2}}^{\theta_2}. \tag{6.16}$$

Theorem 3.1 and Remark 3.2 in Chapter 1 of [27] imply the following assertion.

Theorem 6.8. *Let X and Y be separable Hilbert spaces and $X \subset Y$ with continuous injection. Let $u \in W^1(0,T;X,Y)$. Then, u almost everywhere on $[0,T]$ equals to a function $\tilde{u} \in C^0([0,T];Z)$, where $Z = [X,Y]_{1/2}$ is the intermediate space. Moreover, the trace $u(0) \in Z$ is well defined as the corresponding value of $\tilde{u} \in C^0([0,T];Z)$ at $t=0$.*

The following assertion was proved in [19, Lemma 4].

Lemma 6.9. *Let $s, s' \in \mathbb{R}$, $s' \le s$, and $u \in W^1(0,T;H^s_\#, H^{s'}_\#)$ be real valued.*

i. *Then,*

$$\partial_t \|u\|^2_{H_\#^{(s+s')/2}} = 2\langle \Lambda_\#^{s'} u', \Lambda_\#^s u\rangle_{\mathbb{T}} = 2\langle \Lambda_\#^{s'+s} u', u\rangle_{\mathbb{T}} \tag{6.17}$$

for a.e. $t \in (0,T)$ and also in the distribution sense on $t \in (0,T)$.

ii. *Moreover, for any real-valued $v \in W^1(0,T;H^{-s'}_\#, H^{-s}_\#)$ and $t \in (0,T]$,*

$$\int_0^t \left[\langle u'(\tau), v(\tau)\rangle_{\mathbb{T}} + \langle u(\tau), v'(\tau)\rangle_{\mathbb{T}}\right] d\tau = \langle u(t), v(t)\rangle_{\mathbb{T}} - \langle u(0), v(0)\rangle_{\mathbb{T}}. \tag{6.18}$$

Let us now prove the first Korn inequality for the periodic Sobolev spaces by adapting the proof available for the standard Sobolev spaces, for example, in [35, Theorem 10.1]; compare also [20, Theorem 2.8].

Theorem 6.10. *Let $\mathbf{v} \in \mathbf{H}^s_\#$, $s \in \mathbb{R}$. Then,*

$$\|\nabla \mathbf{v}\|^2_{(H_\#^{s-1})^{n\times n}} \le 2\|\mathbb{E}(\mathbf{v})\|^2_{(H_\#^{s-1})^{n\times n}}. \tag{6.19}$$

Proof. By the norm definition (1.5), we obtain

$$\begin{aligned}
\|\mathbb{E}(\mathbf{v})\|^2_{(H_\#^{s-1})^{n\times n}} &= \sum_{\xi\in\mathbb{Z}^n} \varrho(\boldsymbol{\xi})^{2(s-1)} |\mathcal{F}_{\mathbb{T}}(\mathbb{E}(\mathbf{v}))(\boldsymbol{\xi})|_F^2 \\
&= \sum_{\xi\in\mathbb{Z}^n} \varrho(\xi)^{2(s-1)} \mathcal{F}_{\mathbb{T}}(E_{jk}(\mathbf{v}))(\xi) \cdot \overline{\mathcal{F}_{\mathbb{T}}(E_{jk}(\mathbf{v}))(\xi)} \\
&= \sum_{\xi\in\mathbb{Z}^n} \varrho(\xi)^{2(s-1)} (\pi i)\left(\xi_j \hat{\mathrm{v}}_k + \xi_k \hat{\mathrm{v}}_j\right) \cdot \overline{(\pi i)\left(\xi_j \hat{\mathrm{v}}_k + \xi_k \hat{\mathrm{v}}_j\right)} \\
&= \sum_{\xi\in\mathbb{Z}^n} \varrho(\xi)^{2(s-1)} 2\pi^2 \left(|\xi|^2 |\hat{\mathbf{v}}|^2 + |\xi \cdot \hat{\mathbf{v}}|^2\right) \\
&\ge \frac{1}{2} \sum_{\xi\in\mathbb{Z}^n} \varrho(\xi)^{2(s-1)} |2\pi\xi|^2 |\hat{\mathbf{v}}|^2 \\
&= \frac{1}{2} \sum_{\xi\in\mathbb{Z}^n} \varrho(\xi)^{2(s-1)} |\widehat{\nabla \mathbf{v}}|_F^2 = \frac{1}{2}\|\nabla \mathbf{v}\|^2_{(H_\#^{s-1})^{n\times n}}.
\end{aligned}$$

□

6.6 | Gronwall's Inequalities

Gronwall's inequality is well known and can be found, for example, in [36, Appendix B.2.j], [7, Lemma A.24]. Here, we provide its slightly more general version valid also for arbitrary-sign coefficients.

Lemma 6.11. *Let $\eta : [0,T] \to \mathbb{R}$ be an absolutely continuous function that satisfies the differential inequality*

$$\eta'(t) \le \phi(t)\eta(t) + \psi(t) \quad \textit{for a.e. } t \in [0,T], \tag{6.20}$$

where ϕ and ψ are real integrable functions.

a. *Then,*

$$\eta(t) \le e^{\int_0^t \phi(r)dr}\left[\eta(0) + \int_0^t e^{-\int_0^s \phi(\rho)d\rho}\psi(s)ds\right] \quad \forall\, t \in [0,T]. \tag{6.21}$$

b. *Moreover, for nonnegative ϕ and ψ, (6.21) implies*

$$\eta(t) \le e^{\int_0^t \phi(r)dr}\left[\eta(0) + \int_0^t \psi(s)ds\right] \quad \forall\, t \in [0,T]. \tag{6.22}$$

c. *In particular, if η is nonnegative, while $\psi \equiv 0$ on $[0,T]$ and $\eta(0) = 0$, then $\eta \equiv 0$ on $[0,T]$.*

Proof. Multiplying (6.20) by

$$a(t) := e^{-\int_0^t \phi(r)dr} > 0,$$

we obtain

$$\frac{d}{dt}[a(t)\eta(t)] \le a(t)\psi(t).$$

Integration gives

$$a(t)\eta(t) \le a(0)\eta(0) + \int_0^t a(s)\psi(s)ds.$$

Dividing by $a(t)$, we arrive at

$$\eta(t) \le \frac{1}{a(t)}\left[\eta(0) + \int_0^t a(s)\psi(s)ds\right] \quad \forall\, t \in [0,T],$$

giving (6.21) and thus proving item (a). Items (b) and (c) follow from (6.21). □

Let us slightly generalize and give an alternative proof of [7, Lemma 10.3].

Lemma 6.12. *Let $\eta : [0,T] \to [0,\infty)$ be an absolutely continuous function that satisfies the differential inequality*

$$\eta'(t) + by(t) \le cy(t)\eta(t) + \psi(t), \quad \textit{for a.e. } t \in [0,T]; \quad \eta(0) = \eta_0, \tag{6.23}$$

where $\psi, y \ge 0$ are integrable real functions, while $b, c > 0$ and $\eta_0 \ge 0$ are real constants.

If

$$D := \eta_0 + \int_0^T \psi(\tau)d\tau < \frac{b}{ec} \tag{6.24}$$

then

$$\sup_{0\le\tau\le T} \eta(\tau) < De < \frac{b}{c} \quad \textit{and} \quad \int_0^T y(\tau)d\tau < \frac{1}{c}. \tag{6.25}$$

Proof. By Lemma 6.11(a), inequality (6.23) and condition (6.24) lead to

$$a(t)\eta(t) + b\int_0^t a(s)y(s)ds \le \eta_0 + \int_0^t a(s)\psi(s)ds \le D, \qquad (6.26)$$

where $a(s) := e^{-cY(s)} > 0$ and $Y(s) := \int_0^s y(\tau)d\tau$. Inequality (6.26) implies

$$be^{-cY(t)}Y(t) \le b\int_0^t a(s)y(s)ds \le D < \frac{b}{ec} \quad \forall\, t \in [0,T]. \qquad (6.27)$$

Let us consider the function $f(Y) := e^{-cY}Y$ on the interval $0 \le Y < \infty$. One can elementary obtain that $\max_{0\le Y<\infty} f(Y)$ is reached at $Y = 1/c$ and equals to $1/(ec)$. But due to (6.27), this maximum for $e^{-cY(t)}Y(t)$ is not reached for $t \in [0,T]$, and hence, $Y(T) < 1/c$, giving the second inequality in (6.25).

Further, (6.26) implies that

$$\eta(t) \le \frac{D}{a(t)} = De^{cY(t)} < De < \frac{b}{c} \quad \forall\, t \in [0,T],$$

thus giving the first inequality in (6.25). □

Let us give a generalization of [7, Lemma 10.3] and of Lemma 6.12.

Lemma 6.13. *Let $\eta : [0,T] \to [0,\infty)$ be an absolutely continuous function that satisfies the differential inequality*

$$\eta'(t) + by(t) \le [cy(t) + \phi(t)]\eta(t) + \psi(t), \quad \textit{for a.e. } t \in [0,T];$$
$$\eta(0) = \eta_0, \qquad (6.28)$$

where $\phi, \psi, y \ge 0$ are integrable real functions, while $b, c > 0$ and $\eta_0 \ge 0$ are real constants.

If

$$D := \eta_0 + \int_0^T e^{-\Phi(\tau)}\psi(\tau)d\tau < \frac{b}{c}e^{-1-\Phi(T)}, \qquad (6.29)$$

where $\Phi(s) := \int_0^s \phi(\tau)d\tau$, then

$$\sup_{0\le\tau\le T} \eta(\tau) < \frac{b}{c} \quad \text{and} \quad \int_0^T y(\tau)d\tau < \frac{1}{c}. \qquad (6.30)$$

Proof. By Lemma 6.11(a), inequality (6.28) and condition (6.29) lead to

$$a(t)\eta(t) + b\int_0^t a(s)y(s)ds \le \eta_0 + \int_0^t a(s)\psi(s)ds \le D, \qquad (6.31)$$

where $a(s) := e^{-cY(s)-\Phi(s)} > 0$ and $Y(s) := \int_0^s y(\tau)d\tau$. Inequality (6.31) implies

$$be^{-\Phi(T)}e^{-cY(t)}Y(t) \le b\int_0^t a(s)y(s)ds \le D < \frac{b}{c}e^{-1-\Phi(T)} \quad \forall\, t \in [0,T]. \qquad (6.32)$$

Let us consider the function $f(Y) := e^{-cY}Y$ on the interval $0 \le Y < \infty$. One can elementary obtain that $\max_{0\le Y<\infty} f(Y)$ is reached at $Y = 1/c$ and equals to $1/(ec)$. But due to (6.32), this maximum of $e^{-cY(t)}Y(t)$ is not reached for $t \in [0,T]$, and hence, $Y(T) < 1/c$, giving the second inequality in (6.30).

Further, (6.31) implies that

$$\eta(t) \le \frac{D}{a(t)} = De^{cY(t)+\Phi(t)} < De^{1+\Phi(T)} < \frac{b}{c} \quad \forall\, t \in [0,T],$$

thus giving the first inequality in (6.30). □

Let us give a version of integral Gronwall's inequality implied, for example, by Theorem 1.3 and Remark 1.5 in [37].

Lemma 6.14. *Let u, b, and a be measurable functions in $J = [\alpha, \beta]$, such that $bu, ba \in L_1(J)$. Suppose that $b(t)$ is nonnegative a.e. on J. Suppose*

$$u(t) \le a(t) + \int_\alpha^t b(s)u(s)ds, \quad \text{for a.e. } t \in J.$$

Then,

$$u(t) \le a(t) + \int_\alpha^t a(s)b(s)\exp\left(\int_s^t b(\tau)d\tau\right)ds, \quad \text{for a.e. } t \in J.$$

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Sergey E. Mikhailov: conceptualization, investigation, writing – original draft, methodology, validation, writing – review and editing, formal analysis.

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Conflict of Interest

The author declares no conflicts of interest.

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